Non-real zeros of higher derivatives of real entire functions of infinite order

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June 8, 2005

Abstract

Let f be a real meromorphic function of infinite order in the plane such that f has finitely many poles. Then for each $k \geq 3$, at least one of f and $f^{(k)}$ has infinitely many non-real zeros. Together with a result of Edwards and Hellerstein this establishes the analogue for higher derivatives of a conjecture going back to Wiman around 1911.
MSC 2000: 30D20, 30D35.

1 Introduction

The starting point of this paper is the following theorem, in which the term real entire function denotes an entire function mapping $\mathbb{R}$ into $\mathbb{R}$.

Theorem A. *Let f be a real entire function such that f and f'' have only real zeros. Then f belongs to the Laguerre-Pólya class LP.*

Here the class LP [4, 16, 17, 20, 23] consists of those entire functions g such that g is a locally uniform limit of real polynomials with real zeros, from which it follows that g and all its derivatives have only real zeros.

Following partial results in [16, 17, 20] and elsewhere, Theorem A was proved by Sheil-Small [23] when f has finite order, and for infinite order in [4], and confirmed a conjecture going back to Wiman around 1911 [1, 2, 20].

The present paper is concerned with the analogous problem in which the second derivative f'' is replaced by a higher derivative $f^{(k)}, k \geq 3$. The following theorem will be proved: here a meromorphic function is called real if it maps $\mathbb{R}$ into $\mathbb{R} \cup \{\infty\}$.

Theorem 1.1 *Let $k \geq 3$ be an integer, and let f be a real meromorphic function of infinite order in the plane such that f has finitely many poles. Then at least one of f and $f^{(k)}$ has infinitely many non-real zeros.*

*Research partly carried out during a visit to the Christian-Albrechts-Universität Kiel, supported by a grant from the Alexander von Humboldt Stiftung. The author thanks the Mathematisches Seminar and in particular Walter Bergweiler for their hospitality.

Theorem 1.1 is also true for $k = 2$ [4, Theorems 1.2 and 1.3]. On combination with a result of Edwards and Hellerstein [7, Corollary 5.2], Theorem 1.1 establishes the following analogue of Theorem A.

Theorem 1.2 *Let f be a real entire function such that f and $f^{(k)}$ have only real zeros, for some $k \geq 3$. Then $f \in LP$.*

2 Preliminaries

Definitions 2.1 *For $a \in \mathbb{C}$ and $r > 0$ set*

$$D(a,r) = \{z \in \mathbb{C} : |z-a| < r\}, \quad S(a,r) = \{z \in \mathbb{C} : |z-a| = r\},$$

and

$$A(r,\infty) = \{z \in \mathbb{C} \cup \{\infty\} : r < |z| \leq \infty\}.$$

Further, set

$$H^+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}, \quad D^+(0,r) = D(0,r) \cap H^+, \quad A^+(r,\infty) = A(r,\infty) \cap H^+. \quad (1)$$

The following lemma is standard.

Lemma 2.1 ([26]) *Let u be a non-constant continuous subharmonic function in the plane. For $r > 0$ let $\theta^*(r)$ be the angular measure of that subset of $S(0,r)$ on which $u(z) > 0$, except that $\theta^*(r) = \infty$ if $u(z) > 0$ on the whole circle $S(0,r)$. Then for $r > 0$,*

$$B(r,u) = \max\{u(z) : |z| = r\} \leq \frac{3}{2\pi} \int_0^{2\pi} \max\{u(2re^{it}), 0\} dt \quad (2)$$

and, if $r \leq R/4$ and r is sufficiently large,

$$B(r,u) \leq 9\sqrt{2} B(R,u) \exp\left(-\pi \int_{2r}^{R/2} \frac{ds}{s\theta^*(s)}\right). \quad (3)$$

The inequality (2) follows from Poisson's formula, and (3) from a standard application of a well known estimate for harmonic measure [26, pp.116-7]. □

It will be convenient to use the following standard estimate for harmonic measure.

Lemma 2.2 ([27]) *Let $z_0 \neq 0$ lie in the simply connected domain D, and let r be positive with $r \neq |z_0|$. For $s > 0$ let $\theta(s)$ denote the angular measure of $D \cap S(0,s)$, and let D_r be the component of $D \setminus S(0,r)$ which contains z_0. Then the harmonic measure of $S(0,r)$ with respect to the domain D_r, evaluated at z_0, satisfies*

$$\omega(z_0, S(0,r), D_r) \leq \exp\left(-\frac{1}{\pi}\left|\int_{|z_0|}^{r} \frac{ds}{s\tan(\theta(s)/4)}\right|\right).$$

The next lemma requires the characteristic function in a half-plane as developed by Tsuji [25] and Levin and Ostrovskii [20] (see also [10]). Let g be meromorphic in a domain containing the closed upper half-plane $\overline{H} = \{z \in \mathbb{C} : \mathrm{Im}\, z \geq 0\}$. For $t \geq 1$ let $\mathfrak{n}(t, g)$ be the number of poles of g, counting multiplicity, in $\{z : |z - it/2| \leq t/2, |z| \geq 1\}$, and for $r \geq 1$ set

$$\mathfrak{N}(r, g) = \int_1^r \frac{\mathfrak{n}(t, g)}{t^2} dt.$$

The Tsuji characteristic $\mathfrak{T}(r, g)$ is given by

$$\mathfrak{T}(r, g) = \mathfrak{m}(r, g) + \mathfrak{N}(r, g), \quad \mathfrak{m}(r, g) = \frac{1}{2\pi} \int_{\sin^{-1}(1/r)}^{\pi - \sin^{-1}(1/r)} \frac{\log^+ |g(r \sin\theta e^{i\theta})|}{r \sin^2\theta} d\theta. \tag{4}$$

Lemma 2.3 ([20]) *Let g be meromorphic in $\overline{H}$ such that*

$$\mathfrak{m}(r, g) = O(\log r) \quad \textit{as} \quad r \to \infty, \tag{5}$$

where $\mathfrak{m}(r, g)$ is given by (4). Then

$$\int_R^\infty \frac{m_{0\pi}(r, g)}{r^3} dr = O(R^{-1} \log R) \quad \textit{as} \quad R \to \infty, \tag{6}$$

in which

$$m_{0\pi}(r, g) = \frac{1}{2\pi} \int_0^\pi \log^+ |g(re^{i\theta})| d\theta. \tag{7}$$

Proof. A result of Levin-Ostrovskii [20, p. 332] leads to

$$\int_R^\infty \frac{m_{0\pi}(r, g)}{r^3} dr \leq \int_R^\infty \frac{\mathfrak{m}(r, g)}{r^2} dr = O(R^{-1} \log R) \quad \text{as} \quad R \to \infty,$$

which gives (6). $\square$

The following theorem was proved for families of analytic functions by Schwick [22] and in the meromorphic case in [5], using in both cases but in different ways the rescaling method [28].

Theorem 2.1 ([5, 22]) *Let $k \geq 2$ and let $\mathcal{F}$ be a family of functions meromorphic on a plane domain D such that $ff^{(k)}$ has no zeros in D, for each $f \in \mathcal{F}$. Then the family $\{f'/f : f \in \mathcal{F}\}$ is normal on D.*

Lemma 2.4 *Let $k \geq 2$ and $\eta > 0$ and let g be analytic in $D(0, 2\eta)$ with $g(z)g^{(k)}(z) \neq 0$ there, and let $G = g'/g$. Then*

$$\log M(\eta, G) \leq c_0(1 + \log^+ |G(0)|),$$

in which $c_0 = c_0(\eta) > 0$ depends only on η.

Proof. Let

$$h(z) = g(2\eta z), \quad H(z) = \frac{h'(z)}{h(z)} = 2\eta G(2\eta z)$$

for $|z| < 1$, and use C_j to denote positive absolute constants. Since $h(z)h^{(k)}(z) \neq 0$, Theorem 2.1 implies that

$$\frac{|H'(z)|}{1+|H(z)|^2} \leq C_1 \quad \text{for} \quad |z| \leq \frac{3}{4}.$$

This in turn implies that the Ahlfors-Shimizu characteristic $T_0(r, H)$ satisfies

$$T_0\left(\frac{3}{4}, H\right) \leq C_2.$$

But [12, p.13] now leads to

$$\log M\left(\frac{1}{2}, H\right) \leq C_3 T\left(\frac{3}{4}, H\right) \leq C_3\left(T_0\left(\frac{3}{4}, H\right) + C_4 + \log^+ |H(0)|\right),$$

and this gives the result for G. □

Lemma 2.5 *Let $0 < \sigma < \pi/2$. Let $S \geq 1$ and let g be analytic in $S/64 < |z| < 64S$, $\operatorname{Im} z > 0$, with $g(z)g^{(k)}(z) \neq 0$ there. Set $G = g'/g$. Then*

$$M_1 = \max\{|G(z)| : S/32 \leq |z| \leq 32S, \sigma \leq \arg z \leq \pi - \sigma\}$$

and

$$M_2 = \min\{|G(z)| : S/32 \leq |z| \leq 32S, \sigma \leq \arg z \leq \pi - \sigma\}$$

satisfy

$$\log^+ M_1 \leq c_1(1 + \log S + \log^+ M_2),$$

in which $c_1 > 0$ depends only on σ.

Proof. When $S = 1$ Lemma 2.5 is proved by applying Lemma 2.4 repeatedly. Suppose now that $S > 1$, and set

$$g_S(z) = g(Sz), \quad G_S(z) = \frac{g_S'(z)}{g_S(z)} = SG(Sz).$$

Then

$$|G(Sz)| \leq |G_S(z)| \leq S|G(Sz)|$$

and so Lemma 2.5 is proved. □

Lemma 2.6 *Let $D = D^+(0,1)$ be as defined in (1) and let $w = g(z)$ be a conformal map of D onto the unit disc $D(0,1)$ sending $i/2$ to 0. Then there exists $c > 0$ such that*

$$|g(z) - g(z')| \geq c|z - z'|^2 \quad \textit{for} \quad z, z' \in \partial D. \tag{8}$$

The following elementary proof is included for completeness. The function g is the composition of the map $h(z) = z + 1/z$ with a Möbius transformation of the lower half-plane onto the unit disc. Assuming (8) false there exist sequences $(z_n), (Z_n)$ in ∂D with

$$g(z_n) - g(Z_n) = o(|z_n - Z_n|^2) \quad \text{as} \quad n \to \infty.$$

Without loss of generality (z_n) and (Z_n) converge to $z^* \in \partial D$, and z^* must be ± 1, since otherwise g^{-1} is analytic at $g(z^*)$. Assume that $z^* = 1$ and $|z_n - 1| \leq |Z_n - 1|$. Then

$$1 - \frac{1}{z_n Z_n} = o(|z_n - Z_n|) = o(|Z_n - 1|), \quad \frac{Z_n(z_n - 1)}{Z_n - 1} = -1 + o(1), \quad \arg \frac{z_n - 1}{Z_n - 1} = \pi + o(1),$$

which is impossible. □

The proof of Theorem 1.1 will require Fuchs' small arcs lemma [9] in the form given in [15, p.721]. The term $\log^+ |1/g(0)|$ arises in (9) since [15, p.721] assumes the condition $g(0) = 1$.

Lemma 2.7 ([15]) *Let $R > 0$ and let g be meromorphic in $|z| \leq R$, with $g(0) \neq 0, \infty$. Let η_1, η_2 be positive with $\eta_1 + \eta_2 < 1$. Then there exists a subset E_R of $[0, R(1 - \eta_1)]$, having measure greater than $R(1 - \eta_1 - \eta_2)$, with the following property. If $r \in E_R$ and F_r is a subinterval of $[0, 2\pi]$ of length m then*

$$\int_{F_r} \left| \frac{rg'(re^{i\theta})}{g(re^{i\theta})} \right| d\theta \leq 400 \eta_1^{-2} \eta_2^{-1} \left(T(R, g) + \log^+ \frac{1}{|g(0)|} \right) m \log \frac{2\pi e}{m}. \tag{9}$$

Lemma 2.8 ([6]) *Let $1 < r < R < \infty$ and let g be meromorphic in $|z| \leq R$. Let $I(r)$ be a subset of $[0, 2\pi]$ of Lebesgue measure $\mu(r)$. Then*

$$\frac{1}{2\pi} \int_{I(r)} \log^+ |g(re^{i\theta})| d\theta \leq \frac{11 R \mu(r)}{R - r} \left(1 + \log^+ \frac{1}{\mu(r)} \right) T(R, g).$$

Lemma 2.9 ([13]) *Let $S(r)$ be an unbounded positive non-decreasing function on $[r_0, \infty)$, continuous from the right, of order ρ. Let $A > 1, B > 1$. Then*

$$\overline{\mathrm{logdens}}\, G \leq \rho \left(\frac{\log A}{\log B} \right), \quad G = \{ r \geq r_0 : S(Ar) \geq BS(r) \}.$$

3 Proof of Theorem 1.1: first part

Let $k \geq 3$ be an integer, and let f be a real meromorphic function of infinite order such that f has finitely many poles and f and $f^{(k)}$ have finitely many non-real zeros. Assume without loss of generality that $f^{(m)}(0) \neq 0, \infty$ for all non-negative integers m.

Definitions 3.1 *For $m = 0, \ldots, k - 2$ set*

$$L_m = \frac{f^{(m+1)}}{f^{(m)}}, \quad L = L_{k-2}, \quad F(z) = z - \frac{1}{L(z)} = z - \frac{f^{(k-2)}(z)}{f^{(k-1)}(z)}. \tag{10}$$

The L_m are related by

$$L_{m+1} = L_m + \frac{L_m'}{L_m}. \tag{11}$$

Lemma 3.1 ([18]) *For $m = 0, \ldots, k - 2$ the Tsuji characteristic of L_m satisfies*

$$\mathfrak{m}(r, L_m) \leq \mathfrak{T}(r, L_m) = O(\log r) \quad \textit{as} \quad r \to \infty. \tag{12}$$

Proof. (12) is proved for $m=0$ in [18, Lemma 1] by coupling the method of [12, pp.67-77] with the Tsuji characteristic. This can also be done using a method of G. Frank (see, for example, [8, Theorem 3]), again with the Nevanlinna characteristic replaced by that of Tsuji. The result for $1 \leq m \leq k-2$ then follows from (11) and the analogue for the Tsuji characteristic of the lemma of the logarithmic derivative [4, 10, 20]. □

4 The Levin-Ostrovskii representation for L_m

For $m = 0, \ldots, k-2$ set

$$L_m = \phi_m \psi_m, \tag{13}$$

in which L_m is as in (10) and ψ_m is defined as follows. If $f^{(m)}$ has finitely many real zeros, set $\psi_m = 1$. Assuming next that $f^{(m)}$ has infinitely many real zeros a_p, the a_p are then simple poles of L_m satisfying, without loss of generality,

$$\ldots < a_{p-1} < a_p < a_{p+1} < \ldots .$$

For $|p| \geq p_0$, where p_0 is large, a_p and a_{p+1} are of the same sign, and there is a zero b_p of $f^{(m+1)}$, and hence of L_m, in the interval (a_p, a_{p+1}). Then the product

$$\psi_m(z) = \prod_{|p| \geq p_0} \frac{1 - z/b_p}{1 - z/a_p}$$

converges by the alternating series test, and satisfies

$$0 < \sum_{|p| \geq p_0} \arg \frac{1 - z/b_p}{1 - z/a_p} = \sum_{|p| \geq p_0} \arg \frac{b_p - z}{a_p - z} < \pi \quad \text{for} \quad z \in H^+.$$

Lemma 4.1 *For $m = 0, \ldots, k-2$ the functions ϕ_m and ψ_m in (13) are real meromorphic and satisfy the following:*
(i) ψ_m and ϕ_m have only simple poles, all of which are simple poles of L_m and zeros or poles of $f^{(m)}$;
(ii) ψ_m has only real zeros and poles, all of which are simple;
(iii) all but finitely many real zeros of $f^{(m)}$ are poles of ψ_m, and all non-real zeros of $f^{(m)}$ are poles of ϕ_m;
(iv) all but finitely many poles of ϕ_m are non-real zeros of $f^{(m)}$ and, in particular, ϕ_0 has finitely many poles;
(v) either $\psi_m(H^+) \subseteq H^+$, or $\psi_m \equiv 1$.
Further, for $m = 0, \ldots, k-2$,

$$n(r, \phi_m) \leq \sum_{0 \leq j < m} n(r, 1/\phi_j) + O(1) \quad \textit{as} \quad r \to \infty. \tag{14}$$

Proof. When $m = 0$ the inequality (14) follows from part (iv), the sum on the right-hand-side being interpreted as empty in this case. Now suppose that $1 \leq m \leq k-2$, and that z_0 is a pole of ϕ_m with $|z_0|$ large. Then z_0 is a simple pole of ϕ_m and a non-real zero of $f^{(m)}$, by part (iv). Let p be the least integer with $0 \leq p \leq m$ such that $f^{(p)}(z_0) = 0$. Then $p \geq 1$, since f has finitely many non-real zeros, and so $\phi_{p-1}(z_0) = 0$. This completes the proof of (14). □

5 Estimates for ψ_m

Condition (v) of Lemma 4.1 implies the Carathéodory inequality [19, Ch. I.6, Thm $8'$]

$$\frac{1}{5}|\psi_m(i)|\frac{\sin\theta}{r} < |\psi_m(re^{i\theta})| < 5|\psi_m(i)|\frac{r}{\sin\theta} \quad \text{for} \quad r \geq 1,\, \theta \in (0,\pi). \tag{15}$$

Since the image of H^+ under $\log\psi_m(z)$ contains no disc of radius greater than $\pi/2$, by part (v) of Lemma 4.1, applying Bloch's (or Landau's) theorem yields

$$\left|\frac{\psi_m'(re^{i\theta})}{\psi_m(re^{i\theta})}\right| \leq \frac{c}{r\sin\theta} \quad \text{for} \quad r \geq 1,\, \theta \in (0,\pi), \tag{16}$$

where c is a positive absolute constant. In particular, (15) and (16) imply that

$$m(r,\psi_m) + m(r,1/\psi_m) + m(r,\psi_m'/\psi_m) = O(\log r) \quad \text{as} \quad r \to \infty. \tag{17}$$

6 Estimates for $T(r,\phi_m)$

Define $m_{0\pi}(r,\phi_0)$ by (7). Since (12), (13) and (15) give

$$\mathfrak{m}(r,\phi_0) = O(\log r) \quad \text{as} \quad r \to \infty,$$

in which $\mathfrak{m}(r,\phi_0)$ is defined as in (4), Lemma 2.3 implies that

$$\int_R^\infty \frac{m_{0\pi}(r,\phi_0)}{r^3}dr = O(R^{-1}\log R) \quad \text{as} \quad R \to \infty. \tag{18}$$

But ϕ_0 is a real meromorphic function with finitely many poles, using part (iv) of Lemma 4.1, and so

$$T(r,\phi_0) = 2m_{0\pi}(r,\phi_0) + O(\log r) \quad \text{as} \quad r \to \infty.$$

Combining this relation with (18) yields

$$\int_R^\infty \frac{T(r,\phi_0)}{r^3}dr = O(R^{-1}\log R) \quad \text{as} \quad R \to \infty,$$

and hence, since $T(r,\phi_0)$ is non-decreasing,

$$T(r,\phi_0) = m(r,\phi_0) + O(\log r) = O(r\log r) \quad \text{as} \quad r \to \infty. \tag{19}$$

Let $\rho = \rho(\phi_0)$ be the order of growth of ϕ_0. Then (19) gives

$$\rho = \rho(\phi_0) \leq 1. \tag{20}$$

Lemma 6.1 *For $m = 0,\ldots,k-2$, as $r \to \infty$,*

$$T(r,\phi_0) - O(\log r) \leq m(r,\phi_m) \leq T(r,\phi_m) \leq 2^m T(r,\phi_0) + O(\log r) = O(r\log r). \tag{21}$$

Proof The estimate (21) is proved by induction on m, and is evidently true for $m = 0$, by (19). Assume that $0 \leq p \leq k - 3$ and that (21) holds for $0 \leq m \leq p$. The relations (11) and (13) yield

$$L_{m+1} = \phi_{m+1}\psi_{m+1} = L_m + \frac{\phi_m'}{\phi_m} + \frac{\psi_m'}{\psi_m} = \phi_m\psi_m + \frac{\phi_m'}{\phi_m} + \frac{\psi_m'}{\psi_m}. \tag{22}$$

Now repeated application of (17), (21) and (22) gives, as $r \to \infty$,

$$m(r, \phi_{p+1}) = m(r, \phi_p) + O(\log r) = m(r, \phi_0) + O(\log r) = T(r, \phi_0) + O(\log r). \tag{23}$$

Next, (14) and (21) lead to

$$N(r, \phi_{p+1}) \leq \sum_{m=0}^{p} N(r, 1/\phi_m) + O(\log r) \leq (2^{p+1} - 1)T(r, \phi_0) + O(\log r)$$

as $r \to \infty$, which on combination with (23) completes the induction. □

7 An upper bound for $T(r, f)$ in terms of $T(r, \phi_0)$

Lemma 7.1 *For all large r, and for $m = 0, \ldots, k - 2$,*

$$T(r, f^{(m)}) \leq 2T(2r, f) \leq \exp(20T(16r, \phi_0)). \tag{24}$$

Proof. The following argument from [4] is based on the Wiman-Valiron theory [14]. Since f has finitely many poles there exists a polynomial P_1 such that $f_1 = P_1 f$ is an entire function of infinite order. Let $f_1(z) = \sum_{q=0}^{\infty} \lambda_q z^q$ be the Maclaurin series of f_1. For $r > 0$ define

$$\mu(r) = \max\{|\lambda_q| r^q : q = 0, 1, 2, \ldots\}, \quad \nu(r) = \max\{q : |\lambda_q| r^q = \mu(r)\},$$

to be respectively the maximum term and central index of f_1. By [14, Theorems 10 and 12], there exists a set E_0 of finite logarithmic measure with the following property. Let r be large, not in E_0, and let z_0 be such that $|z_0| = r$ and $|f_1(z_0)| = M(r, f_1)$. Then

$$\frac{f_1'(z)}{f_1(z)} = \frac{\nu(r)}{z}(1 + o(1)) \quad \text{for} \quad z = z_0 e^{it},\ t \in [-\nu(r)^{-2/3}, \nu(r)^{-2/3}].$$

Since

$$\frac{f'(z)}{f(z)} = \frac{f_1'(z)}{f_1(z)} + \frac{O(1)}{z} \quad \text{as} \quad z \to \infty,$$

this leads at once to

$$\int_0^{2\pi} \left| \frac{f'(re^{it})}{f(re^{it})} \right|^{5/6} dt \geq \nu(r)^{1/6} r^{-5/6} \quad \text{as} \quad r \to \infty \quad \text{with} \quad r \notin E_0.$$

But (10), (13) and (15) give, for some positive absolute constant c,

$$\int_0^{2\pi} \left| \frac{f'(re^{it})}{f(re^{it})} \right|^{5/6} dt \leq cM(r, \phi_0)^{5/6} |\psi_0(i)|^{5/6} r^{5/6} \quad \text{as} \quad r \to \infty.$$

It follows that

$$\nu(r) \leq M(r, \phi_0)^5 r^{11} \quad \text{as} \quad r \to \infty \quad \text{with} \quad r \notin E_0.$$

In particular, ϕ_0 must be transcendental, since f_1 has infinite order. If s is large and $2s \notin E_0$, then

$$\log M(s, f_1) \leq \log \mu(2s) + \log 2 \leq \nu(2s) \log 2s + O(1) \leq M(2s, \phi_0)^6 \leq \exp(19T(4s, \phi_0)),$$

using the fact that ϕ_0 has finitely many poles. For r large choose $R \in [r, 2r]$ such that $4R \notin E_0$, so that

$$T(r, f^{(m)}) \leq 2T(2r, f) \leq 2 \log M(2R, f_1) + O(\log r) \leq \exp(20T(8R, \phi_0)),$$

which gives (24). $\square$

8 Pointwise estimates for logarithmic derivatives

Choose $a \in \mathbb{C}$ such that $\phi_0(0) \neq a$ and

$$m(r, 1/(\phi_0 - a)) = o(T(r, \phi_0)) \quad \text{as} \quad r \to \infty; \tag{25}$$

such values a always exist [21, p.281]. Set

$$n(r) = n(r, 1/(\phi_0 - a)), \quad N(r) = N(r, 1/(\phi_0 - a)). \tag{26}$$

The following estimates are consequences of (10), (21), (24) and results of Gundersen [11, Theorem 2 and Theorem 3].

Lemma 8.1 *There exists a set $E_1 \subseteq [1, \infty)$, of finite logarithmic measure, such that, for $m = 0, \ldots, k-2$,*

$$|L_m(z)| \leq T(2s, f^{(m)})^2 \leq \exp(40T(32s, \phi_0)) \quad \textit{for} \quad |z| = s \in [1, \infty) \setminus E_1, \tag{27}$$

and

$$\left| \frac{\phi_m'(z)}{\phi_m(z)} \right| + \left| \frac{\phi_0'(z)}{\phi_0(z) - a} \right| \leq s^{-1+\rho+o(1)} \quad \textit{for} \quad |z| = s \in [1, \infty) \setminus E_1, \tag{28}$$

where $\rho = \rho(\phi_0)$ is as in (20). Further, there exist

$$t_1 \in (3\pi/16, 5\pi/16), \quad t_2 \in (7\pi/16, 9\pi/16), \quad t_3 \in (11\pi/16, 13\pi/16), \tag{29}$$

and $R_0 > 0$ such that, for $s \geq R_0$, $m = 0, \ldots, k-2$, and $n = 1, 2, 3$,

$$|L_m(se^{it_n})| \leq T(2s, f^{(m)})^2 \leq \exp(40T(32s, \phi_0)) \quad \textit{and} \quad \left| \frac{\phi_m'(se^{it_n})}{\phi_m(se^{it_n})} \right| \leq s^{-1+\rho+o(1)}. \tag{30}$$

9 Application of Lemma 2.5

The estimates of Lemma 2.5, which followed from the normal families result Theorem 2.1, will now be used to show that the functions L_m defined in (13) are large in a substantial part of the upper half-plane H^+.

Lemma 9.1 *Let $\delta > 0$ and $C > 1$. Let r be large, with*

$$T(64r, \phi_0) \leq CT(2r, \phi_0). \tag{31}$$

Then for $m = 0, \ldots, k-2$, and for $s \in [r/4, 4r] \setminus E_1$,

$$\log |L_m(z)| \geq C_1 T(2r, \phi_0) \quad \textit{for} \quad |z| = s,\ \delta \leq \arg z \leq \pi - \delta. \tag{32}$$

Further, for $m = 0, \ldots, k-2$ and $n = 1, 2, 3$,

$$\log |L_m(se^{it_n}))| \geq C_1 T(2r, \phi_0) \quad \textit{for} \quad r/4 \leq s \leq 4r. \tag{33}$$

Here t_1, t_2, t_3 and the exceptional set E_1 are as in Lemma 8.1, and the positive constant C_1 depends only on δ and C.

Proof. Let S be a member of the set $[2r, 4r] \setminus E_1$, which is non-empty since r is large and E_1 has finite logarithmic measure. Then (31) gives

$$T(16S, \phi_0) \leq CT(S, \phi_0).$$

Since ϕ_0 is transcendental with finitely many poles Lemma 2.8 now shows that the set

$$I_S = \left\{ \theta \in [0, 2\pi] : \log |\phi_0(Se^{i\theta})| > \frac{1}{2} T(S, \phi_0) \right\}$$

has measure at least 8η, where $\eta > 0$ depends only on C.

Let $\sigma = \min\{\eta, \delta\}$. Then since ϕ_0 is real there exists z satisfying

$$|z| = S, \quad \sigma \leq \arg z \leq \pi - \sigma, \quad \log |\phi_0(z)| > \frac{1}{2} T(S, \phi_0),$$

and hence

$$\log |L_0(z)| = \log \left| \frac{f'(z)}{f(z)} \right| > \frac{1}{4} T(S, \phi_0),$$

using (13), (15) and the fact that S is large. Applying Lemma 2.5 now gives (32) and (33) for $m = 0$. The result for $m = 1, \ldots, k-2$ then follows by repeated application of (16), (22), (28) and (30). $\square$

Lemma 9.2 *Let $\delta, N > 0$. There exists a set $F_0 \subseteq [1, \infty)$ of logarithmic density 1 such that, for $r \in F_0$ and $m = 0, \ldots, k-2$,*

$$|L_m(z)| > |z|^N \quad \textit{and} \quad |F(z) - z| < |z|^{-N} \quad \textit{for} \quad |z| = r,\ \delta \leq \arg z \leq \pi - \delta. \tag{34}$$

Proof. By (10) it suffices to prove the result for the L_m. Let $\eta > 0$. Then by Lemma 2.9 there exist $C_2 > 1$ and a set E_2 of upper logarithmic density at most η such that

$$T(64r, \phi_0) \leq C_2 T(2r, \phi_0)$$

for large r not in E_2. Assume without loss of generality that E_1, which has finite logarithmic measure, is a subset of E_2. Then Lemma 9.1 gives a constant $C_3 > 0$ such that

$$\log |L_m(z)| \geq C_3 T(2r, \phi_0) \quad \text{for} \quad m = 0, \ldots, k-2,\ |z| = r,\ \delta \leq \arg z \leq \pi - \delta,$$

if r is large but not in E_2. Evidently $C_3 T(2r, \phi_0) > N \log r$ for large r, since ϕ_0 is transcendental. Hence the set of r such that (34) holds has lower logarithmic density at least $1 - \eta$, and η may be chosen arbitrarily small. $\square$

10 Singularities of the inverse function of F

Lemma 10.1 *All but finitely many multiple points z_0 of F in $\mathbb{C} \setminus \mathbb{R}$ satisfy the following:*
(i) z_0 is a simple zero of F';
(ii) z_0 is a simple zero of $f^{(k-2)}$, and a simple pole of L and ϕ_{k-2};
(iii) z_0 is a superattracting fixpoint of F.

Proof. By (10) poles of F in $\mathbb{C} \setminus \mathbb{R}$ must be zeros of $f^{(k-1)}$ and all but finitely many of these are simple, since $f^{(k)}$ has finitely many non-real zeros. Hence all but finitely many multiple points of F in $\mathbb{C} \setminus \mathbb{R}$ are zeros of F'. Next, (10) gives

$$F' = \frac{f^{(k-2)} f^{(k)}}{(f^{(k-1)})^2}. \tag{35}$$

Again since $f^{(k)}$ has finitely many non-real zeros, it follows that all but finitely many zeros of F' in $\mathbb{C} \setminus \mathbb{R}$ are zeros of $f^{(k-2)}$, and hence fixpoints of F, and simple poles of L and ϕ_{k-2}, using (10) again. Finally, if z_0 is a zero of $f^{(k-2)}$ of multiplicity $m_0 \geq 2$ then (35) gives $F'(z_0) = (m_0 - 1)/m_0 \neq 0$. $\square$

Proposition 10.1 *F has no finite non-real asymptotic value.*

To prove Proposition 10.1 will require a number of intermediate lemmas and the following classification of asymptotic values [3, 21]. Suppose that the function g is transcendental and meromorphic in the plane and $g(z)$ tends to the finite complex number a^* as z tends to infinity along a path γ. Then the inverse function g^{-1} is said to have a transcendental singularity over a^*. For each positive t, a domain $C(t)$ is uniquely determined as that component of the set $\{z \in \mathbb{C} : |g(z) - a^*| < t\}$ which contains an unbounded subpath of γ. Here $C(t) \subseteq C(s)$ if $0 < t < s$, and the intersection of all the $C(t), t > 0$, is empty. The singularity of g^{-1} over a^* corresponding to γ is then said to be direct if $C(t)$, for some positive t, contains finitely many zeros of $g(z) - a^*$, and indirect otherwise. If the singularity is direct then $C(t)$, for sufficiently small t, contains no zeros of $g(z) - a^*$.

Lemma 10.2 *Let $\alpha \in \mathbb{C} \setminus \mathbb{R}$. Then the inverse function F^{-1} has no direct transcendental singularity over α.*

Proof. Assume that F^{-1} has a direct transcendental singularity over $\alpha \in \mathbb{C} \setminus \mathbb{R}$. This gives a small positive constant η and a component C_0 of the set $\{z \in \mathbb{C} : |F(z) - \alpha| < \eta\}$, such that $F(z) \neq \alpha$ on C_0 but C_0 contains a path tending to infinity on which $F(z) \to \alpha$. If η is chosen small enough then $C_0 \subseteq \mathbb{C} \setminus \mathbb{R}$ and without loss of generality $C_0 \subseteq H^+$. The function

$$u(z) = \log \left| \frac{\eta}{F(z) - \alpha} \right| \quad (z \in C_0), \quad u(z) = 0 \quad (z \in \mathbb{C} \setminus C_0),$$

is then non-negative, non-constant and subharmonic in the plane, and vanishes outside H^+. For large t let $\sigma(t)$ be the angular measure of that subset of $S(0,t)$ on which $u(z) > 0$. Then $\sigma(t) \leq \pi$ for all large t and, if $\delta > 0$, then Lemma 9.2 gives $\sigma(t) \leq 2\delta$ for all large $t \in F_0$, where F_0 has logarithmic density 1. Hence Lemma 2.1 gives for some large r_0, as r tends to infinity,

$$\begin{aligned} \log \left(\frac{1}{2\pi} \int_0^\pi u(4re^{it}) dt \right) &\geq \log B(2r, u) - O(1) \\ &\geq \int_{r_0}^r \frac{\pi ds}{s\sigma(s)} - O(1) \\ &\geq \int_{[r_0, r] \cap F_0} \frac{\pi ds}{2\delta s} - O(1) \\ &\geq \frac{\pi}{2\delta}(1 - o(1)) \log r. \end{aligned}$$

Since δ may be chosen arbitrarily small and

$$u(z) \leq \log^+ |1/(F(z) - \alpha)| + O(1),$$

it follows that

$$\lim_{r \to \infty} \frac{\log m_{0\pi}(r, 1/(F - \alpha))}{\log r} = \infty. \tag{36}$$

But (10) and (12) give, with the notation (4),

$$\mathfrak{m}(r, 1/(F - \alpha)) \leq \mathfrak{T}(r, 1/(F - \alpha)) = O(\log r) \quad \text{as} \quad r \to \infty,$$

which using Lemma 2.3 leads to

$$\int_R^\infty \frac{m_{0\pi}(r, 1/(F - \alpha))}{r^3} dr = O(R^{-1} \log R) \quad \text{as} \quad R \to \infty,$$

contradicting (36). □

Assume for the remainder of this section that F^{-1} has an indirect transcendental singularity over some value in $\mathbb{C} \setminus \mathbb{R}$. Then the argument of [3, p.364] gives the following.

Lemma 10.3 *There exist $\alpha \in \mathbb{C} \setminus \mathbb{R}$ and pairwise distinct values $\beta_j, j = 0, 1, 2, \ldots,$ with $|\beta_j - \alpha| = \eta_j$ small and positive, and pairwise disjoint simply connected domains $U_j \subseteq H^+$ such that:*
(i) F maps U_j univalently onto $D(\alpha, \eta_j)$;
(ii) there exists a simple path $\Gamma_j \subseteq U_j$ tending to infinity, mapped by F onto the half-open line segment $[\alpha, \beta_j)$, with $F(z) \to \beta_j$ as $z \to \infty$ on Γ_j.

Proof. Following [3] take $\alpha \in \mathbb{C} \setminus \mathbb{R}$ such that F^{-1} has an indirect singularity over α and a corresponding path $\gamma \to \infty$ on which $F(z) \to \alpha$. For each $t > 0$ let $C(t)$ be that component of the set $\{z \in \mathbb{C} : |F(z) - \alpha| < t\}$ which contains an unbounded subpath of γ. Since the singularity is indirect each $C(t)$ contains infinitely many zeros of $F(z) - \alpha$. Let T be small and positive. Then it may be assumed without loss of generality that $C(T) \subseteq H^+$, and that $C(T)$ contains no zeros of F', since by Lemma 10.1 the function F has finitely many critical points with $F(z) \in D(\alpha, |\mathrm{Im}\, \alpha|)$.

Let $0 < T_j < T$. Let $z_j \in C(T_j)$ with $F(z_j) = \alpha$, and let η_j be the supremum of positive s such that the branch of F^{-1} mapping α to z_j admits unrestricted analytic continuation in $D(\alpha, s)$. Then $\eta_j < T_j$ since F is not univalent on $C(T_j)$, and F maps a subdomain U_j of $C(T_j)$ univalently onto $D(\alpha, \eta_j)$. By a compactness argument there must be a singularity $\beta_j \in \partial D(\alpha, \eta_j)$ of F^{-1} and, as $w \to \beta_j$ along $[\alpha, \beta_j)$, the preimage $z = F^{-1}(w)$ must tend to infinity along a path Γ_j in U_j.

The U_j are then constructed inductively as follows. Set $T_0 = T/2$ and assume that $\beta_j, \eta_j, \Gamma_j$ and U_j have been determined for $j = 0, \ldots, n$. Let $0 < T_{n+1} < \min\{\eta_0, \ldots, \eta_n\}$. Then for $0 \leq j \leq n$ the component $C(T_{n+1})$ satisfies $C(T_{n+1}) \not\subseteq U_j$ and hence $C(T_{n+1}) \cap U_j = \emptyset$, from which it follows that $U_{n+1} \cap U_j = \emptyset$. $\square$

Choose $j \in \mathbb{Z}$ with $j \geq 0$, and for convenience drop the subscripts on $\beta_j, \eta_j, \Gamma_j, U_j$.

Lemma 10.4 *Let $N > 0$. Then $|F(z) - \beta| < |z|^{-N}$ as $z \to \infty$ on Γ.*

Proof. Let δ be small and positive. For large s let $\theta^*(s)$ denote the angular measure of the intersection of U with the circle $S(0, s)$. Since $U \subseteq H^+$ and F is bounded on U, Lemma 9.2 gives $\theta^*(s) \leq 2\delta$ for all s in a set of logarithmic density 1, and so if r_0 is large it follows that

$$\int_{r_0}^{r} \frac{\theta^*(s) ds}{s} \leq (2\delta + o(1)) \log r$$

as $r \to \infty$. Hence applying the Cauchy-Schwarz inequality gives

$$\left(\log \frac{r}{r_0} \right)^2 \leq (2\delta + o(1)) \log r \int_{r_0}^{r} \frac{ds}{s\theta^*(s)} \quad \text{and} \quad \int_{r_0}^{r} \frac{ds}{4s\theta^*(s)} > 2N \log r \tag{37}$$

as $r \to \infty$, provided δ was chosen small enough. Let $z = G(w)$ be the branch of the inverse function F^{-1} mapping $D(\alpha, \eta)$ onto U. For $z \in \Gamma$ the distance from z to ∂U is at most $|z|\theta^*(|z|)$ and so Koebe's theorem implies that

$$|(w - \beta)G'(w)| \leq 4|z|\theta^*(|z|) \quad \text{for} \quad z = G(w),\, w \in [\alpha, \beta).$$

Hence, for large $z \in \Gamma$ and $w = F(z)$, writing $u = G(v)$ for $v \in [\alpha, w]$ gives, using (37),

$$\begin{aligned} \log \left| \frac{\beta - \alpha}{\beta - F(z)} \right| &= \int_{\alpha}^{w} \frac{|dv|}{|\beta - v|} = \int_{G(\alpha)}^{z} \frac{|du|}{|(\beta - v)G'(v)|} \geq \int_{G(\alpha)}^{z} \frac{|du|}{4|u|\theta^*(|u|)} \\ &\geq \int_{r_0}^{|z|} \frac{ds}{4s\theta^*(s)} > 2N \log |z|. \end{aligned}$$

$\square$

For $z \in \Gamma$ let Γ_z denote the part of Γ joining z to infinity, so that

$$F(\Gamma_z) = \gamma_w = [w, \beta), \quad w = F(z), \quad z = G(w) = F^{-1}(w). \tag{38}$$

(10) gives, as $z \to \infty$ on Γ,

$$\frac{f^{(k-2)}(z)}{f^{(k-1)}(z)} = z-\beta+\beta-F(z), \quad \frac{f^{(k-1)}(z)}{f^{(k-2)}(z)} = \frac{1}{z-\beta}+\mu(z), \quad \mu(z) = O(|z|^{-2}|F(z)-\beta|). \tag{39}$$

Denote positive constants by c, not necessarily the same at each occurrence.

Lemma 10.5 *The function $\mu(z)$ in (39) satisfies*

$$\int_{\Gamma_z} |\mu(u)||du| \leq c|F(z)-\beta|$$

as $z \to \infty$ on Γ.

Proof. Using (38) and (39), write

$$v = F(u), \quad u = G(v), \quad u \in \Gamma_z, \quad \int_{\Gamma_z} |\mu(u)||du| \leq c \int_{[w,\beta)} |u|^{-2}|v-\beta||G'(v)||dv|. \tag{40}$$

Since $U \subseteq H^+$ the function $\log G$ is defined on $D(\alpha, \eta)$ and maps $D(\alpha, \eta)$ univalently onto a domain containing no disc of radius greater than $\pi/2$, and so Koebe's theorem gives

$$\left| \frac{G'(v)}{G(v)} \right| \leq c \frac{1}{|v-\beta|} \quad \text{for} \quad v \in [w, \beta). \tag{41}$$

Using (39), (40) and (41) gives, as $z \to \infty$ on Γ,

$$\int_{\Gamma_z} |\mu(u)||du| \leq c \int_{[w,\beta)} |u|^{-2}|G(v)||dv| = c \int_{[w,\beta)} |G(v)|^{-1}|dv| \leq c \int_{[w,\beta)} |dv| \leq c|w-\beta|.$$

$\square$

Integrating (39) and using Lemma 10.5 leads to, for some constant $A \in \mathbb{C} \setminus \{0\}$,

$$f^{(k-2)}(z) = A(z-\beta)(1+O(|F(z)-\beta|)) = A(z-\beta)+\tau(z), \quad \tau(z) = O(|z(F(z)-\beta)|), \tag{42}$$

as $z \to \infty$ on Γ.

Lemma 10.6 *Let $0 < \sigma < 1$ and $M \in \mathbb{N}$. Then the function $\tau(z)$ in (42) satisfies*

$$\int_{\Gamma_z} |u^M \tau(u)||du| \leq c|F(z)-\beta|^{1-\sigma}$$

as $z \to \infty$ on Γ. In particular, the integral converges.

Proof. Using (40), (41), (42) and Lemma 10.4 leads to, provided N is chosen large enough in Lemma 10.4,

$$\begin{aligned}\int_{\Gamma_z} |u^M \tau(u)||du| &\leq c \int_{[w,\beta)} |u|^{M+1} |v-\beta||G'(v)||dv| \\ &\leq c \int_{[w,\beta)} |u|^{M+1} |G(v)||dv| \\ &= c \int_{[w,\beta)} |u|^{M+2} |dv| \\ &\leq c \int_{[w,\beta)} |v-\beta|^{-\sigma} |dv| \\ &= c|w-\beta|^{1-\sigma}.\end{aligned}$$

□

Lemma 10.7 *As $z \to \infty$ on Γ,*

$$f(z) = \frac{A(z-\beta)^{k-1}}{(k-1)!} + O(|z|^{k-3}) \tag{43}$$

and

$$f'(z) = \frac{A(z-\beta)^{k-2}}{(k-2)!} + O(|z|^{k-4}), \tag{44}$$

where A is as in (42).

Proof. Set

$$g(z) = f(z) - \frac{A(z-\beta)^{k-1}}{(k-1)!}, \quad h(z) = f'(z) - \frac{A(z-\beta)^{k-2}}{(k-2)!}. \tag{45}$$

Then (42) gives

$$g^{(k-2)}(z) = h^{(k-3)}(z) = \tau(z). \tag{46}$$

Fix $z_0 \in \Gamma$ with $|z_0|$ large. Then Taylor's formula and (45) and (46) give a polynomial P_{k-3} of degree at most $k-3$ such that

$$f(z) - \frac{A(z-\beta)^{k-1}}{(k-1)!} = g(z) = P_{k-3}(z) + \int_{z_0}^{z} \frac{(z-u)^{k-3}}{(k-3)!} \tau(u) du = O(|z|^{k-3})$$

as $z \to \infty$ on Γ, using Lemma 10.6, from which (43) follows at once.

Next, if $k = 3$ then (44) is an immediate consequence of (42) and Lemma 10.4, while if $k \geq 4$ then (45) and (46) give a polynomial Q_{k-4} of degree at most $k-4$ such that

$$f'(z) - \frac{A(z-\beta)^{k-2}}{(k-2)!} = h(z) = Q_{k-4}(z) + \int_{z_0}^{z} \frac{(z-u)^{k-4}}{(k-4)!} \tau(u) du = O(|z|^{k-4})$$

as $z \to \infty$ on Γ, using Lemma 10.6 again. □

Lemma 10.8 *As $z \to \infty$ on Γ,*

$$L_0(z) = \frac{f'(z)}{f(z)} = R_\beta(z)(1 + O(|z|^{-2})) = R_\beta(z) + O(|z|^{-3}), \quad R_\beta(z) = \frac{k-1}{z-\beta}. \tag{47}$$

Proof. (47) follows at once from (43) and (44). □

To complete the proof of Proposition 10.1, take a large positive integer n and $r_1 > 0$ such that the region $A^+(r_1, \infty)$, defined as in (1), contains no zeros nor poles of f, and none of the β_j, for $0 \leq j \leq n$. Then by Lemma 10.8 there exist paths Γ_j^* in $A^+(r_1, \infty)$, each tending to infinity, and pairwise disjoint apart from a common starting point z_1, such that, for $j = 0, 1, \ldots, n$,

$$L_0(z) - R_{\beta_j}(z) = O(|z|^{-3}) \quad \text{as} \quad z \to \infty \quad \text{with} \quad z \in \Gamma_j^*. \tag{48}$$

Re-labelling if necessary gives n pairwise disjoint simply connected domains $D_1, \ldots, D_n$ lying in $A^+(r_1, \infty)$, with D_j bounded by Γ_{j-1}^* and Γ_j^*. For $j = 1, \ldots, n$ set

$$H_j(z) = \frac{L_0(z) - R_{\beta_j}(z)}{R_{\beta_{j-1}}(z) - R_{\beta_j}(z)} = 1 + \frac{L_0(z) - R_{\beta_{j-1}}(z)}{R_{\beta_{j-1}}(z) - R_{\beta_j}(z)}, \tag{49}$$

and for $s > 0$ let $\theta_j(s)$ be the angular measure of the intersection of D_j with the circle $S(0, s)$.

Since

$$R_{\beta_{j-1}}(z) - R_{\beta_j}(z) = \frac{(k-1)(\beta_{j-1} - \beta_j)}{(z - \beta_{j-1})(z - \beta_j)} \sim \frac{(k-1)(\beta_{j-1} - \beta_j)}{z^2} \quad \text{as} \quad z \to \infty,$$

(47), (48), (49) and the construction of the domains D_j show that $H_j(z)$ is analytic on the closure of D_j, tends to 0 as z tends to infinity on Γ_j, and tends to 1 as z tends to infinity on Γ_{j-1}.

Let c^* be large and positive, and for $j = 1, \ldots, n$ define

$$u_j(z) = \log^+ \left| \frac{H_j(z)}{c^*} \right| \quad (z \in D_j), \quad u_j(z) = 0 \quad (z \in \mathbb{C} \setminus D_j).$$

Then each u_j is continuous, non-negative and subharmonic in the plane, and unbounded on D_j. Lemma 2.1 gives, for some large r_2 and for $1 \leq j \leq n$, using (49),

$$\begin{aligned}
\int_{r_2}^{r} \frac{\pi ds}{s\theta_j(s)} &\leq \log B(2r, u_j) + O(1) \\
&\leq \log \left(\frac{1}{2\pi} \int_0^\pi u_j(4re^{it}) dt \right) + O(1) \\
&\leq \log (m_{0\pi}(4r, H_j)) + O(1) \\
&\leq \log (m_{0\pi}(4r, L_0) + O(\log r)) + O(1)
\end{aligned}$$

as $r \to \infty$. Hence, for $1 \leq j \leq n$,

$$\int_{r_2}^{r} \frac{\pi ds}{s\theta_j(s)} \leq \log^+ (m_{0\pi}(4r, L_0)) + o(\log r) \quad \text{as} \quad r \to \infty. \tag{50}$$

However, the Cauchy-Schwarz inequality gives

$$n^2 \leq \sum_{j=1}^{n} \theta_j(s) \sum_{j=1}^{n} \frac{1}{\theta_j(s)} \leq \sum_{j=1}^{n} \frac{\pi}{\theta_j(s)}$$

for $s \geq r_2$, which on combination with (50) leads to

$$n^2 \log r \leq n \log^+(m_{0\pi}(4r, L_0)) + o(\log r), \quad m_{0\pi}(r, L_0) \geq r^{n-o(1)} \quad \text{as} \quad r \to \infty. \tag{51}$$

Since n may be chosen arbitrarily large, (51) contradicts (12) and Lemma 2.3. This completes the proof of Proposition 10.1. $\Box$

Proposition 10.1 now permits the following classification of non-real poles of L.

Lemma 10.9 *All but finitely many poles z_0 of L in $\mathbb{C} \setminus \mathbb{R}$ satisfy conditions (i), (ii) and (iii) of Lemma 10.1.*

Proof. Let $z^* \in \mathbb{C} \setminus \mathbb{R}$ be large and a pole of L. Then $f^{(k-2)}(z^*) = 0$ by (10), since f has finitely many poles. Suppose that z^* is a zero of $f^{(k-2)}$ of multiplicity at least 2. Then as in the proof of Lemma 10.1, (10) and (35) show that z^* is an attracting fixpoint of F but not a critical point of F, and z^* lies in a component C^* of the Fatou set of F, such that the iterates F_n of F tend to z^* locally uniformly in C^*, so that $C^* \subseteq \mathbb{C} \setminus \mathbb{R}$ since F is real. Further, the component C^* must contain a non-real singular value of F^{-1}, and using Lemma 10.1 and Proposition 10.1 all but finitely many such singular values are themselves fixpoints of F. Hence all but finitely many zeros of $f^{(k-2)}$ in $\mathbb{C} \setminus \mathbb{R}$ are simple and by (35) are zeros of F'. $\Box$

11 Components of $F^{-1}(D^+(0, R))$

The following lemma is an immediate consequence of Lemma 10.1 and Proposition 10.1.

Lemma 11.1 *There exists a simple path $\Gamma^+ : [0, \infty) \to H^+$ with the following properties:*
(i) All critical values of F in H^+ lie on Γ^+;
(ii) Γ^+ consists of countably many radial segments and arcs of circles $S(0, \rho_j)$, $0 < \rho_j \to \infty$;
(iii) $|\Gamma^+(t)|$ is non-decreasing, with $\lim_{t\to\infty} |\Gamma^+(t)| = \infty$;
(iv) if $D \subseteq H^+ \setminus \Gamma^+$ is a simply connected domain, then all components of $F^{-1}(D)$ are mapped univalently onto D by F.

$\Box$

Lemma 11.2 *Let $0 < R < \infty$ and let $W_R = \{z \in H^+ : F(z) \in D^+(0, R)\}$, where $D^+(0, R)$ is defined as in (1). Let C be a component of W_R. Then there exists an integer k_C such that each value $w \in D^+(0, R)$ is taken k_C times in C, counting multiplicity, and the number of zeros of F' in C, counting multiplicity, is at least $k_C - 1$.*

In the terminology of [24, p.4], $F : C \to D^+(0,R)$ is a proper map of topological degree k_C.
Proof. By the construction of the path Γ^+ in Lemma 11.1, the region $D_R = D^+(0,R) \setminus \Gamma^+$ is simply connected and all components of $F^{-1}(D_R)$ are mapped univalently onto D_R by F.

Claim 1. *There are finitely many components B of $F^{-1}(D_R)$ with $B \subseteq C$.*
If B is any component of $F^{-1}(D_R)$ with $B \subseteq C$ and if $\partial B \cap C$ contains no critical point of F, then using Proposition 10.1 the branch F_B^{-1} of the inverse function of F which maps D_R onto B may be analytically continued into $D^+(0,R)$, and F maps C univalently onto $D^+(0,R)$. Since a critical point of F belongs to the boundary of at most finitely many components B of $F^{-1}(D_R)$, and since F has finitely many critical points over $D^+(0,R)$, by part (iii) of Lemma 10.1, Claim 1 follows.

Let k_C be the number of components $B \subseteq C$ as in Claim 1. Clearly every value $w \in D_R$ is taken k_C times in C, each simply, and it follows from the open mapping theorem that no value $w \in D^+(0,R)$ is taken more than k_C times in C, counting multiplicity.

Claim 2. *Let (z_n) be a sequence in C such that $\lim_{n\to\infty} z_n = z^* \in \partial_\infty C$, where $\partial_\infty C$ denotes the boundary of C in $\mathbb{C} \cup \{\infty\}$. Then every limit point w of the sequence $(F(z_n))$ satisfies $w \in \partial D^+(0,R)$.*
To prove Claim 2, assume without loss of generality that $F(z_n) \to w_0 \in D^+(0,R)$ as $n \to \infty$. Then clearly $z^* = \infty$. Take R^* large and positive such that all w_0 points of F in C and all critical points of F over $D^+(0,R)$ lie in $D(0,R^*)$, and such that $|F(z) - w_0| > \delta > 0$ on $S(0,R^*)$, where $D(w_0,\delta) \subseteq D^+(0,R)$. Let n be large. Then $|z_n| > R^*$ and $F'(z_n) \neq 0$ and the branch of F^{-1} mapping $F(z_n)$ to z_n may be analytically continued throughout $D(w_0,\delta)$. But this gives $z_n' \in C$ with $|z_n'| > R^*$ and $F(z_n') = w_0$, a contradiction.

The argument of [24, Theorem 1, p.5] now shows that every value $w \in D^+(0,R)$ is taken k_C times in C, counting multiplicity, and the Riemann-Hurwitz formula [24, p.7] implies that F has at least $k_C - 1$ critical points in C, all of which must be zeros of F'. $\square$

12 A growth lemma

The next lemma determines certain annuli, corresponding roughly to Pólya peaks [12, p.101], in which the subsequent analysis will take place.

Lemma 12.1 *Let N be a large positive integer and let the positive constants K and ε satisfy*

$$K^{33} = 1 + 2^{-k-4}, \quad 0 < \varepsilon < \frac{1}{512}. \tag{52}$$

Then there exist arbitrarily large $r \in [1,\infty)$ with the following properties:
(i)

$$n(K^{22}r) \leq (1 + 2^{-k-4}) n(r), \tag{53}$$

in which $n(r) = n(r, 1/(\phi_0 - a))$ and a are as in (25) and (26);
(ii)

$$T(64r, \phi_0) \leq d_1 T(2r, \phi_0), \tag{54}$$

in which the positive constant d_1 depends only on the order ρ of ϕ_0;
(iii) for $m = 0, \ldots, k-2$, the L_m satisfy

$$|L_m(z)| \leq \exp(40T(64r, \phi_0)) \tag{55}$$

for $\quad |z| = s \in [r, 2r] \setminus E_1$ *and for*

$$z = se^{it_n}, \quad r \leq s \leq 2r, \quad n = 1, 2, 3, \tag{56}$$

where the exceptional set E_1 and t_1, t_2, t_3 are as in Lemma 8.1;
(iv) for $m = 0, \ldots, k-2$, the estimates

$$|L_m(z)| > |z|^N, \quad |F(z) - z| < |z|^{-N}, \tag{57}$$

hold for

$$|z| = s \in [r, 2r] \setminus E_1, \quad t_1 \leq \arg z \leq t_3, \tag{58}$$

and for z satisfying (56);
(v) there exist a set $J_r \subseteq [r, K^{22}r] \setminus E_1$, and a function $\theta(s) : J_r \to (0, \pi/4)$ such that the estimates

$$|\phi_0(z)| > |z|^{N+3}, \quad |L_m(z)| > |z|^N, \quad \left| \frac{L_m'(z)}{L_m(z)} \right| < |z|, \quad |F(z) - z| < |z|^{-N}, \tag{59}$$

hold for $m = 0, \ldots, k-2$ and

$$|z| = s \in J_r, \quad \theta(s) \leq \arg z \leq \pi - \theta(s); \tag{60}$$

(vi) the function $\theta(s)$ satisfies, for $q = 1, \ldots, 22$,

$$\varepsilon^2 n(r) \int_{[K^{q-1}r, K^q r] \cap J_r} \frac{ds}{s\theta(s)} > T(64r, \phi_0); \tag{61}$$

(vii) for $q = 1, \ldots, 22$ there exists $s_q \in (K^{q-1}r, K^q r) \cap J_r$ such that, for $m = 0, \ldots, k-2$,

$$\left(\int_{-\theta(s_q)}^{\theta(s_q)} + \int_{\pi-\theta(s_q)}^{\pi+\theta(s_q)} \right) s_q \left| \frac{\phi_m'(s_q e^{i\tau})}{\phi_m(s_q e^{i\tau})} \right| + s_q \left| \frac{\phi_0'(s_q e^{i\tau})}{\phi_0(s_q e^{i\tau}) - a} \right| d\tau < 2^{-k-4} n(r). \tag{62}$$

Proof. First, part (iii) follows at once from (27) and (30).

Next, let $\rho \leq 1$ be the order of growth of ϕ_0 as in (20). Then (25) and (26) imply immediately that $n(r)$ also has order ρ. Denote by d_j positive constants depending at most on k and ρ.

For a given $d_1 > 0$, if r satisfies (54) and is large enough then the conclusions of part (iv) are automatically satisfied, using (10), Lemma 9.1 and the fact that ϕ_0 is transcendental.

It remains to show that r can be chosen to satisfy (i), (ii), (v), (vi) and (vii), and to this end the proof of Lemma 12.1 will now be divided into two subcases, depending on ρ.

Case 1: *suppose that $\rho > 0$.*

Then the standard existence result for Pólya peaks [12, p.101] shows that there exist $r_0 > 0$ and arbitrarily large r such that

$$n(t) \leq \left(\frac{t}{r}\right)^{\rho/2} n(r) \quad (r_0 \leq t < r), \quad n(t) \leq \left(\frac{t}{r}\right)^{3\rho/2} n(r) \quad (r \leq t < \infty). \tag{63}$$

But $\rho \leq 1$ by (20), and so (53) follows using (52). Next, (25) and (63) give, for $R \geq r$,

$$T(R, \phi_0) \sim N(R) \leq N(r_0) + 2n(r)\left(\frac{1}{\rho} + \frac{(R/r)^{3\rho/2}}{3\rho}\right),$$

which on combination with (21) yields, for $m = 0, \ldots, k-2$ and large such r,

$$T(64r, \phi_m) \leq d_2 T(64r, \phi_0), \quad T(64r, \phi_0) \leq d_3 n(r) \leq \frac{d_3}{\log 2} N(2r) \sim \frac{d_3}{\log 2} T(2r, \phi_0), \tag{64}$$

where d_3 depends only on ρ. In particular (54) follows from (64).

To complete the proof in this case, set $J_r = [r, K^{22}r] \setminus E_1$, and for each $s \in J_r$ set $\theta(s) = \delta$, with δ a small positive constant independent of r. If z satisfies (60) and r is large enough then (59) follows from (13), (15), (16), (20), (28), (54) and Lemma 9.1. Further, provided δ is chosen small enough, (61) holds using (64) and the fact that E_1 has finite logarithmic measure. Finally, again provided δ is small enough, the existence of s_q as in (62) follows from (64) and Lemma 2.7.

Case 2: *suppose that* $\rho = 0$.
Choose a rational function R_0 with $R_0(z) = O(|z|^{N+3})$ as $z \to \infty$ and such that

$$\phi^*(z) = z^{-N-4}(\phi_0(z) - R_0(z)) \tag{65}$$

is entire. Let A be a large positive constant. Since ϕ_0 and ϕ^* have order 0, there exist by Lemma 2.9 arbitrarily large positive r_1 such that

$$\begin{aligned} n(2Ar_1) &\leq (1 + 2^{-k-4})n(r_1), \\ T(2Ar_1, \phi_0) &\leq 2T(r_1, \phi_0), \\ \log M(4Ar_1, \phi^*) &\leq 2\log M(r_1, \phi^*). \end{aligned} \tag{66}$$

For $2r_1 \leq s \leq 2Ar_1$ set

$$U_s = \{t \in [0, 2\pi) : |\phi^*(se^{it})| > 1\}. \tag{67}$$

If $U_s = [0, 2\pi)$ set $\theta_0(s) = \infty$, and otherwise let $\theta_0(s)$ be the Lebesgue measure of U_s. Then Lemma 2.1 gives

$$\log M(r_1, \phi^*) \leq 9\sqrt{2}\exp\left(-\pi \int_{2r_1}^{2Ar_1} \frac{ds}{s\theta_0(s)}\right) \log M(4Ar_1, \phi^*)$$

and hence, using (66),

$$\exp\left(\pi \int_{2r_1}^{2Ar_1} \frac{ds}{s\theta_0(s)}\right) \leq 18\sqrt{2}. \tag{68}$$

Let

$$J = \{s \in [2r_1, 2Ar_1] \setminus E_1 : U_s = [0, 2\pi)\}. \tag{69}$$

Then for $s \in [2r_1, 2Ar_1] \setminus J$ either $s \in E_1$ or $\theta_0(s) \leq 2\pi$ and so provided r_1 is large enough (68) gives

$$\int_{[2r_1,2Ar_1]\setminus J} \frac{ds}{s} \leq o(1) + 2\pi \int_{2r_1}^{2Ar_1} \frac{ds}{s\theta_0(s)} \leq 4\log(18\sqrt{2}),$$

using the fact that E_1 has finite logarithmic measure. Since A may be chosen arbitrarily large it is then evidently possible using (52) to choose r such that $[r, K^{22}r] \subseteq [r, K^{33}r] \subseteq [r, 64r] \subseteq [2r_1, 2Ar_1]$ and such that

$$\int_{[r,2r]\setminus J} \frac{ds}{s} \leq \frac{1}{2}\log K. \tag{70}$$

For this choice of r, (53) and (54) follow from (66).

Set

$$J_r = [r, K^{22}r] \cap J, \quad \theta(s) = s^{-1/2} \quad (s \in J_r). \tag{71}$$

Then (52), (66) and (70) give, for $q = 1, \ldots, 22$,

$$\varepsilon^2 n(r) \int_{[K^{q-1}r, K^q r]\cap J_r} \frac{ds}{s\theta(s)} \geq \varepsilon^2 n(r)\sqrt{r} \int_{[K^{q-1}r, K^q r]\cap J_r} \frac{ds}{s} \geq \frac{1}{2}\varepsilon^2 n(r)\sqrt{r}\log K > T(64r, \phi_0),$$

provided r_1 is large enough, since $r_1 \leq r < 64r \leq 2Ar_1$ and

$$T(64r, \phi_0) \leq 2T(r, \phi_0) \sim 2N(r) \leq 2n(r)\log r + O(1) = o(n(r)\sqrt{r}),$$

using (25) and (26). This proves (61).

Next, for $q = 1, \ldots, 22$ let s_q be any element of $(K^{q-1}r, K^q r) \cap J_r$, which is non-empty by (70) and (71). Then $s_q \notin E_1$, by (69) and (71), and so (28) and (71) give, since $\rho = 0$,

$$\left(\int_{-\theta(s_q)}^{\theta(s_q)} + \int_{\pi-\theta(s_q)}^{\pi+\theta(s_q)} \right) s_q \left| \frac{\phi_m'(s_q e^{i\tau})}{\phi_m(s_q e^{i\tau})} \right| + s_q \left| \frac{\phi_0'(s_q e^{i\tau})}{\phi_0(s_q e^{i\tau}) - a} \right| d\tau \leq s_q^{o(1)-1/2} < 2^{-k-4} n(r),$$

again provided r_1 is large enough, which proves (62).

It remains only to establish part (v). Assume r_1 is large and that z satisfies (60). Then $|\phi^*(z)| > 1$, by (67), (69) and (71), and so

$$|\phi_0(z)| > s^{N+3}, \tag{72}$$

using (65). Also (15), (16), (28), (69), (71) and the fact that $\rho = 0$ give

$$|\psi_0(z)| > s^{-2}, \quad \left| \frac{\phi_m'(z)}{\phi_m(z)} \right| + \left| \frac{\psi_m'(z)}{\psi_m(z)} \right| \leq 1,$$

for $m = 0, \ldots, k-2$, which on combination with (72) and repeated use of (22) yields

$$|L_0(z)| > s^{N+1}, \quad |L_m(z)| > s^{N+1} - (k-2) > s^N,$$

for $m = 1, \ldots, k-2$. Hence (59) follows using (10). $\square$

13 The number of zeros and poles of the ϕ_m

Retain the notation of Lemma 12.1, including the constants ε, K, N. In what follows all $O(1)$ terms should be understood as being uniformly bounded for large r as in Lemma 12.1. The aim of this section is essentially to show that for such r and for certain s close to r the function ϕ_{k-2} has more zeros than poles in $|z| \leq s$.

Lemma 13.1 *Let r as in Lemma 12.1 be large. Then for $m = 0, \ldots, k-2$ and $q = 1, \ldots, 22$,*

$$n(s_q, 1/\phi_m) - n(s_q, \phi_m) = n(r) + \sigma_q, \quad |\sigma_q| < 2^{-k-2} n(r). \tag{73}$$

Proof. On the two arcs

$$|z| = s_q, \quad \theta(s_q) \leq \pm \arg z \leq \pi - \theta(s_q), \tag{74}$$

the functions ϕ_0, ψ_m and L_m satisfy

$$\phi_0(z) \sim \phi_0(z) - a, \quad L_m(z) \sim L_0(z), \quad |\arg \psi_m(z)| \leq \pi,$$

by (11), (59) and part (v) of Lemma 4.1. Hence the net changes in $\arg \phi_0(z), \arg(\phi_0(z) - a)$ and $\arg \phi_m(z)$ as z describes the two circular arcs in (74) differ by at most $O(1)$. On combination with (62) this gives

$$n(s_q, 1/\phi_m) - n(s_q, \phi_m) = n(s_q, 1/(\phi_0 - a)) - n(s_q, \phi_0) + \sigma_q^*, \quad |\sigma_q^*| < 2^{-k-4} n(r) + O(1).$$

Now (73) follows since ϕ_0 has finitely many poles and, using (26) and (53),

$$0 \leq n(s_q, 1/(\phi_0 - a)) - n(r, 1/(\phi_0 - a)) = n(s_q) - n(r) \leq 2^{-k-4} n(r).$$

$\square$

Lemma 13.2 *Let r as in Lemma 12.1 be large. For $m = 0, \ldots, k-2$ and for $q = 1, \ldots, 22$ the following inequality holds:*

$$n(s_q, \phi_m) \leq (2^m - 1)(1 + 2^{-k-2}) n(r) + O(1). \tag{75}$$

Further, for $m = 0, \ldots, k-2$,

$$n(s_{22}, \phi_m) - n(s_1, \phi_m) \leq (2^m - 1) 2^{-k-1} n(r). \tag{76}$$

Finally,

$$n(s_1, 1/\phi_{k-2}) \geq n(s_{22}, \phi_{k-2}) + \frac{n(r)}{2}. \tag{77}$$

Proof. The proof of (75) is by induction on m, the result for $m = 0$ being obvious since ϕ_0 has finitely many poles. Now suppose that $1 \leq p \leq k-2$ and that (75) holds for $0 \leq m < p$. Then (14) and (73) give

$$\begin{aligned}
n(s_q, \phi_p) &\leq \sum_{0 \leq m < p} n(s_q, 1/\phi_m) + O(1) \\
&\leq \sum_{0 \leq m < p} (n(s_q, \phi_m) + (1 + 2^{-k-2}) n(r)) + O(1) \\
&\leq \sum_{0 \leq m < p} 2^m (1 + 2^{-k-2}) n(r) + O(1) \\
&= (2^p - 1)(1 + 2^{-k-2}) n(r) + O(1),
\end{aligned}$$

so that (75) is proved in full.

Further, (76) is true for $m = 0$, using the fact that ϕ_0 has finitely many poles. Assume that $1 \leq p \leq k-2$ and that (76) is true for $0 \leq m < p$. Then (73) and the same argument as in the proof of (14) give

$$\begin{aligned} n(s_{22}, \phi_p) - n(s_1, \phi_p) &\leq \sum_{0 \leq m < p} (n(s_{22}, 1/\phi_m) - n(s_1, 1/\phi_m)) \\ &\leq \sum_{0 \leq m < p} (n(s_{22}, \phi_m) - n(s_1, \phi_m) + 2^{-k-1} n(r)) \\ &\leq \sum_{0 \leq m < p} 2^m 2^{-k-1} n(r) = (2^p - 1) 2^{-k-1} n(r). \end{aligned}$$

Thus (76) is also proved by induction.

Next, (73) and (76) lead to

$$n(s_1, 1/\phi_{k-2}) \geq n(s_1, \phi_{k-2}) + n(r) - 2^{-k-2} n(r) \geq n(s_{22}, \phi_{k-2}) + n(r) - 2^{-k-2} n(r) - 2^{-3} n(r),$$

which gives (77). $\square$

14 The behaviour of L and F near zeros of ϕ_{k-2}

Assume henceforth that r as in Lemma 12.1 is large.

Lemma 14.1 *There exist positive real numbers λ and Λ depending on r, with λ small and Λ large, and*

$$N_0 \geq \frac{1}{2} n(s_1, 1/\phi_{k-2}) - \frac{n(r)}{32}$$

pairs $\{A_j, B_j\}$ such that with the notation of Definitions 2.1:
(i) A_j is a component of the set $L^{-1}(D^+(0, \lambda))$, mapped univalently onto $D^+(0, \lambda)$ by L;
(ii) B_j is a component of the set $F^{-1}(A^+(\Lambda, \infty))$, mapped univalently onto $A^+(\Lambda, \infty)$ by F;
(iii) $A_j \subseteq B_j \subseteq D^+(0, Kr)$;
(iv) $B_j \cap B_{j'} = \emptyset$ for $j \neq j'$;
(v) $\partial A_j \cap \partial B_j$ contains one zero of L.

Proof. Since

$$L = L_{k-2} = \frac{f^{(k-1)}}{f^{(k-2)}} = \phi_{k-2} \psi_{k-2},$$

by (10) and (13), and since every pole of ψ_{k-2} is simple and a simple pole of L, all zeros of ϕ_{k-2} are zeros of L and poles of F.

Let ζ_ν be the distinct zeros of L in $D(0, Kr)$. Choose λ so small and Λ so large that each ζ_ν lies in a component $C_\nu \subseteq D(0, Kr)$ of the set $L^{-1}(D(0, \lambda))$, and in a component $C_\nu^* \subseteq D(0, Kr)$ of the set $F^{-1}(A(\Lambda, \infty))$. It may be assumed that $C_\nu \subseteq C_\nu^*$, since (10) gives

$$|F(z)| \geq \lambda^{-1} - Kr \quad \text{for} \quad z \in C_\nu.$$

It may be assumed further that Λ is so large that each C_ν^* contains exactly one pole of F, possibly multiple, and $C_\nu^* \subseteq \mathbb{C} \setminus \mathbb{R}$ if ζ_ν is non-real.

Choose $\beta_\nu \in \mathbb{C} \setminus \{0\}$ and $m_\nu \in \mathbb{N}$ such that

$$L(z) = \beta_\nu (z - \zeta_\nu)^{m_\nu} (1 + o(1)) \quad \text{as} \quad z \to \zeta_\nu.$$

Then $L(z)$ and $F(z)$ have positive imaginary part as z tends to ζ_ν with

$$\arg(z - \zeta_\nu) = \tau_q = \frac{1}{m_\nu} \left(\frac{\pi}{2} - \arg \beta_\nu + 2\pi q \right), \quad q = 0, \ldots, m_\nu - 1,$$

and negative imaginary part as z tends to ζ_ν with

$$\arg(z - \zeta_\nu) = \tau_q' = \frac{1}{m_\nu} \left(-\frac{\pi}{2} - \arg \beta_\nu + 2\pi q \right), \quad q = 0, \ldots, m_\nu - 1.$$

Provided λ and $1/\Lambda$ are small enough this gives m_ν components $A' \subseteq C_\nu \subseteq C_\nu^*$ of the set $L^{-1}(D^+(0, \lambda))$, such that:

(a) the A' are separated by the rays $\arg(z - \zeta_\nu) = \tau_q'$;

(b) if δ_1 is positive but small enough then each A' contains precisely one of the radial segments $0 < |z - \zeta_\nu| < \delta_1$, $\arg(z - \zeta_\nu) = \tau_q$.

Moreover, there are m_ν components $B' \subseteq C_\nu^*$ of the set $F^{-1}(A^+(\Lambda, \infty))$, again satisfying conditions (a) and (b). Further, if $A' \subseteq H^+$ then A' is contained in one of the components B', by (10), and if A', A'' are distinct such components in H^+ then the corresponding components B', B'' are distinct, by (b).

Let n_1 be the number of zeros of ϕ_{k-2} in $D(0, Kr) \setminus \mathbb{R}$, and n_2 the number of zeros of ϕ_{k-2} in the interval $(-Kr, Kr)$, in both cases counting multiplicities. If a zero ζ_ν of L lies in $D^+(0, Kr)$ and $|\zeta_\nu|$ is large then ζ_ν is a simple zero of L and a simple pole of F, since $f^{(k)}$ has finitely many non-real zeros. Hence there exist components $A_j \subseteq C_\nu$ and $B_j \subseteq C_\nu^* \subseteq D^+(0, R)$ as in the statement of the lemma, with $\zeta_\nu \in \partial A_j \cap \partial B_j$. The number of distinct pairs $\{A_j, B_j\}$ arising from zeros of ϕ_{k-2} in $D^+(0, Kr)$ is thus

$$n_3 \geq \frac{1}{2} n_1 - O(1) = \frac{1}{2}(n_0 - n_2) - O(1), \quad \text{where} \quad n_0 = n_1 + n_2 \geq n(s_1, 1/\phi_{k-2}), \tag{78}$$

using the fact that $s_1 < Kr$.

Partition the interval $[-Kr, Kr]$ as

$$-Kr = x_0 < \ldots < x_Q = Kr,$$

such that L has no poles on each interval (x_{p-1}, x_p) and such that if $1 \leq p < Q$ then x_p is a pole of L. Then by the construction of ψ_{k-2} in §4, all but $Q - O(1)$ of the x_p are poles of ψ_{k-2}, and ψ_{k-2} has $Q - O(1)$ zeros in the interval $(-Kr, Kr)$. Let M, M' be the number of zeros of L and ψ_{k-2} respectively in the interval $(-Kr, Kr)$, and for $p = 1, \ldots, Q$ let M_p be the number of zeros of L in the interval (x_{p-1}, x_p), in each case counting multiplicity. Since zeros of ϕ_{k-2} are not poles of ψ_{k-2} and zeros of ψ_{k-2} are not poles of ϕ_{k-2}, this gives

$$n_2 + M' = M = \sum_{p=1}^{Q} M_p, \quad M' \geq Q - O(1). \tag{79}$$

Consider a real zero ζ_ν of L in the interval $(-Kr, Kr)$, of multiplicity m_ν. If m_ν is even then there are $m_\nu/2$ pairs of components $\{A_j, B_j\}$ as in the statement of the lemma, with $\zeta_\nu \in \partial A_j \cap \partial B_j$. In this case as x passes through ζ_ν from left to right the sign of $L(x)$ does not change. Next, if m_ν is odd and $L^{(m_\nu)}(\zeta_\nu) > 0$, then there are $(m_\nu + 1)/2$ pairs of components $\{A_j, B_j\}$ as in the statement of the lemma with $\zeta_\nu \in \partial A_j \cap \partial B_j$, and $L(x)$ has a positive sign change at ζ_ν (i.e. $L(x)$ goes from negative to positive as x passes through ζ_ν from left to right). Finally, if m_ν is odd and $L^{(m_\nu)}(\zeta_\nu) < 0$, then there are $(m_\nu - 1)/2$ pairs of components $\{A_j, B_j\}$ as in the statement of the lemma with $\zeta_\nu \in \partial A_j \cap \partial B_j$, and $L(x)$ has a negative sign change at ζ_ν. For $p = 1, \ldots, Q$, let H_p be the number of pairs of components $\{A_j, B_j\}$ as in the statement of the lemma, attached to zeros of L in the interval (x_{p-1}, x_p). Since the number of negative sign changes of L in the interval (x_{p-1}, x_p) exceeds the number of positive sign changes in the same interval by at most 1, it follows that

$$H_p \geq \frac{1}{2}(M_p - 1). \tag{80}$$

Summing over p and using (79) and (80) it follows that there are

$$n_4 \geq \frac{1}{2}\sum_{p=1}^{Q}(M_p - 1) = \frac{1}{2}(M - Q) \geq \frac{1}{2}n_2 - O(1)$$

pairs of components $\{A_j, B_j\}$ as in the statement of the lemma, attached to zeros of L in the interval $(-Kr, Kr)$, and using (78) the total number of pairs is at least

$$n_3 + n_4 \geq \frac{1}{2}(n_1 + n_2) - O(1) = \frac{1}{2}n_0 - O(1) \geq \frac{1}{2}n(s_1, 1/\phi_{k-2}) - O(1),$$

thus completing the proof of the lemma. $\square$

15 Analytic continuation of F^{-1}

Proposition 15.1 *For each component $B_j \subseteq D^+(0, Kr)$ as in Lemma 14.1 let S_j be the infimum of $S > 0$ such that the branch of the inverse function F^{-1} mapping $A^+(\Lambda, \infty)$ onto B_j admits unrestricted analytic continuation in $A^+(S, \infty)$. Let $R_j = \max\{S_j, K^{19}r\}$. Then:*
(i) B_j lies in a component $C_j \subseteq H^+$ of the set $F^{-1}(A^+(R_j, \infty))$ which is mapped univalently onto $A^+(R_j, \infty)$ by F;
(ii) at least

$$N_1 \geq \frac{1}{2}n(s_1, 1/\phi_{k-2}) - \frac{n(r)}{16}$$

of the C_j are such that $C_j \subseteq D^+(0, K^{18}r)$;
(iii) of the N_1 components C_j in (ii) at least

$$N_2 \geq \frac{1}{2}n(s_1, 1/\phi_{k-2}) - \frac{n(r)}{8}$$

have $S_j \leq K^{19}r = R_j$.

The proof of Proposition 15.1 will require a number of intermediate lemmas. The existence of a component C_j as in part (i) of the lemma follows from the definition of S_j and R_j. Further, if $S_j > K^{19}r$ then $R_j = S_j$ and by Proposition 10.1 there must be a critical point z^* of F with $z^* \in \partial C_j \cap H^+$ and $F(z^*) \in S(0, S_j) \cap H^+$ (note that F has finitely many critical values in $\frac{1}{2}S_j < |w| < 2S_j$, $\operatorname{Im} w > 0$, by Lemma 10.1). But all but finitely many critical points z^* of F in H^+ are fixpoints of F, by Lemma 10.1, in which case $|z^*| = |F(z^*)| = S_j > K^{19}r$ and hence $C_j \not\subseteq D^+(0, K^{18}r)$. If the zero of $F(z) - F(z^*)$ at z^* has multiplicity m^*, then z^* belongs to the boundary of at most m^* components $C_{j'}$, and so (iii) follows from (ii).

To prove (ii), it suffices therefore to show that among the N_0 components C_j arising from Lemma 14.1 there are less than $n(r)/32$ components with $C_j \not\subseteq D^+(0, K^{18}r)$. Suppose then that M is an integer with

$$M \geq \frac{n(r)}{256} \tag{81}$$

and that $4M$ of the C_j, without loss of generality $C_1, \ldots, C_{4M}$, are such that $C_j \not\subseteq D^+(0, K^{18}r)$, so that C_j meets $D^+(0, Kr)$ and $A^+(K^{17}r, \infty)$.

Lemma 15.1 *For $Kr \leq s \leq K^{17}r$ let $\theta_j(s)$ be the angular measure of $C_j \cap S(0,s)$. Then there exists $j \in \{1, \ldots, 4M\}$ such that*

$$\varepsilon \int_{[K^{q-1}r, K^q r] \cap J_r} \frac{ds}{s\theta_j(s)} \geq T(64r, \phi_0) \tag{82}$$

for $q = 8$ and $q = 11$, where ε and J_r are as in Lemma 12.1.

Proof. Suppose first that at least M of the C_j, without loss of generality $C_1, \ldots, C_M$, are such that (82) fails for some fixed $q \in \{8, 11\}$.

Let $s \in [K^{q-1}r, K^q r] \cap J_r$. For z in the closure of C_j it follows from the definition of C_j that $F(z)$ satisfies $|F(z)| \geq R_j \geq K^{19}r > Ks$. Hence part (v) of Lemma 12.1 shows that the arc $|z| = s, \theta(s) \leq \arg z \leq \pi - \theta(s)$, meets none of the C_j, since r is large. Thus an application of the Cauchy-Schwarz inequality leads to

$$M^2 \leq \left(\sum_{j=1}^{M} \theta_j(s) \right) \left(\sum_{j=1}^{M} \frac{1}{\theta_j(s)} \right) \leq 2\theta(s) \sum_{j=1}^{M} \frac{1}{\theta_j(s)}.$$

Integrating over $[K^{q-1}r, K^q r] \cap J_r$ then gives, using (61) and the assumption that (82) fails,

$$\begin{aligned} T(64r, \phi_0) &< \varepsilon^2 n(r) \int_{[K^{q-1}r, K^q r] \cap J_r} \frac{ds}{s\theta(s)} \\ &\leq \frac{2\varepsilon^2 n(r)}{M^2} \sum_{j=1}^{M} \int_{[K^{q-1}r, K^q r] \cap J_r} \frac{ds}{s\theta_j(s)} \\ &< \frac{2\varepsilon n(r)}{M} T(64r, \phi_0), \end{aligned}$$

so that $M < 2\varepsilon n(r)$, which contradicts (52) and (81). □

Lemma 15.2 *Assume without loss of generality that* (82) *is satisfied for* $j = 1$ *and for* $q = 8$ *and* $q = 11$. *Let* $u_1 \in C_1$ *be such that* $F(u_1) = 2iR_1$. *Choose integers* p, q *according to*

$$(p, q) = (6, 8) \quad \text{if} \quad |u_1| \geq K^9 r, \quad (p, q) = (17, 11) \quad \text{if} \quad |u_1| < K^9 r,$$

and choose

$$T_1 \in (K^{p-5}r, K^{p-4}r) \setminus E_1, \quad T_2 \in (K^{p-1}r, K^p r) \setminus E_1, \tag{83}$$

where E_1 *is the exceptional set of Lemma* 8.1. *Choose an arc* E *of* ∂C_1 *such that* E *joins* $S(0, T_1)$ *to* $S(0, T_2)$ *and, apart from its endpoints, lies in* $T_1 < |z| < T_2$. *Then*

$$\omega(u_1, E, C_1) \leq \exp\left(-\frac{T(64r, \phi_0)}{\pi\varepsilon}\right). \tag{84}$$

Proof. T_1 and T_2 certainly exist, since E_1 has finite logarithmic measure and r is large. There exists a rational function R^* mapping $A^+(R_1, \infty)$ univalently onto $D(0, 1)$ (see Lemma 2.6). Thus ∂C_1 consists of level curves $|R^*(F(z))| = 1$, and T_1, T_2 can be chosen so that ∂C_1 meets the circles $S(0, T_1)$, $S(0, T_2)$ only finitely often, and never tangentially. Since $R^* \circ F$ maps C_1 univalently onto $D(0, 1)$ each component of ∂C_1 is either a simple curve going to infinity in both directions or a simple closed curve (in which case there is only one component). Hence the arc E exists since $Kr < T_1 < T_2 < K^{17}r$ and C_1 meets $D^+(0, R)$ and $A^+(K^{17}r, \infty)$. Using (82) and the inequality

$$\frac{1}{\theta} \leq \frac{1}{2\tan(\theta/4)} + \frac{1}{\pi} \quad \text{for} \quad 0 < \theta < 2\pi,$$

gives

$$\varepsilon \int_{[K^{q-1}r, K^q r] \cap J_r} \frac{ds}{s \tan(\theta_1(s)/4)} \geq T(64r, \phi_0).$$

By the choice of p and q the arc E and the point u_1 are separated by the annulus $K^{q-1}r \leq |z| \leq K^q r$ and so (84) follows from Lemma 2.2. □

Lemma 15.3 *There exists* $w_0 \in F(E)$ *with* $|w_0| \geq R_1 \geq K^{19}r$ *such that*

$$\left|\frac{1}{F(z)} - \frac{1}{w_0}\right| \leq \exp\left(-\frac{T(64r, \phi_0)}{3\pi\varepsilon}\right) \quad \text{for} \quad z \in E. \tag{85}$$

Proof. The function $F(z)$ maps C_1 univalently onto $A^+(R_1, \infty)$ and so

$$F_1(z) = -\frac{R_1}{F(z)}$$

maps C_1 univalently onto $D^+(0, 1)$, with $F_1(u_1) = i/2$. Choose $z_0 \in E$ and set

$$w_0 = F(z_0), \quad v_0 = F_1(z_0) = -\frac{R_1}{w_0}.$$

Since $R_1 > 1$ and E is mapped by F_1 onto an arc of $\partial D^+(0, 1)$, Lemma 2.6 gives a positive absolute constant c such that

$$\left|\frac{1}{F(z)} - \frac{1}{w_0}\right| < |F_1(z) - v_0| \leq c\omega(u_1, E, C_1)^{1/2}.$$

Using (84) and the fact that r is large gives (85). $\square$

In the remainder of this section d will denote a positive constant, not necessarily the same at each occurrence, but independent of r and the constant ε in (85). By (52) the constant K does not depend on r or ε.

Lemma 15.4 *The arc E of ∂C_1 satisfies*

$$E \cap F_1 = \emptyset \quad \textit{where} \quad F_1 = \{z : T_1 \leq |z| \leq T_2, t_1 \leq \arg z \leq t_3\}, \tag{86}$$

in which t_1, t_3 are as in (29)*. Form a domain $D \subseteq H^+$ such that ∂D is the union of the arc E, the radial segment*

$$E^* = \{se^{it_2} : T_1 \leq s \leq T_2\},$$

and arcs T_1^*, T_2^* of the circles $S(0,T_1), S(0,T_2)$ respectively. Let $\gamma_1, \ldots, \gamma_\mu$ be the poles of L in D, repeated according to multiplicity, and set

$$g(z) = \prod_{\nu=1}^{\mu} (1 - z/\gamma_\nu).$$

Then

$$\mu \leq 2^{k-1} n(r) \leq dT(64r, \phi_0) \quad \textit{and} \quad \log M(K^p r, g) \leq d\mu, \tag{87}$$

and there exists $s^* \in (K^{p-3}r, K^{p-2}r) \setminus E_1$ such that

$$\log |g(z)| \geq -dT(64r, \phi_0) \quad \textit{for} \quad |z| = s^*. \tag{88}$$

Finally, there exists $t_4 \in \{3\pi/8, 5\pi/8\}$ such that

$$s^* e^{it_4} \in D \quad \textit{and} \quad \omega(s^* e^{it_4}, E, D) \geq d. \tag{89}$$

Proof. Part (iv) of Lemma 12.1 and (83) show that $|F(z)| \leq |z| + o(1) \leq K^{17}r + o(1)$ for $z \in \partial F_1$. Thus ∂F_1 does not meet the closure of C_1, on which $|F(z)| \geq R_1 \geq K^{19}r$, and so (86) follows, since E meets $S(0,T_1)$ and $S(0,T_2)$.

Since $D \subseteq H^+$, poles of L in D must be poles of ϕ_{k-2}, by (10) and Lemma 4.1, and so the estimate for μ in (87) follows from (26) and (75). The estimate for $\log M(K^p r, g)$ is elementary, since (83) gives $|\gamma_\nu| \geq T_1 \geq K^{p-5}r \geq K^{-5}|z|$ for $|z| \leq K^p r$.

Next, Cartan's lemma [15, p.366] gives a family Y_0 of discs, having sum of diameters at most

$$12h = \frac{1}{2}(K^{p-2}r - K^{p-3}r),$$

outside which, using (83) and (87),

$$\begin{aligned}
\log |g(z)| &\geq \sum_{\nu=1}^{\mu} (\log |z - \gamma_\nu| - \log |\gamma_\nu|) \\
&\geq \mu(\log h - \log T_2) \\
&\geq \mu(\log h - \log(K^p r)) \\
&\geq -d\mu \\
&\geq -dT(64r, \phi_0).
\end{aligned}$$

Thus to obtain s^* satisfying (88) it suffices to choose $s^* \in (K^{p-3}r, K^{p-2}r) \setminus E_1$ such that the circle $S(0, s^*)$ meets none of the discs of Y_0, which is possible since E_1 is a subset of $[1, \infty)$ of finite logarithmic measure and r is large.

Finally, it is clear from (29) and the construction of D that $s^* e^{it_4} \in D$, for some $t_4 \in \{3\pi/8, 5\pi/8\}$. Suppose without loss of generality that $t_4 = 3\pi/8$, so that by (86) and the construction of D the arc E lies in $0 \leq \arg z < t_1$. Since $t_2 \geq 7\pi/16$ by (29), and since (83) gives

$$T_1 < K^{p-4}r < K^{-1}s^* < Ks^* < K^{p-1}r < T_2,$$

it follows from an elementary comparison that

$$\omega(s^* e^{i3\pi/8}, E, D) \geq \omega(s^* e^{i3\pi/8}, [K^{-1}s^*, Ks^*], D'),$$

where D' is the domain given by

$$D' = \{z \in \mathbb{C} : K^{-1}s^* < |z| < Ks^*, 0 < \arg z < 7\pi/16\}.$$

This proves (89). □

The remainder of the proof of Proposition 15.1 will now be divided into two subcases, depending on the modulus of w_0 in (85).

Case 1. *Suppose that*

$$\left| \frac{1}{w_0} \right| \leq \exp\left(-\frac{T(64r, \phi_0)}{24\pi\varepsilon} \right).$$

Then (85) gives

$$\left| \frac{1}{F(z)} \right| \leq 2\exp\left(-\frac{T(64r, \phi_0)}{24\pi\varepsilon} \right) \quad \text{for} \quad z \in E,$$

and so, using (10) and the fact that ϕ_0 is transcendental,

$$|L(z)| \leq 4\exp\left(-\frac{T(64r, \phi_0)}{24\pi\varepsilon} \right) \leq \exp\left(-\frac{T(64r, \phi_0)}{48\pi\varepsilon} \right) \quad \text{for} \quad z \in E. \tag{90}$$

The function

$$u(z) = \log |L(z)g(z)|$$

is subharmonic in D by Lemma 15.4, and by (87), (90), part (iii) of Lemma 12.1 and the construction of D satisfies

$$u(z) \leq \left(d - \frac{1}{48\pi\varepsilon} \right) T(64r, \phi_0) \quad \text{for} \quad z \in E, \tag{91}$$

and

$$u(z) \leq dT(64r, \phi_0) \quad \text{for} \quad z \in \partial D \setminus E. \tag{92}$$

Since ε may be chosen arbitrarily small in Lemma 12.1, whereas the constants d do not depend on ε, (89), (91), (92) and the two-constants theorem [21, p.42] lead to

$$u(s^* e^{it_4}) \leq -\frac{d}{\varepsilon} T(64r, \phi_0) \tag{93}$$

and so, recalling (88), to $L(s^* e^{it_4}) = o(1)$. Since $t_1 \leq t_4 \leq t_3$ and $s^* \notin E_1$, this contradicts part (iv) of Lemma 12.1.

Case 2. *Suppose that*

$$\left|\frac{1}{w_0}\right| > \exp\left(-\frac{T(64r, \phi_0)}{24\pi\varepsilon}\right).$$

This time (85) implies that

$$\left|\frac{1}{F(z)}\right| > \frac{1}{2}\exp\left(-\frac{T(64r, \phi_0)}{24\pi\varepsilon}\right) \quad \text{and} \quad |F(z)w_0| < 2\exp\left(\frac{T(64r, \phi_0)}{12\pi\varepsilon}\right) \quad \text{for} \quad z \in E.$$

Using (85) again gives

$$\left|z - \frac{1}{L(z)} - w_0\right| = |F(z) - w_0| \leq \exp\left(-\frac{T(64r, \phi_0)}{24\pi\varepsilon}\right) \quad \text{for} \quad z \in E. \tag{94}$$

Since $D \subseteq D(0, T_2) \subseteq D(0, K^{17}r)$ by (83), it follows using Lemma 15.3 that

$$|z - w_0| \geq R_1 - K^{17}r \geq K^{19}r - K^{17}r > 2 \quad \text{for} \quad z \in D \cup \partial D. \tag{95}$$

Combining this with (94) gives $|L(z)| \leq 1$ for $z \in E$ and so multiplying (94) by L leads to

$$|(z - w_0)L(z) - 1| \leq \exp\left(-\frac{T(64r, \phi_0)}{24\pi\varepsilon}\right) \quad \text{for} \quad z \in E.$$

This time set

$$u(z) = \log|((z - w_0)L(z) - 1)g(z)|,$$

so that u is again subharmonic on D and satisfies (91) and (92). Applying the two-constants theorem again gives (93), and so

$$(s^* e^{it_4} - w_0)L(s^* e^{it_4}) - 1 = o(1),$$

in view of (88). Using (95) once more leads to $|L(s^* e^{it_4})| \leq 1$, which again contradicts part (iv) of Lemma 12.1.

A contradiction having been obtained in both cases, the proof of Proposition 15.1 is complete.

16 Completion of the proof of Theorem 1.1

By Proposition 15.1 there are, re-labelling if necessary,

$$N_2 \geq \frac{1}{2}n(s_1, 1/\phi_{k-2}) - \frac{n(r)}{8} \tag{96}$$

pairwise disjoint components $D_1, \ldots, D_{N_2}$ of the set $F^{-1}(A^+(K^{19}r, \infty))$ lying in $D^+(0, K^{18}r)$, each mapped univalently onto $A^+(K^{19}r, \infty)$ by F.

Choose $R \in (K^{20}r, K^{21}r)$ such that F has no poles on $S(0, R)$. For $j = 1, \ldots, N_2$ choose $v_j \in D_j$ with $F(v_j) = K^{20}ri$. Then there exists a component $\Omega_j \subseteq H^+$ of the set $F^{-1}(D^+(0, R))$ with $v_j \in \Omega_j$. Here it is possible that $\Omega_j = \Omega_{j'}$ for $j \neq j'$. However, by Lemma 11.2, F is a proper map of Ω_j onto $D^+(0, R)$, of finite topological degree k_j, and the number of zeros of F' in Ω_j is at least $k_j - 1$, counting multiplicity.

Lemma 16.1 *Let*

$$W = \{z \in H^+ : F(z) \in H^+\}, \quad Y = \{z \in H^+ : L(z) \in H^+\}. \tag{97}$$

Then:
(i) $Y \subseteq W$;
(ii) if x_0 is a real pole of L but not a pole of f then x_0 does not lie in the closure of Y;
(iii) if z_0 is a pole of f and z_0 belongs to the closure of Ω_j and of $\Omega_{j'}$ then $\Omega_j = \Omega_{j'}$.

Proof. Assertion (i) follows directly from (10). To prove (ii) suppose that x_0 is a real pole of L. Then x_0 is a simple pole of L and L is univalent on a disc $D(x_0, \delta_0)$ of small radius δ_0. If x_0 is not a pole of f then L has positive residue at x_0 which gives

$$\lim_{y \to 0+} \operatorname{Im} L(x_0 + iy) = -\infty,$$

so that $\operatorname{Im} L(z) < 0$ on $D(x_0, \delta_0) \cap H^+$ and $D(x_0, \delta_0) \cap Y = \emptyset$. To prove (iii) let z_0 be a pole of f in the closure of Ω_j. Then $F(z_0) = z_0$ and $|F(z_0)| < R$ by the choice of R, so that if z_0 is non-real it follows that $z_0 \in \Omega_j$. On the other hand if z_0 is real then $F'(z_0) > 0$ by (10) and (35) so that, provided δ_0 is small enough, $\operatorname{Im} F(z) > 0$ on $D(z_0, \delta_0) \cap H^+$ and $D(z_0, \delta_0) \cap H^+ \subseteq \Omega_j$, using again the fact that $|F(z_0)| < R$. □

The next lemma gives an upper bound for the number of distinct v_j in a given Ω_J.

Lemma 16.2 *For each Ω_J let:*
l_J be the number of v_j in Ω_J;
m_J be the number of simple zeros of F' in Ω_J which are poles of ϕ_{k-2};
n_J be the number of zeros of F' in Ω_J, counting multiplicity, which either are multiple zeros of F' or are not poles of ϕ_{k-2};
p_J be the number of poles of ϕ_{k-2} in Ω_J which are not simple zeros of F';
q_J be the number of distinct poles of f in the closure of Ω_J.
Then

$$l_J \leq m_J + n_J + p_J + q_J. \tag{98}$$

Proof. Assume that (98) is false for some J. The topological degree k_J of the map $F : \Omega_J \to D^+(0, R)$ is at least l_J, and the number of zeros of F' in Ω_J is at least $k_J - 1$. Hence

$$l_J \leq k_J \leq m_J + n_J + 1 \leq l_J.$$

Thus Ω_J must contain $M = l_J = k_J$ distinct v_j, without loss of generality $v_1, \ldots, v_M$, and precisely $k_J - 1$ zeros of F', counting multiplicity. Let

$$\Omega = \Omega_J \cup \bigcup_{j=1}^{M} D_j.$$

Then Ω is a domain, since $v_j \in \Omega_J \cap D_j$, and $F(\Omega) \subseteq H^+$, so that $\Omega \subseteq W$, where W is defined in (97). Clearly $\partial\Omega \subseteq \partial\Omega_J \cup \bigcup_{j=1}^{M} \partial D_j$. Further,

$$\{z \in \Omega : |F(z)| < R\} = \Omega_J, \quad \{z \in \Omega : K^{19} r < |F(z)| < R\} = \bigcup_{j=1}^{M} (\Omega_J \cap D_j), \tag{99}$$

because each value $w \in H^+$ with $K^{19}r < |w| < R$ is taken M times in Ω_J and precisely once in each $\Omega_J \cap D_j$.

Claim 1. *Let $z^* \in \partial\Omega$. Then $F(z^*) \in \mathbb{R} \cup \{\infty\}$.*
To see this, assume that $F(z^*) \notin \mathbb{R} \cup \{\infty\}$. If $z^* \in \partial D_j$ then $|F(z^*)| = K^{19}r < R$, so that z^* is the limit of a sequence in $D_j \cap \Omega_J$ and so is in the closure of Ω_J, and hence an interior point of Ω_J. This is impossible, and so z^* must belong to $\partial\Omega_J$ and $|F(z^*)| = R$. But then (99) shows that z^* is the limit of a sequence in some $\Omega_J \cap D_j$, so that z^* is in the closure of D_j and is therefore an interior point of D_j, since $|F(z^*)| = R$ and $F(z^*) \notin \mathbb{R} \cup \{\infty\}$. This contradiction completes the proof of Claim 1.

Let X_J be the component of W which contains Ω_J. Then $\Omega \subseteq X_J$. Indeed, $\Omega = X_J$ by Claim 1, since otherwise there exists a path $\gamma \subseteq X_J \subseteq W$ joining a point in Ω to a point in $X_J \setminus \Omega$, and γ must meet the boundary of Ω.

Claim 2. *If Ω is unbounded then $L(z) \to 0$ as $z \to \infty$ in Ω.*
Since F is bounded on Ω_J and each D_j is bounded, it follows that $F(z)$ is bounded as $z \to \infty$ in Ω, which implies using (10) that $L(z) \to 0$. This proves Claim 2.

Choose $t_0 \in (0, \pi)$ such that L has no critical values w with $0 < |w| < \infty, \arg w = t_0$. Each D_j contains by Lemma 14.1 a component A_j of the set $L^{-1}(D^+(0, \lambda))$, where λ is small and positive. This gives M distinct points $V_j \in \Omega$ such that $L(V_j) = \frac{1}{2}\lambda e^{it_0}$. Take the branch of L^{-1} mapping $\frac{1}{2}\lambda e^{it_0}$ to V_j and analytically continue L^{-1} along the half-open ray $w = Se^{it_0}, \lambda/2 \leq S < \infty$. The image $z = L^{-1}(w)$ under this continuation cannot exit Ω, because $Y \subseteq W$, and is bounded because of Claim 2. Thus the continuation is possible along the whole half-open ray, and as $S \to \infty$ the image $z = L^{-1}(w)$ must tend to a pole z_0 of L, which lies in the closure of Ω and of Y.

Since each $D_{j'}$ is a component of the set $F^{-1}(A^+(K^{19}r, \infty))$ lying in $D^+(0, K^{18}r)$, the closure of $D_{j'}$ contains no fixpoint of F, and so z_0 is in the closure of Ω_J. Further, if z_0 is real then since z_0 is in the closure of Y it follows from Lemma 16.1 that z_0 is a pole of f. Suppose, on the other hand, that z_0 is non-real. Then z_0 is a pole of ϕ_{k-2} by Lemma 4.1, and $F(z_0) = z_0 \in H^+$ so that $z_0 \in \Omega$, and again since the $D_{j'}$ contain no fixpoints of F it follows that $z_0 \in \Omega_J$.

Moreover, the continuations from distinct V_j cannot coalesce, because of the choice of t_0, and cannot tend to the same pole of L, because all these poles are simple. This gives at least k_J distinct poles of L, all of which must be poles of ϕ_{k-2} in Ω_J or poles of f in the closure of Ω_J. Hence

$$l_J = k_J \leq m_J + p_J + q_J,$$

contradicting the assumption that (98) is false. $\square$

Recall next from Lemma 10.1 that all but finitely many zeros of F' in H^+ are simple and are poles of ϕ_{k-2}, and by Lemmas 4.1 and 10.9 all but finitely many poles of ϕ_{k-2} in H^+ are simple zeros of F'. Furthermore, if $z_0 \in \Omega_J$ is a pole of ϕ_{k-2} then z_0 is a pole of L and so

$$z_0 = F(z_0) \in D^+(0, R) \subseteq D^+(0, K^{21}r) \subseteq D^+(0, s_{22}),$$

where s_{22} is as defined in Lemma 12.1. Summing over all the distinct Ω_J and using (77), (96), (98) and Lemma 16.1 now leads to

$$\begin{aligned}
\frac{1}{2}n(s_1, 1/\phi_{k-2}) - \frac{n(r)}{8} \leq N_2 &\leq \sum_{\Omega_J} l_J \\
&\leq \sum_{\Omega_J}(m_J + n_J + p_J + q_J) \\
&\leq O(1) + \sum_{\Omega_J} m_J \\
&\leq O(1) + \frac{1}{2}n(s_{22}, \phi_{k-2}) \\
&\leq O(1) + \frac{1}{2}n(s_1, 1/\phi_{k-2}) - \frac{n(r)}{4}.
\end{aligned}$$

If r is large enough this gives a contradiction, and the proof of Theorem 1.1 is complete.

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