The Spin Foam Lectures
2: State sum models

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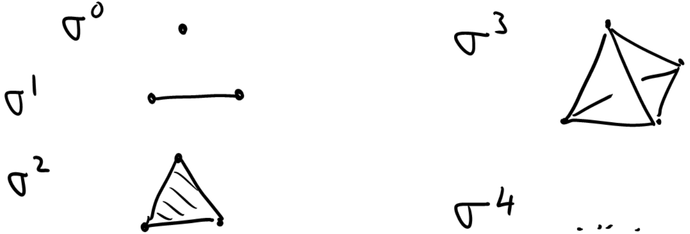
The n-simplex

An n-simplex σ^n is the convex hull of $n+1$ independent points in Euclidean space $\mathbb{R}^d$, $d \geq n$.



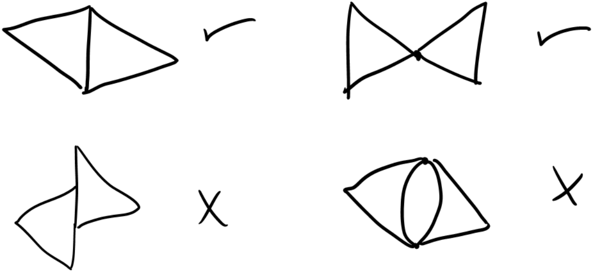
A face of a simplex: $\sigma^k \subset \sigma^n$ determined by a proper subset of the vertices.

Triangulated n-manifold

Triangulated n-manifold: n-simplexes glued together on a face of each to form a manifold M.

$$M = \bigcup_i \sigma^n_{(i)}$$

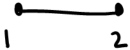
$$\sigma^n_{(i)} \cap \sigma^n_{(j)} = \sigma^k, \text{ a face of each.}$$



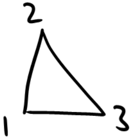
Orientation

Orientation of a simplex: (ordering of the vertices, $\pm$), up to permutation. Each simplex has two orientations.

$$+(1) \neq -(1)$$



$$+(12) = -(21) \neq -(12) = +(21)$$



$$+(123) = +(312) = -(213) \quad \text{etc}$$

Boundary: $\partial +(1234) = +(234) - (134) + (124) - (123)$

State sum model data: states

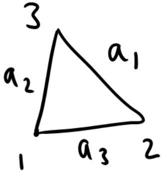
Requirements for a state sum model on an oriented triangulated manifold of dimension n: states and weights.

States:

- The $n-2$ simplex has a set of states S.
- An oriented $n-1$ simplex $+(1,2,3,\ldots)$ has a vector space $H(a_1,a_2,a_3,\ldots)$, depending on $a_i \in S$.

e.g $n=3$

$+(123)$

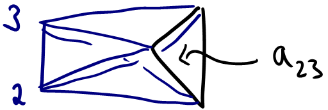


$H(a_1, a_2, a_3)$

Write $H(a_1,a_2,a_3,\ldots) = H(1,2,3,\ldots)$ when labelling is given.

State sum model data: weights

Consider n-simplex $(1,2,3,\ldots)$, labelled with $a_{12}, a_{13}, a_{23}, \ldots \in S$,



- The oriented n simplex $+(1,2,3,\ldots)$ has a weight $W(\sigma^n) \in$ $H(2,3,4,\ldots) \otimes H(1,3,4,\ldots)^{*} \otimes H(1,2,4,\ldots) \otimes \ldots$. dual vector space
- The oriented n simplex $-(1,2,3,\ldots)$ has a weight $W(\sigma^n) \in$ $H(2,3,4,\ldots)^{*} \otimes H(1,3,4,\ldots) \otimes H(1,2,4,\ldots)^{*} \otimes \ldots$.

$\partial + (1234) = +(234) - (134) + (124) - \ldots\ldots$

$\partial - (1234) = -(234) + (134) \ldots$

State sum model

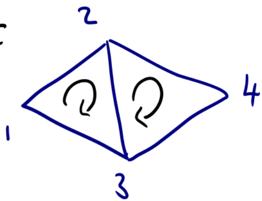
Given:

- M an oriented, triangulated n-manifold with an ordering of the vertices.
- l: set of $n-2$ simplexes $\to S$

Then the partition function is

$$Z(M, l) = \sum_{\substack{\text{interior} \\ n-1 \text{ states}}} \bigotimes_{\sigma^n} W(\sigma^n) \qquad \in H(\partial(M, l)).$$

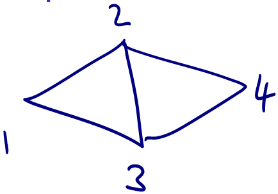
$n=2$



$+(123)$ $\quad$ $-(234)$

$$Z(M) = \sum W(+(123)) \otimes W(-(234))$$

Example



Interior states : $H(23)$

$$W(+(123)) \otimes W(-(234)) \in \underline{H(23)} \otimes H(13)^* \otimes H(12)$$
$$\otimes\, H(34)^* \otimes H(24) \otimes \underline{H(23)^*}$$

$$\xrightarrow[\text{basis/dual basis}]{\Sigma} H(13)^* \otimes H(12) \otimes H(34)^* \otimes H(24)$$
$$\equiv H(\partial(M,\ell))$$

State sum models

- Other geometrical shapes can be used (e.g. cubes).
- Some of the data may be trivial.
- There are generalisations where lower-dimensional simplexes are also labelled.
- State sum models occur in statistical mechanics, lattice gauge theory, mathematics,. . .
- In some cases, $H(\partial M)$ is a Hilbert space.

Simple 2d model

Let $S = \{\bullet\}$. Then

$$W(+(123)) \in H \otimes H^* \otimes H$$

co-

is a product.

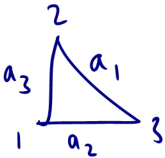
$$W(-(123)) \in H^* \otimes H \otimes H^*$$

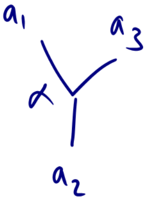
is a ~~co~~product.

3d model from spin networks...

M: 3-manifold
S = set of irreducibles
$H(a_1, a_2, a_3) = \mathrm{Hom}(a_2, a_1 \otimes a_3)$.

α

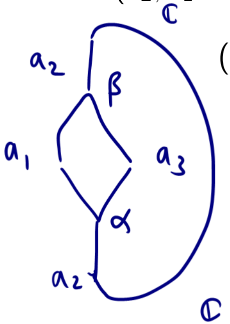




There is a pairing (nondegenerate after equivalence!)

$$\mathrm{Hom}(a_2, a_1 \otimes a_3) \times \mathrm{Hom}(a_1 \otimes a_3, a_2) \to \mathbb{C}$$

$$(\alpha, \beta) \mapsto \mathrm{Tr}(\beta\alpha)$$



$= c$

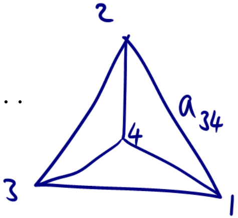
$\mathbb{C} \longrightarrow \mathbb{C}$

$z \longmapsto cz$

So that $H(a_1, a_2, a_3) = \mathrm{Hom}(a_1 \otimes a_3, a_2)^*$

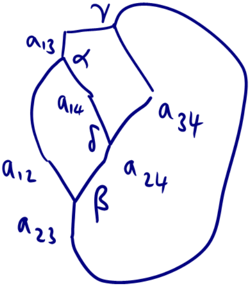
. . .

3-simplex (1234) labelled with $a_{12}, a_{13}, a_{23}, \dots$



$$W(+(1234)) \in H(234) \otimes H(134)^* \otimes H(124) \otimes H(123)^*$$
$$\cong \mathrm{Hom}(a_{12} \otimes a_{14}, a_{13})^* \otimes \mathrm{Hom}(a_{23}, a_{12} \otimes a_{24})^*$$
$$\otimes \mathrm{Hom}(a_{23}, a_{13} \otimes a_{14})^* \otimes \mathrm{Hom}(a_{24}, a_{14} \otimes a_{34})^*$$

$W : \alpha \otimes \beta \otimes \gamma \otimes \delta \mapsto$



$\in \mathbb{C}$

Spherical symmetry

Spherical symmetry for spin networks $\Rightarrow Z(M, l)$ is independent of the ordering of the vertices.

Triangulation dependence

General relativity is independent of the triangulation. Thus two possibilities:

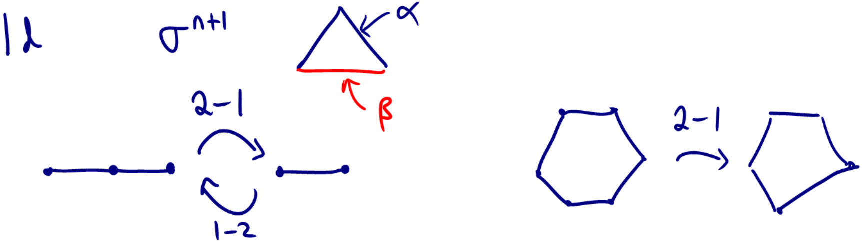
- A quantum gravity model is independent of the triangulation.
- A quantum gravity model becomes independent of the triangulation in the GR limit.

Pachner moves...

In dimension n,

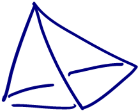
$$\partial\sigma^{n+1} = \alpha \cup \beta$$

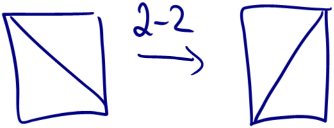
Pachner move: replace α with β

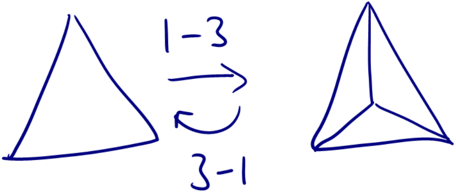


$\underline{2d}$

σ^{n+1}







Triangulation independence

Theorem (Pachner)

Any two triangulations of a compact manifold are related by a finite sequence of Pachner moves.

The lower-dimensional simplexes $\sigma^k, k \leq n-2$ can also be given a weight, $W(\sigma^k) \in \mathbb{C}$.

$W(\sigma^{n-2})$ depends on $l \in S$

Define

$$Z(M) = \sum_{\substack{\text{interior} \\ n-2 \text{ states } l}} Z(M, l) \prod_{\sigma^k, k \leq n-2} W(\sigma^k)$$

Then there exists a choice of $W(\sigma^k) \in \mathbb{C}$ such that $Z(M)$ is independent of the triangulation. Proof: semisimplicity.

for spin network models

Diffeomorphism invariance

Any diffeomorphism $M \to N$ can be approximated by a simplicial map $\Phi \colon M' \to N'$ (vertices to vertices), for some triangulations M' and N' of M and N.
Thus for a triangulation-independent model

$$Z(M) \overset{\text{Pachner}}{=} Z(M') \overset{\phi}{=} Z(N') \overset{\text{Pachner}}{=} Z(N)$$

i.e., the partition function is invariant under diffeomorphisms.

Observables transform covariantly.

Exercises

1. Explain why a 1d state sum model is determined by a linear map $L\colon H \to H$. What is the partition function of a circle triangulated as a polygon with N sides? What property does L have if the model is invariant under Pachner moves?

2. Consider the simple 2d model where there is only one label for vertices. Use a suitable choice of ordering and orientation and the 2-2 Pachner move to show that for a triangulation-independent model, the product on the vector space H is an associative product.
3. Draw the 2-3 and 1-4 Pachner moves in three dimensions.

State Sum Model References

- Barrett and Westbury: Invariants of PL 3-manifolds
- Turaev and Viro: State sum invariants of 3-manifolds and quantum 6j-symbols
- Barrett and Naish-Guzman: The Ponzano-Regge model
- Freidel and Louapre: Ponzano-Regge model revisited I
- Mackaay: Spherical 2-categories and 4-manifold invariants
- Witten: On quantum gauge theories in two dimensions