Logarithmic singularities and the zeros of the second derivative

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Abstract

An estimate is proved for the growth of a meromorphic function near to a logarithmic singularity of the derivative. This estimate is applied to show that if f is meromorphic of finite lower order in the plane, such that the second derivative f'' has finitely many zeros and the multiplicities of the poles z of f grow at most polynomially in $|z|$, then f has finitely many poles. Subsequent results then consider the zeros of linear differential polynomials $F = f^{(k)} + a_{k-1}f^{(k-1)} + \ldots + a_0 f$, where f is transcendental and meromorphic of finite order in the plane, and the coefficients a_j are constants. A.M.S. MSC 2000 Classification: 30D35.

1 Introduction

By a classical theorem of Pólya [18] (see also [8, Theorem 3.6, p.63]), if f is a meromorphic function in the plane with at least two distinct poles then for each sufficiently large k the kth derivative $f^{(k)}$ has at least one zero. Gol'dberg conjectured that the frequency of distinct poles of f is controlled by the frequency of zeros of the derivative $f^{(k)}$, as soon as $k \geq 2$. This is known to be true if all but finitely many poles of f have multiplicity at most $k-1$ [7] (see also [4, 20]); the general case remains open, although it follows from the results of [3, 6, 11] that if two derivatives $f^{(m)}$ and $f^{(n)}$ have finitely many zeros, where $0 \leq m \leq n-2$, then f has finitely many poles. The next result [14] is also strongly supportive of the Gol'dberg conjecture.

Theorem 1.1 ([14]) *Suppose that the function f is meromorphic of finite order in the plane and that $f^{(k)}$ has finitely many zeros, for some $k \geq 2$. Then f has finitely many poles.*

Simple examples show that Theorem 1.1 fails for $k = 1$. The hypothesis that f has finite order is not redundant in Theorem 1.1: indeed there exist meromorphic functions f with infinitely many poles and with f'' zero-free, of arbitrarily slow growth subject to infinite lower order [14].

The proof of Theorem 1.1 in [14] has a number of key steps. Suppose that the function f is meromorphic of finite order in the plane and that f'' has finitely many zeros. Then f' has finitely many critical values and hence finitely many asymptotic values, by a result of Bergweiler and Eremenko [1]. The finite asymptotic values of f' then give rise to logarithmic singularities of the inverse function of f', and [14, Lemma 3.1] gives an estimate for the growth of $f(z)$ for z near to these logarithmic singularities. Here the proof presented in [14] relies on the fact that f has finite order.

The first result of the present paper shows that no growth assumption on f is required for this estimate. To state this result requires the following standard facts from [17, p.287], which are discussed in detail in [14]. Suppose that F is a transcendental meromorphic function in the plane with no asymptotic or critical values in $0 < |w| < d_1 < \infty$. Then every component C_0 of the set $\{z \in \mathbb{C} : |F(z)| < d_1\}$ is

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simply connected, and there are two possibilities. Either (i) C_0 contains a single zero of F of multiplicity k, in which case $F^{1/k}$ maps C_0 univalently onto the disc $B(0, d_1^{1/k})$, or (ii) C_0 contains no zero of F, but instead a path tending to infinity on which $F(z)$ tends to zero. The proposition to be proved concerns components of the second type when F is a derivative.

Proposition 1.1 *Suppose that G is a transcendental meromorphic function in the plane and that G' has no asymptotic or critical values w with $0 < |w| < d_1 < \infty$. Let D be a component of the set $\{z \in \mathbb{C} : |G'(z)| < d_1\}$ on which G' has no zeros, but such that D contains a path tending to infinity on which $G'(z) \to 0$ as $z \to \infty$. Then there exists a positive constant S depending on G and D with the property that if z_1 is in D and $|G'(z_1)| \leq e^{-1}d_1$ then*

$$|G(z_1)| \leq S + \frac{C|z_1 G'(z_1)|}{\log |d_1/G'(z_1)|}, \tag{1}$$

where C is a positive absolute constant, in particular not depending on d_1, G or D.

Proposition 1.1 leads to the following substantial improvement of Theorem 1.1.

Theorem 1.2 *Assume that the function f is meromorphic of finite lower order in the plane and that $f^{(k)}$ has finitely many zeros, for some $k \geq 2$. Assume further that there exists $M \in (0, \infty)$ such that if ζ is a pole of f of multiplicity m_ζ then*

$$m_\zeta \leq M + |\zeta|^M. \tag{2}$$

Then f has finite order and finitely many poles.

It obviously suffices to establish Theorem 1.2 for $k = 2$. The strategy for proving Theorem 1.2 will be to show that the integrated counting function $N(r, f)$ of the poles of f has finite order and that so has f itself, from which the result then follows using Theorem 1.1. The hypothesis (2) may not really be needed but on the other hand seems difficult to dispense with. If it is assumed merely that f has finite lower order and that f'' has finitely many zeros then the present methods give rise to annuli in which f has few distinct poles, but this is not sufficient in order to establish the global estimates for the growth and minimum modulus of f'''/f'' which were the key to Theorem 1.1 in [14].

Proposition 1.1 will be proved in §2. Once this is established the proof of Theorem 1.2 requires only minor modifications of arguments from [14], and these will be outlined in §4, following some background material in §3.

The remainder of this paper will be concerned with the case where the kth derivative $f^{(k)}$ is replaced by a linear differential polynomial

$$L(f) = f^{(k)} + a_{k-1}f^{(k-1)} + \ldots + a_0 f, \tag{3}$$

in which $k \geq 2$ and the coefficients a_j are constants. The determination of those meromorphic functions f in the plane for which f and F have no zeros, where F is defined by (3) with $k \geq 2$, was accomplished for $k \geq 3$ and polynomial a_j in [2] (see also [5]), and for $k = 2$ and rational functions a_j in [12]. This suggests the possibility of an analogue of Theorem 1.1 with $f^{(k)}$ replaced by F. Indeed, it seems natural to conjecture that if f is meromorphic of finite order in the plane and F has finitely many zeros, where $k \geq 2$ and the a_j are constants in (3), then the counting function $\overline{n}(r, f)$ of the distinct poles satisfies $\overline{n}(r, f) = O(r)$ as $r \to \infty$. Such a result would be sharp, as shown by the simple example

$$f(z) = \frac{1}{1 - e^z}, \quad f''(z) - f'(z) = \frac{2e^{2z}}{(1 - e^z)^3}.$$

In this direction the following theorem will be proved in §6.

Theorem 1.3 *Let the function f be meromorphic of finite order in the plane, let $L(f)$ be defined by (3) with the a_j constants, and assume that $L(f)$ has finitely many zeros.*
(i) Suppose that the auxiliary equation

$$x^k + a_{k-1}x^{k-1} + \ldots + a_0 = 0 \tag{4}$$

has a repeated root. Then f has finitely many poles.
(ii) Suppose that (4) has at least two distinct roots α, β. For $r \geq 1$ and $S > 0$ let

$$m_S(r) = \text{the number of distinct poles of } f \text{ in } \{z \in \mathbb{C} : 1 \leq |z| \leq r, \, |\text{ Re}((\beta - \alpha)z)| \geq S\}. \tag{5}$$

Then there exists $S > 0$ such that $m_S(r) = O(r^3)$ as $r \to \infty$.
(iii) Suppose that (4) has at least three roots which are not collinear. Then $\overline{n}(r, f) = O(r^3)$ as $r \to \infty$. In this case, if in addition there exists $M \in (0, \infty)$ such that each pole ζ of f has multiplicity m_ζ satisfying (2), then f either has order at most $M + 3$ or has finitely many poles.

Part (ii) states in effect that if (4) has distinct roots α and β, and $\overline{n}(r, f) \neq O(r^3)$, then most of the distinct poles z of f satisfy $\arg{(\beta - \alpha)z} \sim \pm\pi/2$. When $k \geq 3$ parts (i) and (iii) together give $\overline{n}(r, f) = O(r^3)$ as $r \to \infty$, unless the roots of (4) are collinear and distinct.

2 Proof of Proposition 1.1

Assume the hypotheses of Proposition 1.1. It may be assumed further that 0 is not in D and that $d_1 = 1$, since otherwise $G(z)$ may be replaced by $G(z + b)/d_1$ for some constant b. Now fix $z_0 \in D$. Let $g = G'$ and let $\psi = g^{-1}$ be that branch of the inverse function mapping $w_0 = g(z_0)$ to z_0. Choose v_0 such that $e^{-v_0} = w_0$ and set $\phi(v) = \psi(e^{-v}) = g^{-1}(e^{-v})$. Since $g = G'$ has no zeros in D and $d_1 = 1$ the function ϕ extends to be analytic and univalent on $H = \{v \in \mathbb{C} : \text{ Re}\,(v) > 0\}$ [17, p.287]. Furthermore, $\phi(H) = D$, and D is simply connected. Since $0 \notin D$, an analytic and univalent branch of $\zeta = h(v) = \log \phi(v)$ may be defined on H. For $v \in H$ applying the Koebe one-quarter theorem [19, p.9] to the function

$$\frac{h(v + u \cdot \text{ Re}\,(v)) - h(v)}{\text{Re}\,(v) \cdot h'(v)}$$

gives

$$\left| \frac{d\zeta}{dv} \right| = \left| \frac{\phi'(v)}{\phi(v)} \right| \leq \frac{8\pi}{\text{Re}\,(v)} < \frac{32}{\text{Re}\,(v)} \quad \text{for } v \text{ in } H. \tag{6}$$

Now define a piecewise-linear path L_1 tending to infinity in H as follows. The path L_1 starts at 1 and is parametrized by $s = \text{ Re}\,(v)$ for $s \geq 1$. Moreover, the slope of L_1 is $(-1)^n$ for $r_n < s < r_{n+1}$, $n = 0, 1, 2, \ldots$, where (r_n) is a sequence increasing rapidly to infinity with $r_0 = 1$. It follows that L_1 meets every horizontal line infinitely often.

For v on L_1 let $s = \text{ Re}\,(v)$ and let M_v be the subpath of L_1 joining 1 to v. Then (6) gives

$$|\phi(v)| \;\leq\; |\phi(1)| \exp\left(\int_{M_v} \frac{32}{\text{Re}\,(u)} |du| \right) \leq |\phi(1)| \exp\left(\int_1^s \frac{32\sqrt{2}dt}{t} \right) \leq |\phi(1)| s^{50}.$$

Using (6) again yields $|\phi'(v)| \leq 32|\phi(1)|s^{49}$, which in turn gives

$$\int_{\phi(L_1)} |G'(z)|\, |dz| = \int_{L_1} \exp(-\text{ Re}\,(v)) |\phi'(v)|\, |dv| \leq 32|\phi(1)| \int_1^\infty e^{-s} s^{49} \sqrt{2}\, ds < \infty.$$

This implies that

$$G(z) = O(1) \quad \text{as } z \to \infty \text{ on } \phi(L_1). \tag{7}$$

The rest of the proof follows [14]. Let v_1 be in H with

$$z_1 = \phi(v_1), \quad v_1 = Q + iy, \quad Q = \log|1/G'(z_1)| \geq 1, \tag{8}$$

and let L be the line given by $v = s + iy, s \geq Q$. Integrating (6) leads to, for $s \geq Q$,

$$|\phi(s+iy)| \leq |\phi(Q+iy)| \exp\left(\int_Q^s \frac{32}{t} dt\right) = |\phi(Q+iy)| s^{32} Q^{-32} = |z_1| s^{32} Q^{-32}. \tag{9}$$

Using (6), (8) and (9) gives

$$\begin{aligned} \int_{\phi(L)} |G'(z)|\,|dz| &= \int_L \exp(-\,\text{Re}\,(v))|\phi'(v)|\,|dv| = \int_Q^\infty e^{-s}|\phi'(s+iy)| ds \\ &\leq |z_1| \int_Q^\infty e^{-s} 32 s^{31} Q^{-32} ds \leq C|z_1| e^{-Q} Q^{-1} = \frac{C|z_1 G'(z_1)|}{\log|1/G'(z_1)|} \end{aligned} \tag{10}$$

after repeated integration by parts, where C is a positive absolute constant. But $\phi(L)$ meets $\phi(L_1)$, and so combining (7) with (10) completes the proof of (1).

□

3 Critical points and asymptotic values

Suppose that the function F is meromorphic of finite lower order in the plane, and that F has infinitely many poles, but F' has finitely many zeros. By Hinchliffe's extension [9] to finite lower order of a theorem of Bergweiler and Eremenko [1], all asymptotic values of F give rise to direct transcendental singularities of the inverse function F^{-1} (see [1, 14, 17] for the classification of transcendental singularities as direct or indirect), and by the Denjoy-Carleman-Ahlfors theorem [1, 17] there are finitely many such singularities. Let the finite asymptotic values of F be a_n, repeated according to how often they occur as direct transcendental singularities of F^{-1}. Let δ_0 be small and positive. Then to each a_n corresponds a component U_n of the set $\{z \in \mathbb{C} : |F(z) - a_n| < \delta_0\}$ which contains no zeros of $F - a_n$ but does contain a path tending to infinity on which $F(z)$ tends to a_n.

The following facts are established in detail in [14, Section 4] (see also [13]). Let J be a simple closed polygonal path such that every finite critical or asymptotic value of F lies on J, but is not a vertex of J. Then the complement of J in $\mathbb{C} \cup \{\infty\}$ consists of two simply connected domains B_1 and B_2, such that B_1 is bounded and $\infty \in B_2$. Fix conformal mappings

$$h_m : B_m \to \Delta = B(0,1) = \{w \in \mathbb{C} : |w| < 1\}, \quad m = 1, 2, \quad h_2(\infty) = 0. \tag{11}$$

By the Schwarz reflection principle, if I is a line segment contained in J and not meeting any vertex of J then h_1 and h_2 extend analytically and univalently to a neighbourhood of I and for $m = 1, 2$ there are positive constants b_m, possibly depending on I, such that

$$b_m \leq |h'_m(w)| \leq 1/b_m \quad \text{for} \quad w \in I. \tag{12}$$

Let J' be the set of vertices of J and singularities of F^{-1}, and let $J'' = J \backslash J'$. For each component J^* of J'' choose a line segment I_q contained in J^* and not meeting J': to each of these I_q correspond constants b_1, b_2 as in (12).

Let T be a component of the set $F^{-1}(B_2)$. Then there are two possibilities. The first is that T contains just one pole of F of multiplicity $p \geq 1$ and $v(z) = (h_2(F(z)))^{1/p}$ maps T conformally onto the unit disc. The second possibility is that T contains no pole of F, but instead a path tending to infinity on which $F(z)$ tends to infinity; there are only finitely many components T of this second type. The components of the set $F^{-1}(B_1)$ are simply connected and conformally equivalent under $h_1 \circ F$ to the unit disc.

All but finitely many components of both the sets $F^{-1}(B_1)$ and $F^{-1}(B_2)$ are unbounded. Let S be an unbounded component of $F^{-1}(B_1)$ with no zeros of F' on its boundary. The boundary ∂S consists of finitely many simple curves, each going to infinity in both directions and mapped by F onto an arc of J. As z tends to infinity in one direction along an arc of ∂S, the image $F(z)$ tends to an asymptotic value a_p of F, and z eventually lies in U_p. Moreover,

$$\text{if } z \in S \text{ and } |F(z) - a_p| < \delta_0 \text{ then } z \in U_p, \tag{13}$$

since F is univalent on S. The component S is called type I if there is only one such asymptotic value of F approached along a boundary arc of S, and type II if there are at least two distinct such values. A type I component S cannot separate the plane.

Consider now a pole z_0 of F, of multiplicity p, with $|z_0|$ large, lying in a component T of $F^{-1}(B_2)$. Because F is univalent on each component of $F^{-1}(B_1)$, at least two components of ∂T are of the following form: a piecewise smooth simple curve T^* on which $\arg h_2(F(z))$ is monotone, mapped by F onto an open arc of J whose closure joins two distinct finite asymptotic values of F. Each such curve T^* must form a boundary curve of a type II component of $F^{-1}(B_1)$. In particular there are infinitely many type II components.

The following lemma plays a key role in the proof of Theorem 1.3; it is closely related to [13, Proof of Theorem 1.3] and [14, Lemma 5.4].

Lemma 3.1 *With the assumptions of this section on the function F, let d, τ and ν be positive real numbers. Let r be large, let ζ_0 and s be such that $r \leq |\zeta_0| \leq r^2$ and $0 < s < |\zeta_0|/2$ and suppose that the open disc $B(\zeta_0, s)$ of centre z_0 and radius s contains at least N_1 distinct poles of F, each of multiplicity at most r^ν, where $r^\tau = o(N_1)$. Then there exist distinct finite asymptotic values a_m and a_n of F and a simple path γ of length $o(s)$ with the following properties. The path γ lies in $B(\zeta_0, 65s)$, and $F(z) \in B_1$ for all z on γ. Moreover, the endpoints v_m and v_n of γ satisfy, for $p = m, n$,*

$$|F(v_p) - a_p| < C_1 \exp(-dr^\tau) \quad \textit{and} \quad |F'(v_p)| < C_2 s^{-1} N_1^{1/2} r^{-\tau/2} \exp(-dr^\tau). \tag{14}$$

Here the positive constants C_j are independent of r and N_1.

Lemma 3.1 depends on the following estimate of Keogh [10] for the length of the image of a radial segment.

Lemma 3.2 ([10]) *Suppose that $0 < r < s < 1$ and that $h(z) = \sum_{j=1}^{\infty} a_j z^j$ maps the disc $B(0, s)$ conformally onto a simply connected domain D of finite area $A = A(s)$. Then for each real θ the length $L(r, \theta)$ of the image under h of the line segment $z = te^{i\theta}, 0 \leq t \leq r$, satisfies*

$$L(r,\theta)^2 \leq \left(\frac{A}{\pi}\right) \log \frac{1}{1-(r/s)^2}, \quad L(r,\theta) = O\left(A(s) \log \frac{1}{s-r}\right)^{1/2} \quad \textit{as} \quad r \to 1.$$

□

Proof of Lemma 3.1. In this proof, c will denote positive constants, not necessarily the same at each occurrence, but independent of r and N_1. Choose $4N$ distinct poles $z_1, \ldots, z_{4N}$ lying in $B(\zeta_0, s)$, with

$$N > cN_1, \quad r^\tau = o(N), \tag{15}$$

and with the following property. Each pole z_j lies in an unbounded component D_j of the set $F^{-1}(B_2)$, associated with a type II component E_j of the set $F^{-1}(B_1)$ in the sense that the boundary of D_j shares a component K_j with the boundary of E_j. Here each K_j is a simple piecewise smooth curve going to infinity in both directions and mapped by F onto a fixed sub-arc J_1 of the polygonal path J, the closure of J_1 joining distinct finite asymptotic values a_m and a_n of F. Thus K_j meets both U_m and U_n. Fix a sub-arc J_0 of J_1, one of the line segments I_q chosen following (12).

For $t > 0$ denote by $\theta_j(t)$ the angular measure of the intersection of D_j with the circle $S(\zeta_0, t)$ of centre ζ_0 and radius t. Since the D_j are all unbounded but meet $B(\zeta_0, s)$, it follows that $\theta_j(t) < 2\pi$ for $s \leq t \leq 64s$. Moreover, at least $2N$ of these D_j, say $D_1, \ldots, D_{2N}$, are such that

$$\int_{2s}^{4s} \frac{dt}{t\theta_j(t)} > cN. \tag{16}$$

To see this, suppose that $D_1, \ldots, D_M$ are such that (16) fails.Then

$$M^2 \leq \left(\sum_{j=1}^{M} \theta_j(t) \right) \left(\sum_{j=1}^{M} \frac{1}{\theta_j(t)} \right), \quad cNM \geq \sum_{j=1}^{M} \int_{2s}^{4s} \frac{dt}{t\theta_j(t)} \geq \frac{M^2 \log 2}{2\pi},$$

and this proves the assertion. Define $v_j = (h_2 \circ F)^{1/p_j}$, with p_j the multiplicity of the pole of F at z_j, so that v_j maps D_j conformally onto $\Delta = B(0,1)$ with $v_j(z_j) = 0$, and $p_j \leq r^\nu$. The boundary of D_j contains a sub-path λ_j mapped onto J_0 by F, such that λ_j also forms part of the boundary of E_j. As z describes the arc λ_j, the image $(h_2 \circ F)(z)$ describes an arc of the unit circle of length at least c, using (12), so that $v_j(z)$ describes an arc of the unit circle of length at least c/p_j. This delivers a lower bound for the harmonic measure of λ_j given by

$$\omega(z_j, \lambda_j, D_j) \geq c/p_j \geq cr^{-\nu} \quad \text{for} \quad j = 1, \ldots, 2N. \tag{17}$$

Set $\sigma_j = \lambda_j \backslash B(\zeta_0, 8s)$. Since z_j lies in $B(\zeta_0, s)$, a standard estimate for harmonic measure [21, p.116] and (16) together imply that

$$\omega(z_j, \sigma_j, D_j) \leq c \exp\left(-\pi \int_{2s}^{4s} \frac{dt}{t\theta_j(t)} \right) \leq \exp(-cN). \tag{18}$$

Using (15) then gives

$$\omega(z_j, \lambda_j^*, D_j) \geq c/p_j \geq cr^{-\nu}, \quad \text{where} \quad \lambda_j^* = \lambda_j \cap B(\zeta_0, 8s). \tag{19}$$

Now, λ_j^* is mapped by v_j into a finite union of sub-arcs of the unit circle of total length at least c/p_j and so is mapped by F into a union of sub-arcs of J_0 of total length at least c, using (12) again. For $t > 0$ let $\phi_j(t)$ be the angular measure of the intersection of E_j with the circle $S(\zeta_0, t)$. Reasoning as above, there must be at least N of the E_j, without loss of generality $E_1, \ldots, E_N$, each with the property that

$$\int_{16s}^{32s} \frac{dt}{t\phi_j(t)} > cN. \tag{20}$$

Set $V_1 = h_1 \circ F$. Then V_1 maps each E_j univalently onto Δ, with λ_j^* mapped onto a union μ_j of sub-arcs of the unit circle of total length at least c. Hence

$$\omega(w, \mu_j, \Delta) \geq c(1 - |w|) \tag{21}$$

for $|w| < 1$. Suppose that z lies in $E_j \setminus B(\zeta_0, 64s)$. Then, because λ_j^* lies in $B(\zeta_0, 8s)$, the same harmonic measure estimate [21, p.116] combined with (15) and (20) yields

$$\omega(V_1(z), \mu_j, \Delta) = \omega(z, \lambda_j^*, E_j) \leq c \exp\left(-\pi \int_{16s}^{32s} \frac{dt}{t\phi_j(t)}\right) \leq \exp(-cN) \leq \exp(-3dr^\tau). \tag{22}$$

Now (21) and (22) together imply that $B(\zeta_0, 64s)$ contains the pre-image H_j under V_1 of the disc $B(0, 1 - \exp(-2dr^\tau))$, and this is true for $1 \leq j \leq N$. These regions H_j are disjoint, and so there must be at least one j such that H_j has area at most cs^2/N. Without loss of generality this is true for $j = 1$.

For $p = m, n$ choose $\mu_p \in [0, 2\pi)$ such that $\mu_p = \arg h_1(a_p)$, and let $\Gamma_p \subseteq E_1$ be the pre-image under V_1 of the segment $w = te^{i\mu_p}, 0 \leq t < 1$. Then $\Gamma_p \setminus U_p$ is bounded by (13). Let Γ_p^* be the sub-path of Γ_p on which $|V_1(z)| \leq 1 - \exp(-dr^\tau)$. Then $\Gamma_p^* \subseteq H_1$ and, by Lemma 3.2 and (15), this path Γ_p^* has arc length at most $T_p = csN^{-1/2}r^{\tau/2} = o(s)$. Since Γ_p tends to infinity, Γ_p^* may be extended to a sub-path Γ_p^{**} of Γ_p of length $2T_p$. The fact that V_1 maps Γ_p onto a radial segment then implies that

$$\int_{\Gamma_p^{**} \setminus \Gamma_p^*} |V_1'(t)|\, |dt| \leq \exp(-dr^\tau)$$

and so there exists $v_p \in \Gamma_p^{**} \setminus \Gamma_p^*$ with

$$|V_1(v_p) - h_1(a_p)| \leq \exp(-dr^\tau), \quad |V_1'(v_p)| \leq T_p^{-1} \exp(-dr^\tau).$$

Since $V_1 = h_1 \circ F$ and h_1 extends univalently to a neighbourhood of a_p by the reflection principle, the estimates (14) follow. To define γ let $\zeta_1 = V_1^{-1}(0) \in B(\zeta_0, 64s) \cap E_1$ be the common starting point of Γ_m and Γ_n, and let γ_p be the sub-path of Γ_p joining ζ_1 to v_p. Next, let γ be the simple path formed from γ_m and γ_n. Thus γ has length at most $4T_p = o(s)$ and lies in $B(\zeta_0, 64s + 2T_p) \cap E_1$, which completes the proof of the lemma.

□

4 Proof of Theorem 1.2

The method rests upon elements of the proof of Theorem 1.1 in [14], modified appropriately, and the parts where changes are required will be highlighted. Assume that f satisfies the hypotheses of Theorem 1.2, with $k = 2$, but that f has infinitely many poles. Apply the reasoning of Section 3, with $F = f'$, and retain the notation there, including the finite asymptotic values a_p and components U_p of the set $\{z \in \mathbb{C} : |f'(z) - a_p| < \delta_0\}$. If the positive constant δ_1 is small enough then Proposition 1.1 gives

$$|f(z) - a_p z| \leq \delta_0 |z| \quad \text{for all } z \in U_p \text{ with } |f'(z) - a_p| < \delta_1. \tag{23}$$

It may be assumed further that $0 \in B_1$ and $h_1(0) = 0$, since otherwise f may be replaced by $f(z) - \lambda z$, for some $\lambda \in \mathbb{C}$. A key step is to show that the type II components of $(f')^{-1}(B_1)$ are not too "thin".

Lemma 4.1 *There exists a positive constant C_1 with the following property: if D is a type II component of the set $(f')^{-1}(B_1)$ and $z_0 \in D, f'(z_0) = 0$ then, provided $|z_0|$ is large enough,*

$$B(z_0, C_1|z_0|) \subseteq \left\{ z \in D : |h_1(f'(z))| < \frac{1}{2} \right\}. \tag{24}$$

Proof. This proof is almost identical to that of Lemma 5.3 in [14]. Since $h_1 \circ f$ maps D onto $B(0,1)$, Koebe's theorem implies that to prove (24) it suffices to show that $G'(0) \neq o(|z_0|)$, where G is that branch of the inverse function of $V = h_1 \circ f'$ which maps $B(0,1)$ onto D. Assume that z_0 is large and $G'(0) = o(|z_0|)$. Since D is a type II component, f' has distinct finite asymptotic values a_1, a_2 such that $f'(z)$ tends to a_p as $z \to \infty$ on a boundary arc of D, for $p = 1, 2$. Because $G'(0) = o(|z_0|)$ there exists a path γ^* in D, of length $o(|z_0|)$, joining points η_1, η_2 with $|f'(\eta_p) - a_p| < \delta_1$. Moreover η_p lies in U_p by (13). Integrating f' along γ^* and using (23) then gives a contradiction. □

Lemma 4.2 *Let $L(r) \to \infty$ with $L(r) \leq \frac{1}{8}\log r$ as $r \to \infty$, and for $k > 0$ and large r let*

$$A(k) = \{z : re^{-kL(r)} \leq |z| \leq re^{kL(r)}\}.$$

Then the number N_1 of distinct poles of f in $A(1)$ satisfies

$$N_1 = O(\phi(r)) \quad \textit{as } r \to \infty, \textit{ where } \quad \phi(r) = L(r) + \frac{\log r}{L(r)}. \tag{25}$$

The proof of Lemma 4.2 is the same as that of Lemma 5.4 in [14]. Assume that f has distinct poles $w_1, \ldots, w_{N_1}$ in $A(1)$, where $\phi(r) = o(N_1)$. The only direct application of the hypothesis of finite order in the proof of [14, Lemma 5.4] is to give a positive constant M_1 and an estimate $m_j \leq r^{M_1}$ for the multiplicity m_j of a pole w_j of f in $A(1)$. In the present setting this bound follows at once from (2). Harmonic measure estimates (the same as those used in the proof of Lemma 3.1) then give distinct poles $w_1, \ldots, w_N$ say, where $\phi(r) = o(N)$, to each of which is associated a type II component E_j of the set $(f')^{-1}(B_1)$ with the property that $H_j = \{z \in E_j : |h_1(f'(z))| < 1/2\} \subseteq A(3)$. Combining this with (24) then gives $N = O(L(r))$ and so a contradiction. The detailed proof is omitted since, apart from the minor change already noted, it is identical to that of Lemma 5.4 in [14]. □

Choosing $L(r) = \frac{1}{8}\log r$ now gives

$$\overline{n}(r^{9/8}, f) - \overline{n}(r^{7/8}, f) = O(\log r) \quad \text{and} \quad \overline{n}(r, f) = O(\log r) \quad \text{as} \quad r \to \infty. \tag{26}$$

But then combining (26) with the hypothesis (2) on the multiplicities of poles leads to

$$n(r, f'') \leq 3n(r, f) = O(r^{M+1}) \quad \text{and} \quad N(r, f'') = O(r^{M+1}) \quad \text{as} \quad r \to \infty.$$

Since f'' has finitely many zeros and finite lower order, it follows that f'' and f have finite order. Hence f has finitely many poles by Theorem 1.1, contrary to assumption. This proves Theorem 1.2. □

5 Two lemmas required for Theorem 1.3, and a special case

Lemma 5.1 *Let $A \in \mathbb{C}$ and let the function f be meromorphic in the plane of order $\rho(f) > 1$. Then $g = f' + Af$ has order $\rho(f)$.*

Proof. Set $G(z) = f(z)e^{Az}$. Then G has the same order as f, and so has $G'(z) = g(z)e^{Az}$. □

Lemma 5.2 ([15, 16]) *Let $d_1 > 0$ and let the function G be transcendental and meromorphic in the plane of order $\rho(G) < d_1$. Then there exists an unbounded uncountable set of R such that the length $L(r, R, G)$ of the level curves $|G(z)| = R > 0$ lying in $|z| < r$ satisfies $L(r, R, G) \leq r^{1+d_1/2}$ as $r \to \infty$.*

□

Proposition 5.1 *Let the function f be meromorphic of finite order in the plane such that $f''+f'$ has finitely many zeros. For $r \geq 1$ and $S>0$ let $n_S(r)$ denote the number of distinct poles of f in the set*

$$\{z \in \mathbb{C} : 1 \leq |z| \leq r, \ \ \mathit{Re}\,(z) \geq S\}.$$

Then there exists $S>0$ such that $n_S(r) = O(r^3)$ as $r \to \infty$.

To prove Proposition 5.1 let the function f be meromorphic of finite order ρ in the plane, such that F' has finitely many zeros, where $F = f'+f$. Assume that there exists $S_0 > 0$ such that $n_{S_0}(r) \neq O(r^3)$ as $r \to \infty$. Then obviously $\rho \geq 3$. Moreover F has order ρ by Lemma 5.1 and so the function $e^z F'(z)$ is transcendental, and the reasoning and notation of §3 may be applied to F.

Lemma 5.3 *Choose $r_1 > 1$ such that $e^z F'(z)$ has no zeros in $|z| \geq r_1$. Choose $d_2 > 1 + \rho/2$ and an integer $N_1 > d_2$ and set*

$$G_1(z) = \frac{1}{z^{N_1} e^z F'(z)}. \tag{27}$$

Then there exists $R_1 > M(r_1, G_1)$ such that the length $L(r, R_1, G_1)$ of the level curves $|G_1(z)| = R_1$ lying in $|z| < r$ satisfies $L(r, R_1, G_1) \leq r^{d_2}$ as $r \to \infty$. Moreover the set $\{z \in \mathbb{C} : |G_1(z)| > R_1\}$ has finitely many unbounded components Ω_j, and there exists $T_1 > 0$ such that

$$|e^z f(z) - e^z F(z)| \leq T_1 \tag{28}$$

on each Ω_j. Finally if $|z| > r_1$ and $|G_1(z)| > R_1$ then z lies in one of the Ω_j.

Proof. The existence of R_1 follows from Lemma 5.2, and it remains only to prove (28) since all other assertions are standard. Let Ω be one of the Ω_j and divide the boundary $\partial\Omega$ into its intersections with the annuli $2^n \leq |z| < 2^{n+1}$, $n \geq 0$. Then $\partial\Omega$ satisfies

$$\int_{\partial\Omega} |e^t F'(t)|\,|dt| \leq \sum_{n=0}^{\infty} R_1^{-1} 2^{-nN_1} L(2^{n+1}, R_1, G_1) \leq \sum_{n=0}^{\infty} R_1^{-1} 2^{-nN_1 + d_2(n+1)} + O(1) < \infty. \tag{29}$$

Fix $z_1 \in \Omega$ and let $z \in \Omega$ be arbitrary. Then z_1 may be joined to z by a path Γ_z in the closure of Ω consisting of part of the ray $\arg t = \arg z_1$, part of the circle $S(0,|z|)$ and part of $\partial\Omega$, so that (29) gives

$$\int_{\Gamma_z} |e^t F'(t)|\,|dt| \leq \int_{\partial\Omega} |e^t F'(t)|\,|dt| + \int_1^{\infty} R_1^{-1} t^{-N_1}\,dt + 2\pi R_1^{-1} |z|^{1-N_1} \leq c_1, \tag{30}$$

where c_1 is independent of z. Since $F = f' + f$ integration by parts leads to

$$e^z f(z) - e^{z_1} f(z_1) = \int_{\Gamma_z} e^t F(t)\,dt = e^z F(z) - e^{z_1} F(z_1) - \int_{\Gamma_z} e^t F'(t)\,dt,$$

and so (28) now follows using (30). □

To complete the proof of Proposition 5.1 assume now that there exists a sequence $S_q \to +\infty$ such that $n_{S_q}(r) \neq O(r^3)$ as $r \to \infty$. Let q be large. There exist arbitrarily large r such that the set A_r given by $r \leq |z| \leq 2r$, $\mathrm{Re}\,(z) \geq S_q$ contains at least N_0 distinct poles of f, where $r^3 = o(N_0)$. This implies the existence of $\zeta_0 \in A_r$ such that the disc $B(\zeta_0, 1)$ contains at least N_1 distinct poles of f, where

$$r = o(N_1) \quad \text{and} \quad N_1 \leq r^{\rho+1}. \tag{31}$$

Moreover all these poles have multiplicity at most $r^{\rho+1}$. Applying Lemma 3.1 with $s = \tau = 1$, $d = 5$ and $\nu = \rho + 1$ then gives distinct finite asymptotic values a_m, a_n of F and points v_m, v_n lying in $B(\zeta_0, 65)$ and satisfying (14). Here the constants C_j arising from Lemma 3.1 may depend on f but do not depend on q or r. Since r is large and

$$v_p \in B(\zeta_0, 65) \subseteq \{z \in \mathbb{C} : 2r/3 \leq |z| \leq 3r, \ \text{Re}\,(z) \geq S_q/2\} \tag{32}$$

it follows from (14) and (31) that $|e^{v_p} F'(v_p)| \leq e^{-r}$. In particular $G_1(v_p)$ is large for $p = m, n$, where G_1 is defined by (27). Hence Lemma 5.3 and (14) give

$$e^{v_n} f(v_n) - e^{v_m} f(v_m) = e^{v_n} F(v_n) - e^{v_m} F(v_m) - A_1 = a_n e^{v_n} - a_m e^{v_m} - A_2, \tag{33}$$

in which A_1 and A_2 may depend on q and r but satisfy $|A_2| \leq |A_1| + 1 \leq 2T_1 + 1$. Next, Lemma 3.1 also gives a path γ joining v_m to v_n such that γ has length $o(1)$ and $F(z)$ lies in the bounded domain B_1 for $z \in \gamma$. Hence

$$e^{v_n} f(v_n) - e^{v_m} f(v_m) = \int_\gamma e^t F(t)\, dt = o(|e^{v_m}|). \tag{34}$$

Combining (33) and (34) now yields

$$a_n e^{v_n} = a_m e^{v_m} + A_2 + o(|e^{v_m}|) = a_m e^{v_m}(1 + o(1)) + A_2 = a_m e^{v_n}(1 + o(1)) + A_2.$$

But $a_m \neq a_n$ and $|A_2| \leq 2T_1 + 1$, in which T_1 arises from Lemma 5.3 and is independent of q and r, whereas (32) gives $\text{Re}\,(v_n) \geq S_q/2 \to \infty$ as $q \to \infty$. This contradiction proves Proposition 5.1. □

Corollary 5.1 *Let α, β be distinct complex numbers, and let D denote d/dz. Let the function f be meromorphic of finite order in the plane such that $F = (D - \alpha)(D - \beta)f$ has finitely many zeros, and let $m_S(r)$ be defined by (5). Then there exists $S > 0$ such that $m_S(r) = O(r^3)$ as $r \to \infty$.*

Proof. Write

$$f(z) = e^{\beta z} g((\beta - \alpha)z) = e^{\alpha z} h((\alpha - \beta)z).$$

Then

$$F = (\alpha - \beta)^2 e^{\beta z}(g'' + g')((\beta - \alpha)z) = (\alpha - \beta)^2 e^{\alpha z}(h'' + h')((\alpha - \beta)z).$$

Now apply Proposition 5.1 to g and h.

□

6 Proof of Theorem 1.3

To prove Theorem 1.3 let f and $Q = L(f)$ be as in the hypotheses, write $D = d/dz$, and let the roots of the equation (4) be $x_1, \ldots, x_k$. For the proof of part (i) assume that $x_1 = x_2 = \alpha$, and write

$$g = (D - x_3) \ldots (D - x_k) f, \quad Q = (D - \alpha)^2 g = L(f), \quad h(z) = e^{-\alpha z} g(z), \quad h''(z) = e^{-\alpha z} Q(z). \tag{35}$$

The result then follows on applying Theorem 1.1 to h. Next, part (ii) is proved by assuming without loss of generality that $\alpha = x_1 \neq x_2 = \beta$, and applying Corollary 5.1 to g as defined by (35).

To establish part (iii) assume without loss of generality that (4) has distinct roots $x_1 = \alpha$, $x_2 = \beta_1$, $x_3 = \beta_2$ such that

$$0 < \arg(\beta_2 - \alpha) - \arg(\beta_1 - \alpha) < \pi.$$

Applying part (ii), first with α and β_1, and subsequently with α and β_2, shows that $\overline{n}(r,f) = O(r^3)$ as $r \to \infty$, which proves the first assertion of part (iii). Assume now in addition that there exists $M \in (0,\infty)$ such that the poles ζ of f have multiplicities m_ζ satisfying (2), but that the order ρ of f exceeds $M+3$. Then $N(r,f) = O(r^{M+3+o(1)})$ as $r \to \infty$. Write

$$G(z) = e^{-\alpha z}H(z), \quad H = (D-x_2)\ldots(D-x_k)f, \quad G'(z) = e^{-\alpha z}(H'(z) - \alpha H(z)) = e^{-\alpha z}Q(z),$$

where $Q = L(f)$. Then H and G both have order $\rho > M+3$ by Lemma 5.1, while G' has order ρ and finitely many zeros. By [13, Theorem 1.4], the function G has finitely many poles and so has f. $\square$

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