

NON-REAL ZEROS OF DERIVATIVES OF REAL MEROMORPHIC FUNCTIONS

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ABSTRACT. The main result of the paper determines all real meromorphic functions f of finite order in the plane such that f' has finitely many zeros while f and $f^{(k)}$, for some $k \geq 2$, have finitely many non-real zeros.
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1. INTRODUCTION

This paper concerns non-real zeros of the derivatives of real meromorphic functions in the plane: here a meromorphic function is called real if it maps \mathbb{R} into $\mathbb{R} \cup \{\infty\}$. In the setting of real entire functions [19, 20, 40] the class V_{2p} is defined for $p \geq 0$ to consist of all entire functions $f(z) = g(z) \exp(-az^{2p+2})$, where $a \geq 0$ is real and g is a real entire function with real zeros of genus at most $2p+1$ [16, p.29]. It is well known [27] that V_0 coincides with the Laguerre-Pólya class LP of entire functions which are locally uniform limits of real polynomials with real zeros. With the notation $V_{-2} = \emptyset$ the class U_{2p}^* may then be defined for $p \geq 0$ as the set of entire functions $f = Ph$, where $h \in V_{2p} \setminus V_{2p-2}$ and P is a real polynomial without real zeros [8]; thus each real entire function of finite order with finitely many non-real zeros belongs to U_{2p}^* for some $p \geq 0$. The following results, in which all counts of zeros are with respect to multiplicity, established conjectures of Wiman [1, 2] and Pólya [36] respectively.

Theorem 1.1 ([8, 40]). *Let $p \in \mathbb{N}$ and let $f \in U_{2p}^*$. Then f'' has at least $2p$ non-real zeros.*

Theorem 1.2 ([4]). *Let p be a positive integer and let $f \in U_{2p}^*$. Then the number of non-real zeros of the k th derivative $f^{(k)}$ tends to infinity with k .*

Theorem 1.3 ([5, 31]). *If f is a real entire function of infinite order then $ff^{(k)}$ has infinitely many non-real zeros, for every $k \geq 2$.*

For real meromorphic functions with poles rather less is known. All meromorphic functions f in the plane for which all derivatives $f^{(k)}$ ($k \geq 0$) have only real zeros were determined by Hinkkanen in a series of papers [24, 25, 26]: such functions have at most two distinct poles, by the Pólya shire theorem [16, Theorem 3.6, p.63]. Functions with real poles, for which some of the derivatives have only real zeros, were treated in a number of papers including [21, 22, 38]. In particular the following theorem was proved in [22].

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Theorem 1.4. *Let f be a real meromorphic function in the plane with only real zeros and poles (and at least one of each). Assume that f' has no zeros and that f'' has only real zeros. Then f has one of the forms*

$$A \tan(az + b) + B, \quad \frac{az + b}{cz + d}, \quad A \cdot \frac{(az + b)^2 - 1}{(az + b)^2},$$

where A, B, a, b, c, d are real numbers.

The following result will be proved: here the reality of all but finitely many poles of f is no longer an assumption but turns out to be a conclusion, and the second derivative is shown to be exceptional.

Theorem 1.5. *Let f be a real meromorphic function in the plane, not of the form $f = Re^P$ with R a rational function and P a polynomial. Let $k \geq 2$ be an integer. Assume that:*

- (i) *all but finitely many zeros of f and $f^{(k)}$ are real;*
- (ii) *the first derivative f' has finitely many zeros;*
- (iii) *there exists $M \in (0, \infty)$ such that if ζ is a pole of f of multiplicity m_ζ then*

$$(1) \quad m_\zeta \leq M + |\zeta|^M;$$

- (iv) *if $k = 2$ then f'/f has finite order.*

Then f satisfies

$$(2) \quad f(z) = \frac{R(z)e^{icz} - 1}{AR(z)e^{icz} - A}, \quad \text{where } c \in (0, \infty), A \in \mathbb{C} \setminus \mathbb{R},$$

and

$$(3) \quad R \text{ is a rational function with } |R(x)| = 1 \text{ for all } x \in \mathbb{R}.$$

Moreover, $k = 2$ and all but finitely many poles of f are real.

Conversely, if f is given by (2) and (3) then f satisfies (i) and (ii) with $k = 2$.

If the function f is given by $f = Re^P$ with R a rational function and P a polynomial then obviously f and all its derivatives have finitely many zeros. It is not clear whether the assumptions (iii) and (iv) are really necessary in Theorem 1.5, but they are required for the proof presented below. In the case $k \geq 3$ it will be proved in §4 that if a real meromorphic function f satisfies (i) and (ii) then f'/f has finite order: this will use a method of Frank [10], which is not available for $k = 2$. On the other hand, if f itself has finite order then (1) holds since, with the standard notation of Nevanlinna theory [16],

$$n(r, f) = O(N(2r, f)) = O(T(2r, f)).$$

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2. PRELIMINARIES

For $a \in \mathbb{C}$ and $r > 0$ set $D(a, r) = \{z \in \mathbb{C} : |z - a| < r\}$ and correspondingly let $S(a, r) = \{z \in \mathbb{C} : |z - a| = r\}$. The following lemma is standard [42, pp.116-7].

Lemma 2.1 ([42]). *Let u be a non-constant continuous subharmonic function in the plane. For $r > 0$ let $\theta^*(r)$ be the angular measure of that subset of $S(0, r)$ on*

which $u(z) > 0$, except that $\theta^*(r) = \infty$ if $u(z) > 0$ on the whole circle $S(0, r)$. Then, if $r \leq R/4$ and r is sufficiently large,

$$(4) \quad B(r, u) = \max\{u(z) : |z| = r\} \leq 9\sqrt{2}B(R, u) \exp\left(-\pi \int_{2r}^{R/2} \frac{ds}{s\theta^*(s)}\right).$$

Next, let the function g be meromorphic in a domain containing the closed upper half-plane $\overline{H} = \{z \in \mathbb{C} : \text{Im } z \geq 0\}$. For $t \geq 1$ let $\mathbf{n}(t, g)$ be the number of poles of g , counting multiplicity, in $\{z \in \mathbb{C} : |z - it/2| \leq t/2, |z| \geq 1\}$. The Tsuji characteristic $\mathfrak{T}(r, g)$ [14, 32, 41] is defined for $r \geq 1$ by $\mathfrak{T}(r, g) = \mathfrak{m}(r, g) + \mathfrak{N}(r, g)$, where

$$\mathfrak{m}(r, g) = \frac{1}{2\pi} \int_{\sin^{-1}(1/r)}^{\pi - \sin^{-1}(1/r)} \frac{\log^+ |g(r \sin \theta e^{i\theta})|}{r \sin^2 \theta} d\theta \quad \text{and} \quad \mathfrak{N}(r, g) = \int_1^r \frac{\mathbf{n}(t, g)}{t^2} dt.$$

Lemma 2.2 ([32]). *Let the function g be meromorphic in \overline{H} and assume that $\mathfrak{m}(r, g) = O(\log r)$ as $r \rightarrow \infty$. Then, as $R \rightarrow \infty$,*

$$\int_R^\infty \int_0^\pi \frac{\log^+ |g(re^{i\theta})|}{r^3} d\theta dr = O\left(\frac{\log R}{R}\right).$$

The following theorem was proved in [6, 39] using the rescaling method [43].

Theorem 2.1 ([6, 39]). *Let $k \geq 2$ and let \mathcal{F} be a family of functions meromorphic on a plane domain D such that $ff^{(k)}$ has no zeros in D , for each $f \in \mathcal{F}$. Then the family $\{f'/f : f \in \mathcal{F}\}$ is normal on D .*

The proof of Theorem 1.5 will require well known results of Edrei-Fuchs [7] and Hayman [17] respectively.

Lemma 2.3 ([7]). *Let $1 < r < R < \infty$ and let the function g be meromorphic in $|z| \leq R$. Let $I(r)$ be a subset of $[0, 2\pi]$ of Lebesgue measure $\mu(r)$. Then*

$$\frac{1}{2\pi} \int_{I(r)} \log^+ |g(re^{i\theta})| d\theta \leq \frac{11R\mu(r)}{R-r} \left(1 + \log^+ \frac{1}{\mu(r)}\right) T(R, g).$$

Lemma 2.4 ([17]). *Let $S(r)$ be an unbounded positive non-decreasing function on $[r_0, \infty)$, continuous from the right, of order ρ . Let $A > 1, B > 1$. Then*

$$\overline{\text{logdens}} G \leq \rho \left(\frac{\log A}{\log B}\right), \quad \text{where} \quad G = \{r \geq r_0 : S(Ar) \geq BS(r)\}.$$

The next lemma [9, 37] has found widespread applications in function theory and complex dynamics.

Lemma 2.5 ([9, 37]). *Let the function g be transcendental and meromorphic in the plane such that the set of finite singular values of the inverse function g^{-1} is bounded. Then there exist $L > 0$ and $M > 0$ such that*

$$\left| \frac{z_0 g'(z_0)}{g(z_0)} \right| \geq C \log^+ \left| \frac{g(z_0)}{M} \right| \quad \text{for} \quad |z_0| > L,$$

where C is a positive absolute constant, in particular independent of g, L and M .

3. AN ELEMENTARY LEMMA AND SOME CONSEQUENCES

Lemma 3.1. *Let*

$$g(z) = \frac{1}{1 - e^z}.$$

Then for $k \geq 3$ the function $g^{(k)}$ has infinitely many zeros off the imaginary axis.

Proof. With the notation $X = e^z$, it will be proved by induction that

$$(5) \quad g^{(k)}(z) = \frac{P_k(X)}{(1-X)^{k+1}} = \frac{X^k + A_{k-1}X^{k-1} + \dots}{(1-X)^{k+1}},$$

in which the numerator $P_k(X)$ is a monic polynomial in X of degree k with constant coefficients, and the coefficient of X^{k-1} is denoted by A_{k-1} . Evidently (5) is true for $k = 1$ and $k = 2$, with $A_0 = 0$ and $A_1 = 1$. Assuming that $k \geq 2$ and that (5) holds, differentiation gives

$$\begin{aligned} P_{k+1}(X) &= (1-X)^{k+2}g^{(k+1)}(z) \\ &= (kX^k + (k-1)A_{k-1}X^{k-1} + \dots)(1-X) + \\ &\quad + (k+1)X(X^k + A_{k-1}X^{k-1} + \dots) \\ &= X^{k+1} + X^k(k - (k-1)A_{k-1} + (k+1)A_{k-1}) + \dots \\ &= X^{k+1} + (k + 2A_{k-1})X^k + \dots \end{aligned}$$

This proves (5) with k replaced by $k + 1$, and gives in addition the recurrence relation $A_k = k + 2A_{k-1}$. Since $A_1 = 1$ it follows at once that $A_k \geq k + 2$ for $k \geq 2$. But then for $k \geq 3$ the sum of the roots of $P_k(X)$ is $-A_{k-1} \leq -k - 1$, and so these roots cannot all have modulus 1. \square

Lemma 3.2. *Let the function f be given by (2) and (3). Then all but finitely many zeros of f'' and poles of f are real, and for $k \geq 3$ the function $f^{(k)}$ has infinitely many non-real zeros.*

Proof. The fact that all but finitely many poles of f and zeros of f'' are real is proved in [35], but a slightly different argument is given here for completeness. First, it is evident from (2) and (3) that there exist constants D, E and rational functions S, U, V with

$$(6) \quad f(z) = D + \frac{E}{1 - S(z)e^{icz}} \quad \text{and} \quad f''(z) = \frac{U(z)e^{icz} + V(z)e^{2icz}}{(1 - S(z)e^{icz})^3},$$

where $|S(x)| = 1$ for $x \in \mathbb{R}$. If z_k is a pole of f with $|z_k|$ large then $e^{icz_k} = C + o(1)$, where $C \cdot S(\infty) = 1$ and so $|C| = 1$. Hence z_k lies near a zero x_j of $e^{icz} - C$, and x_j is real. Moreover if $|x_j|$ is large enough then Rouché's theorem gives precisely one pole of f near x_j . Since f is real, so is z_k .

Next, let $x \in \mathbb{R}$. Since $f''(x)$ is real and $|S(x)| = 1$, the representation (6) gives

$$f''(x) = \overline{f''(x)} = \frac{\overline{U(x)}e^{-icx} + \overline{V(x)}e^{-2icx}}{(1 - \overline{S(x)}e^{-icx})^3} = -\frac{S(x)^3\overline{U(x)}e^{2icx} + S(x)^3\overline{V(x)}e^{icx}}{(1 - S(x)e^{icx})^3}$$

so that $S(x)^3\overline{U(x)} = -V(x)$ and $|V(x)/U(x)| = 1$. The same argument as for the poles now shows that all but finitely many zeros of f'' are real (this may also be proved using the Levin-Ostrovskii representation [32] for $-f''/f'$, since the real function $1/f'$ has finitely many poles and finitely many non-real zeros).

To prove the last assertion let $k \geq 3$. Write $S(\infty) = e^{id}$ for some $d \in \mathbb{R}$ and

$$h(z) = \frac{1}{1 - S(z)e^{icz}} = \frac{1}{1 - T(z)e^{i(cz+d)}}, \quad H(z) = \frac{1}{1 - e^{i(cz+d)}} = g(i(cz + d)),$$

in which the function T is rational with $T(\infty) = 1$, and g is as in Lemma 3.1. By (6) it suffices to prove that $h^{(k)}$ has infinitely many non-real zeros. Lemma 3.1 gives a non-real zero w of $H^{(k)}$ and there exist small positive s, t such that

$$|H^{(k)}(z)| \geq s \quad \text{and} \quad s \leq |1 - e^{i(cz+d)}| \leq \frac{1}{s} \quad \text{for } t \leq |z - w| \leq 3t.$$

Let n be a large positive integer. Then the periodicity of H yields

$$h(z) = \frac{1}{1 - e^{i(cz+d)}(1 + o(1))} = \frac{1 + o(1)}{1 - e^{i(cz+d)}} = H(z) + o(1)$$

for $t \leq |z - (w + n2\pi/c)| \leq 3t$. This implies that

$$h^{(k)}(z) = H^{(k)}(z) + o(1) = H^{(k)}(z)(1 + o(1))$$

on the circle $S(w + n2\pi/c, 2t)$, and so $h^{(k)}$ has a zero in $D(w + n2\pi/c, 2t)$ by Rouché's theorem. \square

4. PROOF OF THEOREM 1.5: THE FIRST PART

Let the integer k and the function f be as in the statement of Theorem 1.5. Since f is not of the form $f = Re^P$ with R a rational function and P a polynomial, the logarithmic derivative $L = f'/f$ is transcendental. The first task is to show that f'/f has finite order. For $k = 2$ this is true by hypothesis (iv), and for the case $k \geq 3$ the fact that f and $f^{(k)}$ have finitely many non-real zeros leads at once to the following lemma, which is proved using a method of Frank [10, 12] (see also [6, 11, 13]), but with the Nevanlinna characteristic replaced by that of Tsuji.

Lemma 4.1 ([10]). *Assume that $k \geq 3$. Then the Tsuji characteristic of $L = f'/f$ satisfies*

$$(7) \quad \mathfrak{T}(r, L) = O(\log r) \quad \text{as } r \rightarrow \infty.$$

Lemma 4.2. *The function $L = f'/f$ has finite order.*

Proof. If $k = 2$ there is nothing to prove, so assume that $k \geq 3$. It follows at once from (7) that

$$m(r, 1/L) \leq \mathfrak{T}(r, L) + O(1) = O(\log r) \quad \text{as } r \rightarrow \infty.$$

Hence Lemma 2.2 and the fact that f is real give

$$\int_R^\infty \frac{m(r, 1/L)}{r^3} dr = O\left(\frac{\log R}{R}\right) \quad \text{as } R \rightarrow \infty.$$

But $1/L = f/f'$ has finitely many poles and it now follows that, as $R \rightarrow \infty$,

$$\frac{T(R, 1/L)}{R^2} \leq 2 \int_R^\infty \frac{T(r, 1/L)}{r^3} dr \leq 2 \int_R^\infty \frac{m(r, 1/L)}{r^3} dr + O\left(\frac{\log R}{R^2}\right) = O\left(\frac{\log R}{R}\right).$$

\square

Lemma 4.3. *There exists a set $E_1 \subseteq [1, \infty)$, of upper logarithmic density at most $1/2$, with the following property. If $\sigma \in (0, \pi/2)$ then L satisfies, for all large $r \notin E_1$,*

$$(8) \quad |rL(re^{i\theta})| \leq 1 \quad \text{for } \sigma \leq \pm\theta \leq \pi - \sigma.$$

Proof. Since f is real it is enough to prove this for $\sigma \leq \theta \leq \pi - \sigma$. Set

$$g(z) = \frac{f(z)}{zf'(z)}.$$

Then g is transcendental of finite order, by Lemma 4.2. Lemma 2.4 gives positive real numbers B_1 and r_1 , with r_1 large, such that

$$(9) \quad T(2r, g) \leq B_1 T(r, g)$$

for all $r \geq r_1$ outside a set E_1 of upper logarithmic density at most $1/2$. For $r \geq r_1$ let $I_1(r)$ be that subset of $[0, 2\pi]$ on which $|g(re^{i\theta})| \geq r$ and let $\mu_1(r)$ denote the Lebesgue measure of $I_1(r)$. By (9), Lemma 2.3 and the fact that g is transcendental with finitely many poles, there exists a positive real number τ such that $\mu_1(r) > 5\tau$ for all $r \geq r_1$ with $r \notin E_1$.

Let $\sigma \in (0, \pi/2)$. Theorem 2.1 and the fact that f' has finitely many zeros imply that the family

$$\mathcal{G} = \{G_r(z) = f(rz)/rf'(rz) : r \geq r_1, r \notin E_1\}$$

is a normal family of analytic functions on the domain

$$D^* = \{z \in \mathbb{C} : 1/2 < |z| < 2, 0 < \arg z < \pi\}.$$

Suppose that there exist sequences $s_n \rightarrow \infty$ and θ_n with $s_n \notin E_1$ and $\sigma \leq \theta_n \leq \pi - \sigma$ such that

$$G_{s_n}(e^{i\theta_n}) < 1.$$

Then without loss of generality G_{s_n} converges locally uniformly to G^* on D^* , and $G^* \not\equiv \infty$ on D^* . Thus G^* is analytic on D^* . But then there exists $N_1 > 0$ such that $|G_{s_n}(e^{i\theta})| \leq N_1$ for all large n and for $\tau \leq \theta \leq \pi - \tau$. On the other hand, by the definitions of g and $I_1(r)$ and the fact that f is real there exists for each large n a real number $\phi_n \in (\tau, \pi - \tau)$ such that $|G_{s_n}(e^{i\phi_n})| \geq s_n$. This contradiction proves the lemma. \square

Lemma 4.4. *The function f has finite order.*

Proof. Combining Lemma 4.2 with (1) and the hypothesis that f' has finitely many zeros shows that the zeros and poles of f have finite exponent of convergence. Hence there exist a real meromorphic function Π of finite order and a real entire function h such that

$$(10) \quad f = \Pi e^h, \quad L = \frac{f'}{f} = \frac{\Pi'}{\Pi} + h'.$$

Since L and Π have finite order, so has h . Assume that h is transcendental: if this is not the case there is nothing to prove. Standard estimates for logarithmic derivatives [15] show that there exists a positive real number N_2 such that

$$(11) \quad \log \left| \frac{\Pi'(z)}{\Pi(z)} \right| \leq N_2 \log |z|$$

provided that $|z|$ is large and lies outside a set $E_2 \subseteq [1, \infty)$ of finite logarithmic measure. Let σ be a small positive real number and let N_1 and the set E_1 be as in Lemma 4.3. For large r let

$$I_2(r) = \{\theta \in [0, 2\pi] : \log |h'(re^{i\theta})| \geq (N_2 + 1) \log r\}.$$

Since h' is real, it follows from (8), (10) and (11) that, for all large $r \notin E_3 = E_1 \cup E_2$, the set $I_2(r)$ may be enclosed in a set $I_3(r) \subseteq [0, 2\pi]$ of Lebesgue measure 4σ . Applying Lemma 2.3 to the transcendental entire function h' yields

$$\begin{aligned} T(r, h') &\leq \frac{1}{2\pi} \int_{I_3(r)} \log^+ |h'(re^{i\theta})| d\theta + O(\log r) \\ &\leq 176\sigma \left(1 + \log^+ \frac{1}{4\sigma}\right) T(2r, h') = \delta T(2r, h') \end{aligned}$$

for all large $r \notin E_3 = E_1 \cup E_2$ and so for all r in a set $E_4 \subseteq [1, \infty)$ of lower logarithmic density at least $1/2$. Lemma 2.4 now shows that the order $\rho(h')$ of h' satisfies

$$\frac{1}{2} \leq \underline{\log\text{dens}} E_4 \leq \overline{\log\text{dens}} E_4 \leq \rho(h') \left(\frac{\log 2}{\log 1/\delta} \right).$$

But δ may be made arbitrarily small by choosing σ small enough, and this is a contradiction. \square

5. ASYMPTOTIC VALUES OF f

The next step in the proof of Theorem 1.5 is to show that f has precisely two asymptotic values, both of them finite and non-real. Since f has finite order by Lemma 4.4, and since f' has finitely many zeros, applying a theorem of Bergweiler and Eremenko [3] shows that f has finitely many asymptotic values, each corresponding to a direct transcendental singularity of the inverse function [3, 34].

Lemma 5.1. *Neither 0 nor ∞ is an asymptotic value of f , and f has infinitely many zeros and poles.*

Proof. Let g be f or $1/f$ and assume that ∞ is an asymptotic value of g . Let σ be a small positive real number and let N_1 and the set E_1 be as in Lemma 4.3. Since the inverse function g^{-1} has finitely many singular values, Lemmas 2.5 and 4.3 give a large positive real number N_2 such that, for all large $r \notin E_1$,

$$(12) \quad |g(re^{i\theta})| \leq N_2 \quad \text{for } \sigma \leq \pm\theta \leq \pi - \sigma.$$

Since ∞ is by assumption an asymptotic value of g and therefore a direct transcendental singularity of g^{-1} [3] there exists an unbounded component U of the set $\{z \in \mathbb{C} : |g(z)| > N_2\}$ on which g has no poles, and the function

$$u(z) = \log \left| \frac{g(z)}{N_2} \right| \quad (z \in U), \quad u(z) = 0 \quad (z \in \mathbb{C} \setminus U),$$

is continuous, subharmonic and non-constant in the plane. Let $\theta^*(r)$ be defined as in Lemma 2.1. Then (12) implies that $\theta^*(r) \leq 4\sigma$ for all large $r \notin E_1$, and E_1 has upper logarithmic density at most $1/2$. Let r be large and let $R \geq 4r$. Then (4) gives

$$\log B(R, u) \geq \pi \int_{2r}^{R/2} \frac{ds}{s^{\theta^*(s)}} - O(1) \geq \frac{\pi}{4\sigma} \int_{[2r, R/2] \setminus E_1} \frac{ds}{s} - O(1) \geq \frac{\pi}{12\sigma} \log R$$

as $R \rightarrow \infty$. Since σ may be chosen arbitrarily small, it follows that u has infinite order. But Poisson's formula leads to

$$B(R, u) \leq \frac{3}{2\pi} \int_0^{2\pi} u(2Re^{i\theta}) d\theta \leq 3m(2R, f) + 3m(2R, 1/f) + O(1),$$

and f has finite order, which gives an immediate contradiction. It remains only to observe that f must have infinitely many zeros and poles, by Iversen's theorem. \square

Lemma 5.2. *There exists $\alpha \in \mathbb{C} \setminus \mathbb{R}$ such that the set of asymptotic values of f is precisely $\{\alpha, \bar{\alpha}\}$. Moreover f cannot tend to both α and $\bar{\alpha}$ on paths tending to infinity in the same component of $\mathbb{C} \setminus \mathbb{R}$.*

Proof. By Lemma 5.1, neither 0 nor ∞ is an asymptotic value of f . Suppose that f has exactly one asymptotic value a . Since f has infinitely many poles, $h = (f - a)/f'$ is transcendental with finitely many poles, and by a result of Lewis, Rossi and Weitsman [33] there exists a path γ tending to infinity with

$$\int_{\gamma} \left| \frac{f'(z)}{f(z) - a} \right| |dz| = \int_{\gamma} \left| \frac{1}{h(z)} \right| |dz| < \infty.$$

But then integration shows that $f(z) - a$ tends to a non-zero finite value as z tends to infinity on γ , a contradiction.

Now let ε be small and positive, so small that $|a - a'| \geq 2\varepsilon$ whenever a and a' are distinct singular values of f^{-1} . Let a be an asymptotic value of f . Then there exists a component C_a of the set $\{z \in \mathbb{C} : |f(z) - a| < \varepsilon\}$ containing a path tending to infinity on which $f(z)$ tends to a . By a standard argument [34, p.287] the function $\phi(t) = f^{-1}(a + e^{-t})$ maps the half-plane $\operatorname{Re}(t) > \log 1/\varepsilon$ univalently onto C_a . Furthermore, the component C_a contains infinitely many paths $\gamma_{a,j}$, each tending to infinity and mapped by f onto $L_a = \{a + t : 0 < t < \varepsilon/2\}$. If a is non-real then the $\gamma_{a,j}$ do not meet \mathbb{R} , since ε is small and f is real. On the other hand, if a is real and $\gamma_{a,j}$ meets \mathbb{R} then $\gamma_{a,j} \subseteq \mathbb{R}$ since f is real and has no critical values $w \in L_a$. Moreover if a is real then $f(z)$ also tends to a as z tends to infinity on the path $\overline{\gamma_{a,j}}$.

It follows that if f has at least three distinct asymptotic values, or at least two distinct real asymptotic values, then there exist disjoint simple paths λ_1 and λ_2 tending to infinity with the following properties. Either both paths lie in the upper half-plane H^+ , or both in the lower half-plane H^- , and as z tends to infinity on λ_j the function $f(z)$ tends to $b_j \in \mathbb{C} \setminus \{0\}$ with $b_1 \neq b_2$. Choose a large positive real number R_1 , in particular so large that f has no non-real zeros z with $|z| \geq R_1$. This gives an unbounded domain D_1 with no zeros of f in its closure, bounded by a subpath of λ_1 , a subpath of λ_2 and an arc of the circle $S(0, R_1)$. Since $b_1 \neq b_2$ the Phragmén-Lindelöf principle forces $1/f$ to be unbounded on D_1 , which implies the existence of a direct transcendental singularity of f^{-1} over 0, contradicting Lemma 5.1.

Thus f has exactly two distinct asymptotic values, of which at most one is real. Since f is real the set of asymptotic values of f is $\{\alpha, \bar{\alpha}\}$ for some non-real α , and the last assertion of the lemma follows from the argument of the previous paragraph. \square

6. THE MULTIPLICITIES OF THE POLES OF f

In this section it will be shown that all but finitely many poles of f are simple. It follows from Lemma 5.2 and the fact that f' has finitely many zeros that a simple closed polygonal path J may be chosen with the following properties. J is symmetric with respect to the real axis, and all non-real finite singular values of f^{-1} lie on J . Moreover, $J \cap \mathbb{R} = \{-R, R\}$, where the positive real number R is chosen so that all the finitely many real critical values of f lie in the interval $[-R/2, R/2]$, which in turns lies in the interior domain D_1 of J . Let D_2 be the complement of $J \cup D_1$ in $\mathbb{C} \cup \{\infty\}$. Thus the D_j are simply connected domains, with $0 \in D_1$ and $\infty \in D_2$.

Then each component C of $f^{-1}(D_2)$ is simply connected, and contains exactly one pole of f of multiplicity p , say, and is mapped $p : 1$ onto D_2 by f . This follows from the fact that f^{-1} has no singular values in $D_2 \setminus \{\infty\}$, and may be proved (see [29, p.362] or [30]) by choosing a quasiconformal mapping ϕ which satisfies $\phi(D_2) = \{w \in \mathbb{C} \cup \{\infty\} : |w| > 1\}$ and $\phi(\infty) = \infty$, and writing $\phi \circ f = g \circ \psi$, where g is meromorphic and ψ is quasiconformal, following which the argument from [34, p.287] is applied to g .

Next, let $D_3 = D_1 \setminus (-R, R/2]$. Then D_3 is a simply connected domain containing no singular values of f^{-1} , and all components of $f^{-1}(D_3)$ are conformally equivalent to D_3 under f . Since f' has finitely many zeros a standard argument [5, Lemma 4.2] then shows that each component C of $f^{-1}(D_1)$ contains finitely many components of $f^{-1}(D_3)$, so that f is finite-valent on C . Moreover all but finitely many components of $f^{-1}(D_1)$ are conformally equivalent to D_1 under f . These considerations lead at once to the following lemma.

Lemma 6.1. *All but finitely many poles w of f lie in components $C = C_w$ of $f^{-1}(D_2)$ such that C and all components D of $f^{-1}(D_1)$ with $\partial C \cap \partial D \neq \emptyset$ have the following properties:*

- (a) f' has no zeros in $C \cup \partial C$, and w is the only pole of f in C ;
- (b) f' has no zeros in $D \cup \partial D$;
- (c) the mapping $f : D \rightarrow D_1$ is a conformal bijection;
- (d) f has no non-real zeros in D .

The next lemma is the key to the proof of Theorem 1.5.

Lemma 6.2. *All but finitely many poles of f are simple.*

Proof. Assume that f has infinitely many multiple poles. Then f has a multiple pole w satisfying the conclusions of Lemma 6.1. Let $C = C_w$ be the component of $f^{-1}(D_2)$ which contains w . Then C is simply connected and ∂C consists of finitely many pairwise disjoint piecewise smooth simple curves, each tending to infinity in both directions. Let Γ be a component of ∂C . Then $\Gamma \subseteq \partial D$ for some component D of $f^{-1}(D_1)$, and f is univalent on D and so on Γ by Lemma 6.1(c). As z tends to infinity along Γ in each direction, the image $f(z)$ tends to either α or $\bar{\alpha}$. Here Γ will be called a type A component of ∂C if $f(\Gamma)$ is a component of $J \setminus \{\alpha, \bar{\alpha}\}$. If this is not the case then $f(\Gamma)$ is either $J \setminus \{\alpha\}$ or $J \setminus \{\bar{\alpha}\}$, and Γ will be called type B.

Every type A component Γ of ∂C must meet the real axis, by Lemma 5.2, and so is symmetric with respect to \mathbb{R} , since f is real. Conversely, a component Γ of ∂C which meets the real axis has $\Gamma = \bar{\Gamma}$ and must be type A since the real function f then has asymptotic values α and $\bar{\alpha}$ on Γ .

If Γ is a type B component of ∂C and D is that component of $f^{-1}(D_1)$ which satisfies $\partial D \cap \Gamma \neq \emptyset$ then $\Gamma = \partial D$ by Lemma 6.1(b) and (c). Thus at least one component of ∂C must be type A, and hence at least two. On the other hand ∂C cannot have three type A components, since one would have to separate the other two. It follows that since w is a multiple pole there must be exactly two type A components A_1 and A_2 of ∂C , and at least one type B component B_1 . Here B_1 is the boundary of a component E_1 of $f^{-1}(D_1)$. This component E_1 cannot meet the real axis since otherwise the fact that $B_1 \cap \mathbb{R} = \emptyset$ gives $\mathbb{R} \subseteq E_1$, whereas $A_q \cap \mathbb{R} \neq \emptyset$ for $q = 1, 2$. Now E_1 must contain a zero of f , but cannot contain a non-real zero of f by Lemma 6.1(d). This is a contradiction and Lemma 6.2 is proved. \square

7. COMPLETION OF THE PROOF OF THEOREM 1.5

It now follows from Lemma 4.4, Lemma 6.2 and the fact that f' has finitely many zeros that the Schwarzian derivative

$$(13) \quad S_f = \frac{f'''}{f'} - \frac{3}{2} \left(\frac{f''}{f'} \right)^2 = 2B$$

of f is a rational function, and is not identically zero since f is transcendental. Let

$$(14) \quad B(z) = b^2 z^n (1 + o(1)) \quad \text{as } z \rightarrow \infty, \text{ where } b \in \mathbb{C} \setminus \{0\} \text{ and } n \in \mathbb{Z}.$$

Since f^{-1} has at least two direct transcendental singularities the order of f is at least 1, and so (13) and an application of the Wiman-Valiron theory [18] to $1/f'$ show that $n \geq 0$ in (14).

The following argument is self-contained but uses some methods similar to those of [35]. If u and v are linearly independent solutions of the differential equation

$$(15) \quad w'' + B(z)w = 0$$

on a simply connected domain D on which B has no poles, then there exists a Möbius transformation T_0 such that $f = T_0(u/v)$ on D [28, Chapter 6]. Hence u/v and $(u/v)'$ extend to be meromorphic on \mathbb{C} and, since the Wronskian of u and v is constant, so do v^2 , u^2 and uv .

The equation (15) has $n+2$ distinct critical rays [23], namely those rays $\arg z = \theta$ such that

$$2 \arg b + (n+2)\theta = 0 \pmod{2\pi}.$$

If $\arg z = \theta_0$ is a critical ray and ε and $1/R_0$ are small and positive then in the sectorial region

$$S(R_0, \theta_0, \varepsilon) = \left\{ z \in \mathbb{C} : |z| > R_0, \quad |\arg z - \theta_0| < \frac{2\pi}{n+2} - \varepsilon \right\}$$

there are principal solutions u_1, u_2 of (15) given by [23]

$$u_j(z) \sim B(z)^{-1/4} \exp((-1)^j iZ), \quad Z = \int_{R_0 e^{i\theta_0}}^z B(t)^{1/2} dt \sim \left(\frac{2b}{n+2} \right) z^{(n+2)/2}.$$

Lemma 7.1. *The critical rays of (15) are the positive and negative real axis.*

Proof. Suppose that $\arg z = \theta_0$ is a critical ray, where $\theta_0 \in (0, \pi) \cup (\pi, 2\pi)$. Let δ be small and positive. Then the u_j are such that, without loss of generality, $u_1(z)/u_2(z)$ tends to infinity on the ray $\arg z = \theta_0 + \delta$ and to zero on the ray $\arg z = \theta_0 - \delta$. Since $f = T_1(u_1/u_2)$ for some Möbius transformation T_1 , this gives

distinct asymptotic values of f approached on paths in the same component of $\mathbb{C} \setminus \mathbb{R}$, contradicting Lemma 5.2. \square

It now follows that $n = 0$ and that b may be chosen to be real and positive in (14). Thus (15) has principal solutions satisfying

$$u_j(z) = \exp((-1)^j i(bz + O(\log |z|))) \quad \text{in } \{z \in \mathbb{C} : |z| > R_0, \quad |\arg z| < \pi - \varepsilon\}.$$

Each function $w = w_j = u_j^2$ is meromorphic in the plane and by (15) satisfies

$$2ww'' - (w')^2 + 4Bw^2 = 0,$$

so that w_j has finitely many poles in \mathbb{C} . Applying the Phragmén-Lindelöf principle to the functions $v_j(z) = u_j(z)^2 \exp((-1)^{j+1} 2ibz)$, which are of finite order in the plane with finitely many poles, then shows that v_1 and v_2 are rational functions. Hence

$$\frac{u_2(z)^2}{u_1(z)^2} = S_1(z) \exp(4ibz)$$

with S_1 a rational function, and since u_2/u_1 is meromorphic in the plane it follows that there exists a rational function R such that $u_2(z)/u_1(z) = R(z)e^{2ibz}$. Since f has infinitely many zeros and poles by Lemma 5.1, it may be assumed that

$$f(z) = \frac{R(z)e^{icz} - 1}{AR(z)e^{icz} + A'},$$

where $c = 2b > 0$ and $A, A' \in \mathbb{C} \setminus \{0\}$. But f has infinitely many real zeros and so $R(z)\overline{R(\bar{z})} \equiv 1$. Further, f has asymptotic values $1/A$ and $-1/A'$ and Lemma 5.2 gives $A' = -\bar{A} \notin \mathbb{R}$, which proves (2) and (3). The remaining assertions of Theorem 1.5 now follow from Lemma 3.2.

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