The following is Hayman's version of Fuchs' small arcs lemma, rewritten for the meromorphic, rather than delta-subharmonic, case. [1]

Theorem 0.1 *Let the function f be meromorphic in $|z| \leq R$ with $f(0) = 1$ and Nevanlinna characteristic $T(R)$. Let*

$$\eta_1 > 0, \quad \eta_2 > 0, \quad \eta_1 + \eta_2 < 1. \tag{1}$$

Then there exists $E \subseteq [0, (1-\eta_1)R]$, of measure greater than $R(1-\eta_1-\eta_2)$, with the following property. If $r \in E$ and F is a measurable subset of $[0, 2\pi]$ or $[-\pi, \pi]$ of measure at most δ then

$$I = \int_F \left| \frac{re^{i\theta} f'(re^{i\theta})}{f(re^{i\theta})} \right| d\theta \tag{2}$$

satisfies

$$I \leq \frac{400\delta T(R)}{\eta_1^2 \eta_2} \log \frac{2\pi e}{\delta}. \tag{3}$$

Proof (Fuchs-Hayman). Let

$$R_1 = (1-\eta_1/2)R \geq R/2. \tag{4}$$

Then

$$n(R_1, f) \leq \int_{R_1}^R \frac{n(t,f)\,dt}{t} \left(\log \frac{R}{R_1} \right)^{-1} \leq T(R) \left(\log \frac{1}{1-\eta_1/2} \right)^{-1} \leq \frac{2}{\eta_1} T(R), \tag{5}$$

since

$$\log \frac{1}{1-x} = x + \frac{x^2}{2} + \dots \quad \text{for} \quad 0 < x < 1.$$

In the same way, since $f(0) = 1$,

$$n(R_1, 1/f) \leq \frac{2}{\eta_1} T(R). \tag{6}$$

Denote by

$$c_1, \dots, c_M, \quad \text{where} \quad M \leq \frac{4}{\eta_1} T(R), \tag{7}$$

the zeros and poles of f in $|z| < R_1$, repeated according to multiplicity. Since

$$m(R_1, f) + m(R_1, 1/f) \leq 2T(R_1, f) \leq 2T(R),$$

a standard estimate (e.g. from the book of Jank and Volkmann) gives, for $|z| = r < R_1$,

$$\left| \frac{f'(z)}{f(z)} \right| \leq \frac{4R_1 T(R)}{(R_1 - r)^2} + 2 \sum_{j=1}^M \frac{1}{|z - c_j|}. \tag{8}$$

Apply the method of Cartan's lemma (as in Hayman, *Subharmonic Functions II*, p.366) to the positive numbers $|c_j|$, with repetition as appropriate, and with

$$h = \frac{\eta_2 R}{12}, \quad A = 6. \tag{9}$$

[1] PGNOTES/wflem.tex

This gives a union E_0 of open intervals having sum of lengths less than $2Ah = \eta_2 R$, with the following property. If $r \notin E_0$ and $t > 0$ then the number $\mu(r,t)$ of the moduli $|c_j|$ lying in the open interval $(r-t, r+t)$ satisfies

$$\mu(r,t) \leq \frac{Mt}{eh} \leq \frac{48T(R)t}{e\eta_1\eta_2 R}, \tag{10}$$

using (7) and (9). Hence the set

$$E = [0, R(1-\eta_1)] \setminus E_0 \tag{11}$$

has measure

$$|E| > R(1-\eta_1) - \eta_2 R = R(1-\eta_1-\eta_2). \tag{12}$$

Let $r \in E$ and let F be a union of finitely many pairwise disjoint open subintervals of $[0, 2\pi]$ or $[-\pi, \pi]$ having sum of lengths at most δ, and estimate I as defined by (2). Now (8) gives

$$I \leq \frac{4R_1\delta r T(R)}{(R_1-r)^2} + J \leq \frac{4R_1\delta r T(R)}{R^2(\eta_1/2)^2} + J \leq \frac{16\delta T(R)}{\eta_1^2} + J, \tag{13}$$

where

$$J = \sum_{j=1}^{M} \int_F \frac{2r}{|re^{i\theta} - c_j|}\, d\theta. \tag{14}$$

To estimate J, write

$$K_c = \int_F \frac{2r}{|re^{i\theta} - c|}\, d\theta, \quad c \in \{c_j\} \tag{15}$$

and

$$c = \rho e^{i\alpha}, \quad \rho = |c| > 0, \quad \theta = \alpha + \phi, \quad t = |\rho - r|. \tag{16}$$

Then

$$K_c = \int_{F'} \frac{2r}{|re^{i\phi} - \rho|}\, d\phi,$$

where F' is a union of finitely many pairwise disjoint open intervals having sum of lengths at most δ. It may be assumed that F' is contained in $[-\pi, \pi]$. If $0 \leq \phi \leq \pi/2$ then

$$|re^{i\phi} - \rho| \geq r\sin\phi \geq \frac{2r\phi}{\pi},$$

while if $\pi/2 \leq \phi \leq \pi$ then $|re^{i\phi} - \rho| \geq r$. Thus, for any $\phi \in [-\pi, \pi]$,

$$(2\pi+1)|re^{i\phi} - \rho| \geq 2\pi|re^{i\phi} - \rho| + |\rho - r| \geq 2r|\phi| + t.$$

Hence

$$K_c \leq 2r \int_{F'} \frac{2\pi+1}{2r|\phi| + t}\, d\phi,$$

in which the right-hand side is evidently maximised if $F' = (-\delta/2, \delta/2)$. This gives

$$K_c \leq 4r \int_0^{\delta/2} \frac{2\pi+1}{2r\phi + t}\, d\phi \leq (4\pi+2)\log\left(1 + \frac{\delta R}{t}\right). \tag{17}$$

Now substitute (17) into (14). Here $t = |\rho - r| < R$, since $r, \rho \in [0, R)$. Moreover, the number $\mu(r,t)$ of $|c_j|$ lying in the open interval $(r-t, r+t)$ satisfies (10) for $0 < t < R$, and is M for $t \geq R$. This gives

$$\begin{aligned}
\frac{J}{4\pi+2} &\leq \int_0^R \log\left(1+\frac{\delta R}{t}\right) d\mu(r,t) \\
&= \left[\mu(r,t)\log\left(1+\frac{\delta R}{t}\right)\right]_0^R + \int_0^R \frac{\delta R \mu(r,t)}{t(t+\delta R)}\, dt \\
&\leq M\log(1+\delta) + \int_0^R \frac{M\delta R}{eh(t+\delta R)}\, dt \\
&= M\log(1+\delta) + \frac{M\delta R}{eh}\log\left(1+\frac{1}{\delta}\right) \\
&\leq M\delta\left(1+\frac{12}{e\eta_2}\log\left(1+\frac{1}{\delta}\right)\right) \\
&\leq M\delta\left(1+\frac{12}{e\eta_2}\log\frac{2\pi e}{\delta}\right) \\
&\leq M\delta\left(1+\frac{12}{e\eta_2}\right)\log\frac{2\pi e}{\delta}.
\end{aligned}$$

On recalling (1), (7) and (13) this gives

$$I \leq \frac{16\delta T(R)}{\eta_1^2} + \frac{4T(R)}{\eta_1}(4\pi+2)\delta\left(1+\frac{12}{e\eta_2}\right)\log\frac{2\pi e}{\delta} \leq \frac{L\delta T(R)}{\eta_1^2\eta_2}\log\frac{2\pi e}{\delta},$$

where

$$L = 16 + 4(4\pi+2)\left(1+\frac{12}{e}\right) < 400.$$

This proves (3) when F is a union of finitely many pairwise disjoint open intervals, and by the monotone (or dominated) convergence theorem the inequality extends immediately to the case of a countable union of pairwise disjoint open intervals. Since a set of measure δ is contained in an open set of measure arbitrarily close to δ, the result follows for measurable F.