Integer points of entire functions

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2005

Starting point:

Theorem A (Pólya ca. 1920)

2^z is the slowest growing transcendental entire function taking integer values at $0, 1, 2, \ldots$.

More precisely, let F be entire with

$$\limsup_{r \to \infty} \frac{\log M(r,F)}{r} < \ln 2,$$

such that $F(n) \in \mathbb{Z}$ for $n = 0, 1, 2, \ldots$.

Then F is a polynomial.

Whittaker, *Interpolatory function theory*, 1935.

We note some ingredients of the method.

(a). Forward differences

$$\Delta^0 F(a) = F(a),$$

$$\Delta^1 F(a) = \Delta F(a) = F(a+1) - F(a),$$

$$\Delta^{n+1} F(a) = \Delta^n F(a+1) - \Delta^n F(a).$$

Forward-difference polynomial

$$\begin{aligned} P_n(z) = \; & F(a) + (z-a)\Delta F(a) + \ldots + \\ & + \frac{(z-a)\ldots(z-a-n+1)}{n!}\Delta^n F(a). \end{aligned}$$

$P_n(z) = F(z)$ at $z = a, \ldots, a+n$.

(b). Analogue of Cauchy integral formula: for $a, \ldots, a+n$ and z inside C,

$$F(z) = P_n(z) + \\ + \frac{1}{2\pi i} \int_C \frac{(z-a)\ldots(z-a-n)F(t)}{(t-a)\ldots(t-a-n)(t-z)} dt.$$

(c). General idea of Pólya's proof: apply (b) with $a = 0$ and C the circle $|t| = \lambda n$, $\lambda > 1$, to get

$$F(z) = \lim_{n\to\infty} P_n(z), \quad |z| \le 1.$$

But

$$P_{n+1}(-1) - P_n(-1) = (-1)^{n+1} \Delta^{n+1} F(0) \in \mathbb{Z}.$$

So for large n,

$$\Delta^{n+1} F(0) = 0, \quad P_{n+1} = P_n, \quad F = P_n.$$

Generalisations include:

functions taking integer values on subsets of $\mathbb{N}$ with positive density;
functions taking integer values on a GP (q^n), $q > 1$.
(Gel'fond et al.)

Functions in a half-plane:

Lemma B (JKL 1994)

Let F be analytic with $|F(z)| \le c_1(1 + |z|^N)$ in a half-plane $\operatorname{Re} z > c_2 > 0$, and with $F(n) \in \mathbb{Z}$ for $n \in \mathbb{N}$, $n > c_3$. Then F is a polynomial.

Conjecture C (Osgood-Yang 1976)

Let f, g be entire functions such that

$$g(z) \in \mathbb{Z} \Rightarrow f(z) \in \mathbb{Z}$$

and

$$T(r, f) = O(T(r, g)), \quad r \to \infty. \qquad (1)$$

Then there exists a polynomial P with $f = P \circ g$.

Some growth condition such as (1) is needed e.g.

$$g(z) \in \mathbb{Z} \Rightarrow z \sin(\pi g(z)) = 0$$

but $f(z) = z \sin(\pi g(z))$ is not a function of g.

Origin of conjecture: when is

$$\frac{e^{2\pi i f} - 1}{e^{2\pi i g} - 1}$$

entire?

Conjecture C is true (and rather more)

(JKL 2004, to appear, Bulletin LMS).

Key idea: use *Wiman-Valiron theory*. Let

$$g(z) = \sum_{k=0}^{\infty} a_k z^k$$

be non-constant entire. For $r > 0$ set

$$\mu(r) = \max\{|a_k|r^k : k = 0, 1, 2, \ldots\},$$

$$N = N(r) = \max\{k : |a_k|r^k = \mu(r)\},$$

(maximum term, central index).

For large $r \notin E$, where $\int_E dt/t < \infty$,

Wiman-Valiron theory shows that

$|z_0| = r$, $|g(z_0)| \geq (1 - o(1))M(r, g) \Rightarrow$

$$g(z) \sim g(z_0)(z/z_0)^N, \quad |\log(z/z_0)| \leq N^{-2/3}.$$

Now let f be entire with

$$\log M(r,f) = O(M(r,g)^{1-\delta})$$

and $g(z) \in \mathbb{N} \Rightarrow f(z) \in \mathbb{Z}$.

Pick $r \notin E$ and z_0 so that

$$|z_0| = r, \quad M(r,g) \sim g(z_0) = R \in \mathbb{N},$$

and write $f = F \circ g$ near z_0

(in fact for $|\log(z/z_0)| < \delta_1/N$).

Then $F(n) \in \mathbb{Z}$ for $n \in \mathbb{N}$ near R

(in fact for $|\log(n/R)| < \delta_2$).

Also $\log|F(w)| \leq R^{1-\delta_3}$ for $|\log(w/R)| < \delta_4$.

Approximate $F(w)$ by a forward-difference polynomial $P_r(w)$ near R.

Pólya method gives bounds for $|P_r|$ and $\deg P_r$, and $f(z) - P_r(g(z))$ extremely small near z_0:

$$\begin{aligned} |f(z) - P_r(g(z))| &< \exp(-\delta_5 R) \\ &< \exp(-\delta_6 M(r, g)) \end{aligned}$$

on a small arc $|z| = r$, $|\arg(z/z_0)| < \delta_7/N$.

Apply Poisson to $\log|f - P_r(g)|$ to get $f(z) - P_r(g(z))$ small in $|z| \leq r/2$.

Normal families $\Rightarrow$

subsequence of (P_r) converges to entire H

and $f = H \circ g$ in $\mathbb{C}$.

Theorem D (JKL 2004)

Let f, g be entire, $\delta > 0$ and $J \subseteq [1, \infty)$ s.t.:

(i) $\log M(r, f) = O(M(r, g)^{1-\delta})$ as $r \to \infty$;

(ii) $\int_J dt/t = \infty$;

(iii) for $r \in J$, we have $f(z) \in \mathbb{Z}$ for

$$g(z) \in \mathbb{N},\ |g(z)| > (1-\delta)M(|z|, g),\ \left|\log\left|\frac{z}{r}\right|\right| \leq \delta.$$

Then there exists an entire H with $f = H \circ g$.

Note: (iii) weaker than $g(z) \in \mathbb{Z} \Rightarrow f(z) \in \mathbb{Z}$.

If (i) is replaced by $T(r, f) = O(T(r, g))$

then H is a polynomial ($\Rightarrow$ Conjecture C).

If entire f, g and $\delta > 0$ and $J \subseteq [1, \infty)$

satisfy (ii) and (iii) *and (iii) with f, g interchanged*

then $f = \pm g + q$ for some $q \in \mathbb{Z}$.

Meromorphic case:

if g is entire and f has poles,

but none near the max modulus points z_0 *of* g

then a version of Theorem D holds.

If g has poles there is an easy result:

Theorem E (JKL 2004)

Let f, g be meromorphic on a domain U such that g has a pole $a \in U$ and $g(z) \in \mathbb{N} \Rightarrow f(z) \in \mathbb{Z}$ on U. Then f is a polynomial in g.

Pf. Take $D \subseteq U$, $a \in \partial D$, mapped univalently to a half-plane $\mathrm{Re}\, w > c_1$ by g. Write $f = F \circ g$ on D. Then $F(n) \in \mathbb{Z}$ for $n \in \mathbb{N}$, $n > c_1$.

F is a polynomial by Lemma B.