Transcendental singularities for a meromorphic function with logarithmic derivative of finite lower order

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July 24, 2018

Abstract

It is shown that two key results on transcendental singularities for meromorphic functions of finite lower order have refinements which hold under the weaker hypothesis that the logarithmic derivative has finite lower order.
MSC 2000: 30D35. Keywords: meromorphic function, direct and indirect transcendental singularities, logarithmic derivative.

1 Introduction and results

Suppose that f is a transcendental meromorphic function on $\mathbb{C}$ such that, as z tends to infinity along a path γ in the plane, $f(z)$ tends to some $\alpha \in \mathbb{C}$. Then, for each $t > 0$, an unbounded subpath of γ lies in a component $C(t)$ of the set $\{z \in \mathbb{C} : |f(z) - \alpha| < t\}$. Here $C(t) \subseteq C(s)$ if $0 < t < s$, and the intersection $\bigcap_{t>0} C(t)$ is empty [2]. The path γ then determines a transcendental singularity of the inverse function f^{-1} over the asymptotic value α and each $C(t)$ is called a neighbourhood of the singularity [2, 18]. Two transcendental singularities over α are distinct if they have disjoint neighbourhoods for some $t > 0$. Following [2, 18], a transcendental singularity of f^{-1} over $\alpha \in \mathbb{C}$ is said to be direct if $C(t)$, for some $t > 0$, contains finitely many points z with $f(z) = \alpha$, in which case there exists $t_1 > 0$ such that $C(t)$ contains no α-points of f for $0 < t < t_1$. A direct singularity over $\alpha \in \mathbb{C}$ is logarithmic if there exists $t > 0$ such that $\log t/(f(z) - \alpha)$ maps $C(t)$ conformally onto the right half plane. If, on the other hand, $C(t)$ contains infinitely many α-points of f, for every $t > 0$, then the singularity is called indirect: a well known example is given by $f(z) = z^{-1} \sin z$, with $\alpha = 0$ and γ the positive real axis $\mathbb{R}^+$. Transcendental singularities of f^{-1} over ∞ and their corresponding neighbourhoods may be defined and classified using $1/f$, and the asymptotic and critical values of f together comprise the singular values of f^{-1}.

If f has finite (lower) order of growth, as defined in terms of the Nevanlinna characteristic function $T(r, f)$ [8, 18], then the number of direct singularities is controlled by the celebrated Denjoy-Carleman-Ahlfors theorem [9, 18].

Theorem 1.1 (Denjoy-Carleman-Ahlfors theorem) *Let f be a transcendental meromorphic function in the plane of finite lower order μ. Then the number of direct transcendental singularities of f^{-1} is at most $\max\{1, 2\mu\}$.*

A key consequence of Theorem 1.1 is that a transcendental entire function of finite lower order μ has at most 2μ finite asymptotic values [9]. A result of Bergweiler and Eremenko [2] shows that the critical values of a meromorphic function of finite (lower) order have a decisive influence on indirect transcendental singularities.

Theorem 1.2 ([2]) *Let f be a transcendental meromorphic function in the plane of finite lower order.*
(a) If f^{-1} has an indirect transcendental singularity over $\alpha \in \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ then each neighbourhood of the singularity contains infinitely many zeros of f' which are not α-points of f; in particular, α is a limit point of critical values of f.
(b) If f has finitely many critical values then f^{-1} has finitely many transcendental singularities, and all transcendental singularities are logarithmic.

Theorem 1.2 was proved in [2] for f of finite order, and was extended to finite lower order, using essentially the same method, by Hinchliffe [11]. Part (b) follows from part (a) combined with Theorem 1.1 and a well known classification theorem from [18, p.287], which shows in particular that any transcendental singularity of the inverse function over an isolated singular value is logarithmic. Theorem 1.2 was employed in [2] to prove a long-standing conjecture of Hayman [7] concerning zeros of $ff'-1$, and has found many subsequent applications, including to zeros of derivatives [12]. The reader is referred to [3, 19] for further striking results on singularities of the inverse, both restricted to entire functions but independent of the order of growth.

The starting question of the present paper concerns the extent to which Theorems 1.1 and 1.2 hold under the weaker hypothesis that $f^{(k)}/f$ has finite lower order for some $k \in \mathbb{N} = \{1, 2, \ldots\}$. The obvious example $f(z) = \exp(\exp(z))$ shows that f'/f can have finite order despite f having infinite lower order; here f^{-1} has infinitely many direct (indeed logarithmic) singularities over 0 and ∞, and one over 1. Furthermore, if $k \in \mathbb{N}$ and A_k is a transcendental entire function then the lemma of the logarithmic derivative [8] shows that every non-trivial solution of

$$w^{(k)} - A_k(z)w = 0 \tag{1.1}$$

has infinite lower order, even if A_k has finite order. Clearly each of $\exp(\exp(z))$ and $\exp(z^{-1}\sin z)$ satisfies an equation of form (1.1) with coefficient of finite order. Note further that if f is a transcendental meromorphic function in the plane and f'/f has finite lower order then it is easy to prove by induction that so has $A_k = f^{(k)}/f$ for every $k \geq 1$, using the formula $A_{k+1} = A_k A_1 + A_k'$, whereas the example

$$f(z) = e^{-z/2}\sin(e^z), \quad \frac{f'(z)}{f(z)} = -\frac{1}{2} + e^z\cot(e^z), \quad \frac{f''(z)}{f(z)} = \frac{1}{4} - e^{2z},$$

shows that f''/f can have finite order despite f'/f having infinite lower order.

Theorem 1.3 *Let f be a transcendental meromorphic function in the plane such that f^{-1} has $n \geq 1$ distinct direct transcendental singularities over finite non-zero values. Let $k \in \mathbb{N}$ and let μ be the lower order of $A_k = f^{(k)}/f$. Then the following statements hold.*
(i) There exists a set $F_0 \subseteq [1, \infty)$ of finite logarithmic measure such that

$$\lim_{r \to +\infty, r \notin F_0} \frac{\log\left(\min\{|A_k(z)| : |z| = r\}\right)}{\log r} = -\infty. \tag{1.2}$$

(ii) If $n \geq 2$ then $n \leq 2\mu$.
(iii) If $n = 1$ and there exist $\kappa > 0$ and a path γ tending to infinity in the complement of the neighbourhood $C(\kappa)$ of the singularity, then $\mu \geq 1/2$.

Theorem 1.3 will be deduced from a version of the Wiman-Valiron theory for meromorphic functions with direct tracts developed in [4], and part (ii) is sharp, by Example I in Section 2. Furthermore, if g is a transcendental entire function of lower order less than $1/2$ then the inverse function of $f = 1 - 1/g$ has a direct singularity over 1; in this case A_k obviously has lower order less than $1/2$ but the $\cos \pi\lambda$ theorem [9, Ch. 6] implies that every neighbourhood of the singularity contains circles $|z| = r$ with r arbitrarily large, so that a path γ as in (iii) cannot exist.

Theorem 1.4 *Let f be a transcendental meromorphic function in the plane such that $f^{(k)}/f$ has finite lower order for some $k \in \mathbb{N}$. Assume that f^{-1} has an indirect transcendental singularity over $\alpha \in \widehat{\mathbb{C}}$. Then each neighbourhood of the singularity contains infinitely many zeros of $f'f^{(k)}$ which are not α-points of f.*

Theorem 1.4 will be proved using a modification of methods from [2, 11].

Corollary 1.1 *Let f be a transcendental meromorphic function in the plane, with finitely many critical values, such that f'/f has finite lower order. Then f^{-1} has finitely many transcendental singularities over finite non-zero values, and f has finitely many asymptotic values. Moreover, all transcendental singularities of f^{-1} are logarithmic.*

Corollary 1.1 follows from Theorems 1.3 and 1.4, coupled with [18, p.287].

Corollary 1.2 *Let f be a transcendental meromorphic function in the plane such that f''/f has lower order $\mu < \infty$ and f'/f and f''/f' have finitely many zeros. Then f''/f' is a rational function and f has finite order and finitely many poles.*

To prove Corollary 1.2 observe that all but finitely many zeros of $f'f''$ are zeros of f. Thus f^{-1} has no indirect singularities, by Theorem 1.4, and hence f has finitely many asymptotic values, in view of Theorem 1.3. Since f evidently has finitely many critical values, the result follows via [12, Theorem 2]. The condition $\mu < \infty$ holds if f'/f has finite lower order, and is not redundant, because of an example in [12]. $\square$

The last result of this paper is related to the following theorem from [14].

Theorem 1.5 ([14]) *Let M be a positive integer and let f be a transcendental meromorphic function in the plane with transcendental Schwarzian derivative*

$$S_f(z) = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2, \tag{1.3}$$

such that: (i) f has finitely many critical values and all multiple points of f have multiplicity at most M; (ii) the inverse function of f has finitely many transcendental singularities.

Then the following three conclusions hold: (a) f has infinitely many multiple points; (b) the inverse function of S_f does not have a direct transcendental singularity over ∞; (c) the value ∞ is not Borel exceptional for S_f.

Conclusion (a) is a result of Elfving [6] and Rolf Nevanlinna [17, 18], but was proved in [14] by a completely different method. The following example shows that under the hypotheses of Theorem 1.5 the inverse of the Schwarzian can have a direct transcendental singularity over a finite value: write

$$g(z) = \sinh z, \quad S_g(z) = 1 - \frac{3 \tanh^2 z}{2},$$

so that S_g^{-1} has two logarithmic singularities over $-1/2$. However, assumptions (i) and (ii) of Theorem 1.5 imply that f belongs to the Speiser class $\mathcal{S}$ [1, 2] consisting of all meromorphic functions in the plane for which the inverse function has finitely many singular values. For $f \in \mathcal{S}$, the following result excludes direct singularities of the inverse of S_f over 0.

Theorem 1.6 *Let f be a transcendental meromorphic function in the plane belonging to the Speiser class $\mathcal{S}$, with transcendental Schwarzian derivative S_f. Then the inverse function of S_f does not have a direct transcendental singularity over 0.*

The example $f(z) = \tan^2 \sqrt{z}$ from [5] shows that for $f \in \mathcal{S}$ it is possible for 0 to be an asymptotic value of S_f. Here direct computation shows that $f''(z)/f'(z)$ tends to 0 as $z \to \infty$ in the left half plane, and so does $S_f(z)$.

The author thanks the referees for helpful comments.

2 Examples illustrating Theorems 1.3 and 1.4

Example I. A function extremal for Theorem 1.3(ii), but not for Theorem 1.1, is given by

$$f(0) = 1, \quad \frac{f'(z)}{f(z)} = \frac{\pi z}{\sin \pi z}.$$

Here f is meromorphic in the plane, having at each non-zero integer n a zero or pole of multiplicity $|n|$, depending on the sign and parity of n. Hence $N(r, f)$ and $N(r, 1/f)$ have order 2. Because

$$0 < \alpha = \int_0^{+\infty} \frac{\pi y}{\sinh \pi y}\, dy = \int_0^{+\infty} \frac{\pi y}{\pi y + (\pi y)^3/6 + \ldots}\, dy < \int_0^1 1\, dy + \int_1^{\infty} \frac{6}{\pi^2 y^2}\, dy < \pi$$

and f'/f is even, f has distinct asymptotic values $e^{\pm i\alpha}$, approached as z tends to infinity along the imaginary axis. As f'/f has finite order and f has no finite non-zero critical values, both of these singularities of f^{-1} are direct by Theorem 1.4. □

Example II. Define g by

$$g(0) = 1, \quad \frac{g'(z)}{g(z)} = A_1(z) = \frac{1}{\pi \cos \sqrt{z}}.$$

The zeros of $\cos \sqrt{z}$ occur where $\sqrt{z} = b_n = (2n+1)\pi/2$, with $n \in \mathbb{Z}$, and the residue of A_1 at b_n^2 is $\pm(2n+1)$. Thus g is meromorphic in $\mathbb{C}$, with zeros and poles in $\mathbb{R}^+$ and no finite non-zero critical values. Integration along the negative real axis shows that g has a non-zero real asymptotic value α, and g^{-1} has a logarithmic singularity over α by Corollary 1.1. This gives

$\delta > 0$ and a simply connected component C of $\{z \in \mathbb{C} : |g(z) - \alpha| < \delta\}$ with $(-\infty, R) \subseteq C$ for some $R < 0$. Moreover, C is symmetric with respect to $\mathbb{R}$, since g is real meromorphic, so that $C \cap \mathbb{R}^+$ is bounded, and g is extremal for Theorem 1.3(iii). □

Example III. Let $F(z) = \exp(-z/2 - (1/4)\sin 2z)\cos z$, so that F''/F is entire of finite order. Then $F(z)$ tends to 0 along $\mathbb{R}^+$ and this singularity of F^{-1} is evidently indirect. □

Example IV. Define entire functions A_1 and v by

$$v(0) = 1, \quad \frac{v'(z)}{v(z)} = A_1(z) = \frac{1 - \cos z}{z^2} = \frac{1}{2} + \ldots .$$

Then there exists $\alpha \in \mathbb{R}^+$ such that $v(x) \to \exp(\pm\alpha)$ as $x \to \pm\infty$ on $\mathbb{R}$ and, since A_1 does not satisfy (1.2), Theorem 1.3 implies that v^{-1} has no direct singularities over finite non-zero values. Because all critical points of v are real, all but finitely many of them belong to neighbourhoods of the indirect singularities over $\exp(\pm\alpha)$, and so v^{-1} has no other indirect singularities, by Theorem 1.4. Thus applying [18, p.287] again, in conjunction with Iversen's theorem, shows that v^{-1} has logarithmic singularities over the omitted values 0 and ∞. □

Example V. Let $h(z) = \exp(\sin z - z)$, so that $A_1 = h'/h$ is entire of finite order but does not satisfy (1.2). Since $h(z)$ tends to 0 along $\mathbb{R}^+$, and to ∞ on the negative real axis, with $h'(2\pi n) = 0$ for all $n \in \mathbb{Z}$, these singularities of h^{-1} are direct but not logarithmic. □

3 Preliminaries

The following well known estimate may be found in Theorem 8.9 of [9].

Lemma 3.1 ([9]) *Let $D_1, \ldots, D_n$ be $n \geq 2$ pairwise disjoint plane domains. If $u_1, \ldots, u_n$ are non-constant subharmonic functions on $\mathbb{C}$ such that u_j vanishes outside D_j then*

$$\liminf_{r \to \infty} \frac{h(r)}{r^{n/2}} > 0, \quad h(r) = \max_{1 \leq j \leq n} B(r, u_j), \quad B(r, u_j) = \sup\{u_j(z) : |z| = r\}. \tag{3.1}$$

□

For $a \in \mathbb{C}$ and $R > 0$ denote by $D(a, R)$ the open disc of centre a and radius R, and by $S(a, R)$ its boundary circle.

Lemma 3.2 *To each $k \in \mathbb{N}$ corresponds $d_k \in (0, \infty)$ with the following property. Suppose that $0 < R < \infty$ and $w = h(z)$ maps the domain $U \subseteq \mathbb{C}$ conformally onto $D(a, R)$, with inverse function $F : D(a, R) \to U$. Then there exists an analytic function $V_k : D(a, R) \to \mathbb{C}$ with*

$$h^{(k)}(z) F'(w)^k = V_k(w), \quad |V_k(w)| \leq \frac{d_k}{(R - |w - a|)^{k-1}} \quad \textit{as} \quad |w - a| \to R- . \tag{3.2}$$

Proof. Assume that $a = 0$ and initially that $R = 1$. It is clear that (3.2) holds for $k = 1$, with $V_1(w) = 1$. If (3.2) holds for k then it follows that

$$h^{(k+1)}(z) F'(w)^{k+1} = V_k'(w) - k h^{(k)}(z) F'(w)^{k-1} F''(w) = V_k'(w) - k V_k(w) \frac{F''(w)}{F'(w)}.$$

Since $F''(w)/F'(w) = O(1-|w|)^{-1}$ as $|w| \to 1-$ by [10, p.5, (1.6)], applying Cauchy's estimate for derivatives to V_k proves the lemma by induction when $R = 1$. In the general case write $w = h(z) = RH(z) = Rv$ and $z = F(w) = G(v)$ so that, as $|w| \to R-$,

$$|h^{(k)}(z)F'(w)^k| = R^{1-k}|H^{(k)}(z)G'(v)^k| \leq \frac{d_k R^{1-k}}{(1-|v|)^{k-1}} = \frac{d_k}{(R-|w|)^{k-1}}.$$

□

Lemma 3.3 *Let $M \in \mathbb{N}$ and $s > 2^{24}$ and let $E_1, \ldots, E_N$ be $N \geq 24M$ pairwise disjoint domains in $\mathbb{C}$, and for $t > 0$ let $\phi_j(t)$ be the angular measure of $S(0,t) \cap E_j$. Then at least $N - 12M$ of the E_j satisfy*

$$\int_{[4s^{1/2}, s/4]} \frac{\pi\, dt}{t\phi_j(t)} > M \log s \quad \textit{and} \quad \int_{[4s, s^2/4]} \frac{\pi\, dt}{t\phi_j(t)} > M \log s. \tag{3.3}$$

Proof. This is a standard application as in [9, Ch. 8] or [2] of the Cauchy-Schwarz inequality, which gives

$$\frac{L^2}{t} \leq \frac{1}{t}\left(\sum_{j=1}^{L} \phi_j(t)\right)\left(\sum_{j=1}^{L} \frac{1}{\phi_j(t)}\right) \leq 2\sum_{j=1}^{L} \frac{\pi}{t\phi_j(t)} \tag{3.4}$$

for $M \leq L \leq N$ and $t > 0$. If $s > 2^{24}$ and either inequality of (3.3) fails for $L \geq 6M$ of the E_j, without loss of generality for $j = 1, \ldots, L$, then integrating (3.4) yields a contradiction via

$$2LM \log s < 6LM \log \frac{\sqrt{s}}{16} \leq L^2 \log \frac{\sqrt{s}}{16} \leq 2LM \log s.$$

□

Lemma 3.4 ([1]) *Let h be a transcendental meromorphic function in the plane belonging to the Speiser class $\mathcal{S}$. Then there exist positive constants C, R and M such that*

$$\left|\frac{zh'(z)}{h(z)}\right| \geq C \log^+ \left|\frac{h(z)}{M}\right| \quad \textit{for } |z| \geq R. \tag{3.5}$$

4 Proof of Theorem 1.3

Let f be a transcendental meromorphic function in the plane such that f^{-1} has $n \geq 1$ direct singularities over (not necessarily distinct) finite non-zero values $a_1, \ldots, a_n$. Let $k \in \mathbb{N}$; then $A_k = f^{(k)}/f$ does not vanish identically. There exist a small positive δ and non-empty components D_j of $\{z \in \mathbb{C} : |f(z) - a_j| < \delta\}$, for $j = 1, \ldots n$, such that $f(z) \neq a_j$ on D_j, so that D_j immediately qualifies as a direct tract for $g_j = \delta/(f - a_j)$ in the sense of [4, Section 2]. Here δ may be chosen so small that if $n \geq 2$ then these D_j are pairwise disjoint. For each j, define a non-constant subharmonic function u_j on $\mathbb{C}$ by

$$u_j(z) = \log |g_j(z)| = \log \left|\frac{\delta}{f(z) - a_j}\right| \quad (z \in D_j), \quad u_j(z) = 0 \quad (z \notin D_j).$$

Then [4, Theorem 2.1] implies that, with $B(r, u_j)$ as in (3.1),

$$\lim_{r\to+\infty} \frac{B(r,u_j)}{\log r} = +\infty, \quad \lim_{r\to+\infty} a(r,u_j) = +\infty, \quad a(r,u_j) = rB'(r,u_j). \tag{4.1}$$

Lemma 4.1 *There exists a set $F_0 \subseteq [1,\infty)$, of finite logarithmic measure, such that for each $s \in [1,\infty) \setminus F_0$ and each j there exists z_j with*

$$|z_j| = s, \quad A_k(z_j) = \frac{f^{(k)}(z_j)}{f(z_j)} = O\left(\exp(-B(s,u_j)/2)\right). \tag{4.2}$$

Proof. Fix τ with $1/2 < \tau < 1$ and apply the version of Wiman-Valiron theory developed in [4] for meromorphic functions with direct tracts. By [4, Theorem 2.2 and Lemma 6.10] there exists a set $F_0 \subseteq [1,\infty)$ of finite logarithmic measure such that every $s \in [1,\infty) \setminus F_0$ has the following two properties: first, $a(s,u_j)$ is large, by (4.1), but satisfies

$$a(s,u_j) \leq B(s,u_j)^2; \tag{4.3}$$

second, for each j there exists z_j with $|z_j| = s$ and $u(z_j) = B(s,u_j)$ such that

$$\frac{f(z)-a_j}{f(z_j)-a_j} \sim \left(\frac{z}{z_j}\right)^{-a(s,u_j)} \quad \text{for} \quad |z-z_j| < \frac{s}{a(s,u_j)^\tau}. \tag{4.4}$$

A standard application of Cauchy's estimate for derivatives in (4.4) now gives

$$\left(\frac{f'}{f-a_j}\right)^{(p)}(z) = O\left(\frac{a(s,u_j)}{s}\right)^{p+1} \quad \text{for} \quad p = 0,\ldots,k-1 \quad \text{and} \quad |z-z_j| < \frac{s}{2a(s,u_j)^\tau}.$$

It follows via [8, Lemma 3.5] that

$$\frac{f^{(k)}(z_j)}{f(z_j)} = \frac{f^{(k)}(z_j)}{f(z_j)-a_j} \cdot \frac{f(z_j)-a_j}{f(z_j)} = O\left(\frac{a(s,u_j)^k \exp(-B(s,u_j))}{s^k}\right),$$

which, by (4.3), yields (4.2) for large enough $s \notin F_0$. □

Combining (4.1) with (4.2) for $j = 1$ leads to (1.2). To prove the remaining assertions it may be assumed that A_k has finite lower order μ. Choose a positive sequence (r_m) tending to infinity such that

$$T(8r_m, A_k) < r_m^{\mu+o(1)}. \tag{4.5}$$

Let m be large and let $w_1,\ldots,w_{q_m}$ be the zeros and poles of A_k in $r_m/4 \leq |z| \leq 4r_m$, repeated according to multiplicity: then (4.5) and standard estimates yield

$$q_m \leq n(4r_m, A_k) + n(4r_m, 1/A_k) \leq \frac{2}{\log 2} T(8r_m, A_k) + O(1) \leq r_m^{\mu+o(1)}. \tag{4.6}$$

Let U_m be the union of the discs $D(w_j, r_m^{-\mu})$. Since the sum of the radii of the discs of U_m is $o(r_m)$ by (4.6), there exists a set $E_m \subseteq [r_m/2, 2r_m]$, of linear measure at least r_m, and so logarithmic measure $l_m \geq 1/2$, such that for $r \in E_m$ the circle $|z| = r$ does not meet U_m. A standard application of the Poisson-Jensen formula [8] on the disc $|\zeta| \leq 4r_m$ then yields

$$|\log |A_k(z)|| \leq r_m^{\mu+o(1)} \quad \text{for} \quad |z| \in E_m. \tag{4.7}$$

Since m is large and $l_m \geq 1/2$, there exists $s_m \in E_m \setminus F_0$.

Suppose now that $n = 1$ and there exist $\kappa > 0$ and a path γ tending to infinity in the complement of the neighbourhood $C(\kappa)$ of the singularity, or that $n \geq 2$. Then (3.1) holds, by [9, Theorem 6.4] when $n = 1$, and by Lemma 3.1 when $n \geq 2$. Combining (3.1) and (4.2), with $s = s_m \geq r_m/2$, yields points z_j with $|z_j| = s_m$ and, for at least one j,

$$A_k(z_j) = O\left(\exp(-B(s_m, u_j)/2)\right) = O\left(\exp\left(-s_m^{n/2-o(1)}\right)\right).$$

On combination with (4.7) this forces $2\mu \geq n$. □

5 Indirect singularities

Proposition 5.1 *Let f be a transcendental meromorphic function in the plane such that $f^{(k)}/f$ has finite lower order μ for some $k \in \mathbb{N}$. Assume that f^{-1} has an indirect transcendental singularity over $\alpha \in \mathbb{C} \setminus \{0\}$. Then for each $\delta > 0$ the neighbourhood $C(\delta)$ of the singularity contains infinitely many zeros of $f' f^{(k)}$.*

The proof of Proposition 5.1 will take up the whole of this section. The method is adapted from those in [2, 11], but some complications arise, in particular when $k \geq 2$. Assume throughout that f and α are as in the hypotheses but $C(\varepsilon)$, for some small $\varepsilon > 0$, contains finitely many zeros of $f' f^{(k)}$. It may be assumed that $\alpha = 1$, and that $C(\varepsilon)$ contains no zeros of $f' f^{(k)}$. Choose positive integers $N_1, N_2, \ldots, N_9$ with $5\mu + 12 < N_1$ and N_{j+1}/N_j large for each j.

Lemma 5.1 *For each $j \in \{1, \ldots, N_9\}$ there exist $z_j \in C(\varepsilon)$ and $a_j \in \mathbb{C}$ with $0 < r_j = |1 - a_j| < \varepsilon/2$, as well as a simply connected domain $D_j \subseteq C(\varepsilon)$, with the following properties. The a_j are pairwise distinct and the D_j pairwise disjoint. Furthermore, the function f maps D_j univalently onto $D(1, r_j)$, with $z_j \in D_j$ and $f(z_j) = 1$. Moreover, $0 \notin D_j$ but D_j contains a path σ_j tending to infinity, which is mapped by f onto the half-open line segment $[1, a_j)$, with $f(z) \to a_j$ as $z \to \infty$ on σ_j.*

This is proved exactly as in [2]. If $0 < T_j < \varepsilon/2$ and $z_j \in C(T_j)$ is such that $f(z_j) = 1$, let r_j be the supremum of $t > 0$ such that the branch of f^{-1} mapping 1 to z_j admits unrestricted analytic continuation in $D(1, t)$. Then $r_j < T_j$ because f is not univalent on $C(T_j)$, and there is a singularity a_j of f^{-1} with $|1 - a_j| = r_j$; moreover, a_j must be an asymptotic value of f. The z_j and T_j are then chosen inductively: for the details see [2] (or [13, Lemma 10.3]). □

Lemma 5.2 *Let the z_j, a_j, σ_j and D_j be as in Lemma 5.1. For $t > 0$, let $t\theta_j(t)$ be the length of the longest open arc of $S(0, t)$ which lies in D_j. Then f satisfies, as z tends to infinity on σ_j,*

$$\log \frac{r_j}{|f(z) - a_j|} \geq \int_{|z_j|}^{|z|} \frac{dt}{4t\theta_j(t)}. \tag{5.1}$$

Proof. Let $z = H(w)$ be the branch of f^{-1} mapping $D(1, r_j)$ onto D_j. For $z \in \sigma_j$ the distance from z to ∂D_j is at most $|z|\theta_j(|z|)$. Thus Koebe's quarter theorem [10, Ch. 1] implies that

$$|(w - a_j)H'(w)| \leq 4|z|\theta_j(|z|) \quad \text{for} \quad z = H(w),\, w \in [1, a_j).$$

Hence, for large $z \in \sigma_j$ and $w = f(z)$, writing $u = H(v)$ for $v \in [1, w]$ gives (5.1) via

$$\log \frac{r_j}{|f(z) - a_j|} = \int_{[1,w]} \frac{|dv|}{|a_j - v|} = \int_{H([1,w])} \frac{|du|}{|(a_j - v)H'(v)|} \geq \int_{H([1,w])} \frac{|du|}{4|u|\theta_j(|u|)}.$$

□

Since $N_1 > 5\mu$ there exists a positive sequence (s_n) tending to infinity such that

$$T(s_n^5, f^{(k)}/f) + T(s_n^5, f/f^{(k)}) \leq s_n^{N_1}. \tag{5.2}$$

Set

$$G(z) = z^N \frac{f^{(k)}(z)}{f(z)}, \quad N = N_5. \tag{5.3}$$

Applying [15, Lemma 4.1] to $1/G$ (with $\psi(t) = t$ in the notation of [15]) gives a small positive η such that G has no critical values w with $|w| = \eta$ and such that the length $L(r, \eta, G)$ of the level curves $|G(z)| = \eta$ lying in $D(0, r)$ satisfies

$$L(s_n^4, \eta, G) = O(s_n^6 T(e^8 s_n^4, G)^{1/2}) = O(s_n^{6+N_1/2}) \leq s_n^{N_1} \quad \text{as } n \to \infty, \tag{5.4}$$

using (5.2) and the fact that $N_1 > 12$. Assume henceforth that n is large.

Lemma 5.3 *At least N_8 of the domain D_j and paths σ_j, without loss of generality $D_1, \ldots, D_{N_8}$ and $\sigma_1, \ldots, \sigma_{N_8}$, are such that*

$$|f(z) - a_j| \leq s_n^{-N_7} \quad \textit{for } z \in \sigma_j \textit{ with } |z| \geq s_n/4. \tag{5.5}$$

Proof. By Lemma 3.3 it may be assumed that, for $j = 1, \ldots, N_8$,

$$\int_{[4s_n^{1/2}, s_n/4]} \frac{\pi\, dt}{t\theta_j(t)} > N_8 \log s_n,$$

which, on combination with Lemma 5.2, leads to (5.5). □

Lemma 5.4 *Let $w_1, \ldots, w_{q_n}$ be the zeros and poles of G in $s_n^{1/4} \leq |z| \leq s_n^4$, repeated according to multiplicity. Then*

$$q_n \leq n(s_n^4, 1/G) + n(s_n^4, G) = o\left(s_n^{N_1}\right) \tag{5.6}$$

and there exist t_n, T_n satisfying

$$s_n^{1/2} - 1 \leq t_n \leq s_n^{1/2}, \quad s_n^2 \leq T_n \leq s_n^2 + 1, \tag{5.7}$$

such that

$$\max\{|\log|G(z)|| : z \in S(0, t_n) \cup S(0, T_n)\} \leq s_n^{N_1+1}. \tag{5.8}$$

Proof. (5.6) follows from (5.2). Let U_n be the union of the discs $D(w_q, s_n^{-N_1-1})$: these discs have sum of radii at most s_n^{-1} and so since n is large there exist t_n, T_n satisfying (5.7) such that the circles $S(0,t_n), S(0,T_n)$ do not meet U_n. Hence the Poisson-Jensen formula gives (5.8). □

Lemma 5.5 *Define sets E, K_n and L_n by $E = \{z \in \mathbb{C} : |G(z)| < \eta\}$ and*

$$K_n = \{z \in \mathbb{C} : t_n < |z| < T_n\}, \quad L_n = \{z \in \mathbb{C} : s_n/4 < |z| < 4s_n\}.$$

Then the number of components E_q of $E \cap K_n$ which meet L_n is at most $s_n^{N_1}$.

Proof. If the closure F_q of E_q lies in K_n then E_q must contain a zero of G, whereas if $F_q \not\subseteq K_n$ then $\partial E_q \cap K_n$ has arc length at least $s_n/8$. Thus the lemma follows from (5.4) and (5.6). □

Lemma 5.6 *Let u lie on σ_j with $s_n/4 \leq |u| \leq 4s_n$. Then, with d_k as in Lemma 3.2, there exists v on σ_j such that:*

$$|u| \leq |v| \leq |u| + s_n^{-N_3}; \quad |f(v) - a_j| \leq |f(u) - a_j|; \quad |f^{(k)}(v)| \leq k^k d_k s_n^{kN_3} |f(u) - a_j|. \quad (5.9)$$

Proof. Starting at u, follow σ_j in the direction in which $|f(z) - a_j|$ decreases. Then σ_j describes an arc γ joining the circles $S(0,|u|)$ and $S(0,|u| + s_n^{-N_3})$, such that the first two inequalities of (5.9) hold for all $v \in \gamma$. Since f maps D_j univalently onto $D(1, r_j)$, the inverse function H of f maps a proper sub-segment I of the half-open line segment $J = [f(u), a_j)$ onto γ. Assume that the last inequality of (5.9) fails for all $v \in \gamma$. Then Lemma 3.2 yields, on I,

$$|H'(w)| \leq k^{-1} s_n^{-N_3} |f(u) - a_j|^{-1/k} (r_j - |w-1|)^{1/k-1}.$$

Since 1, $f(u)$ and a_j are collinear, a contradiction arises via

$$\begin{aligned} s_n^{-N_3} \leq \left| \int_I H'(w) dw \right| &\leq \int_I k^{-1} s_n^{-N_3} |f(u) - a_j|^{-1/k} (r_j - |w-1|)^{1/k-1} \, |dw| \\ &< \int_J k^{-1} s_n^{-N_3} |f(u) - a_j|^{-1/k} (r_j - |w-1|)^{1/k-1} \, |dw| \\ &= \int_{|f(u)-1|}^{r_j} k^{-1} s_n^{-N_3} |f(u) - a_j|^{-1/k} (r_j - t)^{1/k-1} \, dt \\ &= s_n^{-N_3} |f(u) - a_j|^{-1/k} (r_j - |f(u) - 1|)^{1/k} = s_n^{-N_3}. \end{aligned}$$

□

Lemma 5.7 *Let E_p be a component of $E \cap K_n$ which meets L_n, and suppose that there exists $j = j(p)$ such that E_p contains k points $\zeta_1, \ldots, \zeta_k \in D_j$ each with $|f(\zeta_q) - a_j| \leq s_n^{-N_7}$. Assume further that $|\zeta_q - \zeta_{q'}| \geq s_n^{-N_3}$ for $q \neq q'$. Then $|f(z) - a_j| \leq s_n^{-N_2}$ for all $z \in E_p$, and $E_p \subseteq C(\varepsilon)$.*

Proof. Let $M_0 = \sup\{|f(z)| : z \in E_p\}$; then $M_0 < +\infty$ since poles of f in $\mathbb{C} \setminus \{0\}$ are poles of G, by (5.3), and $|G(z)| \leq \eta$ on the closure of E_p. Choose $u_0 \in E_p$ with $|f(u_0)| \geq M_0/2$. There exists a polynomial P, of degree at most $k-1$, such that

$$f(z) = P(z) + \int_{u_0}^{z} \frac{(z-t)^{k-1}}{(k-1)!} f^{(k)}(t) \, dt \quad \text{on } E_p.$$

The length of the boundary of E_p is at most $2s_n^{N_1}$ by (5.4). Hence each $z \in E_p$ can be joined to u_0 by a path in the closure of E_p, of length at most $4s_n^{N_1}$, and so

$$|f(z) - P(z)| \leq M_0 \eta t_n^{-N_5} (2T_n)^{k-1} 4 s_n^{N_1} \leq M_0 s_n^{-N_4}, \tag{5.10}$$

by (5.3) and (5.7). In particular this gives $|P(\zeta_q) - a_j| \leq (1 + M_0) s_n^{-N_4}$ for each q. For z in E_p, Lagrange's interpolation formula leads to

$$\begin{aligned} |P(z) - a_j| &= \left| \sum_{q=1}^{k} (P(\zeta_q) - a_j) \prod_{\nu \neq q} \frac{z - \zeta_\nu}{\zeta_q - \zeta_\nu} \right| \\ &\leq k(1 + M_0) s_n^{-N_4} (2T_n)^{k-1} s_n^{(k-1)N_3} \leq (1 + M_0) s_n^{-N_3}. \end{aligned} \tag{5.11}$$

Setting $z = u_0$ in (5.11) then delivers $M_0 \leq 2|P(u_0)| \leq 2|a_j| + o(1 + M_0)$ and so $M_0 \leq 5$. Now combining (5.10) with (5.11) yields $|f(z) - a_j| \leq s_n^{-N_2}$ and hence $|f(z) - 1| < \varepsilon$ on E_p. Since E_p meets $D_j \subseteq C(\varepsilon)$, this gives $E_p \subseteq C(\varepsilon)$. □

For each $j \in \{1, \ldots, N_8\}$ choose $\lambda = s_n^{N_2}$ points $u_{j,1}, \ldots, u_{j,\lambda}$ on σ_j, each with $s_n/2 \leq |u_{j,\kappa}| \leq s_n$ and such that $|u_{j,\kappa+1}| \geq |u_{j,\kappa}| + 2s_n^{-N_3}$. Applying Lemma 5.6 with $u = u_{j,\kappa}$ gives points $v_{j,\kappa} \in \sigma_j$ with $s_n/2 \leq |u_{j,\kappa}| \leq |v_{j,\kappa}| \leq |u_{j,\kappa}| + s_n^{-N_3} \leq 2s_n$ and, using (5.3), (5.5) and (5.9),

$$|f(v_{j,\kappa}) - a_j| \leq s_n^{-N_7}, \quad |G(v_{j,\kappa})| \leq 2|v_{j,\kappa}|^{N_5} |f^{(k)}(v_{j,\kappa})| \leq s_n^{-N_6} < \eta. \tag{5.12}$$

These points $v_{j,\kappa}$ satisfy $|v_{j,\kappa+1}| \geq |v_{j,\kappa}| + s_n^{-N_3}$, and each lies in a component of $E \cap K_n$ which meets L_n. Since there are $s_n^{N_2}$ of these $v_{j,\kappa}$ for each j, but at most $s_n^{N_1}$ available components E_p by Lemma 5.5, it must be the case that for each j there are at least k points $v_{j,\kappa}$ lying in the same component E_p. Lemma 5.7 then implies that $E_p \subseteq C(\varepsilon)$ and $f(z) = a_j + o(1)$ on E_p.

Thus for $j = 1, \ldots, N_8$ the following exist: a component $C_j = E_{p_j} \subseteq C(\varepsilon)$ of $E \cap K_n$ which meets L_n and on which $f(z) = a_j + o(1)$; a point $v_j \in C_j$ such that, by (5.12),

$$s_n/2 \leq |v_j| \leq 2s_n, \quad |G(v_j)| \leq s_n^{-N_6}. \tag{5.13}$$

Since $C_j \subseteq C(\varepsilon)$, the function $\log |1/G(z)|$ is subharmonic on C_j. Moreover, because $j' \neq j$ gives $f(z) \to a_{j'} \neq a_j$ as $z \to \infty$ on $\sigma_{j'}$, the C_j are pairwise disjoint and none of them contains a circle $S(0, t)$ with $t \in [t_n, T_n]$. For $t > 0$ let $\phi_j(t)$ be the angular measure of $C_j \cap S(0, t)$. Then (5.7) and [20, p.116] give a harmonic measure estimate

$$\omega(v_j, C_j, S(0, T_n) \cup S(0, t_n)) \leq c_1 \exp\left(-\pi \int_{2|v_j|}^{T_n/2} \frac{dt}{t\phi_j(t)}\right) + c_1 \exp\left(-\pi \int_{2t_n}^{|v_j|/2} \frac{dt}{t\phi_j(t)}\right),$$

for $j = 1, \ldots, N_8$, in which c_1 is a positive absolute constant. By Lemma 3.3 and (5.7), there exists at least one j for which $\omega(v_j, C_j, S(0, T_n) \cup S(0, t_n)) \leq 2c_1 s_n^{-N_7}$. For this choice of j the two constants theorem [18] delivers, using (5.8), (5.13) and the fact that $|G(z)| = \eta$ on $\partial C_j \cap K_n$,

$$N_6 \log s_n \leq \log \frac{1}{|G(v_j)|} \leq \log \frac{1}{\eta} + 2c_1 s_n^{-N_7 + N_1 + 1},$$

a contradiction since n is large. □

6 Proof of Theorem 1.4

This is almost identical to the corresponding proof in [2], but with Theorem 1.3 standing in for the Denjoy-Carleman-Ahlfors theorem. Suppose that f, k and α are as in the hypotheses but there exists $\varepsilon > 0$ such that in the neighbourhood $C(\varepsilon)$ of the singularity the function $f' f^{(k)}$ has finitely many zeros which are not α-points of f: it may be assumed that there are no such zeros. On the other hand, because the singularity is indirect, f must have infinitely many α-points in $C(\varepsilon)$. Since $f^{(k)}/f$ has finite lower order, f^{-1} cannot have infinitely many direct transcendental singularities over finite non-zero values, by Theorem 1.3. Set $A(\varepsilon) = \{w \in \mathbb{C} : 0 < |w - \alpha| < \varepsilon\}$ if $\alpha \in \mathbb{C}$, with $A(\varepsilon) = \{w \in \mathbb{C} : |w| > 1/\varepsilon\}$ if $\alpha = \infty$. In either case it may be assumed that ε is so small that $A(\varepsilon) \subseteq \mathbb{C} \setminus \{0\}$ and there is no w in $A(\varepsilon)$ such that f^{-1} has a direct transcendental singularity over w.

Take $z_0 \in C(\varepsilon)$, with $f(z_0) = w_0 \neq \alpha$, and let g be that branch of f^{-1} mapping w_0 to z_0. If g admits unrestricted analytic continuation in $A(\varepsilon)$ then, exactly as in [2], the classification theorem from [18, p.287] shows that z_0 lies in a component C_0 of the set $\{z \in \mathbb{C} : f(z) \in A(\varepsilon) \cup \{\alpha\}\}$ which contains at most one point z with $f(z) = \alpha$, so that $C(\varepsilon) \not\subseteq C_0$. But any $z_1 \in C(\varepsilon)$ can be joined to z_0 by a path λ on which $f(z) \in A(\varepsilon) \cup \{\alpha\}$, which gives $\lambda \subseteq C_0$ and hence $C(\varepsilon) \subseteq C_0$, a contradiction.

Hence there exists a path $\gamma : [0, 1] \to A(\varepsilon)$, starting at w_0, such that analytic continuation of g along γ is not possible. This gives rise to $S \in [0, 1]$ such that, as $t \to S-$, the image $z = g(\gamma(t))$ either tends to infinity or to a zero $z_2 \in C(\varepsilon)$ of f' with $f(z_2) = \gamma(S) \in A(\varepsilon)$, the latter impossible by assumption. It follows that setting $z = \sigma(t) = g(\gamma(t))$, for $0 \leq t < S$, defines a path σ tending to infinity in $C(\varepsilon)$, on which $f(z) \to w_1 \in A(\varepsilon)$ as $z \to \infty$. But then there exists $\delta > 0$ such that an unbounded subpath of σ lies in a component $C' \subseteq C(\varepsilon)$ of the set $\{z \in \mathbb{C} : |f(z) - w_1| < \delta\}$, with δ so small that $f' f^{(k)}$ has no zeros on C'. Further, the singularity over w_1 must be indirect, since direct singularities over values in $A(\varepsilon)$ have been excluded, and this contradicts Proposition 5.1.

□

7 A result needed for Theorem 1.6

Theorem 7.1 ([16], Theorem 1) *Let u be a subharmonic function in the plane such that $B(r) = \sup\{u(z) : |z| = r\}$ satisfies $\lim_{r\to\infty}(\log r)^{-1}B(r) = +\infty$. Then there exist $\delta_0 > 0$ and a simple path $\gamma : [0, \infty) \to \mathbb{C}$ with $\gamma(t) \to \infty$ as $t \to +\infty$ and the following properties:*

$$(i) \quad \lim_{z\to\infty, z\in\gamma} \frac{u(z)}{\log |z|} = +\infty; \quad (ii) \quad \text{if } \lambda > 0 \text{ then } \int_\gamma \exp(-\lambda u(z))\, |dz| < \infty; \qquad (7.1)$$

(iii) if $z = \gamma(t)$ then $u(\gamma(s)) \geq \delta_0 u(z)$ for all $s \geq t$.

Conclusion (iii) and the fact that γ may be chosen to be simple are not stated in [16, Theorem 1], but both are implicit in the proof. Here $\gamma = \gamma_1 \cup \gamma_2 \cup \ldots$ is constructed in [16, Section 3] so that, for some fixed $\delta_1 \in (0, 1)$, each $\gamma_k : [k - 1, k] \to \mathbb{C}$ is a simple path from $a_k \in D_k$ to $a_{k+1} \in \partial D_k$, where D_k is the component of $\{z \in \mathbb{C} : u(z) < (1 - \delta_1)^{-1}u(a_k)\}$ containing a_1. By [16, (3.2) and (3.3)], the γ_k are such that $0 < \delta_1 u(a_k) \leq u(z) < (1 - \delta_1)^{-1}u(a_k)$

on $\lambda_k = \gamma_k \setminus \{a_{k+1}\}$ and $u(a_{k+1}) \geq (1-\delta_1)^{-1} u(a_k) > u(a_k)$. Hence if $z = \gamma(t) \in \lambda_k$ then $u(\gamma(s)) \geq \delta_1 u(a_k) \geq \delta_1(1-\delta_1) u(\gamma(t))$ for all $s \geq t$. If the whole path γ is not simple, take the least $k \geq 2$ such that $\Gamma_k = \gamma_1 \cup \ldots \cup \gamma_k$ is not simple. Then there exists a maximal $t \in [k-1, k]$ such that $u_k = \gamma_k(t)$ lies in the compact set Γ_{k-1}, and $t < k$ since $\gamma_k(k) = a_{k+1} \in \partial D_k$. Replacing Γ_k by the part of Γ_{k-1} from a_1 to u_k, followed by the part of γ_k from u_k to a_{k+1}, does not affect conclusions (i), (ii) and (iii), and the argument may then be repeated.

□

Theorem 7.1 leads to the following result.

Proposition 7.1 *Let $N \in \mathbb{N}$ and let A be a transcendental meromorphic function in the plane such that the inverse function of A has a direct transcendental singularity over 0. Then there exist a path γ tending to infinity in $\mathbb{C}$ and linearly independent solutions U, V of*

$$w'' + A(z)w = 0 \tag{7.2}$$

on a simply connected domain containing γ, such that U and V satisfy, as $z \to \infty$ on γ,

$$U(z) = z + \frac{O(1)}{z^N}, \quad U'(z) = 1 + \frac{O(1)}{z^N}, \quad V(z) = 1 + \frac{O(1)}{z^N}, \quad V'(z) = \frac{O(1)}{z^N}. \tag{7.3}$$

To prove Proposition 7.1, observe first that, as in the proof of Theorem 1.3, there exist a small positive δ and a non-empty component D of $\{z \in \mathbb{C} : |A(z)| < \delta\}$ such that $A(z) \neq 0$ on D, as well as a non-constant subharmonic function u on $\mathbb{C}$ given by

$$u(z) = \log \left| \frac{\delta}{A(z)} \right| \quad (z \in D), \quad u(z) = 0 \quad (z \notin D).$$

Then u satisfies the hypotheses of Theorem 7.1, by [4, Theorem 2.1], and so there exists a path $\gamma : [0, \infty) \to D$ as in conclusions (i), (ii) and (iii). In particular, (iii) implies that

$$\text{if } z = \gamma(t) \text{ then } |A(\gamma(s))| \leq \delta^{1-\delta_0} |A(z)|^{\delta_0} \text{ for all } s \geq t. \tag{7.4}$$

Choose a simply connected domain Ω on which A has no poles, such that $\gamma \subseteq \Omega$. By (7.1) it may be assumed that $|A(t)|^{-1/4} \geq |t|^2 \geq 4$ on γ, and that

$$\int_\gamma |t|^2 |A(t)| \, |dt| \leq \int_\gamma |t|^2 |A(t)|^{1/2} \, |dt| \leq \int_\gamma |A(t)|^{1/4} \, |dt| < \frac{1}{4}. \tag{7.5}$$

Lemma 7.1 *Let v be a solution of (7.2) on Ω. Then $v(z) = O(|z|)$ as $z \to \infty$ on γ.*

Proof. This is a standard argument along the lines of Gronwall's lemma. Let y_0 be the starting point of γ. Differentiating twice shows that there exist constants a_1, b_1 such that, on Ω,

$$v(z) = a_1 z + b_1 - \int_{y_0}^{z} (z-t) A(t) v(t) \, dt.$$

If $\phi(z) = v(z)/z$ is unbounded on γ there exist $\zeta_n \to \infty$ on γ such that $\phi(\zeta_n) \to \infty$ and $|\phi(t)| \leq |\phi(\zeta_n)|$ on the part of γ joining y_0 to ζ_n. If n is large then (7.5) delivers a contradiction via

$$|\phi(\zeta_n)| \leq |a_1| + |b_1| + |\phi(\zeta_n)| \int_{y_0}^{z} (1 + |t|) |tA(t)| \, |dt| \leq |a_1| + |b_1| + \frac{|\phi(\zeta_n)|}{2}.$$

□

Lemma 7.2 *(a) Let $N \in \mathbb{N}$. Then on γ every solution v_j of (7.2) has*

$$v_j(z) = \alpha_j z + \beta_j + \int_z^\infty (z-t)A(t)v_j(t)\,dt, \quad \alpha_j, \beta_j \in \mathbb{C}, \tag{7.6}$$

the integration being from z to infinity along γ. Moreover, v_j satisfies, as $z \to \infty$ on γ,

$$v_j(z) - \alpha_j z - \beta_j = \frac{O(1)}{z^N}, \quad v_j'(z) - \alpha_j = \frac{O(1)}{z^N}. \tag{7.7}$$

(b) If v_1, v_2 are linearly independent solutions of (7.2) on Ω then $|\alpha_1| + |\alpha_2| > 0$ in (7.6), and if $\alpha_2 = 0$ then $\beta_2 \neq 0$.

Proof. First, (7.6) follows from (7.5) and Lemma 7.1. Next, (7.1), (7.4), (7.5), (7.6) and Lemma 7.1 imply that, as $z \to \infty$ on γ,

$$\begin{aligned} |v_j(z) - \alpha_j z - \beta_j| &\leq |z| \int_z^\infty (1+|t|)|A(t)|O(|t|)\,|dt| \\ &\leq |z|\, \delta^{(1-\delta_0)/2} |A(z)|^{\delta_0/2} \int_z^\infty (1+|t|)|A(t)|^{1/2} O(|t|)\,|dt| = \frac{O(1)}{z^N}, \\ |v_j'(z) - \alpha_j| &= \left| \int_z^\infty A(t) v_j(t)\,dt \right| \\ &\leq \delta^{(1-\delta_0)/2} |A(z)|^{\delta_0/2} \int_z^\infty |A(t)|^{1/2} O(|t|)\,|dt| = \frac{O(1)}{z^N}. \end{aligned}$$

Finally, suppose that v_1, v_2 are linearly independent solutions of (7.2) on Ω but the conclusion of (b) fails. Then $v_1(z)v_2'(z) - v_1'(z)v_2(z) \to 0$ as $z \to \infty$ on γ, by (7.7), contradicting the fact that $W(v_1, v_2)$ is a non-zero constant by Abel's identity. □

Now fix linearly independent solutions v_1, v_2 of (7.2) on Ω. Then α_1, α_2 cannot both vanish in (7.6). On the other hand, it is possible to ensure that one of α_1, α_2 is 0, by otherwise considering $\alpha_2 v_1 - \alpha_1 v_2$. Hence it may be assumed that $\alpha_1 = 1$, while $\alpha_2 = 0$ and $\beta_2 = 1$. Now write $U = v_1$ and $V = v_2$, so that Lemma 7.2 gives (7.3). □

8 Proof of Theorem 1.6

Assume that f and S_f are as in the hypotheses but that the inverse function of S_f has a direct transcendental singularity over 0. Then evidently so has that of $A = S_f/2$, and it is well known that (1.3) implies that f is locally the quotient of linearly independent solutions of (7.2). Now Proposition 7.1 gives linearly independent solutions U, V of (7.2) satisfying (7.3) on a path γ tending to infinity. Moreover, $h = U/V$ has the form $h = T \circ f$, for some Möbius transformation T, and so $h \in \mathcal{S}$, whereas $h(z) \sim z$ and $zh'(z)/h(z) = O(1)$ on γ, contradicting (3.5). □

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