G12MAN Mathematical Analysis: Chapters 12-14

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These are the notes for Chapters 12-14, the second part of the module.

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CHAPTER 12. Functions on the real line

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  We first recall some terminology.
- A function $f : A \to B$ is SURJECTIVE (or ONTO) if $f(A) = B$ i.e. if for every y in B there is at least one x in A such that $f(x) = y$.
- A function f is INJECTIVE (or ONE-ONE, also written one-to-one) on A if f takes different values at different points in A i.e. if the following holds. For all x_1, x_2 in A, if $f(x_1) = f(x_2)$ then $x_1 = x_2$.

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- We begin with a function which is nowhere continuous.
- Set $f(x) = 1$ if x is rational and $f(x) = -1$ if x is irrational. This is a perfectly good function but it is worth noting that you cannot draw its graph.
- To see, for instance, that f is not continuous at 0, just put $x_n = \sqrt{2}/n$. Then x_n tends to 0, but $f(x_n) = -1$ and so we clearly don't have $\lim_{n\to\infty} f(x_n) = f(0) = 1$.
  In fact, this function has no limits of any kind whatsover.

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- Let I be any real interval (it could be $[a, b]$, $(a, b]$, $(-\infty, b]$, any interval at all), and let f be a real-valued function defined on I. We say that (on I) the function f is:
  strictly increasing if $f(x) < f(y)$ for all $x, y \in I$ with $x < y$;
  non-decreasing if $f(x) \leq f(y)$ for all $x, y \in I$ with $x < y$;
  non-increasing if $f(x) \geq f(y)$ for all $x, y \in I$ with $x < y$;
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  strictly decreasing if $f(x) > f(y)$ for all $x, y \in I$ with $x < y$.
- If any of the above hold, we say that f is monotone on I. Now we look at one-sided limits for these functions.

- **Theorem 12.1:** *Let f be a non-decreasing real function on (a, b). Then $\lim_{x\to a+} f(x)$ and $\lim_{x\to b-} f(x)$ both exist.*
  If $a < c < b$, then

  $$\lim_{x\to c-} f(x) \leq f(c) \leq \lim_{x\to c+} f(x).$$

  If f is non-decreasing on $(a, +\infty)$ then $\lim_{x\to +\infty} f(x)$ exists.

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- The proofs of these assertions are all easy, once we've decided what the limit should be.
- The proofs should remind you of a theorem about sequences.

- To handle $\lim_{x \to b-} f(x)$, we let

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- We use the definition of one-sided limit as given in G11ACF. We need to show that $f(x_n) \to L$ as $n \to \infty$, for every sequence (x_n) which converges to b with $x_n < b$ for all n.

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- To do this we will split into the cases where L is or is not finite.
  Take any such sequence (x_n).

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- Since M can be chosen arbitrarily large we must have $\lim_{n\to\infty} f(x_n) = \infty$.

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- But this gives us $f(x_n) \geq f(t) > L - \varepsilon$ for all $n \geq N$.
- We also have $f(x_n) \leq L$ for all $n \geq N$, because L is an upper bound for C. So in fact, for all $n \geq N$,

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- Since ε can be chosen arbitrarily small we must have $\lim_{n\to\infty} f(x_n) = L$.

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- However, if $g(x) = x$ for $x < 0$ and $g(x) = 1$ for $x \geq 0$, does $\lim_{x \to 0} g(x)$ (the two-sided limit) exist?

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- Let f be a continuous real-valued function on the closed interval $[a, b] \subseteq \mathbb{R}$, and assume that $f(a) < T < f(b)$.
- We will make two sequences (x_n) and (y_n) so that

$$a \leq x_n \leq y_n \leq b, \quad y_n - x_n = \frac{b-a}{2^n}, \quad f(x_n) \leq T, \quad f(y_n) \geq T.$$

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- We start by setting $x_0 = a$ and $y_0 = b$.

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- In either case we have $y_{n+1} - x_{n+1} = (y_n - x_n)/2$ and $f(x_{n+1}) \leq T$, $f(y_{n+1}) \geq T$.

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- In either case we have $y_{n+1} - x_{n+1} = (y_n - x_n)/2$ and $f(x_{n+1}) \leq T$, $f(y_{n+1}) \geq T$.
- Thus our sequences (x_n) and (y_n) are constructed *inductively*.

- Now the intervals $I_n = [x_n, y_n]$ are closed and bounded non-empty sets (in $\mathbb{R}$), and

$$[a, b] = I_0 \supseteq I_1 \supseteq I_2 \supseteq \dots$$

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- Since $x_n \leq c \leq y_n$ and $y_n - x_n \to 0$ we have $x_n \to c$, $y_n \to c$.
- Since f is continuous we have $T \geq f(x_n) \to f(c)$ and $T \leq f(y_n) \to f(c)$.

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- Since f is continuous we have $T \geq f(x_n) \to f(c)$ and $T \leq f(y_n) \to f(c)$.
- So we must have $f(c) = T$. This is:

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- **Theorem 12.2 (the intermediate value theorem):** *Let f be a real-valued function which is continuous on the closed real interval $[a, b]$. If $f(a) < T < f(b)$, or $f(b) < T < f(a)$, then there exists c in (a, b) such that $f(c) = T$.*

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- To handle the case where $f(b) < T < f(a)$, we apply the first case (just proved) to $-f$ and $-T$.
- This IVT is one of the most powerful theorems in calculus/analysis. For example any continuous function $f : [a, b] \to [a, b]$ must have a fixpoint (i.e. a solution of $f(x) = x$) in $[a, b]$. Why?

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- The IVT also allows us to determine what kind of function can be continuous and injective on an interval.

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- I assert that f is strictly increasing on I. Suppose not: then there exist x, y with $a \leq x < y \leq b$ such that $f(x) \geq f(y)$, which implies that $f(x) > f(y)$. We consider two cases.

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- **Case 1:** If $f(y) < f(a)$ then $f(y) < f(a) < f(b)$ and by the IVT there must be some c in (y, b) such that $f(c) = f(a)$, contradicting the fact that f is injective.

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- **Case 2:** If $f(y) \geq f(a)$ then $f(y) > f(a)$, and so
  $f(x) > f(y) > f(a)$.
  But then the IVT gives d in (a, x) such that $f(d) = f(y)$, which again contradicts the fact that f is injective.

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- We've just proved this when I is a closed interval $[a, b]$ and $f(a) < f(b)$.
- The case where I is a closed interval $[a, b]$ and $f(a) > f(b)$ follows by looking at $-f$.
- Now suppose that we have *any* interval I and f is neither strictly increasing nor strictly decreasing on I. Then there must exist t, u, v, w in I such that $t < u$, $v < w$, but $f(t) < f(u)$ and $f(v) > f(w)$.
  Just choose a *closed* interval J contained in I such that t, u, v, w all belong to J. By the first part this is impossible.

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- However, for functions which are *onto* (aka surjective), we have the following.
- **Theorem 12.4:** *Let I and J be intervals in $\mathbb{R}$ (not necessarily bounded) and let the function $f : I \to J$ be non-decreasing and onto. Then f is continuous on I.*
- To prove Theorem 12.4 take any β in I, and any sequence (x_n) in I with limit β. We have to show that $\lim_{n\to\infty} f(x_n) = f(\beta)$. We assume for simplicity that J is an *open* interval (the other cases are OPTIONAL).

- To do this, take $\varepsilon > 0$. Because $f(\beta)$ lies in the open interval J, we can find A and B in J such that

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- This gives, for all $n \geq N$, since f is non-decreasing,

$$f(\beta) - \varepsilon < f(s) \leq f(x_n) \leq f(t) < f(\beta) + \varepsilon,$$

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$$f(\beta) - \varepsilon < f(s) \leq f(x_n) \leq f(t) < f(\beta) + \varepsilon,$$

and hence $|f(x_n) - f(\beta)| < \varepsilon$.
- Since ε can be chosen arbitrarily small, we must have $\lim_{n\to\infty} f(x_n) = f(\beta)$.

We will review the concept of differentiability from G11CAL and look at some important consequences.

13.1 Differentiability on the real line

- The real-valued function f is differentiable at $a \in \mathbb{R}$ if there exists a real number $f'(a)$ such that

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

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- We can rewrite this as

$$\frac{f(x) - f(a)}{x - a} = f'(a) + \varepsilon(x),$$

and so

$$f(x) = f(a) + f'(a)(x - a) + \varepsilon(x)(x - a).$$

where $\varepsilon(x) \to 0$ as $x \to a$.

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- The formula

$$f(x) = f(a) + f'(a)(x-a) + \varepsilon(x)(x-a),$$

where $\varepsilon(x) \to 0$ as $x \to a$, can be interpreted as follows. To approximate $f(x)$ for x near a, we can use the *linear* function $g(x) = f(a) + f'(a)(x-a)$, and this approximation will be very good if x is close enough to a.

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- Thus differentiability is really about whether you can approximate $f(x)$ by a linear function. This idea also has the advantage that you can generalise it to higher dimensions.
- We also see at once that $f(x) \to f(a)$ as $x \to a$.
- **Theorem 13.1:** *If the real-valued function f is differentiable at $a \in \mathbb{R}$, then f is continuous at a.*
The converse is false, as the example $f(x) = |x|$, $a = 0$ shows.

- **Example 1:** Define f by

$$f(x) = x^2 \sin(1/x^2) \quad (x \neq 0), \quad f(0) = 0.$$

For $x \neq 0$, the product rule and chain rule give us

$$f'(x) = 2x \sin(1/x^2) - 2x^{-1} \cos(1/x^2).$$

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$$f'(x) = 2x \sin(1/x^2) - 2x^{-1} \cos(1/x^2).$$

- Does $f'(0)$ exist? For $x \neq 0$ we have

$$\frac{f(x) - f(0)}{x - 0} = x \sin(1/x^2) \to 0 \quad \text{as } x \to 0.$$

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So $f'(0) = 0$.

- Note that $f'(x)$ is not bounded as $x \to 0$ and so not continuous at 0, so $f''(0)$ cannot exist.

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- But $f''(0)$ does not exist, as

$$\lim_{x \to 0+} \frac{f'(x) - f'(0)}{x - 0} = \lim_{x \to 0+} \frac{2x}{x} = 2,$$

$$\lim_{x \to 0-} \frac{f'(x) - f'(0)}{x - 0} = \lim_{x \to 0-} \frac{3x^2}{x} = 0 \neq 2.$$

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- Since $\sum_{n=0}^{\infty} 2^{-n}$ converges and $|\cos((21)^n \pi x)| \leq 1$ on $\mathbb{R}$, the series $W(x)$ converges on $\mathbb{R}$ and is continuous, by the Weierstrass M-test (Theorem 11.2).

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- The effect of the powers $(21)^n$ is to make the graph of $\cos((21)^n \pi x)$ so steep that the graph of W turns out to have no tangent.

13.2 Differentiability on the real line

Figure 1 shows a partial sum of the Weierstrass function.

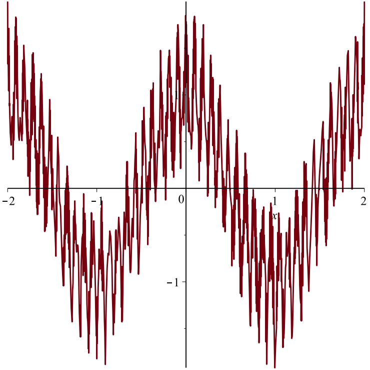

Figure: Plot of the function $\sum_{n=0}^{20} 2^{-n} \cos((21)^n \pi x)$ (MAPLE)

- A slightly easier example of a continuous nowhere differentiable function (due to van der Waerden in 1930) is given in detail in **Optional additional material for G12MAN** on the Moodle page.

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- Now we define

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Note that $|f_n(x)| < 10^{-n}$ for all x and for all $n \geq 0$.

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$$f(x) = \sum_{n=0}^{\infty} f_n(x).$$

  Note that $|f_n(x)| < 10^{-n}$ for all x and for all $n \geq 0$.
- So the sum converges, and is continuous by the Weierstrass M-test.

Figure 2 shows f_0 and f_1 for $0 \leq x \leq 1$.

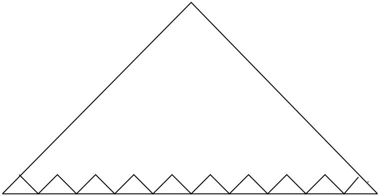

Figure: The functions f_0, f_1 for $0 \leq x \leq 1$

13.2 Differentiability on the real line

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- Then for $n > q$ both x and y_q are integer multiples of $1/10^n$, so we have $f_n(x) = f_n(y_q) = 0$.

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- For $s \leq n \leq q$ our x is an integer multiple of $1/10^n$, but y_q is not, so $f_n(y_q) - f_n(x) = 1/10^{q+1} = y_q - x$.

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- For $0 \leq n < s$ then since we move a distance $1/10^{q+1}$ from x to y_q, we get that f_n cannot change by more than $1/10^{q+1}$ and so

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- Then for $n > q$ both x and y_q are integer multiples of $1/10^n$, so we have $f_n(x) = f_n(y_q) = 0$.
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$$f_n(y_q) - f_n(x) \geq -1/10^{q+1} = -(y_q - x).$$

- So as $q \to \infty$ we have $y_q \to x$ and

$$\frac{f(y_q) - f(x)}{y_q - x} = \sum_{n=0}^{q} \frac{f_n(y_q) - f_n(x)}{y_q - x} \geq (q + 1 - s) - s \to \infty.$$

- **Example 4:** Let

$$h(x) = x \quad (x < 0), \quad h(x) = \sin x \quad (x \geq 0).$$

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- A student writes:
  For $x < 0$ we have $h'(x) = 1$ and for $x > 0$ we have $h'(x) = \cos x$. Since

$$\lim_{x \to 0-} 1 = \lim_{x \to 0+} \cos x = 1$$

  we have $h'(0) = 1$.

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we have $h'(0) = 1$.
- Is this correct?

The following theorem is very useful:
Theorem 13.2: *Let $a < c < b$ and let the real-valued function f be continuous on (a, b) and differentiable on (a, c) and on (c, b). Assume that*

$$\lim_{x \to c-} f'(x) = L, \quad \lim_{x \to c+} f'(x) = M.$$

(i) If $L = M \in \mathbb{R}$ then $f'(c)$ exists and equals M.
(ii) If $f'(c)$ exists then $L = M = f'(c) \in \mathbb{R}$.
Note that this result will use L'Hôpital's rule from G11ACF, which depends on Rolle's theorem, but we will prove Rolle's theorem later on.

- First we prove (i), so suppose $L = M \in \mathbb{R}$. Then we have

$$\lim_{x \to c} f'(x) = L = M.$$

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- So L'Hôpital's rule gives

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c} \frac{f'(x)}{1} = L = M.$$

- Next we prove (ii), so suppose $f'(c)$ exists. Then (by definition) we have

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c) \in \mathbb{R}.$$

13.2 Differentiability on the real line

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  and so $L = f'(c)$.
- Similarly we obtain $f'(c) = M$.

Theorem 13.3 (the product rule etc.): *Suppose that the real-valued functions f and g are differentiable at $a \in \mathbb{R}$, and that $\lambda \in \mathbb{R}$. Then:*
(i) $(f+g)'(a) = f'(a) + g'(a)$;
(ii) $(\lambda f)'(a) = \lambda f'(a)$;
(iii) $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$;
(iv) if $g(a) \neq 0$, then $(1/g)'(a) = -g'(a)/g(a)^2$.
The proofs are omitted (and so OPTIONAL).
See **Optional additional material for G12MAN** if you want to read them.

- Slightly harder is the chain rule:
  Theorem 13.4: *If the real-valued function g is differentiable at $a \in \mathbb{R}$ and the real-valued function f is differentiable at $b = g(a)$, then $h = f(g)$ is differentiable at a and $h'(a) = g'(a)f'(b)$.*

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$$g(x) = g(a) + (x - a)(g'(a) + \varepsilon(x))$$

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- We can write

$$g(x) = g(a) + (x - a)(g'(a) + \varepsilon(x))$$

  where $\varepsilon(x) \to 0$ as $x \to a$.
- Similarly,

$$f(y) = f(b) + (y - b)(f'(b) + \rho(y))$$

  where $\rho(y) \to 0$ as $y \to b$. We put $\rho(b) = 0$ and combine these as follows.

- If x is close to a then $g(x)$ will be close to b (since g is continuous at a) and so

$$h(x)-h(a) = f(g(x))-f(g(a)) = (g(x)-b)(f'(b)+\rho(g(x))).$$

13.3 Differentiability on the real line

- If x is close to a then $g(x)$ will be close to b (since g is continuous at a) and so

$$h(x)-h(a) = f(g(x))-f(g(a)) = (g(x)-b)(f'(b)+\rho(g(x))).$$

- Thus

$$\begin{aligned} h(x)-h(a) &= (x-a)(g'(a)+\varepsilon(x))(f'(b)+\rho(g(x))) \\ &= (x-a)g'(a)f'(b)+(x-a)\delta(x). \end{aligned}$$

- If x is close to a then $g(x)$ will be close to b (since g is continuous at a) and so

$$h(x)-h(a) = f(g(x))-f(g(a)) = (g(x)-b)(f'(b)+\rho(g(x))).$$

- Thus

$$\begin{aligned} h(x) - h(a) &= (x-a)(g'(a)+\varepsilon(x))(f'(b)+\rho(g(x))) \\ &= (x-a)g'(a)f'(b) + (x-a)\delta(x). \end{aligned}$$

- Here

$$\delta(x) = \varepsilon(x)f'(b) + \varepsilon(x)\rho(g(x)) + g'(a)\rho(g(x))$$

tends to 0 as $x \to a$. This gives $h'(a) = g'(a)f'(b)$.

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13.4 Differentiability on the real line

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- A local minimum is defined similarly.
- If a is a local maximum or local minimum and f is differentiable at a, then $f'(a) = 0$.
- Say a is a local maximum. If x is in U and $x > a$, then $(f(x) - f(a))/(x - a) \leq 0$, so $f'(a) \leq 0$. Similarly, if x is in U and $x < a$, then $(f(x) - f(a))/(x - a) \geq 0$, so $f'(a) \geq 0$.

- Suppose that $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b] \subseteq \mathbb{R}$ and differentiable on (a, b), with $f(a) = f(b)$.

13.4 Differentiability on the real line

- Suppose that $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b] \subseteq \mathbb{R}$ and differentiable on (a, b), with $f(a) = f(b)$.
- If $f(x) = f(a)$ for all x in $[a, b]$ then obviously $f'(c) = 0$ for all $c \in (a, b)$.

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- If $f(x) = f(a)$ for all x in $[a, b]$ then obviously $f'(c) = 0$ for all $c \in (a, b)$.
- If $f(x) > f(a)$ for some x in $[a, b]$ then f has a maximum at some $c \in (a, b)$ (by the maximum theorem 10.3).
  Then c is a local maximum and $f'(c) = 0$.

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  Then c is a local maximum and $f'(c) = 0$.
- If $f(x) < f(a)$ for some x in $[a, b]$ then f has a minimum at some $c \in (a, b)$, and c is a local minimum and $f'(c) = 0$.

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- Suppose that $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b] \subseteq \mathbb{R}$ and differentiable on (a, b), with $f(a) = f(b)$.
- If $f(x) = f(a)$ for all x in $[a, b]$ then obviously $f'(c) = 0$ for all $c \in (a, b)$.
- If $f(x) > f(a)$ for some x in $[a, b]$ then f has a maximum at some $c \in (a, b)$ (by the maximum theorem 10.3).
  Then c is a local maximum and $f'(c) = 0$.
- If $f(x) < f(a)$ for some x in $[a, b]$ then f has a minimum at some $c \in (a, b)$, and c is a local minimum and $f'(c) = 0$.
- So in all three cases there exists $c \in (a, b)$ with $f'(c) = 0$.
  This is *Rolle's theorem*.

- **Theorem 13.5 (the mean value theorem)** *Suppose that* $f : [a, b] \to \mathbb{R}$ *is continuous on* $[a, b] \subseteq \mathbb{R}$ *and differentiable on* (a, b). *Then there exists* c *in* (a, b) *such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

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$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

- To prove this, set

$$g(x) = f(x) - (x - a)\left(\frac{f(b) - f(a)}{b - a}\right).$$

Then $g(a) = f(a) = g(b)$, and by Rolle's theorem there must be some $c \in (a, b)$ such that $g'(c) = 0$.

- **Theorem 13.6** *Suppose that the real-valued function f is differentiable on the open interval $I \subseteq \mathbb{R}$. Then the following all hold:*
  (i) f is strictly increasing on I if $f'(x) > 0$ for all x in I;
  (ii) f is non-decreasing on I iff $f'(x) \geq 0$ for all x in I;
  (iii) f is constant on I iff $f'(x) = 0$ for all x in I;
  (iv) f is non-increasing on I iff $f'(x) \leq 0$ for all x in I;
  (v) f is strictly decreasing on I if $f'(x) < 0$ for all x in I;
  (vi) f is injective on I if $f'(x) \neq 0$ for all $x \in I$.

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  (vi) f is injective on I if $f'(x) \neq 0$ for all $x \in I$.
- All of these follow from the definition of f' and the mean value theorem.
- The function $f(x) = x^3$ is strictly increasing but $f'(0) = 0$. Thus (i) is not "if and only if".

- **Example A:** Show that $g(x) = x/(1+x^2)$ is strictly increasing on $[0, 1]$.
  This is not obvious, as g is an increasing function divided by an increasing function.

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  This is not obvious, as g is an increasing function divided by an increasing function.
- **Example B:** Show that $(1+x)^{-1/2} > 1 - x/2$ for $x > 0$.

13.6 Differentiability on the real line

- We saw from the IVT that a continuous function $f : [a, b] \to \mathbb{R}$ must satisfy the intermediate value property: *if $f(a) < T < f(b)$ or $f(a) > T > f(b)$ then f takes the value T at some $c \in (a, b)$.*

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- A non-continuous function may fail to have this property e.g.

$$g(x) = -1 \quad (x < 0), \quad g(x) = 1 \quad (x \geq 0)$$

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- A non-continuous function may fail to have this property e.g.

$$g(x) = -1 \quad (x < 0), \quad g(x) = 1 \quad (x \geq 0)$$

  never takes the value 0.
- We've seen in this chapter that a derivative can fail to be continuous.
- But can a derivative fail to have the intermediate value property?

- **Theorem 13.6:** *Let the real-valued function f be differentiable at every point in $[a, b] \subseteq \mathbb{R}$. If $f'(a) < T < f'(b)$ or $f'(a) > T > f'(b)$ then f' takes the value T at some $c \in (a, b)$.*

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- To see this, we can assume $T = 0$ (else look at $f(x) - Tx$).
- We can also assume that $f'(a) < 0 < f'(b)$ (else look at $-f$).

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- To see this, we can assume $T = 0$ (else look at $f(x) - Tx$).
- We can also assume that $f'(a) < 0 < f'(b)$ (else look at $-f$).
- We assume that f' is never 0 on (a, b) and seek a contradiction.

- Clearly f is continuous on $I = [a, b]$.

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13.6 Differentiability on the real line

- Clearly f is continuous on $I = [a, b]$.
- By the mean value theorem, f is injective on I. Why?
- By Theorem 12.3, f is either strictly increasing on I, or strictly decreasing on I.
- But if f is strictly increasing on I, then $f'(a) \geq 0$.
  If f is strictly decreasing on I, then $f'(b) \leq 0$.
  Both give a contradiction.

CHAPTER 14. The Riemann integral

- Suppose that we have a bounded real-valued function f on the closed interval $I = [a, b] \subseteq \mathbb{R}$.

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CHAPTER 14. The Riemann integral

- Suppose that we have a bounded real-valued function f on the closed interval $I = [a, b] \subseteq \mathbb{R}$.
- We need to define what is meant by the integral $\int_a^b f(x)\,dx$, and to determine for which f it exists.
- It may be tempting to define the integral as the "area under the curve", but it is not obvious that the area exists.
  The function f may give a very messy curve, such as the continuous, nowhere differentiable function in Chapter 13.
  Moreover, it is not obvious what to do if f changes sign infinitely often, as does, for example, $x\sin(1/x)$.

CHAPTER 14. The Riemann integral

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- We need to define what is meant by the integral $\int_a^b f(x)\,dx$, and to determine for which f it exists.
- It may be tempting to define the integral as the "area under the curve", but it is not obvious that the area exists. The function f may give a very messy curve, such as the continuous, nowhere differentiable function in Chapter 13. Moreover, it is not obvious what to do if f changes sign infinitely often, as does, for example, $x \sin(1/x)$.
- The idea is to "approximate" the area from above and below. Throughout this chapter, $-\infty < a < b < \infty$.

- Let f be a bounded real-valued function on the closed interval $[a, b] = I \subseteq \mathbb{R}$. Assume that $|f(x)| \leq M < \infty$ for all x in I.

14.1 The Riemann integral

- Let f be a bounded real-valued function on the closed interval $[a, b] = I \subseteq \mathbb{R}$. Assume that $|f(x)| \leq M < \infty$ for all x in I.
- A PARTITION P of I means a finite set $\{x_0, ..., x_n\}$ such that

$$a = x_0 < x_1 < .... < x_n = b.$$

The points x_j are called the vertices of P.

14.1 The Riemann integral

- For a partition $P = \{x_0, ..., x_n\}$ of I, we define

$$M_k(P, f) = M_k(f) = \sup\{f(x) : x_{k-1} \leq x \leq x_k\} \leq M$$

and

$$m_k(P, f) = m_k(f) = \inf\{f(x) : x_{k-1} \leq x \leq x_k\} \geq -M.$$

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- Further, we define the UPPER SUM

$$U(P, f) = \sum_{k=1}^{n} M_k(f)(x_k - x_{k-1})$$

and the LOWER SUM

$$L(P, f) = \sum_{k=1}^{n} m_k(f)(x_k - x_{k-1}).$$

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- Further, we define the UPPER SUM
$$U(P,f) = \sum_{k=1}^{n} M_k(f)(x_k - x_{k-1})$$
and the LOWER SUM
$$L(P,f) = \sum_{k=1}^{n} m_k(f)(x_k - x_{k-1}).$$
- Note that $-M(b-a) \leq L(P,f) \leq U(P,f) \leq M(b-a)$.

Figure 3 shows a Riemann upper sum.

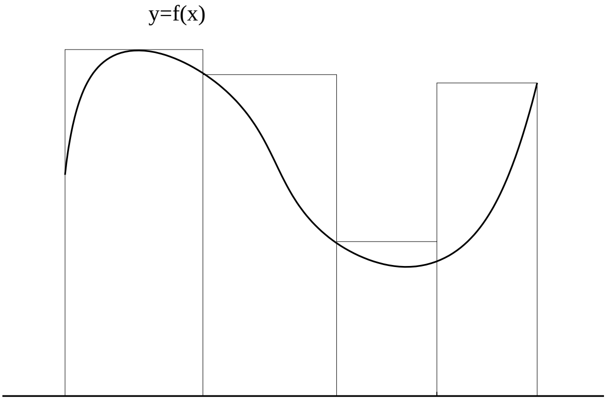

Figure: A Riemann upper sum

Figure 4 shows a Riemann lower sum.

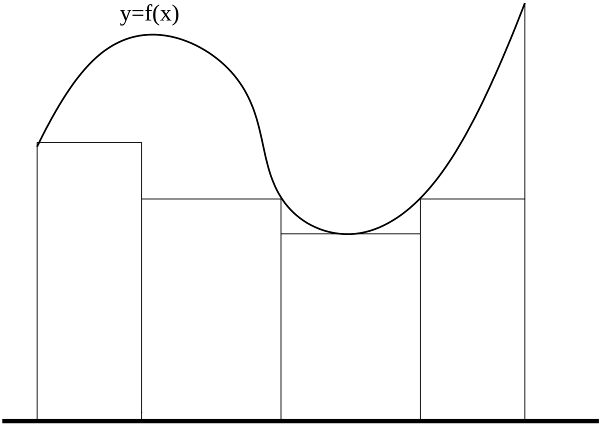

Figure: A Riemann lower sum

- Now we introduce the idea of refinements.
  If P and Q are partitions of $[a, b]$ then Q is a *refinement* of P if every vertex of P is a vertex of Q (crudely put, $P \subseteq Q$).

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  If P and Q are partitions of $[a, b]$ then Q is a *refinement* of P if every vertex of P is a vertex of Q (crudely put, $P \subseteq Q$).
- If you draw for yourself a simple curve, it is not hard to convince yourself that refining P tends to increase $L(P, f)$ and decrease $U(P, f)$.
  The proof is OPTIONAL, but see the next slide.

In Figure 5 an extra vertex has been introduced, and the lower sum has increased.

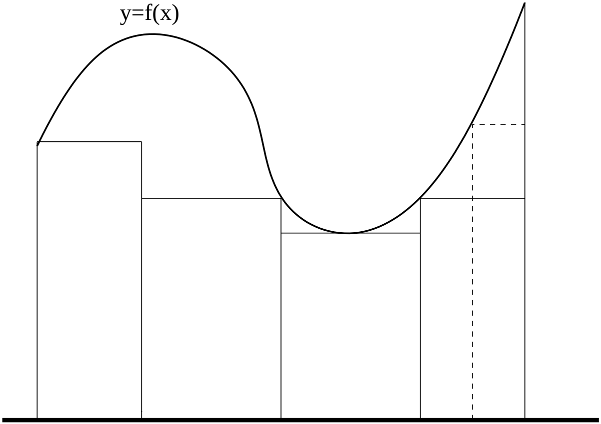

Figure: A Riemann lower sum and the effect of refinement

- **Lemma 14.1** *Let f be a bounded real-valued function on $I = [a, b]$.*
  (i) If P, Q are partitions of I and Q is a refinement of P, then

$$L(P, f) \leq L(Q, f), \quad U(P, f) \geq U(Q, f).$$

  (ii) If P_1 and P_2 are any partitions of I, then $L(P_1, f) \leq U(P_2, f)$.

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- The proof of (i) is OPTIONAL, but not hard. To get (ii) we just let P be the partition obtained by taking all the vertices of P_1 and all those of P_2, in order.

14.2 The Riemann integral

- **Lemma 14.1** *Let f be a bounded real-valued function on $I = [a, b]$.*
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  $$L(P, f) \leq L(Q, f), \quad U(P, f) \geq U(Q, f).$$
  (ii) If P_1 and P_2 are any partitions of I, then $L(P_1, f) \leq U(P_2, f)$.
- The proof of (i) is OPTIONAL, but not hard. To get (ii) we just let P be the partition obtained by taking all the vertices of P_1 and all those of P_2, in order.
- Since P is a refinement of P_1 and of P_2,
  $$L(P_1, f) \leq L(P, f) \leq U(P, f) \leq U(P_2, f).$$

- Let f be bounded, real-valued on $I = [a, b]$ as before, with $|f(x)| \leq M < \infty$ there. We define the UPPER INTEGRAL of f from a to b as

$$\overline{\int_a^b} f(x)\,dx = \inf\{U(P, f) : P \text{ is a partition of } I\}.$$

This exists and is finite, because all the upper sums are bounded below by $-M(b-a)$.

14.3 The Riemann integral

- Let f be bounded, real-valued on $I = [a, b]$ as before, with $|f(x)| \leq M < \infty$ there. We define the UPPER INTEGRAL of f from a to b as

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This exists and is finite, because all the upper sums are bounded below by $-M(b-a)$.

- Similarly we define the LOWER INTEGRAL

$$\underline{\int_a^b} f(x)\,dx = \sup\{L(P, f) : P \text{ is a partition of } I\}.$$

Again this exists and is finite, because all the lower sums are bounded above by $M(b-a)$.

- Since $U(P, f) \leq M(b-a)$ for every P we get

$$\overline{\int_a^b} f(x)\, dx \leq M(b-a).$$

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- The lower integral is never greater than the upper integral, since $L(P, f) \leq U(Q, f)$ for partitions P, Q.

14.4 The Riemann integral

- We say that f is Riemann integrable on I if

$$\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx,$$

and, if so, we denote their common value by $\int_a^b f(x)\,dx$.

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- Note that we then get

$$-M(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a), \quad \left|\int_a^b f(x)\,dx\right| \leq M(b-a).$$

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- As usual in integration, it does not matter whether you write $f(x)\,dx$ or $f(t)\,dt$ etc.

- Define f on $I = [0, 1]$ as follows. Let $H(x) = 1$ if x is a rational number of form $p/10^q$ with p and q non-negative integers, and $H(x) = 0$ otherwise.
  Is H Riemann integrable?

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- Let $P = \{x_0, ..., x_n\}$ be any partition of I. So $0 = x_0 < x_1 < \ldots < x_n = 1$.

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- Let $P = \{x_0, ..., x_n\}$ be any partition of I. So $0 = x_0 < x_1 < \ldots < x_n = 1$.
- Clearly each sub-interval $[x_{k-1}, x_k]$ contains a point where $H(x) = 1$, and so $M_k(H) = 1$.
  This gives $U(P, H) = \sum_{k=1}^{n}(x_k - x_{k-1}) = 1$ and so the upper integral is 1.

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  This gives $U(P,H) = \sum_{k=1}^n (x_k - x_{k-1}) = 1$ and so the upper integral is 1.
- Similarly, each $[x_{k-1}, x_k]$ contains a point where $H(x) = 0$. So we have $m_k(H) = 0$, all lower sums are 0, and the lower integral is 0.

14.5 The Riemann integral

- Define f on $I = [0,1]$ as follows. Let $H(x) = 1$ if x is a rational number of form $p/10^q$ with p and q non-negative integers, and $H(x) = 0$ otherwise.
  Is H Riemann integrable?
- Let $P = \{x_0, ..., x_n\}$ be any partition of I. So $0 = x_0 < x_1 < \ldots < x_n = 1$.
- Clearly each sub-interval $[x_{k-1}, x_k]$ contains a point where $H(x) = 1$, and so $M_k(H) = 1$.
  This gives $U(P, H) = \sum_{k=1}^{n}(x_k - x_{k-1}) = 1$ and so the upper integral is 1.
- Similarly, each $[x_{k-1}, x_k]$ contains a point where $H(x) = 0$. So we have $m_k(H) = 0$, all lower sums are 0, and the lower integral is 0.
- So $\int_0^1 H(x)\, dx$ does not exist.

- In the other direction, suppose that $G : [0, 1] \to [0, M]$ is a bounded non-negative function which is 0 except at finitely many points. Is G Riemann integrable?

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- Now let n be a positive integer and let $P = \{0, 1/n, ..., (n-1)/n, 1\}$.

14.5 The Riemann integral

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- Now let n be a positive integer and let $P = \{0, 1/n, ..., (n-1)/n, 1\}$.
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- So $U(P,G) \leq 2NM/n$ and since N and M do not depend on n we can make this as small as we like.

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- So $U(P, G) \leq 2NM/n$ and since N and M do not depend on n we can make this as small as we like.
- So the upper integral (infimum of the upper sums) is 0, and $\int_0^1 G(x)\, dx = 0$.

14.6 The Riemann integral

- Suppose f is a non-decreasing real function on $I = [a, b]$.

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- Since f is non-decreasing we have $M_k(f) = f(x_k)$ and $m_k(f) = f(x_{k-1})$.
- Thus

$$\begin{aligned} U(P,f) - L(P,f) &= \sum_{k=1}^{n} (M_k(f) - m_k(f))(x_k - x_{k-1}) \\ &= \sum_{k=1}^{n} (f(x_k) - f(x_{k-1}))(x_k - x_{k-1}) \\ &\leq \sum_{k=1}^{n} (f(x_k) - f(x_{k-1}))\,\varepsilon \\ &= (f(x_n) - f(x_0))\varepsilon = (f(b) - f(a))\varepsilon. \end{aligned}$$

- Thus

$$
\begin{aligned}
\underline{\int_a^b} f(x)\,dx &\leq \overline{\int_a^b} f(x)\,dx \leq U(P,f) \\
&\leq L(P,f) + (f(b) - f(a))\,\varepsilon \\
&\leq \underline{\int_a^b} f(x)\,dx + (f(b) - f(a))\,\varepsilon .
\end{aligned}
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- Thus

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- Since ε can be chosen arbitrarily small we have

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14.6 The Riemann integral

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- Since ε can be chosen arbitrarily small we have

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- An almost identical argument works for a non-increasing function f.
- **Theorem 14.1:** *Every monotone real function on* $[a,b]$ *is Riemann integrable on* $[a,b]$.

14.7 The Riemann integral

- What happens if $f : [a, b] \to \mathbb{R}$ is continuous?

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- I claim that for every $m \in \mathbb{N}$ there exist $s_m, t_m \in [a, b]$ with

$$|s_m - t_m| < \frac{1}{m} \quad \text{and} \quad |f(s_m) - f(t_m)| \geq D = \frac{C}{2(b-a)}.$$

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- I claim that for every $m \in \mathbb{N}$ there exist $s_m, t_m \in [a, b]$ with

$$|s_m - t_m| < \frac{1}{m} \quad \text{and} \quad |f(s_m) - f(t_m)| \geq D = \frac{C}{2(b-a)}.$$

- **If** I can prove this claim, then by the Bolzano-Weierstrass theorem there exists a convergent subsequence (s_{m_k}) of (s_m), tending as $k \to \infty$ to $u \in [a, b]$. But then $t_{m_k} \to u$ also and

$$D = \frac{C}{2(b-a)} \leq \quad |f(s_{m_k}) - f(t_{m_k})| \to |f(u) - f(u)| = 0,$$

a contradiction.

- To prove the claim take a partition $P = \{x_0, \ldots, x_n\}$ of $[a, b]$ with $x_k - x_{k-1} < 1/m$ for every k. Then we get:

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0 < C &= \overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx \\
&\leq U(P, f) - L(P, f) = \sum_{k=1}^{n} (M_k(f) - m_k(f))(x_k - x_{k-1}) \\
&\leq \left(\sum_{k=1}^{n} (x_k - x_{k-1}) \right) \max_k \{M_k(f) - m_k(f)\} \\
&= (b - a) \max_k \{M_k(f) - m_k(f)\}.
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14.7 The Riemann integral

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- $$\begin{aligned} 0 < C &= \overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx \\ &\le U(P,f) - L(P,f) = \sum_{k=1}^{n} (M_k(f) - m_k(f))(x_k - x_{k-1}) \\ &\le \left(\sum_{k=1}^{n} (x_k - x_{k-1}) \right) \max_k \{M_k(f) - m_k(f)\} \\ &= (b-a) \max_k \{M_k(f) - m_k(f)\}. \end{aligned}$$
- So there must be some k with $M_k(f) - m_k(f) \ge C/(b-a) = 2D$.

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- We choose $s_m, t_m \in [x_{k-1}, x_k]$ with $f(s_m)$ close to $m_k(f)$ and $f(t_m)$ close to $M_k(f)$. This gives

$$|s_m - t_m| \leq x_k - x_{k-1} < \frac{1}{m}, \quad |f(s_m) - f(t_m)| \geq D,$$

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  as asserted.
- **Theorem 14.2:** *Every continuous real-valued function on $[a, b]$ is Riemann integrable on $[a, b]$.*

- **Theorem 14.3:** *Let $a < c < b$ and let $f : [a, b] \to \mathbb{R}$ be a bounded function which is Riemann integrable on $[a, c]$ and $[c, b]$. Then f is Riemann integrable on $[a, b]$ and*

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

14.8 The Riemann integral

- **Theorem 14.3:** *Let $a < c < b$ and let $f : [a, b] \to \mathbb{R}$ be a bounded function which is Riemann integrable on $[a, c]$ and $[c, b]$. Then f is Riemann integrable on $[a, b]$ and*

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

- The proof is not hard, but is OPTIONAL.

- Similarly, we have

  Theorem 14.4: *Let $a < c < b$ and let $f : [a, b] \to \mathbb{R}$ be a bounded function which is Riemann integrable on $[a, b]$. Then f is Riemann integrable on $[a, c]$ and $[c, b]$.*

- Similarly, we have
  Theorem 14.4: *Let $a < c < b$ and let $f : [a, b] \to \mathbb{R}$ be a bounded function which is Riemann integrable on $[a, b]$. Then f is Riemann integrable on $[a, c]$ and $[c, b]$.*
- Again the proof is not hard, but is OPTIONAL.

14.9 The Riemann integral

- Suppose that $f : [a, b] \to \mathbb{R}$ is a bounded real function and that f is Riemann integrable on $[a, b]$.
  For $a \leq x \leq b$ set $F(x) = \int_a^x f(t)\, dt$. What kind of a function is $F(x)$?

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- So for $a \leq x < y \leq b$ we get

$$|F(y) - F(x)| = \left| \int_x^y f(t)\,dt \right| \leq M(y - x).$$

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- So F is certainly *continuous*.

- Is $F(x) = \int_a^x f(t)\,dt$ differentiable?

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- Not necessarily: see the problem sheets.
- But suppose that f is continuous at $c \in (a, b)$.
- Let $a \le x < y \le b$: then

$$\int_x^y f(t)\,dt \le \sup\{f(t) : x \le t \le y\}\,(y - x),$$

$$\int_x^y f(t)\,dt \ge \inf\{f(t) : x \le t \le y\}\,(y - x).$$

Now divide by $y - x$.

- We get

$$\inf\{f(t) : x \leq t \leq y\} \leq \frac{\int_x^y f(t)\,dt}{y-x} =$$

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- If I let x and y tend to c with $x < y$ then the first and last terms tend to $f(c)$. Thus $F'(c) = f(c)$.
- **Theorem 14.5** *Suppose that $f : [a,b] \to \mathbb{R}$ is a bounded real function and is Riemann integrable on $[a,b]$. For $a \le x \le b$ set $F(x) = \int_a^x f(t)\,dt$.*
  If f is continuous at $c \in (a,b)$ then $F'(c) = f(c)$.

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- If I let x and y tend to c with $x < y$ then the first and last terms tend to $f(c)$. Thus $F'(c) = f(c)$.
- **Theorem 14.5** *Suppose that* $f : [a,b] \to \mathbb{R}$ *is a bounded real function and is Riemann integrable on* $[a,b]$. *For* $a \leq x \leq b$ *set* $F(x) = \int_a^x f(t)\,dt$.
  If f *is continuous at* $c \in (a,b)$ *then* $F'(c) = f(c)$.
- This is usually called the *first fundamental theorem of the calculus*.

- The *second fundamental theorem of the calculus* is better known, since it is the key to integration by antiderivatives.

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- Suppose that F, f are real-valued functions on $[a, b]$, that F is continuous and f is Riemann integrable on $[a, b]$, and that $F'(x) = f(x)$ for all x in (a, b).

14.10 The Riemann integral

- The *second fundamental theorem of the calculus* is better known, since it is the key to integration by antiderivatives.
- Suppose that F, f are real-valued functions on $[a, b]$, that F is continuous and f is Riemann integrable on $[a, b]$, and that $F'(x) = f(x)$ for all x in (a, b).
- Take any partition $P = \{x_0, \ldots, x_n\}$ of $[a, b]$.

14.10 The Riemann integral

- The *second fundamental theorem of the calculus* is better known, since it is the key to integration by antiderivatives.
- Suppose that F, f are real-valued functions on $[a, b]$, that F is continuous and f is Riemann integrable on $[a, b]$, and that $F'(x) = f(x)$ for all x in (a, b).
- Take any partition $P = \{x_0, \ldots, x_n\}$ of $[a, b]$.
- Then the mean value theorem gives points c_k with

$$\begin{aligned} F(b) - F(a) &= \sum_{k=1}^{n} (F(x_k) - F(x_{k-1})) \\ &= \sum_{k=1}^{n} f(c_k)(x_k - x_{k-1}) \quad (x_{k-1} < c_k < x_k). \end{aligned}$$

- But $m_k(f) \leq f(c_k) \leq M_k(f)$. So we get

$$F(b) - F(a) \leq \sum_{k=1}^{n} M_k(f)(x_k - x_{k-1}) = U(P, f),$$

$$F(b) - F(a) \geq \sum_{k=1}^{n} m_k(f)(x_k - x_{k-1}) = L(P, f).$$

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- Taking the inf over upper sums $U(P, f)$ gives $F(b) - F(a) \le \int_a^b f(t)\, dt$.

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- Taking the inf over upper sums $U(P, f)$ gives $F(b) - F(a) \leq \int_a^b f(t)\, dt$.
- Taking the sup over lower sums $L(P, f)$ gives $F(b) - F(a) \geq \int_a^b f(t)\, dt$.

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- Taking the inf over upper sums $U(P, f)$ gives $F(b) - F(a) \leq \int_a^b f(t)\, dt$.
- Taking the sup over lower sums $L(P, f)$ gives $F(b) - F(a) \geq \int_a^b f(t)\, dt$.
- This proves the last (but most important) theorem of this chapter.

- **Theorem 14.6 (second fundamental theorem of the calculus):** *Suppose that F and f are real-valued functions on $[a, b]$, that F is continuous and f is Riemann integrable on $[a, b]$, and that $F'(x) = f(x)$ for all x in (a, b). Then*

$$\int_a^b f(x)\,dx = F(b) - F(a).$$