G12MAN Mathematical Analysis: Chapters 1-11

Professor J K Langley

September 25, 2014

CHAPTER 1. Module information for G12MAN

- Module lecturer: Prof. J K Langley

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- Room B46, Mathematics Building, (95) 14964

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- jkl@maths.nott.ac.uk, james.langley@nottingham.ac.uk

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- For office hours, optional booklist etc. see the Moodle page. The main teaching resource for this module is these notes and there is no book which is strongly recommended.

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- Lectures: see Moodle page
- Problem classes: these will be roughly fortnightly. There will also be some group tutorials. Participation in all of these is vital in order to absorb and master the concepts involved. There will also be opportunities to hand in non-assessed coursework in order to gain practice and feedback. Details will appear on the Moodle page.

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  The module is much more about *ideas and concepts* than techniques. It follows on from G11ACF in the first year, so you need to be familiar with the material there.
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- (b) properties of sets in higher dimensional space $\mathbb{R}^d$;
- (c) properties of functions on subsets of $\mathbb{R}^d$ (such as continuity).

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- There will also be one class test (counting 10%): details will appear on the Moodle page.
- If re-assessment is required this will normally be 100% examination (usually in August).

1. The G12MAN examination

In the G12MAN examination you will need to be able to state and use the definitions, facts and theorems from the module. While it is unlikely that you would be asked to reproduce the proof of a theorem as given in the slides, the ideas and concepts featuring in the proofs may be required in order to answer questions on the examination. Thus everything in the slides, lectures and problem sets is potentially examinable, except where it is explicitly stated otherwise (e.g. if a topic is marked "optional"). It will also be assumed that you are familiar with material from the Core, in particular G11ACF and G11CAL.

In the examination all answers should be justified fully, supported where appropriate by stating in full facts or theorems from the module. The only exception will be where the question explicitly states that justification is not required. Marks will be lost for careless or disjointed presentation.

1. The G12MAN examination

While some marks will be available for stating definitions and/or theorems, to score a good mark on this module you will need to be able to prove statements. The 2012-13 and 2013-14 G12MAN examinations will give you an idea of what to expect, but it is important to note that it takes time and practice to absorb concepts such as those in G12MAN, and merely studying past exam questions is very unlikely to be adequate preparation.

CHAPTER 2. Types of limits

We will review some of the basic limits from G11ACF, and in particular look at what happens when these are combined.

2.1 Types of limits

Commonly used limits include the following (all in G11ACF).

- The limit of a sequence e.g.

$$\lim_{n\to\infty} \frac{3n+4}{4n+7} = \lim_{n\to\infty} \frac{3+4/n}{4+7/n} = \frac{3+0}{4+0} = \frac{3}{4}$$

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- The limit of a function e.g. what is

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- What happens if we mix these ideas: are the following the same?

$$(A) \lim_{n\to+\infty} \left(\lim_{x\to 1-} x^n \right) \quad ; \quad (B) \lim_{x\to 1-} \left(\lim_{n\to+\infty} x^n \right).$$

2.2 Types of limits

More commonly used limits:

- The derivative of a function e.g. the derivative of $f(x) = x^2$ at $a \in \mathbb{R}$ is

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{x^2 - a^2}{x - a} = \lim_{x \to a} (x + a) = 2a.$$

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- What happens if we combine this concept with that of sequences?
  For $x \in \mathbb{R}$ and $n \in \mathbb{N} = \{1, 2, \ldots\}$ set

$$f_n(x) = \frac{nx}{1 + n^2 x^2} \quad , \quad f(x) = \lim_{n \to \infty} f_n(x).$$

  Does $f_n'(0)$ tend to $f'(0)$ as $n \to \infty$?

Figure 1 shows one of these functions.

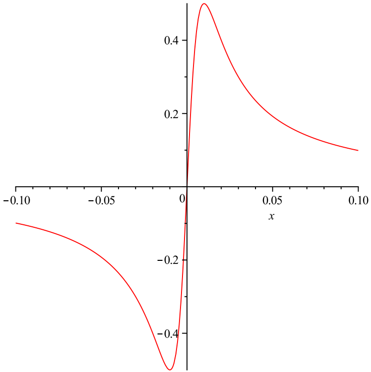

Figure: Plot of the function f_{100} (MAPLE)

Commonly used limits include also:

- The sum of a series e.g.

$$T = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) = \lim_{N \to \infty} \sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+1} \right).$$

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- This gives

$$\begin{aligned} T &= \lim_{N \to \infty} \left(1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \ldots + \frac{1}{N} - \frac{1}{N+1} \right) \\ &= \lim_{N \to \infty} \left(1 - \frac{1}{N+1} \right) = 1. \end{aligned}$$

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- If we take infinite sums involving *functions* we can get some very counterintuitive examples.

- Suppose we take $-\pi < x < \pi$ and look at

$$S(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin nx}{n} = \sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \dots .$$

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- This is an example of a *Fourier sine series*. These feature in the module G12DEF and are among the most important tools in applied mathematics (waves, temperature distribution etc.).

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- This is an example of a *Fourier sine series*. These feature in the module G12DEF and are among the most important tools in applied mathematics (waves, temperature distribution etc.).
- The sum of the series $S(x)$ for $-\pi < x < \pi$ is not obvious! For the determination with proof of the sum you can look at the document
  Optional additional material for G12MAN
  on the module Moodle page.

Figure 2 shows a partial sum.

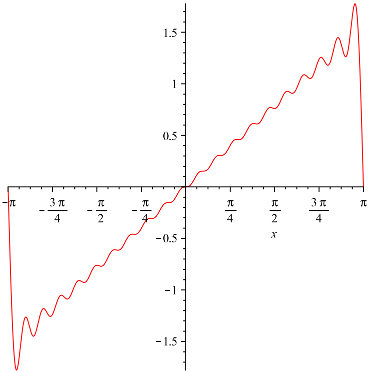

Figure: Plot of the sum of the sine series $S(x)$ up to the $n = 20$ term (MAPLE)

But what is important here is whether the following are the same:

$$(A) \quad \lim_{x \to \pi-} \left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin nx}{n} \right) \quad ;$$

$$(B) \quad \sum_{n=1}^{\infty} \left(\lim_{x \to \pi-} \frac{(-1)^{n+1} \sin nx}{n} \right) .$$

2.3 Types of limits

- It turns out that in this example the key fact is that the coefficients are $c_n = (-1)^{n+1}/n$ and
$$\sum_{n=1}^{\infty} |c_n| = 1 + \frac{1}{2} + \frac{1}{3} + \dots$$
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- In fact, if (a_n) is a real sequence such that

$$\sum_{n=1}^{\infty} |a_n|$$

converges, then we always have

$$\lim_{x \to \pi-} \left(\sum_{n=1}^{\infty} a_n \sin nx \right) = \sum_{n=1}^{\infty} a_n \sin n\pi = 0.$$

We will see this later in the module, in the section on the Weierstrass M-test.

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- Intuition tells you that if you interchange the order of limits you should get the same answer, but the above examples make it clear that this is wrong.
- Analysis gives us tools to determine what does work and why.

We will review briefly the idea of convergence, and consider the important topic of subsequences.

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- A real sequence (x_n) just means a non-terminating list of real numbers

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- For example $x_n = 2^{1/n}$, $n = 1, 2, \ldots$.
- A *convergent* sequence (x_n) approaches a (finite) limit $A \in \mathbb{R}$ as $n \to \infty$ e.g. $2^{1/n} \to 1$, and $y_n = 2^{1/n} - 1 \to 0$.

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- More precisely, if we are given a permitted error (tolerance) $\varepsilon > 0$, then x_n is within ε of A for for all sufficiently large n.
- More precisely still, to each positive real number ε corresponds an integer $N = N(\varepsilon)$ such that $|x_n - A| < \varepsilon$ for all $n \geq N$.

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- A real sequence (x_n) is *non-decreasing* for $n \geq N$ if we have $x_{n+1} \geq x_n$ for all $n \geq N$ i.e.

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- Saying that (x_n) is non-increasing just means that $(-x_n)$ is non-decreasing: equivalently $x_{n+1} \leq x_n$ for all $n \geq N$.

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- For example
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  is obviously non-decreasing for $n \geq 1$, and it tends to a limit as $n \to \infty$.
- In fact $X_n \to \pi^2/6$, which can be shown e.g. using Fourier series (G12DEF).

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- When we do this we are forming what is called a *subsequence*.

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- For example, if we start with $x_n = 1/n$ then

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  is a subsequence,
- but

$$\frac{1}{2}, \quad \frac{1}{6}, \quad \frac{1}{3}, \quad \frac{1}{8}, \ldots$$

  is not (why not?).

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- Now let's take *any* real sequence (x_n) $(n = p, p+1, \ldots)$.
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  We consider two cases.
- **Case I: suppose first that every one of these sets E_q has a maximum element**
  (i.e. there exists $r \geq q$ such that $x_n \leq x_r$ for all $n \geq q$).

- Now let's take *any* real sequence (x_n) $(n = p, p+1, \ldots)$.
- For each $q \geq p$, look at the set

$$E_q = \{x_n : n \geq q\} = \{x_q, x_{q+1}, \ldots\}.$$

  We consider two cases.
- **Case I: suppose first that every one of these sets E_q has a maximum element**
  (i.e. there exists $r \geq q$ such that $x_n \leq x_r$ for all $n \geq q$).
- Clearly, if $p \leq q < Q$ then $E_Q \subseteq E_q$ and so the maximum element of E_Q is not greater than the maximum element of E_q.

- So (still in Case I) we take $n_1 \geq p$ such that x_{n_1} is the maximum element of E_p.

3.2 Another look at sequences in $\mathbb{R}$

- So (still in Case I) we take $n_1 \geq p$ such that x_{n_1} is the maximum element of E_p.
- Then we look at the set E_{1+n_1} and we let x_{n_2} be the maximum element of this set.

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- This gives us $n_2 \geq 1 + n_1 > n_1$ and $x_{n_2} \leq x_{n_1}$.

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- Then we look at the set E_{1+n_2} and we let x_{n_3} be the maximum element of this set.
  Again, $n_3 \geq 1 + n_2 > n_2$ and $x_{n_3} \leq x_{n_2}$.

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  Again, $n_3 \geq 1 + n_2 > n_2$ and $x_{n_3} \leq x_{n_2}$.
- Repeating this gives us a *non-increasing* subsequence x_{n_k} $(k = 1, 2, \ldots)$ of (x_n).
- To be precise, once we have chosen n_k, we choose n_{k+1} so that $x_{n_{k+1}}$ is the maximum element of E_{1+n_k}.

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- Take this E_q, and let $n_1 = q$.
- Since E_q has no maximum element, there is a member of E_q which is greater than x_{n_1}. This must be x_{n_2} for some $n_2 > n_1$.

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- Next, we can find a member of E_q which is greater than *all of*

$$x_q = x_{n_1}, \quad x_{1+n_1}, \ldots, \quad x_{n_2}.$$

Let this element be x_{n_3}: then we must have $n_3 > n_2$.

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- This member of E_q must be $x_{n_{k+1}}$ for some $n_{k+1} > n_k$.
- This process gives us a (strictly) increasing subsequence of (x_n).

3.2 Another look at sequences in $\mathbb{R}$

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- In particular, suppose that (x_n) is a *bounded* real sequence i.e. there exists a real number M such that $|x_n| \leq M$ for all $n \geq p$.
- Then our monotone subsequence is also bounded, and so cannot tend to $+\infty$ or $-\infty$.

- So as a special case we have a (famous) theorem.

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- For example, the sequence $\sin n$ $(n = 1, 2, \ldots)$ goes (2 d.p.)

$$0.84, 0.91, 0.14, -0.75, -0.95, -0.28, 0.66, 0.99, \ldots$$

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  and looks quite random, but even this has a convergent subsequence.
- Which real numbers are the limit of a subsequence of $(\sin n)$? See **Optional additional material for G12MAN** for answer.

CHAPTER 4. Distance in $\mathbb{R}^d$

This chapter will look at the standard properties of distance in $\mathbb{R}^d$, which will later be used in connection with sequences and functions.

4.1 Distance in $\mathbb{R}^d$

- The well known modulus function on $\mathbb{R}$ is given by

$$|x| = \sqrt{x^2} = \begin{cases} x, & \text{if } x \geq 0; \\ -x, & \text{otherwise.} \end{cases}$$

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- This generalises naturally to higher dimensions.

4.2 Distance in $\mathbb{R}^d$

Figure 3 shows a simple example in $\mathbb{R}^2$.

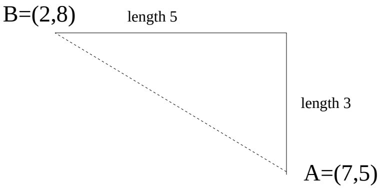

Figure: Straight line between two points

The Euclidean "as the crow flies" distance from A to B is obviously $\sqrt{3^2+5^2} = \sqrt{34} \approx 5.83$. If we can travel only horizontally and vertically we get $3+5=8>\sqrt{34}$ (sometimes called the "taxicab distance").

4.3 Distance in $\mathbb{R}^d$

- We will consider points $\mathbf{x}$ in $\mathbb{R}^d$ given by

$$\mathbf{x} = (x_1, \ldots, x_d).$$

  Here the coordinates x_j are real numbers.

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$$\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + \ldots + x_d^2} = \sqrt{\sum_{j=1}^{d} x_j^2}.$$
- It is easy to see that, for each k,
$$\begin{aligned} x_k^2 &\leq x_1^2 + x_2^2 + \ldots + x_d^2 = |x_1|^2 + |x_2|^2 + \ldots + |x_d|^2 \\ &\leq (|x_1| + |x_2| + \ldots + |x_d|)^2. \end{aligned}$$

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- Taking square roots now gives a useful inequality: for each k,
$$|x_k| \leq \|\mathbf{x}\| \leq \sum_{j=1}^{d} |x_j|.$$

4.4 Distance in $\mathbb{R}^d$

- Now suppose we take two points in $\mathbb{R}^d$ given by

$$\mathbf{x} = (x_1, \ldots, x_d), \quad \mathbf{y} = (y_1, \ldots, y_d).$$

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- We just define the distance from $\mathbf{x}$ to $\mathbf{y}$ (points in $\mathbb{R}^d$) to be

$$\text{dist}\{\mathbf{x}, \mathbf{y}\} = \|\mathbf{x} - \mathbf{y}\| = \sqrt{\sum_{j=1}^{d} (x_j - y_j)^2}.$$

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- We then have

$$\|\mathbf{x} - \mathbf{y}\| \leq \sum_{j=1}^{d} |x_j - y_j|$$

  i.e. the distance from $\mathbf{x}$ to $\mathbf{y}$ is not greater than the taxicab distance.

4.6 Distance in $\mathbb{R}^d$

- A fundamental inequality is the *triangle inequality* for $\mathbb{R}^d$:

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  An (optional) proof using the Cauchy-Schwarz inequality from G11ACF is given in **Optional additional material for G12MAN**.

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$$\|\mathbf{x}\| = \|\mathbf{y}+(\mathbf{x}-\mathbf{y})\| \leq \|\mathbf{y}\| + \|\mathbf{x}-\mathbf{y}\|,$$

$$\|\mathbf{x}-\mathbf{y}\| \geq \|\mathbf{x}\| - \|\mathbf{y}\|.$$

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$$\|\mathbf{x}-\mathbf{y}\| \geq \|\mathbf{x}\| - \|\mathbf{y}\|.$$

- This is sometimes called the second triangle inequality.

- We can interpret the second triangle inequality

$$\|\mathbf{x} - \mathbf{y}\| \geq \|\mathbf{x}\| - \|\mathbf{y}\|$$

as saying that the distance from $\mathbf{x}$ to $\mathbf{y}$ is at least the distance from $\mathbf{x}$ to $\mathbf{0}$ minus the distance from $\mathbf{y}$ to $\mathbf{0}$.

CHAPTER 5. Sequences in $\mathbb{R}^d$

This chapter will look at sequences $(\mathbf{x}_n)$ in which each $\mathbf{x}_n$ is a point in $\mathbb{R}^d$.

5.1 Sequences in $\mathbb{R}^d$

- By a sequence $(\mathbf{x}_n)$ in $\mathbb{R}^d$ we mean a non-terminating list

$$\mathbf{x}_p, \quad \mathbf{x}_{p+1}, \quad \mathbf{x}_{p+2}, \ldots$$

of points in $\mathbb{R}^d$.

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- We sometimes write $(\mathbf{x}_n)$ in terms of its coordinates as

$$\mathbf{x}_n = (x_{n,1}, x_{n,2}, \dots, x_{n,d}).$$

  Each coordinate $x_{n,j}$ then forms a real sequence $(j = 1, \dots, d)$.

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  Each coordinate $x_{n,j}$ then forms a real sequence ($j = 1, \ldots, d$).
- For example, what happens as $n \to \infty$ for $(\mathbf{x}_n)$ as follows?

$$\mathbf{x}_n = \left(\frac{\ln n}{\sqrt{n}}, \quad n^{1/n} \right).$$

5.2 Sequences in $\mathbb{R}^d$

- Suppose we take a sequence $(\mathbf{x}_n)$ and a point $\mathbf{a}$ in $\mathbb{R}^d$. Our basic inequality for points in $\mathbb{R}^d$ gives, for each k,

$$|x_{n,k} - a_k| \leq \|\mathbf{x}_n - \mathbf{a}\| \leq \sum_{j=1}^{d} |x_{n,j} - a_j|$$

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- We say that a sequence $(\mathbf{x}_n)$ in $\mathbb{R}^d$ *converges* if there exists $\mathbf{a} \in \mathbb{R}^d$ such that

$$\lim_{n\to\infty} \|\mathbf{x}_n - \mathbf{a}\| = 0$$

i.e. the distance from $\mathbf{x}_n$ to $\mathbf{a}$ tends to 0.

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- This holds if and only if each of the coordinate sequences $(x_{n,k})$ has limit a_k.

- Again let's take a sequence $(\mathbf{x}_n)$ in $\mathbb{R}^d$.
  Our basic inequality gives, for each k,

$$|x_{n,k}| \leq \|\mathbf{x}_n\| \leq \sum_{j=1}^{d} |x_{n,j}|.$$

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- We say that a sequence $(\mathbf{x}_n)$ $(n = p, p+1, \ldots)$ in $\mathbb{R}^d$ is *bounded* if there exists a real number M such that

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- This will lead to an important theorem for sequences in $\mathbb{R}^d$.

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- Take a bounded sequence in $\mathbb{R}^d$ given in terms of its coordinates as

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  Does $(\mathbf{x}_n)$ have a convergent subsequence?
- Recall that a convergent subsequence of $(y_n) \subseteq \mathbb{R}$ means that we take $n_1 < n_2 < n_3 < \ldots$ in such a way that $\lim_{k\to\infty} y_{n_k}$ exists.

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- Take a bounded sequence in $\mathbb{R}^d$ given in terms of its coordinates as

$$\mathbf{x}_n = (x_{n,1}, x_{n,2}, \ldots, x_{n,d}).$$

- So each coordinate sequence $(x_{n,j})$ is bounded, and has a convergent subsequence by the Bolzano-Weierstrass theorem (Chapter 3).
  Does $(\mathbf{x}_n)$ have a convergent subsequence?
- Recall that a convergent subsequence of $(y_n) \subseteq \mathbb{R}$ means that we take $n_1 < n_2 < n_3 < \ldots$ in such a way that $\lim_{k\to\infty} y_{n_k}$ exists.
- We can think of this as discarding some members of the sequence to leave a convergent sequence e.g. for $y_n = (-1)^n$ we can discard the even n to leave $-1, -1, -1, \ldots$..

- So start with our bounded sequence

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5.3 Sequences in $\mathbb{R}^d$

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- By discarding some n we form a subsequence in which the first coordinate converges.
- This gives a sequence $\mathbf{y}_n$ for which $\lim_{n\to\infty} y_{n,1}$ exists (in $\mathbb{R}$).
- This sequence $\mathbf{y}_n$ is still bounded (the same M will do) so we take another subsequence so that the second coordinate converges.

  Keep repeating this. After d steps we get a convergent subsequence of $(\mathbf{x}_n)$.

5.3 Sequences in $\mathbb{R}^d$

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  in which the first coordinate converges.
- Now if we take $n = 5, 10, 15, 20, \ldots$ in $\mathbf{y}_n$ we get

$$\mathbf{y}_{5k} = \mathbf{x}_{10k} = (1, \cos(k2\pi)) = (1, 1),$$

  a convergent subsequence.

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- Sequences in $\mathbb{R}^d$ aren't in themselves very important, because as we have seen they can be reduced to real sequences by looking at the coordinates.
- However, they are extremely useful when we look at properties of *sets* in $\mathbb{R}^d$.

CHAPTER 6. Sets and frontiers in $\mathbb{R}^d$

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- This consists of all $\mathbf{y}$ whose distance from $\mathbf{x}$ is less than r: hence the name *ball*. The label *open* will be explained later.

6.1 Sets and frontiers in $\mathbb{R}^d$

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- For example, what is H^c in $\mathbb{R}^2$?

6.2 Sets and frontiers in $\mathbb{R}^d$

- Given a set $E \subseteq \mathbb{R}^d$, we look at its *frontier*. An alternative name is *boundary*, but I will use frontier to avoid confusion with a set being "bounded".
  Informally the frontier consists of all points which lie on the border between E and E^c. They may or may not lie in E.

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- Notice that
  $$F = \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\| \leq 1\}$$
  seems to have the same frontier C, and so does
  $$G = \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\| > 1\}.$$

- To define the frontier mathematically, take (in $\mathbb{R}^2$)

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- Take any $\mathbf{y}$ with $\|\mathbf{y}\| = 1$. Now, as $n \to \infty$,

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- So $\mathbf{y}$ is the limit of a sequence in E, and of a sequence in E^c.

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- So we define the frontier as follows.
  Let E be a subset of $\mathbb{R}^d$. The frontier ∂E of E (with respect to $\mathbb{R}^d$) is the set of all $\mathbf{y} \in \mathbb{R}^d$ such that $\mathbf{y}$ is the limit of a sequence in E, *and* of a sequence in E^c.

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  this gives the right frontier.
- We will generally leave out the phrase "with respect to $\mathbb{R}^d$", since it will be clear which dimension we are working in from the set E.
  The frontier is also commonly called the "boundary".
  This concept can give surprising results, however: what is the frontier of $\mathbb{Q} \subseteq \mathbb{R}$?

Having defined the frontier of a set, we now consider points which lie in a set but not on its frontier.

7.1 Interior points of sets in $\mathbb{R}^d$

- We start with a useful fact. Let E be a subset of $\mathbb{R}^d$, and let $\mathbf{x} \in \mathbb{R}^d$, not necessarily in E.

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- Then for each $n \in \mathbb{N}$ there exists $\mathbf{y}_n \in B(\mathbf{x}, 1/n) \cap E$ i.e. there exists $\mathbf{y}_n \in E$ with $\|\mathbf{y}_n - \mathbf{x}\| < 1/n$.

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- Thus $(\mathbf{y}_n)$ is a sequence in E with limit $\mathbf{x}$.

- Suppose next that, for some real $r > 0$, the open ball $B(\mathbf{x}, r)$ does not meet E i.e. $B(\mathbf{x}, r) \cap E = \emptyset$.

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- Then there cannot exist a sequence $(\mathbf{y}_n)$ in E with limit $\mathbf{x}$; if we had such a sequence then we would get, for all sufficiently large n,

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- We combine these observations as a useful lemma.

7.1 Interior points of sets in $\mathbb{R}^d$

- **Lemma 7.1:** *Let E be a subset of $\mathbb{R}^d$, and let $\mathbf{x} \in \mathbb{R}^d$. Then the following are equivalent:*
  (A) for every real $r > 0$, the open ball $B(\mathbf{x}, r)$ meets E;
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  (A) for every real $r > 0$, the open ball $B(\mathbf{x}, r)$ meets E;
  (B) there exists a sequence in E with limit $\mathbf{x}$.
- *Stated another way (using the contrapositive), the following are equivalent:*
  (C) there exists a real $r > 0$ such that the open ball $B(\mathbf{x}, r)$ does not meet E;
  (D) there is no sequence in E with limit $\mathbf{x}$.

7.2 Interior points of sets in $\mathbb{R}^d$

- Let A be a subset of $\mathbb{R}^d$. We define the *interior points* of A as follows. A point $\mathbf{x} \in \mathbb{R}^d$ is an interior point of A if there exists a real $r > 0$ such that $B(\mathbf{x}, r) \subseteq A$.

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- If we take $E = A^c = \mathbb{R}^d \setminus A$ in Lemma 7.1 then we get the following.

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- Informally, this says that $\mathbf{x}$ and all its close neighbours lie in A.
- If we take $E = A^c = \mathbb{R}^d \setminus A$ in Lemma 7.1 then we get the following.
- **Lemma 7.1A:** *Let A be a subset of $\mathbb{R}^d$, and let $\mathbf{x} \in \mathbb{R}^d$. Then the following are equivalent:*
  (i) $\mathbf{x}$ is an interior point of A;
  (i') there exists an open ball $B(\mathbf{x}, r)$ which does not meet A^c;
  (ii) there exists no sequence in A^c with limit $\mathbf{x}$.

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- We just saw that if $\mathbf{x} \in \mathbb{R}^d$ is an interior point of A then there is no sequence in A^c with limit $\mathbf{x}$.
- Thus an interior point of A cannot belong to the frontier ∂A.
- On the other hand, if $\mathbf{y} \in A$ is not an interior point of A, then there exists a sequence in A^c with limit $\mathbf{y}$, so $\mathbf{y}$ is in the frontier ∂A. This yields:

- **Lemma 7.2:** *Let A be any subset of $\mathbb{R}^d$. Then*

$$A = (int\, A) \cup (\partial A \cap A),$$

where $int\, A$ denotes the set of interior points of A, and the two sets $int\, A$ and $\partial A \cap A$ are disjoint.

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- This leads to an important class of sets, called *open sets*.

CHAPTER 8. Open sets in $\mathbb{R}^d$

We look at open subsets of $\mathbb{R}^d$, which play an important role in analysis.

8.1 Open sets in $\mathbb{R}^d$

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- This is equivalent to the condition that $\partial A \cap A = \emptyset$, which is the same as the condition that $\partial A \subseteq A^c = \mathbb{R}^d \setminus A$.

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- This means that for every $\mathbf{x} \in A$ there exists $r > 0$ with $B(\mathbf{x}, r) \subseteq A$.
- This is equivalent to the condition that $\partial A \cap A = \emptyset$, which is the same as the condition that $\partial A \subseteq A^c = \mathbb{R}^d \setminus A$.
- The name *open* will be justified to some extent when we meet closed sets.
  Imagine also that you own a field, but none of its boundary edge. Can you prevent your neighbour(s) from stepping on your property?

- For example,
$$H = \{(u, v) \in \mathbb{R}^2 : u > 0\}$$
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8.1 Open sets in $\mathbb{R}^d$

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- For $\mathbf{x} \in \mathbb{R}^d$ the open ball
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- In fact, if $\|\mathbf{y} - \mathbf{x}\| = s < r$ and $\mathbf{z} \in \mathbb{R}^d$ then

$$\|\mathbf{z} - \mathbf{y}\| < r - s \Rightarrow \|\mathbf{z} - \mathbf{x}\| \leq \|\mathbf{z} - \mathbf{y}\| + \|\mathbf{y} - \mathbf{x}\| < (r - s) + s = r$$

  so $B(\mathbf{y}, r - s) \subseteq B(\mathbf{x}, r)$ (draw a sketch!).

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so $B(\mathbf{y}, r - s) \subseteq B(\mathbf{x}, r)$ (draw a sketch!).
- Alternatively, it's not hard to work out what $\partial B(\mathbf{x}, r)$ is.

8.1 Open sets in $\mathbb{R}^d$

- Which of the following sets are open subsets of the given space?

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- We take

$$(a, b] \subseteq \mathbb{R}; \quad \mathbb{Q} \subseteq \mathbb{R}; \quad \{\mathbf{x} = (u, v) \in \mathbb{R}^2 : \|\mathbf{x}\| \leq 1, u > 0\}.$$

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- To be precise, suppose that U_t is an open subset of $\mathbb{R}^d$, for every t belonging to some set T.
- We then look at

$$W = \bigcup_{t \in T} U_t = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{x} \text{ is in at least one of the } U_t\}.$$

- Is W open? This will be true if and only if every $\mathbf{x} \in W$ is an interior point of W, i.e. to each $\mathbf{x} \in W$ corresponds $r > 0$ with $B(\mathbf{x}, r) \subseteq W$.

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- But if we take $\mathbf{x} \in W$, then $\mathbf{x}$ lies in one of the U_t, and because U_t is open we get $r > 0$ with $B(\mathbf{x}, r) \subseteq U_t$.

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- But $U_t \subseteq W$, and so we have $B(\mathbf{x}, r) \subseteq W$ as required.

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- Thus W *is* open:
  Theorem 8.1: *the union of any family of open subsets of $\mathbb{R}^d$ is an open subset of $\mathbb{R}^d$.*
- Notice that we did not use the frontier here, and the frontier of a union may be tricky to determine.

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  then each U_n is an open ball and so open.
- But the intersection $\bigcap_{n \in \mathbb{N}} U_n$ of all the U_n is just $\{\mathbf{0}\}$, and this is not an open set.
- So the intersection of infinitely many open sets may fail to be open.

8.3 Open sets in $\mathbb{R}^d$

- But now let's take *finitely many* open sets, say $V_1, \ldots, V_n$ (in $\mathbb{R}^d$) and look at their intersection

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- If I now let r be the minimum of the r_j then $r > 0$ and $B(\mathbf{x}, r) \subseteq B(\mathbf{x}, r_j) \subseteq V_j$ for $j = 1, \ldots, n$, and so $B(\mathbf{x}, r) \subseteq W$.

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- **Theorem 8.2:** *the intersection of finitely many open subsets of $\mathbb{R}^d$ is an open subset of $\mathbb{R}^d$.*

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- **Theorem 8.2:** *the intersection of finitely many open subsets of $\mathbb{R}^d$ is an open subset of $\mathbb{R}^d$.*
- You can see from the proof why this fails for infinitely many V_j: an infinite set of positive real numbers is not guaranteed to have a positive lower bound.

CHAPTER 9. Closed sets in $\mathbb{R}^d$

Closed subsets of $\mathbb{R}^d$ are again very important from the point of view of analysis.

9.1 Closed sets in $\mathbb{R}^d$

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- A set $A \subseteq \mathbb{R}^d$ is called closed if $B = A^c = \mathbb{R}^d \setminus A$ is open.
- Since $\partial A = \partial B$, this is equivalent to the condition that $B \cap \partial A = \emptyset$, and so equivalent to the condition that $\partial A \subseteq A$.

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- A set $A \subseteq \mathbb{R}^d$ is called closed if $B = A^c = \mathbb{R}^d \setminus A$ is open.
- Since $\partial A = \partial B$, this is equivalent to the condition that $B \cap \partial A = \emptyset$, and so equivalent to the condition that $\partial A \subseteq A$.
- Closed sets can be characterised in terms of sequences.

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- There exists $\mathbf{x} \in B$ and a sequence in $B^c = A$ with limit $\mathbf{x}$ (using Lemma 7.1A).

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- A is not closed.
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- There exists $\mathbf{x} \in B$ which is not an interior point of B.
- There exists $\mathbf{x} \in B$ and a sequence in $B^c = A$ with limit $\mathbf{x}$ (using Lemma 7.1A).
- There exists a convergent sequence in A whose limit is not in A.

Taking contrapositives we get:

Theorem 9.1: *Let A be a subset of $\mathbb{R}^d$. Then A is closed if and only if every convergent sequence $(\mathbf{x}_n)$ in A satisfies $\lim_{n\to\infty} \mathbf{x}_n \in A$.*

So a closed set A is closed in the sense that a convergent sequence in A cannot escape to a limit outside A.

- Let A be a subset of $\mathbb{R}^d$. We say that A is bounded if there exists a positive real number M such that $\|\mathbf{x}\| < M$ for every $\mathbf{x} \in A$ i.e. $A \subseteq B(\mathbf{0}, M)$.

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- Suppose that A is closed **and** bounded, and take a sequence $(\mathbf{x}_n)$ in A.
- Then $(\mathbf{x}_n)$ is a bounded sequence and so has a convergent subsequence (Bolzano-Weierstrass).

9.3 Closed sets in $\mathbb{R}^d$

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- Suppose that A is closed **and** bounded, and take a sequence $(\mathbf{x}_n)$ in A.
- Then $(\mathbf{x}_n)$ is a bounded sequence and so has a convergent subsequence (Bolzano-Weierstrass).
- Because A is closed the convergent subsequence must have limit in A.

- Suppose on the other hand that A is not closed.
  Then by Theorem 9.1 there exists a convergent sequence $(\mathbf{x}_n)$ in A with $\mathbf{y} = \lim_{n\to\infty} \mathbf{x}_n$ not in A.
  Here any subsequence of $(\mathbf{x}_n)$ must also have limit $\mathbf{y} \notin A$.

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- What happens if A is not bounded?
  Then we can find $\mathbf{y}_n \in A$ with $\|\mathbf{y}_n\| > n \to \infty$, and this sequence cannot have a convergent subsequence.

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- What happens if A is not bounded?
  Then we can find $\mathbf{y}_n \in A$ with $\|\mathbf{y}_n\| > n \to \infty$, and this sequence cannot have a convergent subsequence.
- Combining these, we get another important theorem.

Theorem 9.2: *Let A be a subset of $\mathbb{R}^d$. Then the following are equivalent:*
(i) The set A is closed and bounded.
(ii) Every sequence in A has a convergent subsequence with limit in A.
This is sometimes called the *Heine-Borel theorem*.
A set A which satisfies condition (ii) is called *sequentially compact*.

9.4 Closed sets in $\mathbb{R}^d$

- Suppose that $E_1, E_2, \ldots$ are non-empty closed and bounded subsets of $\mathbb{R}^d$, with

$$E_1 \supseteq E_2 \supseteq E_3 \supseteq \ldots .$$

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- Take $\mathbf{x}_j \in E_j$. Then $(\mathbf{x}_n)$ is a sequence in the closed and bounded set E_1.

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- Take $\mathbf{x}_j \in E_j$. Then $(\mathbf{x}_n)$ is a sequence in the closed and bounded set E_1.
- Then $(\mathbf{x}_n)$ has a convergent subsequence (say $(\mathbf{x}_{n_k})$, with $1 \leq n_1 < n_2 < \ldots$). Let $\mathbf{a} = \lim_{k \to \infty} \mathbf{x}_{n_k}$.

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- Now take $m \in \mathbb{N}$. Since $n_k \geq m$ for all $k \geq m$ we have $\mathbf{x}_{n_k} \in E_m$ for all $k \geq m$.

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- Then $(\mathbf{x}_n)$ has a convergent subsequence (say $(\mathbf{x}_{n_k})$, with $1 \leq n_1 < n_2 < \ldots$). Let $\mathbf{a} = \lim_{k \to \infty} \mathbf{x}_{n_k}$.
- Now take $m \in \mathbb{N}$. Since $n_k \geq m$ for all $k \geq m$ we have $\mathbf{x}_{n_k} \in E_m$ for all $k \geq m$.
- Hence $\mathbf{a} \in E_m$, since E_m is closed, and so

$$\mathbf{a} \in \bigcap_{j=1}^{\infty} E_j .$$

- **Theorem 9.3 (nested closed and bounded sets theorem)** *Suppose that $E_1, E_2, \ldots$ are non-empty closed and bounded subsets of $\mathbb{R}^d$, with*

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Then the intersection

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- Is this still true if we drop the word "bounded"?

CHAPTER 10. Continuous functions on sets in $\mathbb{R}^d$

We will consider continuous functions defined on (subsets of) $\mathbb{R}^d$ and their properties.

10.1 Continuous functions on sets in $\mathbb{R}^d$

- If we have a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ then f is continuous at $a \in \mathbb{R}$ if
$$\lim_{x \longrightarrow a} f(x) = f(a).$$

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- More precisely, given any positive real number ε, there exists a positive real number δ such that $|f(x) - f(a)| < \varepsilon$ for all x with $|x - a| < \delta$.

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- More precisely, given any positive real number ε, there exists a positive real number δ such that $|f(x) - f(a)| < \varepsilon$ for all x with $|x - a| < \delta$.
- Here ε can be thought of as εrror, or tolεrance, and δ as δisplacement, or δistance.

10.1 Continuous functions on sets in $\mathbb{R}^d$

- We will extend this first to functions defined on an open set $U \subseteq \mathbb{R}^d$.

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- We will extend this first to functions defined on an open set $U \subseteq \mathbb{R}^d$.
- An open set U has the advantage that if $\mathbf{a} \in U$ then all points $\mathbf{x}$ which are sufficiently close to $\mathbf{a}$ also lie in U.
- Later we will consider functions on *any* $E \subseteq \mathbb{R}^d$.

10.2 Continuous functions on sets in $\mathbb{R}^d$

- So suppose we have a function $\mathbf{f} : U \to \mathbb{R}^q$, where $U \subseteq \mathbb{R}^d$ is open.

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- So suppose we have a function $\mathbf{f} : U \to \mathbb{R}^q$, where $U \subseteq \mathbb{R}^d$ is open.
- We say that $\mathbf{f}$ is continuous at $\mathbf{a} \in U$ if the following is true.

10.2 Continuous functions on sets in $\mathbb{R}^d$

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- Given any positive real number ε, there exists a positive real number δ such that $\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a})\| < \varepsilon$ for all $\mathbf{x}$ with $\|\mathbf{x} - \mathbf{a}\| < \delta$.

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- Again this just says that the distance from $\mathbf{f}(\mathbf{x})$ to $\mathbf{f}(\mathbf{a})$ is as small as we like, provided the distance from $\mathbf{x}$ to $\mathbf{a}$ is small enough.
- Keep in mind, though, that the first distance $\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a})\|$ is with respect to $\mathbb{R}^q$, and the second distance $\|\mathbf{x} - \mathbf{a}\|$ is with respect to $\mathbb{R}^d$.

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- Since $(\mathbf{x}_n)$ converges to $\mathbf{a}$ there exists an integer N such that $\|\mathbf{x}_n - \mathbf{a}\| < \delta$ for all $n \geq N$.
- This gives $\|\mathbf{f}(\mathbf{x}_n) - \mathbf{f}(\mathbf{a})\| < \varepsilon$ for all $n \geq N$.
- Since we can do this for any $\varepsilon > 0$, we conclude that $\mathbf{f}(\mathbf{x}_n)$ converges to $\mathbf{f}(\mathbf{a})$.

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- So for each $n \in \mathbb{N}$ we can find $\mathbf{x}_n$ with $\|\mathbf{x}_n - \mathbf{a}\| < 1/n$ but $\|\mathbf{f}(\mathbf{x}_n) - \mathbf{f}(\mathbf{a})\| \geq \varepsilon$.

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- So $(\mathbf{x}_n)$ converges to $\mathbf{a}$, but $(\mathbf{f}(\mathbf{x}_n))$ does not converge to $\mathbf{f}(\mathbf{a})$.
- Combining this reasoning gives the following theorem.

10.3 Continuous functions on sets in $\mathbb{R}^d$

Theorem 10.1: *Let* $\mathbf{f} : U \to \mathbb{R}^q$ *be a function, where* $U \subseteq \mathbb{R}^d$ *is open, and let* $\mathbf{a} \in U$*. Then the following are equivalent:*
(i) $\mathbf{f}$ *is continuous at* $\mathbf{a}$*;*
(ii) for every sequence $(\mathbf{x}_n)$ *in* $\mathbb{R}^d$ *which converges to* $\mathbf{a}$*, the image sequence* $(\mathbf{f}(\mathbf{x}_n))$ *converges to* $\mathbf{f}(\mathbf{a})$*.*

10.4 Continuous functions on sets in $\mathbb{R}^d$

- Suppose next that we write our function $\mathbf{f} : U \to \mathbb{R}^q$ in terms of its coordinates as

$$\mathbf{f} = (f_1, \ldots, f_q),$$

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- Then from Chapter 5 we know that $(\mathbf{f}(\mathbf{x}_n))$ converges to $\mathbf{f}(\mathbf{a})$ if and only if each coordinate sequence $(f_j(\mathbf{x}_n))$ converges to $f_j(\mathbf{a})$.

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- So $\mathbf{f}$ is continuous at $\mathbf{a}$ if and only if all the f_j are.
- So the continuity of our function $\mathbf{f} = (f_1, \ldots, f_q)$ is equivalent to the continuity of all of its coordinate functions f_j.

- Now consider

$$f(u,v) = \frac{u^4 v^2}{u^8 + v^4} \quad (\text{if } (u,v) \neq (0,0)), \quad f(0,0) = 0.$$

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- So if we fix one variable (u or v) and regard f as a function of the other variable then f is continuous.
- But is f really continuous as a function of *both* variables?

- If we let $(u, v) \to (0, 0)$ along a straight line $u = mv$ (where m is a non-zero constant) then we get

$$f(mv, v) = \frac{m^4 v^6}{m^8 v^8 + v^4} = \frac{m^4 v^2}{m^8 v^4 + 1} \to 0 \quad \text{as } v \to 0.$$

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- But if we let $(u,v) \to (0,0)$ along the curve $v = u^2$ we get

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- So f is NOT continuous at $(0, 0)$.

- The conclusion is that in deciding whether a function $\mathbf{f} = (f_1, \ldots, f_q)$ is continuous it is sufficient to look at each coordinate f_j of the image separately, but it is NOT sufficient to consider $\mathbf{f}$ with respect to each coordinate variable x_k separately.

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- Nor is it enough to look at $\mathbf{f}$ as $\mathbf{x}$ approaches a point along straight lines.
- In general it can be quite tricky to determine whether a function of several variables is continuous, but the next example shows how this can be done in some cases.

- We change v^2 to v^3 in the numerator to get

$$g(u,v) = \frac{u^4 v^3}{u^8 + v^4} \quad (\text{if } (u,v) \neq (0,0)), \quad g(0,0) = 0.$$

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$$|u^4 v^3| \leq |u|^4 |u|^6 = |u|^{10}, \quad |g(u,v)| \leq \frac{|u|^{10}}{|u|^8} = |u|^2 \to 0 = g(0,0).$$

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- If we let $(u,v) \to (0,0)$ with $|v| > |u|^2$ then $|u| < |v|^{1/2}$ and

$$|u^4 v^3| < |v|^2 |v|^3 = |v|^5, \quad |g(u,v)| \leq \frac{|v|^5}{|v|^4} = |v| \to 0 = g(0,0).$$

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- So this time $g(u,v) \to g(0,0)$ as $(u,v) \to (0,0)$ in any fashion, and so g is continuous at $(0,0)$.
- If you try this method on the previous example you will see that it is inconclusive.

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- Let $E \subseteq \mathbb{R}^d$, and let $\mathbf{f} : E \to \mathbb{R}^q$ be a function. Then $\mathbf{f}$ is said to be continuous on E if the following is true.
For every convergent sequence $(\mathbf{x}_n)$ in E with limit $\mathbf{a} \in E$ we have $\lim_{n\to\infty} \mathbf{f}(\mathbf{x}_n) = \mathbf{f}(\mathbf{a})$.

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- When E is open this is, by Theorem 10.1, equivalent to $\mathbf{f}$ being continuous, in terms of the original definition, at every point in E.

10.6 Continuous functions on sets in $\mathbb{R}^d$

- An important case is where E is a closed and bounded subset of $\mathbb{R}^d$. Let $E \subseteq \mathbb{R}^d$ be closed and bounded, and let $\mathbf{f} : E \to \mathbb{R}^q$ be a continuous function on E.

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- Let $(\mathbf{y}_n)$ be a sequence in $\mathbf{f}(E)$. Then each $\mathbf{y}_n$ is $\mathbf{f}(\mathbf{x}_n)$ for some $\mathbf{x}_n \in E$.

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- This means we have $n_1 < n_2 < n_3 < \ldots$ and $\mathbf{x}_{n_k} \to \mathbf{a}$ as $k \to \infty$.
- Since $\mathbf{f}$ is continuous on E we get $\mathbf{f}(\mathbf{x}_{n_k}) \to \mathbf{f}(\mathbf{a})$ as $k \to \infty$.

10.6 Continuous functions on sets in $\mathbb{R}^d$

- An important case is where E is a closed and bounded subset of $\mathbb{R}^d$. Let $E \subseteq \mathbb{R}^d$ be closed and bounded, and let $\mathbf{f} : E \to \mathbb{R}^q$ be a continuous function on E.
- Let $(\mathbf{y}_n)$ be a sequence in $\mathbf{f}(E)$. Then each $\mathbf{y}_n$ is $\mathbf{f}(\mathbf{x}_n)$ for some $\mathbf{x}_n \in E$.
- Since E is closed and bounded the sequence $(\mathbf{x}_n)$ has a convergent subsequence with limit $\mathbf{a} \in E$, by the Heine-Borel theorem (Theorem 9.2).
- This means we have $n_1 < n_2 < n_3 < \ldots$ and $\mathbf{x}_{n_k} \to \mathbf{a}$ as $k \to \infty$.
- Since $\mathbf{f}$ is continuous on E we get $\mathbf{f}(\mathbf{x}_{n_k}) \to \mathbf{f}(\mathbf{a})$ as $k \to \infty$.
- Thus $(\mathbf{y}_n)$ has a convergent subsequence $(\mathbf{y}_{n_k})$ with limit $\mathbf{f}(\mathbf{a}) \in \mathbf{f}(E)$.

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- **Theorem 10.2** *Let $E \subseteq \mathbb{R}^d$ be closed and bounded, and let $\mathbf{f} : E \to \mathbb{R}^q$ be a continuous function on E. Then the image $\mathbf{f}(E) = \{\mathbf{f}(\mathbf{x}) : \mathbf{x} \in E\}$ is a closed and bounded subset of $\mathbb{R}^q$.*

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- Let $E \subseteq \mathbb{R}^d$ be closed and bounded and non-empty, and let $f : E \to \mathbb{R}$ be a continuous real-valued function on E.

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- In this theorem the special case where $q = 1$ plays a key role.
- Let $E \subseteq \mathbb{R}^d$ be closed and bounded and non-empty, and let $f : E \to \mathbb{R}$ be a continuous real-valued function on E.
- The theorem tells us that f is bounded on E. Does f have a maximum on E?

- To answer this, let $A = f(E)$. Then A is a bounded non-empty subset of $\mathbb{R}$.

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- Since M is the *least* upper bound there are members of A as close as we like to M, and so we can find $\mathbf{x} \in E$ with $f(\mathbf{x})$ as close as we like to M.

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- Let $M = \sup A$ be the least upper bound i.e. the least real number which is greater than or equal to every member of A. Such a number M exists (G11ACF).
- Since M is the *least* upper bound there are members of A as close as we like to M, and so we can find $\mathbf{x} \in E$ with $f(\mathbf{x})$ as close as we like to M.
- So there must be some $\mathbf{x} \in E$ with $f(\mathbf{x}) = M$, because if not then the function

$$g(\mathbf{x}) = \frac{1}{M - f(\mathbf{x})}$$

is continuous, real-valued and unbounded on the closed and bounded set E, contradicting the fact that $g(E)$ must be closed and bounded.

- We deduce:
  Theorem 10.3 (the maximum theorem for continuous functions): *Let $E \subseteq \mathbb{R}^d$ be closed and bounded and non-empty, and let $f : E \to \mathbb{R}$ be a continuous real-valued function on E. Then f has a maximum on E i.e. there exists $\mathbf{x}_0 \in E$ with $f(\mathbf{x}) \leq f(\mathbf{x}_0)$ for every $\mathbf{x} \in E$.*

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- Note that x_0 does not have to be unique here.
- Applying the theorem to $-f$, we also get that f has a minimum on E.
- The particular case where $E = [a, b]$ is a closed and bounded interval in $\mathbb{R}$ is one of the most important theorems in calculus.

- To close the chapter, let's take a function $\mathbf{f} : \mathbb{R}^d \to \mathbb{R}^q$.

10.7 Continuous functions on sets in $\mathbb{R}^d$

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- Let $V \subseteq \mathbb{R}^q$ be open, and consider the pre-image

$$\mathbf{f}^{-1}(V) = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{f}(\mathbf{x}) \in V\}.$$

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- We are not assuming here that $\mathbf{f}$ is injective or has an inverse function: the notation $\mathbf{f}^{-1}(V)$ just means all points which are mapped into V.

10.7 Continuous functions on sets in $\mathbb{R}^d$

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- Since $\mathbf{f}$ is continuous, there exists $r > 0$ such that $\|\mathbf{x} - \mathbf{x}_0\| < r$ implies that

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- But this says that if $\|\mathbf{x} - \mathbf{x}_0\| < r$ then $\mathbf{f}(\mathbf{x}) \in B(\mathbf{y}_0, s) \subseteq V$, i.e. $\mathbf{x} \in \mathbf{f}^{-1}(V)$. Hence $B(\mathbf{x}_0, r) \subseteq \mathbf{f}^{-1}(V)$.

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- Take $\mathbf{x}_0 \in \mathbb{R}^d$ and $\mathbf{y}_0 = \mathbf{f}(\mathbf{x}_0)$, and $\varepsilon > 0$.
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- Take $\mathbf{x}_0 \in \mathbb{R}^d$ and $\mathbf{y}_0 = \mathbf{f}(\mathbf{x}_0)$, and $\varepsilon > 0$.
- Then $V = B(\mathbf{y}_0, \varepsilon)$ is open, and therefore so is $\mathbf{f}^{-1}(V)$.
- Since $\mathbf{x}_0 \in \mathbf{f}^{-1}(V)$, we get $\delta > 0$ such that $B(\mathbf{x}_0, \delta) \subseteq \mathbf{f}^{-1}(V)$.

- Conversely, suppose we know that $\mathbf{f}^{-1}(V)$ is open, for every open subset V of $\mathbb{R}^q$.
- Take $\mathbf{x}_0 \in \mathbb{R}^d$ and $\mathbf{y}_0 = \mathbf{f}(\mathbf{x}_0)$, and $\varepsilon > 0$.
- Then $V = B(\mathbf{y}_0, \varepsilon)$ is open, and therefore so is $\mathbf{f}^{-1}(V)$.
- Since $\mathbf{x}_0 \in \mathbf{f}^{-1}(V)$, we get $\delta > 0$ such that $B(\mathbf{x}_0, \delta) \subseteq \mathbf{f}^{-1}(V)$.
- This tells us that $\|x - \mathbf{x}_0\| < \delta$ implies $\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}_0)\| < \varepsilon$, and since we can do this for any $\varepsilon > 0$ our $\mathbf{f}$ must be continuous.

- We deduce:
  Theorem 10.4: *Let* $\mathbf{f} : \mathbb{R}^d \to \mathbb{R}^q$ *be a function. Then the following are equivalent:*
  (a) $\mathbf{f}$ *is continuous on* $\mathbb{R}^d$*;*
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10.7 Continuous functions on sets in $\mathbb{R}^d$

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- There is a version of this for functions on subsets of $\mathbb{R}^d$, but it is not quite so easy to state.
- This idea gives a useful alternative approach to continuity (avoiding ε and δ) and this is developed in the module G13MTS (Metric and Topological Spaces).

We saw in Chapter 2 that taking sequences of functions may lead to unexpected consequences. This chapter will introduce a strong form of convergence called *uniform* convergence.

11.1 Convergence of sequences of functions

- We start with an example. For $n \in \mathbb{N}$ and $x \in [0, 1] \subseteq \mathbb{R}$ set $f_n(x) = x^n$.

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- Thus although the functions f_n are continuous on $[0,1]$, the limit function f is not.
- We saw similar phenomena in Chapter 2.

- The problem is that the condition

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- We will define a stronger form of convergence of functions, which does preserve many good properties.

11.2 Convergence of sequences of functions

- Let $E \subseteq \mathbb{R}^d$ and let $\mathbf{f}_n$ $(n = p, p+1, \ldots)$ and $\mathbf{f}$ be functions from E into $\mathbb{R}^q$. We say that $\mathbf{f}_n$ converges uniformly to $\mathbf{f}$ on E if
$$\sup\{\|\mathbf{f}_n(\mathbf{x}) - \mathbf{f}(\mathbf{x})\| : \mathbf{x} \in E\} \to 0 \quad \text{as } n \to \infty.$$

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- If this is true then for each $\mathbf{y} \in E$ we have
$$\|\mathbf{f}_n(\mathbf{y}) - \mathbf{f}(\mathbf{y})\| \leq \sup\{\|\mathbf{f}_n(\mathbf{x}) - \mathbf{f}(\mathbf{x})\| : \mathbf{x} \in E\} \to 0 \quad \text{as } n \to \infty.$$
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Thus uniform convergence on E implies pointwise convergence on E.
- But for our first example
$$\sup\{|f_n(x) - f(x)| : 0 \leq x \leq 1\} = \sup\{x^n : 0 \leq x < 1\} = 1 \not\to 0,$$
and so in this example the convergence is NOT uniform.

- Let $E \subseteq \mathbb{R}^d$ and suppose that *continuous* functions $\mathbf{f}_n : E \to \mathbb{R}^q$ converge uniformly to $\mathbf{f} : E \to \mathbb{R}^q$ on E.

11.3 Convergence of sequences of functions

- Let $E \subseteq \mathbb{R}^d$ and suppose that *continuous* functions $\mathbf{f}_n : E \to \mathbb{R}^q$ converge uniformly to $\mathbf{f} : E \to \mathbb{R}^q$ on E.
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- To do this it is enough to take $\varepsilon > 0$ and find an integer M such that $\|\mathbf{f}(\mathbf{x}_m) - \mathbf{f}(\mathbf{a})\| < \varepsilon$ for all $m \geq M$.

11.3 Convergence of sequences of functions

- First of all, since $\mathbf{f}_n \to \mathbf{f}$ uniformly on E, we can find $N \in \mathbb{N}$ such that

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- So we can find $M \in \mathbb{N}$ such that

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- Now for $m \geq M$ we use the triangle inequality to write

$$\begin{aligned}
\|\mathbf{f}(\mathbf{x}_m) - \mathbf{f}(\mathbf{a})\| &= \|\mathbf{f}(\mathbf{x}_m) - \mathbf{f}_N(\mathbf{x}_m) + \\
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  Theorem 11.1: *Let $E \subseteq \mathbb{R}^d$ and suppose that the functions $\mathbf{f}_n : E \to \mathbb{R}^q$ converge uniformly to $\mathbf{f} : E \to \mathbb{R}^q$ on E. If the $\mathbf{f}_n$ are continuous on E then so is $\mathbf{f}$.*
- Put briefly, the uniform limit of continuous functions is continuous.

11.4 Convergence of sequences of functions

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- Let $E \subseteq \mathbb{R}^d$ and suppose that the functions $\mathbf{f}_k : E \to \mathbb{R}^q$ satisfy

$$\|\mathbf{f}_k(\mathbf{x})\| \leq M_k \quad \text{for all } k \geq p \text{ and all } \mathbf{x} \in E,$$

and suppose that

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- On E we look at the sum

$$\mathbf{F}(\mathbf{x}) = \sum_{k=p}^{\infty} \mathbf{f}_k(\mathbf{x})$$

and the partial sums

$$\mathbf{F}_n(\mathbf{x}) = \sum_{k=p}^{n} \mathbf{f}_k(\mathbf{x}).$$

- For $n \geq p$ and $\mathbf{x} \in E$ we can write, by the triangle inequality,

$$\begin{aligned} \|\mathbf{F}(\mathbf{x}) - \mathbf{F}_n(\mathbf{x})\| &= \lim_{m\to\infty} \|\mathbf{F}_m(\mathbf{x}) - \mathbf{F}_n(\mathbf{x})\| \\ &= \lim_{m\to\infty} \left\| \sum_{k=n+1}^{m} \mathbf{f}_k(\mathbf{x}) \right\| \\ &\leq \lim_{m\to\infty} \sum_{k=n+1}^{m} \|\mathbf{f}_k(\mathbf{x})\| \\ &\leq \lim_{m\to\infty} \sum_{k=n+1}^{m} M_k \\ &= \sum_{k=n+1}^{\infty} M_k. \end{aligned}$$

- So this gives us

$$\sup\{\|\mathbf{F}(\mathbf{x}) - \mathbf{F}_n(\mathbf{x})\| : \mathbf{x} \in E\} \leq \sum_{k=n+1}^{\infty} M_k$$

and the RHS tends to 0 as $n \to \infty$, because the series $\sum_{k=p}^{\infty} M_k$ converges!

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  and the RHS tends to 0 as $n \to \infty$, because the series $\sum_{k=p}^{\infty} M_k$ converges!
- Hence the series $\mathbf{F}$ converges uniformly on E, in the sense that the partial sums $\mathbf{F}_n$ converge uniformly to $\mathbf{F}$ on E. This is:

Theorem 11.2 (the Weierstrass M-test): *Let $E \subseteq \mathbb{R}^d$ and suppose that the functions $\mathbf{f}_k : E \to \mathbb{R}^q$ satisfy*

$$\|\mathbf{f}_k(\mathbf{x})\| \leq M_k \quad \text{for all } k \geq p \text{ and all } \mathbf{x} \in E,$$

where

$$\sum_{k=p}^{\infty} M_k < \infty.$$

Then the series

$$\mathbf{F}(\mathbf{x}) = \sum_{k=p}^{\infty} \mathbf{f}_k(\mathbf{x})$$

converges uniformly on E. In particular, if the $\mathbf{f}_k$ are continuous on E then so is $\mathbf{F}$.

- As a special case we can take our example from Chapter 2: if $a_n \in \mathbb{R}$ and $\sum_{n=1}^{\infty} |a_n| < \infty$, look at

$$U(x) = \sum_{n=1}^{\infty} a_n \sin nx.$$

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$$U(x) = \sum_{n=1}^{\infty} a_n \sin nx.$$

- What is $\lim_{x \to \pi-} U(x)$?

- Unfortunately, even uniform convergence has its limitations. For example, as $n \to \infty$,

$$f_n(x) = \frac{\sin(n^2 x)}{n}$$

converges uniformly on $\mathbb{R}$ to 0 (since $|f_n(x) - 0| \leq 1/n$ on $\mathbb{R}$).

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- But is it true that the derivative of the limit equals the limit of the derivative?

11.6 A space filling curve

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- (iii) we have $\phi(t) = 1$ for $2/3 \leq t \leq 1$.

11.6 A space filling curve

Figure 4 shows the standard way to define the Schoenberg function ϕ.

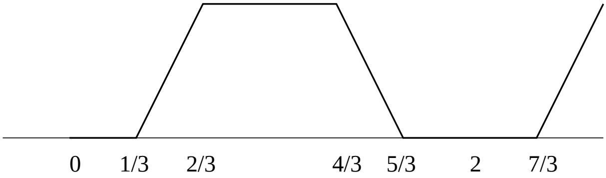

Figure: Part of the graph of the periodic function ϕ

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- Now we set

$$f_1(t) = \sum_{n=1}^{\infty} \frac{\phi(3^{2n-2} t)}{2^n}, \quad f_2(t) = \sum_{n=1}^{\infty} \frac{\phi(3^{2n-1} t)}{2^n}.$$

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- Where does this curve lie? We have

$$0 \leq f_j(t) \leq \sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \ldots = 1,$$

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  so our curve lies in the unit square $[0, 1] \times [0, 1]$.
- How much of this square does it occupy?

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- To answer this question take the point $(a, b) \in [0, 1]^2$ with $a, b \in [0, 1]$ and write a and b in binary as

$$a = \frac{a_1}{2} + \frac{a_2}{4} + \frac{a_3}{8} + \dots, \quad b = \frac{b_1}{2} + \frac{b_2}{4} + \frac{b_3}{8} + \dots,$$

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- Now let

$$c = 2\left(\frac{a_1}{3} + \frac{b_1}{9} + \frac{a_2}{27} + \frac{b_2}{81} + \dots\right) = 2\sum_{n=1}^{\infty} \frac{c_n}{3^n}.$$

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- Here $c_{2n} = b_n$ and $c_{2n-1} = a_n$ for $n \in \mathbb{N}$.
- Note that

$$0 \leq c \leq 2\left(\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \dots\right) = \frac{2}{3}\left(\frac{1}{1 - 1/3}\right) = 1.$$

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  is an even integer.
- So since ϕ has period 2 we get that

$$\phi(3^k c) = \phi(d_k), \quad d_k = 2 \sum_{n=k+1}^{\infty} \frac{c_n}{3^{n-k}} = 2\sum_{m=1}^{\infty} \frac{c_{k+m}}{3^m}.$$

- Now c_{k+1} is either 0 or 1. If c_{k+1} is 0 then

$$0 \leq d_k = 2 \sum_{m=1}^{\infty} \frac{c_{k+m}}{3^m} = 2 \sum_{m=2}^{\infty} \frac{c_{k+m}}{3^m} \leq 2 \left(\frac{1}{9} + \frac{1}{27} + \ldots \right) = \frac{1}{3},$$

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- So in either case we have $\phi(3^k c) = c_{k+1}$.

- But then, for $n \in \mathbb{N}$,

$$\phi(3^{2n-2}c) = c_{2n-1} = a_n, \quad \phi(3^{2n-1}c) = c_{2n} = b_n.$$

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- This means that $\mathbf{f}(c)$ is the point (a, b) and so $\mathbf{f}([0,1])$ occupies the whole unit square $[0,1]^2$.
- So a square can be filled up using a curve, and this highly counter-intuitive fact ends the first half of the module.