The Spin Foam Lectures
3: Spin foam models

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v3

Spin networks for $\mathrm{SU}(2)$

$$X \cong X^*$$

$$\mathcal{S} = \text{irreducibles} = \{X_j \mid j = 0, \tfrac{1}{2}, 1, \dots\}$$

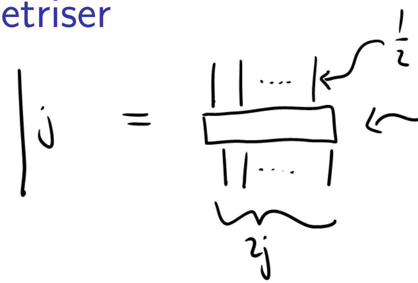
vector space dim $X_j = 2j+1$ $\qquad X_{\frac{1}{2}} \cong \mathbb{C}^2$

$\epsilon_{X_{\frac{1}{2}}} : X_{\frac{1}{2}} \otimes X_{\frac{1}{2}} \to \mathbb{C}$ $\qquad$ matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $\quad (A = \pm 1)$

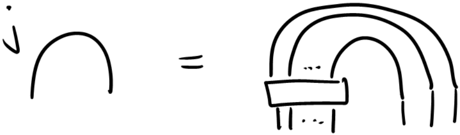
antisymmetric

$$X_j = \mathrm{Sym}\left(\otimes^{2j} X_{\frac{1}{2}} \right)$$

Symmetriser



symmetriser $\frac{1}{2j!}\sum_{\text{permutations}}$



symmetric or antisymmetric

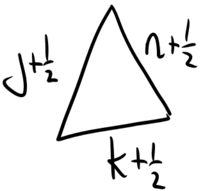
$\dim X_j =$

$= (-1)^{2j}(2j+1)$

Intertwiners

$$\dim \mathrm{Hom}(X_j, X_k \otimes X_n) = 0 \text{ or } 1$$

$\dim = 1$ when



is a Euclidean triangle

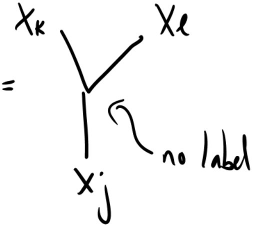
(Clebsch-Gordan rules)

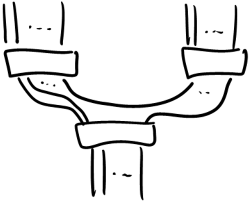
$$\rightsquigarrow \sum_{\ell} Z(M,\ell)$$ is a sum over Euclidean metrics on M

X_j labels an edge $\rightsquigarrow$ length of edge is $j+\frac{1}{2}$.

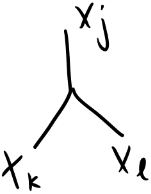
$\rightsquigarrow$ Quantum gravity $+++$ signature metric

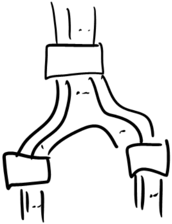
... Canonical intertwiner

$\alpha =$ 

$=$ 

Unique
- no crossings

$\beta =$ 

$=$ 

const x dual
basis
vector

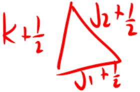
Weights

Up to normalisation $W(\sigma^3) =$ "6j-symbol" depending on $j_1 \cdots j_6$

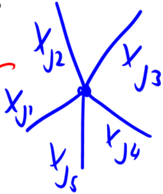
$$W(\sigma^1) = \dim_q X_j = (-1)^{2j}(2j+1)$$

$W(\sigma^0) = \cdots\cdots$ problem. If M has interior vertices

$$\sum_{\ell} Z(M,\ell) \prod_{\sigma^1} W(\sigma^1) \quad \text{diverges}$$



all $j \to \infty$

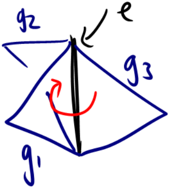


Group variables

Reformulate $Z(M,\ell) = \prod_{\text{triangles } t} \int_{SU(2)} dg_t \prod_{\text{edges } e} (2j_e+1)\,\mathrm{Tr}_{j_e}(h_e)$

 variable $g \in SU(2)$

 g^{-1}



$h_e = g_1 g_2 g_3$

g's form a connection on M (like LGT)

Ponzano-Regge partition function

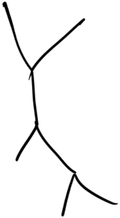
Use $\sum_j (2j+1) \, Tr_j(h) = \delta_I(h_e)$ $\quad I \in SU(2)$ identity

$\rightsquigarrow \sum_\ell Z(M,\ell) \prod W(\sigma') = \prod_{\substack{\text{triangles} \\ t}} \int dg_t \prod_{\text{edges}} \delta_I(h_e)$

Divergence due to too many δ.

Gauge fixing: remove excess delta functions on a maximal tree of edges in M

... is well defined



Theorem $Z(M) = \prod \int dg \prod_{\substack{\text{edges} \\ \text{except on a tree}}} \delta(h_e)$

is well-defined if twisted $H^2 = 0$ for all flat $SU(2)$ connections on M

Then $Z(M) = \displaystyle\int_{\text{flat connections } \rho} \text{Reidemeister Torsion}(\rho)$

Problem: investigate case $H^2 \neq 0$

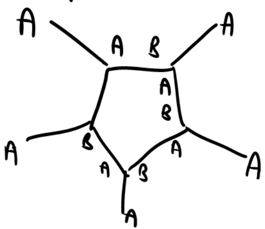
BF functional integral

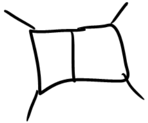
$$Z(M) = \int \mathcal{D}A\,\mathcal{D}B\; e^{i\int B_\ell \wedge \underbrace{[dA^i + A^j \wedge A^k c^i_{jk}]}_{F^i_A}}$$

A^i connection 1-form $\quad c^i_{jk}$ Lie alg structure constants

B_ℓ 1-forms $\quad F$: curvature

One-loop exact





$= 0$

Equivalence to Ponzano-Regge

$$Z_{F.I}(M) = \int_{\text{classical solns}} \text{1-loop det} = \int_{\text{flat connections}} \text{Reidemeister torsion} \qquad (H^2=0)$$

- For $su(2)$, BF action is 1st order gravity $\int R$
  A = spin connection, B = frame field "e"
- Theory is renormalizable
- Appears that very high-energy modes in F.I decouple from observables / Ponzano-Regge

Problem: make this precise.

Spin networks for $U_q sl2$

$$q = e^{i\pi/r} = A^2 \qquad r \in \mathbb{Z},\ r \geq 3. \qquad \text{Fix } r.$$

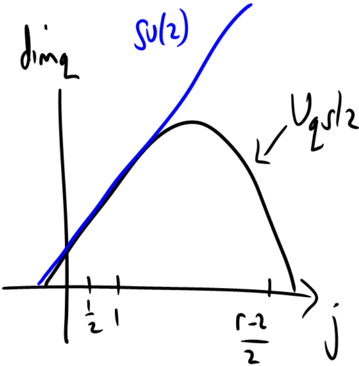
Representations like $SU(2)$, but

$$\mathcal{S} = \text{irreducibles} = \left\{ X_j \,\middle|\, j = 0, \tfrac{1}{2}, 1, \ldots \tfrac{r-2}{2} \right\}$$

$$\epsilon_{X_{\frac{1}{2}}} = \begin{pmatrix} 0 & A \\ -A^{-1} & 0 \end{pmatrix} \qquad \text{(c.f. Exercises)}$$

$$\dim_q X_{\frac{1}{2}} = -A^2 - A^{-2}$$

Quantum dimensions for $\mathrm{U}_q\mathrm{sl}2$

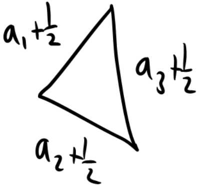


$$\dim_q X_j = (-1)^{2j} \frac{\sin\frac{\pi}{r}(2j+1)}{\sin \pi/r}$$

Clebsch–Gordan:

$\dim H(a_1 a_2 a_3) = 1$ or 0

1 when



spherical triangle

$\rightsquigarrow \sum_{\ell}$ is

quantum gravity with const. curvature metrics

Turaev-Viro model

$$W(\sigma^1) = \dim_q X_j \qquad W(\sigma^3): \text{ 6j q-symbols}$$

$$W(\sigma^0) = \frac{1}{\sum_{j=0}^{r-2k} (\dim_q X_j)^2}$$

$$Z(M) = \sum_{\ell} Z(M, \ell) \prod W(\sigma)$$

is well-defined & independent of triangulation

Turaev-Viro: discussion

- No formulation in terms of group variables g_t
- BF action: extra term cosmological term

$$\int \mathcal{D}A\,\mathcal{D}B\, e^{i\int B_\ell \wedge F^\ell + \Lambda B_m \wedge B_n \wedge B_p\, \varepsilon^{mnp}}$$

$\Lambda \sim \frac{1}{r^2}$. No longer 1-loop exact.

Problem: Understand limit $r \to \infty$ precisely

(Heuristically, TV $\to$ PR)

Exercises

1. Calculate $\dim_q X_j$ for the irreducible representation X_j of $\mathrm{SU}(2)$. One way of doing this is to find a recursion relation for the symmetriser on $2j$ copies of $X_{1/2}$. Alternatively, express the diagrammatic trace in terms of the ordinary matrix trace and a sign.

2. Calculate the Ponzano-Regge partition function of S^3 by using two tetrahedra glued together on all faces (this isn't a triangulation but pretend it is). Use the formula involving integration over $\mathrm{SU}(2)$ and delta functions. Which deltas should be removed to make the integral finite? Assume $\int \mathrm{d}g = 1$.

3d Spin Foam References

- Major: A Spin Network Primer
- Ponzano and Regge: Semiclassical limit of Racah coefficients
- Barrett and Naish-Guzman: The Ponzano-Regge model
- Gurau: The Ponzano-Regge asymptotic of the 6j symbol: an elementary proof
- Dowdall, Gomes, Hellmann: Asymptotic analysis of the Ponzano-Regge model for handlebodies.
- Turaev and Viro: State sum invariants of 3-manifolds and quantum 6j-symbols
- Kauffman and Lins: Temperley-Lieb Recoupling Theory and Invariants of 3-Manifolds
- Taylor and Woodward: 6j symbols for $U_q(sl_2)$ and non-Euclidean tetrahedra