RECENT RESEARCH

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1. The Wiman conjecture

Wiman conjectured around 1911 that if f is an entire function, real on the real axis, such that f and f'' have only real zeros, then f belongs to the Laguerre-Pólya class LP, which consists of all entire functions g which are locally uniform limits of real polynomials with real zeros. For such functions, all derivatives $g^{(k)}$ have only real zeros. Wiman's conjecture was proved in [16] for functions of very rapid growth, and by Sheil-Small [18] for functions of finite order. The remaining case (the so-called "small infinite order") was settled in a joint paper of Bergweiler, Eremenko and Langley [2]. Langley subsequently extended the result to higher derivatives [12]. On combination with [2] this shows that if f is a real entire function of infinite order then either f or $f^{(k)}$ has infinitely many non-real zeros, for every integer $k \geq 2$.

Pólya conjectured around 1943 that if $f = Pg$, where P is a real polynomial and g is a real entire function with real zeros but not in the Laguerre-Pólya class LP, then the number of non-real zeros of $f^{(k)}$ tends to infinity with k. For functions of finite order this has recently been proved by Bergweiler and Eremenko [1], and the case of infinite order follows at once from [12].

2. Integer points of entire and meromorphic functions

A result of Pólya [20, p.55] states that the function 2^z is the slowest growing transcendental entire function taking integer values at the non-negative integers. The following related conjecture was advanced in 1976 [17].

Conjecture 0.1 *Let f and g be non-constant entire functions, such that $T(r, f) = O(T(r, g))$ as $r \to \infty$, and such that $g(z) \in \mathbb{Z}$ implies $f(z) \in \mathbb{Z}$. Then there exists a polynomial G such that $G(\mathbb{Z}) \subseteq \mathbb{Z}$ and $f = G \circ g$.*

Conjecture 0.1 was established in a strong form by Langley [10]. It turns out that it suffices for $f(z)$ to take integer values when $g(z) \in \mathbb{N}$ and $g(z)$ is close to the maximum modulus $M(|z|, g)$, and for this to take place for $|z|$ in a reasonably thick subset of $(0, \infty)$. An extension to meromorphic functions has been proved in [11].

3. Value distribution and difference operators

The recent interest in meromorphic solutions of difference equations (see [7] for references) suggests the possibility of developing for difference operators an analogue of the value distribution theory for derivatives of meromorphic functions [8, Chapter 3]. In this direction Bergweiler and

Langley [3] determined conditions under which the first difference $\Delta f(z) = f(z+1) - f(z)$ must have zeros. The results are analogous to the sharp theorem from [6, 9] that if f is a transcendental meromorphic function of lower order $\lambda(f) < 1$ then f' has infinitely many zeros. However, in a departure from the theory for derivatives, it is shown in [3] that results for differences depend on upper growth and not lower.

4. Critical points of discrete potentials

It was conjectured in [5] that if

$$f(z) = \sum_{k=1}^{\infty} \frac{a_k}{z - z_k}, \tag{1}$$

where $a_k > 0$ and

$$z_k \in \mathbb{C}, \quad \lim_{k\to\infty} z_k = \infty, \quad \sum_{z_k \neq 0} \left| \frac{a_k}{z_k} \right| < \infty, \tag{2}$$

then f has infinitely many zeros in $\mathbb{C}$. This conjecture has a physical interpretation in terms of the existence of equilibrium points of the electrostatic field arising from a system of infinite wires, each carrying a charge density a_k and perpendicular to the complex plane at z_k, and analogous conjectures involving distributions of point charges in space appear also in [5]. Results in the direction of these conjectures may be found in [5, 6, 14, 15, 19].

An analogous conjecture for the unit disc was advanced in [4]: if f is given by (1), where

$$|z_k| < 1, \qquad \lim_{k\to\infty} z_k = 1, \quad a_k > 0, \quad \sum_{k=1}^{\infty} a_k < \infty,$$

then f has infinitely many zeros in $B(0,1)$. It was shown in [13] that this conjecture is false. On the other hand if the z_k converge sufficiently rapidly to 1 from within a suitable Stolz angle, then f must indeed have infinitely many zeros in the disc [13, Theorem 1.1].

References

[1] W. Bergweiler and A. Eremenko, Proof of a conjecture of Pólya on the zeros of successive derivatives of real entire functions, preprint 2005.

[2] W. Bergweiler, A. Eremenko and J.K. Langley, Real entire functions of infinite order and a conjecture of Wiman, Geometric and Functional Analysis 13 (2003), 975-991.

[3] W. Bergweiler and J.K. Langley, Zeros of differences of meromorphic functions, to appear, Math. Proc. Camb. Phil. Soc.

[4] J. Borcea and M. Peña, Equilibrium points of logarithmic potentials induced by positive charge distributions I: generalised de Bruijn-Springer relations, Trans. Amer. Math. Soc., to appear.

[5] J. Clunie, A. Eremenko and J. Rossi, On equilibrium points of logarithmic and Newtonian potentials, J. London Math. Soc. (2) 47 (1993), 309-320.

[6] A. Eremenko, J.K. Langley and J. Rossi, On the zeros of meromorphic functions of the form $\sum_{k=1}^{\infty} \frac{a_k}{z-z_k}$, J. d'Analyse Math. 62 (1994), 271-286.

[7] R.G. Halburd and R. Korhonen, Nevanlinna theory for the difference operator, Ann. Acad. Sci. Fenn. 31 (2006), 463-478.

[8] W.K. Hayman, Meromorphic functions, Oxford at the Clarendon Press, 1964.

[9] J.D. Hinchliffe, The Bergweiler-Eremenko theorem for finite lower order, Result. Math. 43 (2003), 121-128.

[10] J.K. Langley, Integer points of entire functions, Bulletin London Math. Soc. 38 (2006), 239-249.

[11] J.K. Langley, Integer points of meromorphic functions, Comput. Methods Funct. Theory 5 (2005), 253-262.

[12] J.K. Langley, Nonreal zeros of higher derivatives of real entire functions of infinite order, J. d'Analyse Math. 97 (2005), 357-396.

[13] J.K. Langley, Equilibrium points of logarithmic potentials on convex domains, to appear, Proc. Amer. Math. Soc.

[14] J.K. Langley and J. Rossi, Meromorphic functions of the form $f(z) = \sum_{n=1}^{\infty} a_n/(z-z_n)$, Rev Mat. Iberoamericana 20 (2004), 285-314.

[15] J.K. Langley and J. Rossi, Critical points of certain discrete potentials, Complex Variables 49 (2004), 621-637.

[16] B.Ja. Levin and I.V. Ostrovskii, The dependence of the growth of an entire function on the distribution of zeros of its derivatives. Sibirsk. Mat. Zh. 1 (1960) 427–455. English transl., AMS Transl. (2) 32 (1963) 323–357.

[17] C.F. Osgood and C.C. Yang, On the quotient of two integral functions, J. Math. Anal. Appl. 54 (1976), 408-418.

[18] T. Sheil-Small, On the zeros of derivatives of real entire functions and Wiman's conjecture, Ann. of Math. 129 (1989), 179-183.

[19] R. Trickey, On critical points and zeros of certain discrete potentials, Comput. Methods Funct. Theory 6 (2006), 29-36.

[20] J.M. Whittaker, Interpolatory function theory, Cambridge Tract No. 33, Cambridge University Press 1935.

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