Zeros of derivatives of real meromorphic functions

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Abstract

Two results are proved for real meromorphic functions in the plane. First, a lower bound is given for the distance between distinct non-real poles when the function and its second derivative have finitely many non-real zeros and the logarithmic derivative has finite lower order. Second, if the function has finitely many non-real zeros, and one of its higher derivatives has finitely many zeros in the plane, and if the multiplicities of non-real poles grow sufficiently slowly, then the function is a rational function multiplied by the exponential of a polynomial.
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1 Introduction

There is a long history of research into the zeros of entire functions which are real on the real axis, and the effect of differentiation on their location. This work includes the resolution of conjectures of Pólya and Wiman in a series of papers by several authors [2, 3, 5, 6, 20, 21, 28, 29, 32, 40, 45]. In particular, it is now known that for a real entire function f the absence of non-real zeros of f and $f^{(k)}$ for some $k \geq 2$ implies that f belongs to the Laguerre-Pólya class, and that the number of non-real zeros of the kth derivative of any real entire function either is zero for all sufficiently large k or tends to infinity with k.

The picture is less complete for real meromorphic functions, that is, functions meromorphic in the plane mapping $\mathbb{R}$ into $\mathbb{R} \cup \{\infty\}$, but results proved over recent decades may be found in [22, 23, 24, 25, 26, 33, 34, 35, 36, 37, 43], including the following.

Theorem 1.1 ([36, 37]) *Let f be a real meromorphic function in the plane, such that f has finitely many zeros and non-real poles, and assume that f'' has finitely many non-real zeros. Then f satisfies*

$$f = Se^P \text{ with } S \text{ a rational function and } P \text{ a polynomial.} \tag{1}$$

Theorem 1.2 ([33, 34]) *Let f be a real meromorphic function in the plane, not of the form (1). Let μ and k be integers with $1 \leq \mu < k$. Assume that all but finitely many zeros of f and $f^{(k)}$ are real, and that $f^{(\mu)}$ has finitely many zeros. Then $\mu = 1$ and $k = 2$ and f satisfies*

$$f(z) = \frac{R(z)e^{icz} - 1}{AR(z)e^{icz} - \overline{A}}, \quad \text{where } c \in (0, \infty),\ A \in \mathbb{C} \setminus \mathbb{R}, \tag{2}$$

and

$$R \text{ is a rational function with } |R(x)| = 1 \text{ for all } x \in \mathbb{R}. \tag{3}$$

Moreover, all but finitely many poles of f are real.

Conversely, if f is given by (2) and (3) then f satisfies the hypotheses with $\mu = 1$ and $k = 2$.

Theorem 1.1 was proved in [22, 43] under the additional assumption that all zeros of f' are real. Furthermore, Theorem 1.2 was established in [23], but only for $\mu = 1$ and $k = 2$ and subject to the additional hypothesis that poles of f are real, which in Theorem 1.2 appears instead as a conclusion. It seems likely that if $k \geq 2$ and f is a real meromorphic function in the plane, such that f and $f^{(k)}$ have finitely many non-real zeros, then f has in some sense relatively few distinct non-real poles. This is true if in addition the non-real poles of f have bounded multiplicities, in the sense that there then exists $B > 0$ such that if z_0 and z_1 are non-real poles of f then $|z_1 - z_0| > B|\operatorname{Im} z_0|$ (see Lemma 4.3 below). The first result of the present paper shows that, at least if $k = 2$ and f'/f has finite lower order, this additional hypothesis on the multiplicities of non-real poles may be dispensed with.

Theorem 1.3 *Let f be a real meromorphic function in the plane such that f'/f has finite lower order and f and f'' have finitely many non-real zeros. If z_0 is a non-real pole of f and $|z_0|$ is sufficiently large then f has no poles in the set*

$$\left\{ z \in \mathbb{C} : 0 < |z - z_0| < \frac{|\operatorname{Im} z_0|}{16} \right\}.$$

It follows easily from Theorem 1.3 that if $0 < \beta < \pi/2$ then the number of distinct non-real poles of f with $|z| \leq r$, $\beta \leq |\arg z| \leq \pi - \beta$ is $O(\log r)$ as $r \to \infty$. The second main result of this paper deals with real meromorphic functions having real zeros, such that some higher derivative $f^{(k)}$ has finitely many zeros, rather than f as in Theorem 1.1 or an intermediate derivative $f^{(\mu)}$ as in Theorem 1.2.

Theorem 1.4 *Let $k \geq 2$ and let f be a real meromorphic function in the plane satisfying the following conditions:*
(i) the function f has finitely many non-real zeros;
(ii) the function $f^{(k)}$ has finitely many zeros in the plane;
(iii) there exists a positive real number M such that if z_0 is a non-real pole of f of multiplicity m_0 then

$$m_0 \leq M + |z_0|^M; \tag{4}$$

(iv) if $k = 2$ then f'/f has finite lower order or the non-real poles of f have bounded multiplicities. Then f satisfies (1).

It seems plausible that Theorem 1.4 might hold with no need for conditions (iii) and (iv), but they are required for the present method, which depends on showing that f has finite order, so that f has finitely many poles by (ii) and the main result of [31]. On the other hand for each $k \geq 2$ an example is given in [31] of a real meromorphic function, with infinitely many poles, all real and simple, such that $f^{(k)}$ has no zeros in the plane. Thus hypothesis (i) is not redundant.

2 Proof of Theorem 1.3

Let f be as in the hypotheses of Theorem 1.3, and write

$$L = \frac{f'}{f}, \quad F(z) = z - \frac{1}{L(z)}, \quad F' = \frac{ff''}{(f')^2}. \tag{5}$$

There is no loss of generality in assuming that L and F are transcendental, since otherwise f has finitely many poles in the plane. If $M > 0$ is sufficiently large then F' has no zeros and F has no multiple poles in

$$H_M^+ = \{z \in \mathbb{C} : |z| > M, \operatorname{Im} z > 0\}.$$

Lemma 2.1 *There exist at most finitely many $\alpha \in \mathbb{C}$ such that $F(z)$ tends to α as z tends to infinity along a path in $\mathbb{C} \setminus \mathbb{R}$.*

Proof. Assume that this is not the case. Then since f and F are real there exist pairwise distinct $\alpha_1, \ldots, \alpha_n$ in $\mathbb{C}$ such that $F(z)$ tends to α_j as z tends to infinity on a path γ_j tending to infinity in the open upper half-plane H_0^+, and here n may be chosen arbitrarily large. Assume without loss of generality that $|F(z) - \alpha_j| < \varepsilon$ for all z on γ_j, and that $|\alpha_j - \alpha_{j'}| > 2\varepsilon$ for $j \neq j'$, where ε is small and positive. Next take $\delta \in (0, \varepsilon)$ and a large $N > M$ such that $|F(z) - \alpha_j| > 2\delta$ for $j = 1, \ldots, n$ and for all z on the circle $S(0, N)$. Here and subsequently $S(a, r)$ denotes the circle of centre a and radius r, and $D(a, r)$ the corresponding open disc. It may now be assumed that each γ_j starts on $S(0, N) \cap H_0^+$, but otherwise lies in H_N^+, and that γ_{j+1} separates γ_j from γ_{j+2} in H_N^+, for $j = 1, \ldots, n-2$. It then follows that for $j = 2, \ldots, n-1$ an unbounded subpath of γ_j lies in a component C_j of the set $\{z \in \mathbb{C} : |F(z) - \alpha_j| < \delta\}$ with the property that the closure E_j of C_j lies in H_N^+. Since $N > M$ it follows that F' has no zeros in E_j and, since F has finite lower order, [27, Theorem 2] (see also [1, Theorem 1]) implies that F^{-1} has a direct transcendental singularity over α_j, for $j = 2, \ldots, n-1$. But n may be chosen arbitrarily large, and this contradicts the Denjoy-Carleman-Ahlfors theorem. □

Proposition 2.1 *Let $R > 0$ be large and let C be a component of the set*

$$W_R = \{z \in H_0^+ : F(z) \in H_R^+\}.$$

Then C contains at most finitely many poles of f.

The proof of Proposition 2.1 is lengthy and will be deferred to the next section. Assuming Proposition 2.1, the proof of Theorem 1.3 is completed as follows. Let $S > R$. Then

$$F^{-1}(\mathbb{R} \cup \{\infty\}) \cap S(0, S), \quad F^{-1}(S(0, R)) \cap S(0, S)$$

are both finite since F is transcendental. Thus $S(0, S) \cap \partial W_R$ is finite, where ∂V denotes the boundary of a set $V \subseteq \mathbb{C}$ with respect to the finite plane, and so only finitely many components of W_R meet $S(0, S)$. It follows using Proposition 2.1 that all but finitely many poles of f in H_0^+ lie in components D of W_R with $D \subseteq H_S^+$. Let z_0 with $|z_0| > 2S$ be such a pole, with multiplicity m_0. Then $z_0 \in D_0 \subseteq H_S^+$, where D_0 is a component of W_R. Since R is large it follows using Lemma 2.1 and analytic continuation of F^{-1} that $F : D_0 \to H_R^+$ is a conformal bijection. Thus

$$G = F^{-1} : H_R^+ \to D_0 \subseteq H_S^+ \subseteq H_R^+$$

is conformal and z_0 is an attracting fixpoint of G. By Schwarz' lemma G has no other fixpoints in H_R^+ and so f has no other poles in D_0. Now

$$D\left(z_0, \frac{\operatorname{Im} z_0}{2}\right) \subseteq D\left(z_0, \frac{|z_0|}{2}\right) \cap H_0^+ \subseteq H_0^+ \setminus D(0, S) \subseteq H_R^+$$

and

$$G(z_0) = z_0, \quad G'(z_0) = \frac{m_0}{m_0 + 1} \geq \frac{1}{2}.$$

Hence Koebe's one-quarter theorem implies that

$$D\left(z_0, \frac{\operatorname{Im} z_0}{16}\right) \subseteq G\left(D\left(z_0, \frac{\operatorname{Im} z_0}{2}\right)\right) \subseteq G(H_R^+) \subseteq D_0.$$

This completes the proof of Theorem 1.3, subject to Proposition 2.1, which will be proved in the next section.

3 Proof of Proposition 2.1

With the notation of §2, let D be a component of W_R which contains infinitely many poles of f. Since R is large it follows from Lemma 2.1 and analytic continuation of F^{-1} that $F : D \to H_R^+$ is a conformal bijection, but here it need not be the case that $D \subseteq H_R^+$.

Lemma 3.1 *There exist at most finitely many components Γ of ∂D with $\Gamma \subseteq H_0^+$.*

Proof. Let Γ be such a component. Then Γ does not contain any Jordan curve Γ_1. To see this observe that as z describes such a Γ_1 the image $F(z)$ must describe the whole extended boundary $\partial_\infty H_R^+ = \partial H_R^+ \cup \{\infty\}$, since F is univalent on D and hence on ∂D; this gives $\partial D = \Gamma_1$, which contradicts the fact that D is simply connected by conformal equivalence. Thus Γ is a simple curve going to infinity in both directions. As z tends to infinity in some direction along Γ, the image $F(z)$ travels monotonely along $\partial_\infty H_R^+$ and must tend to an asymptotic value α of F. Since Γ lies in H_0^+ and F is univalent on ∂D, it follows from Lemma 2.1 that there are only finitely many possible α and finitely many such components Γ. □

Lemma 3.2 *There exist at most finitely many $\alpha \in \mathbb{C}$ such that $L(z)$ tends to α as z to infinity along a path in D.*

Proof. Assume that this is not the case. Then there exist pairwise distinct $\alpha_1, \ldots, \alpha_n \in \mathbb{C} \setminus \{0\}$ such that $L(z) \to \alpha_j$ as z tends to infinity along a path γ_j in D. Here it may be assumed that the γ_j are simple and pairwise disjoint apart from a common starting point in D, and n may be chosen arbitrarily large. Let N_0 be the number of components Γ of ∂D with $\Gamma \subseteq H_0^+$; then N_0 is finite by Lemma 3.1. This gives, without loss of generality, at least $n - 1 - N_0$ domains D_j bounded by γ_j and γ_{j+1}, the closures of which lie in D. Since $L(z) \neq 0$ on D, this gives in turn at least $n-1-N_0$ direct transcendental singularities of L^{-1} over 0, which contradicts the Denjoy-Carleman-Ahlfors theorem since L has finite lower order. $\Box$

Now use Lemma 3.2 to choose $\theta \in (0, \pi)$ with the following property. If $\alpha \in \mathbb{C} \setminus \{0\}$ and $\arg \alpha = \pm\theta$ then α is not a critical value of L and there is no path tending to infinity in D on which $L(z)$ tends to α. Next choose positive constants T, X with

$$T - X > R, \quad X \sin \theta > R. \tag{6}$$

By assumption there exist infinitely many pairwise distinct poles $z_1, z_2, \ldots$ of f in D. Now continue L^{-1} along the ray $\arg w = \theta$, starting at infinity, so that the image $L^{-1}(w)$ starts at z_j, and let t_j be the infimum of positive t such that this continuation extends to $\{w : \arg w = \theta, t < |w| < \infty\}$. Since L is transcendental there are only finitely many points z on $S(0,T)$ at which $\arg L(z) = \theta$, and so it may be assumed that for each j the image $z = L^{-1}(w)$ under this continuation stays in H_T^+. If $|w| < 1/X$ then (5) and (6) give

$$|F(z)| \geq \text{Im}\, F(z) \geq \text{Im} \left(-\frac{1}{L(z)} \right) > X \sin \theta > R.$$

On the other hand if $|w| \geq 1/X$ then (5) and (6) yield

$$|F(z)| \geq |z| - X > T - X > R.$$

Further, as $L^{-1}(w)$ is continued with $\arg w = \theta$ and $|w|$ decreasing, $\text{Im}\, F(z)$ is at least $(1/|w|) \sin \theta$ and cannot tend to 0. It follows that the image $z = L^{-1}(w)$ stays in D; thus $t_j = 0$ and the continuation is possible along the whole ray $\arg w = \theta$, $|w| > 0$, by the choice of θ. Moreover as $w \to 0$ with $\arg w = \theta$ the image $z = L^{-1}(w)$ tends either to infinity or to a zero of L, and so a pole of F, on ∂D. But F is univalent on D and so has at most one pole on ∂D.

Hence there exist arbitrarily many pairwise disjoint simple paths σ_j tending to infinity and lying in D, such that σ_j starts at z_j and is mapped univalently by L onto the set $\{w = te^{i\theta} : 0 < t \leq \infty\}$, with $L(z) \to 0$ as z tends to infinity on σ_j. These paths σ_j can then be extended to simple paths τ_j in D which are pairwise disjoint except that they all have the same starting point $z^* \in D$. Now applying Lemma 3.1 shows that there exist arbitrarily many pairwise disjoint domains $\Omega_k \subseteq D$, each bounded by two of the τ_j, and so by two of the σ_j and a bounded simple path $\lambda_k \subseteq D$. Since F has no poles in D there exists $r_k > 0$ such that $|L(z)| \geq r_k$ on λ_k.

For each Ω_k use Lemma 3.2 to choose $P_k \in (0, r_k)$ such that the circle $S(0, P_k)$ contains no critical values of L and no $\alpha \in \mathbb{C}$ such that $L(z)$ tends to α as z to infinity along a path in D. Choose $u_k \in \partial\Omega_k$ with $L(u_k) = P_k e^{i\theta}$, and continue $z = L^{-1}(w)$ along $S(0, P_k)$ so that the continuation takes z into Ω_k. By the choice of P_k and the fact that $\Omega_k \subseteq D$ this continuation leads to $v_k \in \Omega_k$ with $L(v_k) = P_k e^{-i\theta}$. The choice of θ then implies that it is possible to continue $L^{-1}(w)$ along the half-ray $w = te^{-i\theta}$, in the direction of decreasing t, so that the resulting image $z = L^{-1}(w)$ starts at v_k and remains in $\Omega_k \subseteq D$. Since $L(z) \neq 0$ on D this gives a path tending to infinity in Ω_k on which $L(z)$ tends to 0 with $\arg L(z) = -\theta$, and hence an unbounded component V_k of the set $\{z \in \mathbb{C} : \text{Im}\,(1/L(z)) > 2/P_k\}$, such that $V_k \cup \partial V_k \subseteq \Omega_k \subseteq D$. Each function

$$u_k(z) = \text{Im}\, \frac{1}{L(z)} \quad (z \in V_k), \quad u_k(z) = \frac{2}{P_k} \quad (z \notin V_k),$$

is then non-constant and subharmonic in the plane, again since $L(z) \neq 0$ on D. Since there are arbitrarily many of these components V_k, with disjoint closures, a standard application of the Phragmén-Lindelöf principle [19] gives a large z in one of the V_k, and so in D, with

$$\operatorname{Im} \frac{1}{L(z)} > |z|^2,$$

and hence $\operatorname{Im} F(z) < 0$ by (5), which is evidently a contradiction. This completes the proof of Proposition 2.1 and hence of Theorem 1.3.

Remark: the reader will observe that the hypothesis that L has finite order is only used twice in the proof of Proposition 2.1 and Theorem 1.3, namely in Lemmas 2.1 and 3.2.

4 Lemmas required for Theorem 1.4

The following theorem was proved for analytic functions by Schwick [44] and in the meromorphic case in [4], using the rescaling method [47].

Theorem 4.1 ([4, 44]) *Let $k \geq 2$ and let $\mathcal{F}$ be a family of functions meromorphic on a plane domain D such that $ff^{(k)}$ has no zeros in D, for each $f \in \mathcal{F}$. Then the family $\{f'/f : f \in \mathcal{F}\}$ is normal on D.*

Lemma 4.1 *Let $k \geq 2$ and $\rho, \sigma, \tau \in (0, \pi/2)$ and let $K_0 \in (0, \infty)$. Then there exists $K_1 \in (0, \infty)$, depending only on k, ρ, σ, τ and K_0, with the following property. If g is a meromorphic function on the domain $D = \{z \in \mathbb{C} : 1/2 < |z| < 2, \, 0 < \arg z < \pi\}$ such that g and $g^{(k)}$ have no zeros in D, and if*

$$\min\{|g'(e^{i\theta})/g(e^{i\theta})| : \rho \leq \theta \leq \pi - \rho\} \leq K_0, \tag{7}$$

then $|g'(e^{i\theta})/g(e^{i\theta})| \leq K_1$ for all $\theta \in [\sigma, \pi - \sigma]$ outside a set of Lebesgue measure at most τ.

Proof. Suppose that no such K_1 exists. Then there exist sequences p_n and g_n such that p_n is positive and tends to infinity, while g_n is meromorphic with $g_n g_n^{(k)} \neq 0$ on D, and such that (7) holds with $g = g_n$, but

$$\{\theta : \sigma \leq \theta \leq \pi - \sigma, \, |g_n'(e^{i\theta})/g_n(e^{i\theta})| > p_n\}$$

has measure greater than τ. By Theorem 4.1 it may be assumed that the functions $G_n = g_n'/g_n$ converge locally uniformly on D, and the limit function G is not identically infinite by (7). Thus G has finitely many poles on the arc $I = \{e^{i\theta} : \sigma \leq \theta \leq \pi - \sigma\}$, and G is bounded on $I \setminus U$, where U is a union of finitely many open arcs of total angular measure at most $\tau/2$. Since G_n converges uniformly to G on $I \setminus U$, this is a contradiction. $\square$

The next two lemmas, the first of which is due to Nicks [42, Lemma 6.3] (see also [41, Lemma 6.33]), involve the Tsuji characteristic $\mathfrak{T}(r, v) = \mathfrak{m}(r, v) + \mathfrak{N}(r, v)$ of a meromorphic function v on the closed upper half-plane [3, 14, 46].

Lemma 4.2 ([41, 42]) *Let $b > 0$ and let the function u be meromorphic on the closed upper half-plane such that any pair $\{z_1, z_2\}$ of distinct zeros of u in the open upper half-plane satisfies $|z_1 - z_2| \geq b \operatorname{Im} z_1$. If the zeros of u have bounded multiplicities then the Tsuji counting function of the zeros of u satisfies $\mathfrak{N}(r, 1/u) = O(\log r)$ as $r \to \infty$.*

Lemma 4.3 *Let m_1 be a positive integer and let f be a real meromorphic function in the plane such that f and $f^{(k)}$ have finitely many non-real zeros, for some $k \geq 2$.*
(a) If all but finitely many non-real poles of f have multiplicity at most m_1, and if w_1 and w_2 are distinct non-real poles of f with $|w_1|$ large, then $|w_1 - w_2| \geq b_1 |\operatorname{Im} w_1|$, where $b_1 > 0$ depends only on k and m_1.
(b) If $k \geq 3$, then f satisfies

$$\overline{\mathfrak{N}}(r, f) \leq \mathfrak{T}(r, f'/f) = O(\log r) \quad \textit{as} \quad r \to \infty. \tag{8}$$

Furthermore, the inequality (8) also holds for $k = 2$ if f'/f has finite lower order or the non-real poles of f have bounded multiplicities.

Proof. The proof of part (a) uses ideas from [8, 41, 42]. Let z_0 be a non-real pole of f of multiplicity $m_0 \leq m_1$ with $|z_0|$ large. Set

$$R_0 = \frac{|\operatorname{Im} z_0|}{2}, \quad g(z) = f(z_0 + R_0 z), \quad G(z) = \frac{g(z)}{g'(z)}.$$

Then $G(0) = 0$ and g and $g^{(k)}$ have no zeros in $D(0,1)$. By Theorem 4.1 there exists $\delta \in (0, 1/4]$, independent of z_0 and f, such that

$$|G(w)| \leq 1 \quad \text{for} \quad |w| \leq 2\delta. \tag{9}$$

Now suppose that $z_1 \neq z_0$ is a pole of f in $D(z_0, \delta R_0)$, and set $w_1 = (z_1 - z_0)/R_0 \in D(0, \delta)$. Then

$$h(w) = \frac{G(w)}{w(w - w_1)}$$

is analytic on $|w| \leq 2\delta$ and (9) gives $|h(w)| \leq 1/2\delta^2$ for $|w| = 2\delta$. Thus the maximum principle implies that

$$\frac{1}{m_0} = |G'(0)| = |w_1 h(0)| \leq \frac{|w_1|}{2\delta^2} \quad \text{and hence} \quad |z_1 - z_0| \geq \frac{2\delta^2 R_0}{m_0} \geq \frac{\delta^2 |\operatorname{Im} z_0|}{m_1}.$$

This proves part (a), with $b_1 = \delta^2/m_1 < \delta/2$.

Next, if $k \geq 3$ then (8) is proved using Frank's Wronskian method exactly as in [9, 11] (see also [4, 10, 12]), but with Tsuji functionals replacing the corresponding Nevanlinna functionals. This method is not available when $k = 2$, but in this case if f is as in part (a) then $u = f/f'$ satisfies the hypotheses of Lemma 4.2, and the same is true by Theorem 1.3 if f'/f has finite lower order. Since $u' = 1 - (ff'')/(f')^2$ this yields

$$\mathfrak{N}(r, 1/u) + \mathfrak{N}(r, 1/(u'-1)) \leq 2\overline{\mathfrak{N}}(r, 1/f) + \overline{\mathfrak{N}}(r, f) + \mathfrak{N}(r, 1/f'') = O(\log r)$$

as $r \to \infty$. Applying Hayman's alternative [16, Ch. 3], using Tsuji rather than Nevanlinna functionals, then gives (8). □

The proof of Theorem 1.4 will require Fuchs' small arcs lemma [13] as given in [19, p.721].

Lemma 4.4 ([19]) *Let $R > 0$ and let the function g be meromorphic in $|z| \leq R$, with $g(0) = 1$. Let η_1, η_2 be positive with $\eta_1 + \eta_2 < 1$. Then there exists a subset E_R of $[0, R(1 - \eta_1)]$, having measure greater than $R(1-\eta_1-\eta_2)$, with the following property. If $r \in E_R$ and F_r is a subinterval of $[0, 2\pi]$ of length m then*

$$\int_{F_r} \left| \frac{\partial \log |g(re^{i\theta})|}{\partial \theta} \right| d\theta \leq \frac{400m}{\eta_1^2 \eta_2} \left(\log \frac{2\pi e}{m} \right) T(R, g).$$

The next lemma is [7, Lemma III, p.322].

Lemma 4.5 ([7]) *Let $1 < r < R < \infty$ and let the function g be meromorphic in $|z| \leq R$. Let $I(r)$ be a subset of $[0, 2\pi]$ of Lebesgue measure $\mu(r)$. Then*

$$\frac{1}{2\pi} \int_{I(r)} \log^+ |g(re^{i\theta})| \, d\theta \leq \frac{11R\mu(r)}{R - r} \left(1 + \log^+ \frac{1}{\mu(r)} \right) T(R, g).$$

Lemma 4.6 ([17]) *Let $S(r)$ be an unbounded positive non-decreasing function on $[r_0, \infty)$, continuous from the right, of order ρ. Let $A > 1, B > 1$. Then*

$$\overline{\mathrm{logdens}} \left(\{ r \geq r_0 : S(Ar) \geq BS(r) \} \right) \leq \rho \left(\frac{\log A}{\log B} \right).$$

Lemma 4.6 is from [17, Lemma 4, p.103] and, although it is stated in [17] only for a characteristic function, the lemma holds for $S(r)$, with the order given by [16, p.16]

$$\rho = \limsup_{r \to \infty} \frac{\log S(r)}{\log r}.$$

The following is a refinement of lemmas from [30, 38].

Lemma 4.7 *Let the function f be transcendental and meromorphic in the plane and let $k \in \mathbb{N}$. Let E be an unbounded subset of $[1, \infty)$ with the following property. For each $r \in E$ there exist real $\theta_1(r) < \theta_2(r) \leq \theta_1(r) + 2\pi$ and an arc $\Omega_r = \{ re^{i\theta} : \theta_1(r) \leq \theta \leq \theta_2(r) \}$ such that*

$$\lim_{r \to \infty, r \in E} r^{2k} M(\Omega_r, f^{(k)}/f) = 0, \quad \textit{where} \quad M(\Omega_r, g) = \max\{ |g(z)| : z \in \Omega_r \}. \tag{10}$$

Let $N = N(r)$ satisfy $0 \leq \log N(r) \leq o(\log r)$ as $r \to \infty$ in E. Then f satisfies, for all sufficiently large $r \in E$,

$$\left| \frac{zf'(z)}{f(z)} \right| \leq kN(r)$$

for all $z \in \Omega_r$ outside a union $U(r)$ of open discs having sum of radii at most $r(k-1)/N(r)$.

Proof. The result is trivial if $k = 1$ so assume that $k \geq 2$. Let $r \in E$ be large and take $z_r \in \Omega_r$ with

$$|f(z_r)| = M_r = M(\Omega_r, f). \tag{11}$$

There exists a polynomial $P = P_r$ of degree at most $k-1$ such that, for $z \in \Omega_r$,

$$f(z) = P(z) + \int_{z_r}^{z} \frac{(z-t)^{k-1}}{(k-1)!} f^{(k)}(t)\, dt, \quad f'(z) = P'(z) + \int_{z_r}^{z} \frac{(z-t)^{k-2}}{(k-2)!} f^{(k)}(t)\, dt.$$

It then follows from (10) and (11) that

$$|P(z_r)| = M_r \quad \text{and} \quad |f(z) - P(z)| + |f'(z) - P'(z)| \le r^{-k} M_r \quad \text{for} \quad z \in \Omega_r. \tag{12}$$

Write $P(z) = P_1(z)P_2(z)$ where P_1 is the product of the terms $z - c_j$ over all zeros c_j of P with $|c_j| < 2r$, and is 1 if there are no such c_j, while P_2 is a polynomial with all its zeros, if any, lying in $|z| \ge 2r$. Let $s \ge 0$ be the degree of P_1, and assume that $z \in \Omega_r$ lies outside the union $U(r)$ of the open discs of centre c_j and radius $r/N(r)$. Then $M(\Omega_r, P_2'/P_2) \le (k-1-s)/r$ and

$$M_r \le (3r)^s M(\Omega_r, P_2) \le (3N)^s |P_1(z)| \exp(2\pi(k-1))|P_2(z)| = (3N)^s \exp(2\pi(k-1))|P(z)|.$$

On combination with (12) and the fact that $\log N(r) = o(\log r)$ this yields

$$\left| \frac{zf'(z)}{f(z)} \right| = \left| \frac{zP'(z) + o(|P(z)|)}{P(z)(1+o(1))} \right| \le ((k-1) + o(1))N(r).$$

□

5 Proof of Theorem 1.4

Let the function f be as in the hypotheses, and assume that $L = f'/f$ is transcendental, since otherwise there is nothing to prove. Since $f^{(k)}$ has finitely many zeros the function

$$G = \frac{f}{f^{(k)}} \tag{13}$$

has finitely many poles.

Lemma 5.1 *Suppose that G is a rational function. Then f satisfies (1).*

Proof. Poles of f are zeros of G, as are all but finitely many zeros of f, and it follows that f has finitely many zeros and poles. Now a standard application of the Wiman-Valiron theory [18, Theorem 12, p.341] to (13) shows that f has finite order. □

Assume henceforth that G is transcendental. Lemma 4.3 gives (8), and (13) and standard properties of the Tsuji characteristic yield $\mathfrak{T}(r, G) = O(\log r)$ as $r \to \infty$.

Lemma 5.2 *The function G has order at most 1.*

Proof. Let $R \to \infty$. Then a result of Levin and Ostrovskii [40, p. 332] (see also [3] and [14, Ch. 6]) implies that

$$\int_R^\infty \frac{1}{r^3\, 2\pi} \int_0^\pi \log^+ |G(re^{i\theta})|\, d\theta\, dr \le \int_R^\infty \frac{\mathfrak{m}(r, G)}{r^2}\, dr \le \int_R^\infty \frac{\mathfrak{T}(r, G)}{r^2}\, dr = O\left(\frac{\log R}{R} \right).$$

Since G is real and has finitely many poles this yields

$$\int_R^\infty \frac{m(r,G)}{r^3}\,dr = O\left(\frac{\log R}{R}\right), \quad \frac{T(R,G)}{2R^2} \leq \int_R^\infty \frac{T(r,G)}{r^3}\,dr = O\left(\frac{\log R}{R}\right).$$

□

By Lemma 5.2 and (4) the zeros and non-real poles of f have finite exponent of convergence, and f has a representation

$$f = \frac{W}{g}, \tag{14}$$

in which W is a real meromorphic function of finite order and g is a real entire function with real zeros. The logarithmic derivative g'/g then has a Levin-Ostrovskii factorisation [40] (see also [3])

$$\frac{g'}{g} = \phi\psi, \tag{15}$$

in which ϕ is a real entire function and ψ is real meromorphic, such that either $\psi \equiv 1$ or ψ maps the upper half-plane into itself. In either case ψ satisfies [39, Ch. I.6, Thm $8'$]

$$\frac{1}{5}|\psi(i)|\frac{\sin\theta}{r} < |\psi(re^{i\theta})| < 5|\psi(i)|\frac{r}{\sin\theta} \quad \text{for} \quad r \geq 1,\ \theta \in (0,\pi). \tag{16}$$

Lemma 5.3 *The function ϕ in (15) has order at most 1.*

Proof. It follows from (14), (15) and (16) that

$$T(r,\phi) = m(r,\phi) \leq m(r,g'/g) + O(\log r) \leq m(r,f'/f) + O(\log r). \tag{17}$$

Using (8) and (17) in the same inequality of Levin and Ostrovskii as in the proof of Lemma 5.2 then yields, as $R \to \infty$,

$$\begin{aligned} \frac{T(R,\phi)}{2R^2} \leq \int_R^\infty \frac{T(r,\phi)}{r^3}\,dr &\leq \int_R^\infty \frac{1}{r^3\,\pi}\int_0^\pi \log^+|L(re^{i\theta})|\,d\theta\,dr + O\left(\frac{\log R}{R^2}\right) \\ &\leq 2\int_R^\infty \frac{\mathfrak{m}(r,L)}{r^2}\,dr + O\left(\frac{\log R}{R^2}\right) = O\left(\frac{\log R}{R}\right). \end{aligned}$$

□

Lemma 5.4 *The function ϕ in (15) is a polynomial.*

Proof. Let σ be small and positive and denote by C positive constants, not necessarily the same at each occurrence, but always independent of σ and $r \in [1,\infty)$. Since G and ϕ have finite order there exists by Lemma 4.6 a a set $E_1 \subseteq [1,\infty)$, of positive lower logarithmic density, such that

$$T(4r,\phi) \leq CT(r,\phi) \quad \text{and} \quad T(4r,G) \leq CT(r,G) \tag{18}$$

for $r \in E_1$. The fact that W has finite order gives the estimate [15]

$$\left|\frac{W'(z)}{W(z)}\right| \leq |z|^C \tag{19}$$

for $|z|$ outside a set E_2 of finite logarithmic measure. Now let $r \in E_1$ be large and apply Lemma 4.4 to $G_1 = GR_1$, with R_1 a rational function chosen so that $G_1(0) = 1$. With $\eta_1 = 1/2$ and $\eta_2 = 1/8$ and $R = 4r$ this yields $s \in [r, 2r] \setminus E_2$ such that if F_s is an interval of length 3σ then

$$\int_{F_s} \left| \frac{\partial \log |G(se^{i\theta})|}{\partial \theta} \right| d\theta \leq C\, T(4r, G)\, \sigma \log \frac{2\pi e}{3\sigma} \leq C T(s, G)\, \sigma \log \frac{2\pi e}{3\sigma},$$

using (18). Since G is transcendental and σ is small this gives an arc of the circle $S(0, s)$, of angular measure 6σ, centred at a point z_s with $2 \log |G(z_s)| \geq T(s, G)$, on which $|G(z)| \geq s^{3k}$. The fact that G is real then implies that $|G(z)| \geq s^{3k}$ on a subarc I_s of $\{z \in S(0, s) : \sigma \leq \arg z \leq \pi - \sigma\}$ of angular measure at least σ. Next, applying Lemma 4.7 shows that there exists $z \in I_s$ with $|zL(z)| \leq C$. Now Lemma 4.1, applied to the function $f(sz)$, gives $|zL(z)| \leq C$ for all $z \in S(0, s)$ with $\sigma \leq \arg z \leq \pi - \sigma$, apart from a set of angular measure at most σ. Since $s \notin E_2$, this yields, on combination with (14), (15), (16) and (19), the estimate $|\phi(z)| \leq s^C$ for all $z \in S(0, s)$ apart from a set of angular measure at most 6σ, so that (18) and Lemma 4.5 give

$$T(s, \phi) \leq C \log s + C\sigma \left(1 + \log^+ \frac{1}{6\sigma}\right) T(2s, \phi) \leq C \log s + C\sigma \left(1 + \log^+ \frac{1}{6\sigma}\right) T(s, \phi).$$

Provided σ was chosen small enough, it follows that ϕ is a polynomial as asserted. □

Lemma 5.5 *The function g in (14) has finite order, and so has f.*

Proof. The fact that g has finite order is a standard consequence of (15) and Lemma 5.4: see [3, Lemma 5.1] or [32, Lemma 7.1]. Once g has finite order it follows from (14) that so has f. □

Since f has finite order and $f^{(k)}$ has finitely many zeros the main result of [31] shows that f has finitely many poles, and $f^{(k)} = Te^Q$, with T a rational function and Q a non-constant real polynomial. The proof now follows an argument from [34]. Suppose first that Q has degree 1. In this case integrating k times shows that $f(z) = T_1(z) + T_2(z)e^{a_1 z}$ with T_1 a polynomial, $T_2 \not\equiv 0$ a rational function, and $a_1 \in \mathbb{R} \setminus \{0\}$. The fact that L is transcendental forces $T_1 \not\equiv 0$ and elementary considerations show that f has infinitely many non-real zeros, which is a contradiction.

Assume henceforth that Q has degree at least 2. Since $G = f/f^{(k)}$ has order at most 1, by Lemma 5.2, it follows that $f = Gf^{(k)} = \Pi e^Q$, where $\Pi = GT$ is meromorphic with finitely many poles and with order at most 1, and that L has order at most 1. Let $\varepsilon \in (0, 1)$ be small. Then Gundersen's estimates for logarithmic derivatives [15] yield

$$\left| \frac{L^{(j)}(z)}{L(z)} \right| + \left| \frac{\Pi'(z)}{\Pi(z)} \right| \leq |z|^\varepsilon, \quad L(z) \sim Q'(z) \quad \text{and} \quad \frac{1}{G(z)} = \frac{f^{(k)}(z)}{f(z)} \sim L(z)^k \sim Q'(z)^k$$

for $j = 1, \ldots, k$ and $|z| = r \in [1, \infty)$ lying outside a set of finite logarithmic measure. Hence G, which has finitely many poles, must be a rational function, and consequently so must Π, which contradicts the assumption that f is not of form (1). This completes the proof of Theorem 1.4.

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References

[1] W. Bergweiler and A. Eremenko, On the singularities of the inverse to a meromorphic function of finite order, *Rev. Mat. Iberoamericana* 11 (1995), 355-373.

[2] W. Bergweiler and A. Eremenko, Proof of a conjecture of Pólya on the zeros of successive derivatives of real entire functions, *Acta Math.* **197** (2006), 145-166.

[3] W. Bergweiler, A. Eremenko and J.K. Langley, Real entire functions of infinite order and a conjecture of Wiman, *Geometric and Functional Analysis* **13** (2003), 975-991.

[4] W. Bergweiler and J.K. Langley, Nonvanishing derivatives and normal families, *J. Analyse Math.* 91 (2003), 353-367.

[5] T. Craven, G. Csordas, and W. Smith, Zeros of derivatives of entire functions, *Proc. Amer. Math. Soc.* **101** (1987), 323–326.

[6] T. Craven, G. Csordas, and W. Smith, The zeros of derivatives of entire functions and the Pólya-Wiman conjecture, *Annals of Math.* (2) **125** (1987), 405–431.

[7] A. Edrei and W.H.J. Fuchs, Bounds for the number of deficient values of certain classes of meromorphic functions, *Proc. London Math. Soc.* (3) 12 (1962), 315-344.

[8] A.N. Fletcher, J.K. Langley and J. Meyer, Nonvanishing derivatives and the MacLane class $\mathcal{A}$, *Illinois J. Math.* 53 (2009) 379-390.

[9] G. Frank, Eine Vermutung von Hayman über Nullstellen meromorpher Funktionen, *Math. Zeit.* 149 (1976), 29-36.

[10] G. Frank and S. Hellerstein, On the meromorphic solutions of nonhomogeneous linear differential equations with polynomial coefficients, *Proc. London Math. Soc.* (3) 53 (1986), 407-428.

[11] G. Frank, W. Hennekemper and G. Polloczek, Über die Nullstellen meromorpher Funktionen und deren Ableitungen, *Math. Ann.* 225 (1977), 145-154.

[12] G. Frank and J.K. Langley, Pairs of linear differential polynomials, *Analysis* 19 (1999), 173-194.

[13] W.H.J. Fuchs, Proof of a conjecture of G. Pólya concerning gap series, *Ill. Math. J.* 7 (1963), 661-667.

[14] A.A. Gol'dberg and I. V. Ostrovskii, *Distribution of values of meromorphic functions*, Nauka, Moscow, 1970 (Russian). English translation, *Translations of Mathematical Monographs* 236, Amer. Math. Soc. Providence 2008.

[15] G. Gundersen, Estimates for the logarithmic derivative of a meromorphic function, plus similar estimates, *J. London Math. Soc.* (2) 37 (1988), 88-104.

[16] W.K. Hayman, *Meromorphic functions*, Clarendon Press, Oxford, 1964.

[17] W.K. Hayman, On the characteristic of functions meromorphic in the plane and of their integrals, *Proc. London Math. Soc.* (3) 14A (1965), 93-128.

[18] W.K. Hayman, The local growth of power series: a survey of the Wiman-Valiron method, *Canad. Math. Bull.* 17 (1974) 317-358.

[19] W.K. Hayman, *Subharmonic functions Vol. 2*, Academic Press, London, 1989.

[20] S. Hellerstein and J. Williamson, Derivatives of entire functions and a question of Pólya, *Trans. Amer. Math. Soc.* **227** (1977) 227–249.

[21] S. Hellerstein and J. Williamson, Derivatives of entire functions and a question of Pólya, II, *Trans. Amer. Math. Soc.* **234** (1977) 497–503.

[22] S. Hellerstein and J. Williamson, The zeros of the second derivative of the reciprocal of an entire function, *Trans. Amer. Math. Soc.* **263** (1981), 501-513.

[23] S. Hellerstein, L.-C. Shen and J. Williamson, Real zeros of derivatives of meromorphic functions and solutions of second order differential equations, *Trans. Amer. Math. Soc.* **285** (1984), 759-776.

[24] A. Hinkkanen, Reality of zeros of derivatives of meromorphic functions, *Ann. Acad. Sci. Fenn.* **22** (1997), 1-38.

[25] A. Hinkkanen, Zeros of derivatives of strictly non-real meromorphic functions, *Ann. Acad. Sci. Fenn.* **22** (1997), 39-74.

[26] A. Hinkkanen, Iteration, level sets, and zeros of derivatives of meromorphic functions, *Ann. Acad. Sci. Fenn.* **23** (1998), 317-388.

[27] J.D. Hinchliffe, The Bergweiler-Eremenko theorem for finite lower order, Result. Math. 43 (2003), 121-128.

[28] H. Ki and Y.-O. Kim, One On the number of nonreal zeros of real entire functions and the Fourier-Pólya conjecture, *Duke Math. J.* **104** (2000), 45–73.

[29] Y.-O. Kim, A proof of the Pólya-Wiman conjecture, *Proc. Amer. Math. Soc.* **109** (1990), 1045-1052.

[30] J.K. Langley, On differential polynomials, fixpoints and critical values of meromorphic functions, *Result. Math.* 35 (1999), 284-309.

[31] J.K. Langley, The second derivative of a meromorphic function of finite order, *Bulletin London Math. Soc.* 35 (2003), 97-108.

[32] J.K. Langley, Non-real zeros of higher derivatives of real entire functions of infinite order, *J. Analyse Math.* **97** (2005), 357-396.

[33] J.K. Langley, Non-real zeros of derivatives of real meromorphic functions, *Proc. Amer. Math. Soc.* **137** (2009), 3355-3367.

[34] J.K. Langley, Zeros of derivatives of meromorphic functions, *Comput. Methods Funct. Theory* **10** (2010), 421-439.

[35] J.K. Langley, Real meromorphic functions and linear differential polynomials, *Science China (Mathematics)* **53** (2010), 739-748.

[36] J.K. Langley, Non-real zeros of real differential polynomials, *Proc. Roy. Soc. Edinburgh Sect. A.* **141** (2011), 631-639.

[37] J.K. Langley, Non-real zeros of derivatives of real meromorphic functions of infinite order, *Math. Proc. Camb. Phil. Soc.* **150** (2011), 343-351.

[38] J.K. Langley, Value distribution of differences of meromorphic functions, *Rocky Mountain J. Math.* **41** (2011), 275-291.

[39] B.Ja. Levin, *Distribution of zeros of entire functions,* GITTL, Moscow, 1956. 2-nd English transl., AMS, Providence RI, 1980.

[40] B.Ja. Levin and I.V. Ostrovskii, The dependence of the growth of an entire function on the distribution of zeros of its derivatives. *Sibirsk. Mat. Zh.* **1** (1960) 427–455. English transl., AMS Transl. (2) **32** (1963) 323–357.

[41] D.A. Nicks, Value distribution of meromorphic functions and their derivatives, Ph.D. thesis, University of Nottingham, 2010.

[42] D.A. Nicks, Non-real zeroes of real entire functions and their derivatives, to appear, *J. Analyse Math.*

[43] J. Rossi, The reciprocal of an entire function of infinite order and the distribution of the zeros of its second derivative, *Trans. Amer. Math. Soc.* **270** (1982), 667-683.

[44] W. Schwick, Normality criteria for families of meromorphic functions, *J. Analyse Math.* 52 (1989), 241-289.

[45] T. Sheil-Small, On the zeros of the derivatives of real entire functions and Wiman's conjecture, *Annals of Math*. **129** (1989) 179–193.

[46] M. Tsuji, On Borel's directions of meromorphic functions of finite order, I, *Tôhoku Math. J.* 2 (1950) 97–112.

[47] L. Zalcman, Normal families: New perspectives, *Bull. Amer. Math. Soc.* 35 (1998) 215–230.

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