The following example shows that the finite order case of Theorem 1.4 of [1] is sharp.

Lemma 0.1 *There exist negative real numbers* $a_1, a_3, \ldots$ *with the following property. Let* $p \geq 0$ *be an integer. Then the polynomial*

$$L_p(z) = a_{2p+1}z^{2p+1} + \ldots + a_1 z = \sum_{k=0}^{p} a_{2k+1}z^{2k+1}$$

is such that

$$Q_p(z) = L_p'(z) + L_p(z)^2 + 1 = \sum_{k=p+1}^{2p+1} b_{p,k}z^{2k}, \quad b_{p,k} \in (0, +\infty),$$

is an even polynomial, of degree $4p+2$*, with a zero of multiplicity* $2p+2$ *at* 0*.*

Proof. Let $a_1 = -1$ and for $p = 0$ set $L_0(z) = -z$, so that $Q_0(z) = L_0'(z) + L_0(z)^2 + 1 = z^2$ and $b_{0,1} = 1$. Now suppose that $0 \leq p \in \mathbb{Z}$ and that $a_1, \ldots, a_{2p+1}$ have been determined so that L_p and Q_p have the asserted properties. Let $a \in \mathbb{R}$ and set

$$\begin{aligned} L_{p+1}(z) &= az^{2p+3} + L_p(z), \\ Q_{p+1}(z) &= L_{p+1}'(z) + L_{p+1}(z)^2 + 1 \\ &= (2p+3)az^{2p+2} + L_p'(z) + a^2z^{4p+6} + 2az^{2p+3}L_p(z) + L_p(z)^2 + 1 \\ &= (2p+3)az^{2p+2} + a^2z^{4p+6} + 2az^{2p+3}L_p(z) + Q_p(z). \end{aligned}$$

Choose $a_{2p+3} = a$ to satisfy $(2p+3)a + b_{p,p+1} = 0$. This forces $a < 0$, so that $2az^{2p+3}L_p(z)$ is an even polynomial of degree $4p+4$, with non-negative real coefficients and a zero of multiplicity $2p+4$ at the origin. Moreover, Q_{p+1} is also even, of degree $4p+6 = 4(p+1)+2$, and the coefficients $b_{p+1,k}$ of z^{2k} in Q_{p+1} satisfy the following. First, if $k \leq p+1$ then $b_{p+1,k} = 0$, while if $p+2 \leq k \leq 2p+1$ then $b_{p+1,k} \geq b_{p,k} > 0$, so that Q_{p+1} has a zero of multiplicity $2p+4 = 2(p+1)+2$ at the origin. Second, $b_{p+1,2p+2} = 2aa_{2p+1} > 0$ and $b_{p+1,2p+3} = a^2 > 0$, and the lemma is proved by induction. □

Since $a_{2p+1} < 0$, the function f_p defined by $f_p(0) = 1$ and $f_p'/f_p = L_p$ belongs to U_{2p}, and $f_p'' + f_p = \left(L_p' + L_p^2 + 1\right) f_p$ has at most $4p+2-(2p+2) = 2p$ non-real zeros, and hence exactly $2p$, by [1, Theorem 1.4].

References

[1] J.K. Langley, Non-real zeros of linear differential polynomials, *J. Analyse Math.* 107 (2009), 107-140.