The Eremenko-Lyubich class and differential equations

Jim Langley, Nottingham, January 2019

All functions f in this talk assumed meromorphic in $\mathbb{C}$ (and transcendental unless stated otherwise).

Singular values of the inverse f^{-1} are:
critical values (values taken at multiple points);
asymptotic values (α s.t. $f(z) \to \alpha$ on a path $\gamma_\alpha \to \infty$).

Speiser class $\mathcal{S}$: finitely many singular values (e.g. $e^z - 1$).

Eremenko-Lyubich class $\mathcal{B}$: the set of finite singular values is bounded (e.g. $(e^z - 1)/z$).

This talk describes two problems in DEs in which $\mathcal{B}$ plays a role.

Part I: escaping to infinity in finite time

For the DE $\dot{z} = \dfrac{dz}{dt} = f(z)$, trajectories are paths

$$z(t), \quad z'(t) = f(z(t)) \neq \infty,$$

for t in some maximal interval $(\alpha, \beta) \subseteq \mathbb{R}$.

Example 1:
If $f(z) = cz$, $c \in \mathbb{C} \setminus \{0\}$, $z(0) \neq 0$ then $z(t) = z(0)e^{ct}$.

$\mathrm{Re}\, c > 0$ implies $z(t) \to \infty$ as $t \to \beta = +\infty$;
$\mathrm{Re}\, c < 0$ implies $z(t) \to 0$ as $t \to \beta = +\infty$;
if $\mathrm{Re}\, c = 0$ then trajectories are circles, with period $2\pi i/c$.

Example 2:

if $f(z) = e^z$ then $e^{-z(t)} = e^{-z(0)} - t$.

(a) If $T = e^{-z(0)} \in \mathbb{R}^+$ (i.e. $\operatorname{Im} z(0)/2\pi \in \mathbb{Z}$) then $t \to \beta = T$ gives $e^{-z(t)} \to 0$ and $z(t) \to \infty$.

(b) If $T \notin \mathbb{R}^+$ then $\beta = +\infty$ and $e^{-z(t)} \to \infty$ as $t \to +\infty$.

Example 3:

if $f(z) = z^2$ and $z(0) \neq 0$ then

$$\frac{1}{z(t)} = \frac{1}{z(0)} - t = T - t.$$

(a) If $z(0) \in \mathbb{R}^+$ then $z(t) \to \infty$ as $t \to \beta = T = 1/z(0)$, following $\mathbb{R}^+$ outwards towards ∞.

(b) If $z(0) \notin \mathbb{R}^+$ then $z(t) \to 0$ as $t \to \beta = +\infty$.

Theorem 1 (King and Needham 1994) *If f is a rational function with a pole of order $n \geq 2$ at ∞ then $\dot{z} = f(z)$ has $n-1$ trajectories tending to infinity in finite time (i.e. $z(t) \to \infty$ as $t \to \beta < +\infty$).*

Proof. As $z \to \infty$ write $f(z) = c_1 z^n + \dots$ and

$$\begin{aligned} \int_\infty^z \frac{1-n}{f(t)}\,dt &= c_2 z^{1-n} + \dots = \phi(z)^{1-n}, \\ w &= \phi(z) = c_3 z + \dots \quad \text{(univalent near } \infty\text{)}, \\ \dot{w} &= w^n. \end{aligned}$$

Question: what happens if f is transcendental? Must there be trajectories tending to infinity in finite time?

Theorem 2 (JKL ca. 2015) *Let f be transcendental with finitely many poles. Then $\dot{z} = f(z)$ has infinitely many (pairwise disjoint) trajectories tending to infinity in finite time.*

Proof uses Wiman-Valiron theory.

Sketch of method. Wiman-Valiron gives large r and z_r with $|z_r| = r, |f(z_r)| = M(r, f)$, such that

$$f(z) \sim (z/z_r)^{N_r} f(z_r), \quad N_r \to +\infty,$$

on a small neighbourhood A_r of z_r. A change of variables

$$w = F(z) = \int^z \frac{dt}{f(t)} \sim c_r z^{1-N_r},$$

maps small logarithmic "rectangles" V_r in A_r onto
$S_r < |w| < T_r, 0 < \arg w < 2\pi$,
where T_r and S_r/T_r are small.

$\dot{z} = f(z)$ becomes locally $\dot{w} = 1$, with horizontal trajectories.
A "bifurcation" argument gives $z(0) \in \partial V_r$ such that
$z(t)$ tend to a limit (∞ or a pole) as $t \to \beta \leq 2S_r$.

When f has infinitely many poles, it is possible for all trajectories to be bounded e.g.

let g be transcendental entire of order

$$\rho(g) = \limsup_{r \to \infty} \frac{\log T(r, g)}{\log r} < \frac{1}{2}.$$

Then $\dot{z} = f(z) = -ig(z)/g'(z)$ gives, for each trajectory,

$$i \log g(z(t)) = t + C, \quad \log |g(z(t))| = \operatorname{Im} C.$$

But $\rho(g) < 1/2$ implies that $\min\{|g(z)| : |z| = r\}$ is unbounded as $r \to \infty$, so no trajectory tends to infinity.

However, suppose f is meromorphic and f^{-1} has a logarithmic singularity over ∞.

This means there exist $M > 0$ and a component U_M of $\{z \in \mathbb{C} : |f(z)| > M\}$ such that $v = \log f(z)$ maps U_M conformally onto $\operatorname{Re} v > \log M$.

This will automatically hold if $f \in \mathcal{B}$ and ∞ is an asymptotic value (Nevanlinna).

Example: $f(z) = e^{-z^2} \tan z$ in the upper half-plane.

Theorem 3 (JKL ca. 2012) *Let f be transcendental meromorphic such that f^{-1} has a logarithmic singularity over ∞. Then $\dot{z} = f(z)$ has infinitely many trajectories tending to infinity in finite time.*

Sketch of method.
$v = \log f(z)$ maps U_M to $\operatorname{Re} v > \log M$.

Let $z = \phi(v)$ be the inverse. Then $\dot{z} = f(z)$ becomes

$$\dot{v} = e^v \phi'(v)^{-1},$$

in which $\phi'(v)$ varies slowly on the half-plane.
Find trajectories on which $\operatorname{Re} v(t)$ and $e^{v(t)} = f(z(t))$ tend to infinity in finite time.

Open question: suppose f is transcendental meromorphic and f^{-1} has a *direct* singularity over ∞.

This means there exist $M > 0$ and a component U_M of $\{z \in \mathbb{C} : |f(z)| > M\}$ on which $f(z) \neq \infty$.

Must $\dot{z} = f(z)$ have infinitely many trajectories tending to infinity in finite time?

Wiman-Valiron is available (Bergweiler-Rippon-Stallard 2008). The method for entire f gives trajectories each tending to a limit as t tends to some $\beta < +\infty$.
But could these limits all be poles of f outside U_M?

Part II: the (former) Bank-Laine conjecture

This involves zeros of of solutions of

$$y'' + A(z)y = 0$$

where A is entire.
Let f_1, f_2 be linearly independent solutions and set

$$E = f_1 f_2, \quad \lambda(E) = \limsup_{r \to +\infty} \frac{\log^+ N(r, 1/E)}{\log r}$$

(measures the frequency of zeros z_k of $f_1 f_2$).
$\lambda(E)$ is called the *exponent of convergence*: indeed

$$\lambda(E) = \inf \left\{ \lambda > 0 : \sum_{z_k \neq 0} |z_k|^{-\lambda} < +\infty \right\}.$$

Theorem 4 (Bank and Laine 1982) *Let A be entire.*
(i) If A is a polynomial of degree $n > 0$ then $\lambda(E) = \rho(E) = (n+2)/2$.
(ii) If $\lambda(E) < \rho(A) < +\infty$ then $\rho(A) \in \mathbb{N} = \{1, 2, \ldots\}$.
(iii) If A is transcendental and $\rho(A) < 1/2$ then $\lambda(E) = +\infty$.

Used the *Bank-Laine equation* (WLOG $W(f_1, f_2) = 1$)

$$4A = \left(\frac{E'}{E}\right)^2 - 2\,\frac{E''}{E} - \frac{1}{E^2}.$$

(iii) was extended to $\rho(A) \leq 1/2$ (Rossi/Shen 1985).

Examples:

$$E = e^P, \quad 4A = -P'^2 - 2P'' - E^{-2},$$
$$E(z) = \frac{1}{\pi} \exp(2\pi i z^2) \sin(\pi z).$$

Scarcity of examples plus Theorem 4(ii)
($\lambda(E) < \rho(A) < +\infty \Rightarrow \rho(A) \in \mathbb{N} = \{1, 2, \ldots\}$)
led to

Conjecture 1 (The Bank-Laine conjecture 1980s)
Let A be a transcendental entire function of order $\rho(A)$ and let f_1, f_2 be linearly independent solutions of

$$y'' + A(z)y = 0.$$

If $E = f_1 f_2$ and $\lambda(E)$ is finite then $\rho(A) \in \mathbb{N} \cup \{\infty\}$.

Theorem 5 (Bergweiler and Eremenko 2015) *The Bank-Laine conjecture is false: for every $\rho > 1/2$ there exists an entire A with $\rho(A) = \rho$ such that*

$$y'' + A(z)y = 0$$

has LI solutions f_1, f_2 with $\lambda(E) < +\infty$, $E = f_1 f_2$.

Highly complicated method constructs $U = \dfrac{f_1}{f_2}$ by combining functions of form $P_m(e^z)e^{e^z}$ via quasiconformal surgery.

Before stating the next theorem we look at one approach to Theorem 4(i).

Suppose A is a non-constant polynomial,
$A(z) = a_n z^n (1 + o(1))$ as $z \to \infty$.
The following shows that $\lambda(E) = (n+2)/2$
(proved by Bank-Laine 1982, via a different method).

The $n+2$ *critical rays* are given by $\arg z = \theta^*$, where $a_n e^{i(n+2)\theta^*} \in \mathbb{R}^+$.

Apply the *Liouville transformation*

$$Y(Z) = A(z)^{1/4} y(z),$$

$$Z = \int_{z_1}^{z} A(t)^{1/2} \, dt \sim \frac{2a_n^{1/2} z^{(n+2)/2}}{n+2},$$

in sectors symmetric about each critical ray.

The equation

$$y'' + A(z)y = 0, \qquad (1)$$

reduces to a sine-type equation

$$\frac{d^2Y}{dZ^2} + \left(1 + \frac{O(1)}{Z^2}\right) Y = 0,$$

with solutions asymptotic to $e^{\pm iZ}$ (Einar Hille 1920s).
This gives solutions $U^{\pm}(z) = A(z)^{-1/4}e^{\pm iZ}(1+o(1))$ of (1) in a sector straddling the critical ray.
Any linear combination $CU^{+} + DU^{-}$ ($CD \neq 0$) has a lot of zeros in the sector.
So if $\lambda(E) < (n+2)/2$ then, in *every* sector,

$$E(z) = f_1(z)f_2(z) \sim c_1 A(z)^{-1/4}e^{iZ}c_2 A(z)^{-1/4}e^{-iZ}$$
$$\sim c_1c_2A(z)^{-1/2} \to 0 \qquad \text{contradiction!}$$

Now suppose $A \in \mathcal{B}$ is transcendental entire.

Then f^{-1} has a logarithmic singularity over ∞

i.e. there exist $M > 0$ and a component U_M of the set $\{z \in \mathbb{C} : |f(z)| > M\}$ such that $v = \log f(z)$ maps U_M conformally onto $H_M = \{v \in \mathbb{C} : \mathrm{Re}\, v > \log M\}$. Let $z = \phi(v)$ be the inverse.

Then the Liouville transformation

$$Z = \int^z A(t)^{1/2}\, dt = \int^v e^{u/2} \phi'(u)\, du$$

can be defined on subdomains of H_M.
Since ϕ' varies slowly, this leads to:

Theorem 6 (JKL 2018) *The Bank-Laine conjecture is true for A in class $\mathcal{B}$. In fact, let $A \in \mathcal{B}$ be transcendental entire. Let $E = f_1 f_2$, where f_1, f_2 are LI solutions of*

$$y'' + A(z)y = 0.$$

Then exactly one of the following holds.
(I) The functions A and E satisfy $\rho(A) = \rho(E) = 1$.
(II) There exists $d > 0$ such that the zeros of E satisfy

$$n(r, 1/E) > \exp\left(dr^{1/2}\right) \quad \textit{as} \quad r \to +\infty,$$

and in particular $\rho(E) = \lambda(E) = +\infty$.

$A(z) = \cos\sqrt{z}$ shows that $1/2$ is sharp in (II).

The *same* example shows that (I) and (II) can occur. Let $A(z) = -e^{2z} - 1/4$. Then we get solutions

$$f_1(z) = e^{-z/2} \exp\left(-e^z\right), \quad f_2(z) = e^{-z/2} \exp\left(e^z\right),$$

$$f_1(z) f_2(z) = e^{-z}, \quad \rho(f_1 f_2) = \rho(A) = 1,$$

as well as solutions

$$g_1(z) = e^{-z/2} \sinh\left(e^z\right), \quad g_2(z) = e^{-z/2} \cosh\left(e^z\right),$$

with $\lambda(g_1 g_2) = +\infty$.