The Spin Foam Lectures
4: 4d Spin foam models

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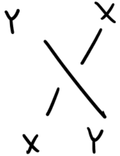
4d models

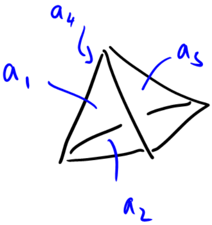
Braided spin networks determine 4d state sum models.

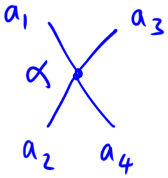


$\rightsquigarrow$ canonical mappings $X \otimes Y \to Y \otimes X$

$$S = \{\text{irreducibles}\}$$

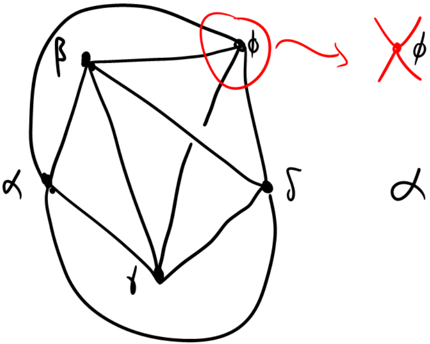
$$H(a_1, a_2, a_3, a_4) = \mathrm{Hom}(a_2 \otimes a_4, a_1 \otimes a_3) \ni \alpha$$





4-simplex weight

Braided spin network on $\partial\sigma^4 \cong S^3$.



$$\alpha \otimes \beta \otimes \gamma \otimes \delta \otimes \phi \to \mathbb{C}$$

"15j-symbol"

Examples

- Lie group e.g. $\mathrm{SU}(2) \longrightarrow$ Ooguri model
- Quantum group e.g. $\mathrm{U}_q\mathrm{sl}2 \longrightarrow$ Crane-Yetter model

- $Z(M)$ triangulation-independent
- Not gravity

BF theory (heuristic)

$$\int \mathrm{D}A\,\mathrm{D}B\ \exp\left(i\int B_l\wedge(\mathrm{d}A+A\wedge A)^l+\Lambda B_m\wedge B^m\right)$$

A^l: connection 1-form

B_l: 2-form

$\Lambda=0$ Ooguri.

$\rightsquigarrow$ flat connections

$\Lambda=\frac{1}{r}$ C-Y.

Problem: make this precise.

Gravity action

Use $\mathrm{SO}(3,1)$, $A^i \longrightarrow A^{ab}$, antisymmetric (Also SO(4) version)

E-H Holst

$$\int *(e_a \wedge e_b) \wedge F^{ab} + \frac{1}{\gamma}(e_a \wedge e_b) \wedge F^{ab}$$

e_a 1-form frame field, $*B_{ab} = \frac{1}{2}\epsilon_{ab}{}^{cd}B_{cd}$

γ: Immirzi parameter

Compare with BF action

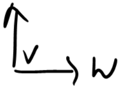
$$\int B_{ab} \wedge F^{ab} = \int *(C_{ab}) \wedge F^{ab} + \frac{1}{\gamma} C_{ab} \wedge F^{ab}$$

When is $C_{ab} = e_a \wedge e_b$? Constraint

When is $C_{ab} = e_a \wedge e_b$?

1. Restricted to every 2-surface σ

$$C^{ab}(\sigma) * C_{ab}(\sigma) = 0$$



$$\left[\epsilon_{abcd} v^a w^b v^c w^d = 0 \right]$$

2. Restricted to every 3-surface Σ

$\exists$ vector n^a such that $C_{ab}(\Sigma) n^a = 0$

Necessary & sufficient.

1. Implies constraint on $B = *C + \frac{1}{\gamma} C$ on every σ^2

$$B(\sigma^2) \cdot B(\sigma^2) = \left(1 - \frac{1}{\gamma^2}\right) C(\sigma^2) \cdot C(\sigma^2)$$

$$B(\sigma^2) \cdot * B(\sigma^2) = -2/\gamma \; C(\sigma^2) \cdot C(\sigma^2)$$

Casimirs on quantization

2. Implies $C_{ab}(\Sigma) \in$ subgroup of $SO(3,1) \cong SO(3)$

Quantization of constraints

$\mathrm{SO}(3,1)$ has irreducibles $X_{(p,k)}$, $k \in \{0, \frac{1}{2}, 1, \dots\}$, $p \in \mathbb{R}$.

1. Relation on Casimirs C_1 & C_2 of $so(3,1)$

$\rightsquigarrow p = \gamma k \qquad S = \{X_{(\gamma k, k)}\}$

2. Define $H(a_1, a_2, a_3, a_4) \subset \mathrm{Hom}_{so(3,1)}(a_2 \otimes a_4, a_1 \otimes a_3)$

$a_i = X_{(\gamma k_i, k_i)}$

$\uparrow$ injection

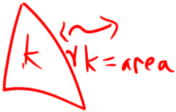
$\mathrm{Hom}(X_{k_2} \otimes X_{k_4}, X_{k_1} \otimes X_{k_3})$

Inject into $\mathrm{Hom}_{so(3,1)}$ & average over $so(3,1)$

4d gravity model

Some choice

$$Z(M) = \sum_{I} Z(M, I) \prod_{k \le n-2} W(\sigma^k)$$



depends on the triangulation.

Problem: Determine if a suitable limit "$G\hbar \to 0$" gives a triangulation-independent theory.

"$G\hbar \to 0$" means observed areas large (in Planck units)

- Seems necessary to refine triangulation in limit
- Possible limits: GR, flat space, non-geometric

Regge calculus

is ~~D~~sicrete general relativity on a triangulated manifold (any n)

n-simplex:

ℓ^2 for every edge $\longleftrightarrow$ flat metric on the simplex

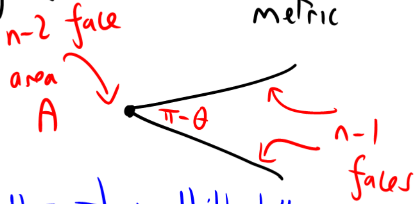
On manifold: ℓ^2 for every edge $\longleftrightarrow$ piecewise flat metric

Regge action: $S(M) = \sum S(\sigma)$

$$S(\sigma^n) = \sum \theta_i A_i$$

$$S(\sigma^{n-1}) = \sum -\pi A_i$$

$$S(\sigma^{n-2}) = 2\pi A$$



"Einstein-Hilbert"

Asymptotics of 4d gravity model

$$W(\sigma^4) \overset{k \text{ large}}{\sim} \sum e^{i\gamma \sum_{\substack{\text{triangles} \\ t}} k_t \theta_t} + \text{other terms}$$

$\rightsquigarrow$ Regge action with area of $t = \gamma k_t$

Other terms: from copy of Ooguri $\subset$ Gravity model

No propagating degrees of freedom – Harmless?

c.f. 3d asymptotics of 6j-symbols.

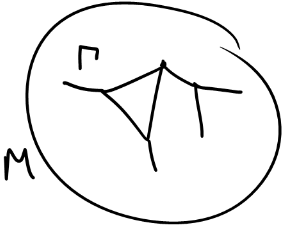
Observables

$$Z(M; f) = \sum_l Z(M, l) \prod W(\sigma) f(l)$$

In 3d $\quad l$: edges of $M \to S$

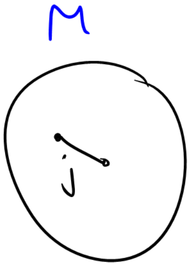
$\rightsquigarrow$ graph Γ of edges that f depends on non-trivially

$\rightsquigarrow$ diffeomorphism invariant of (M, Γ_f)



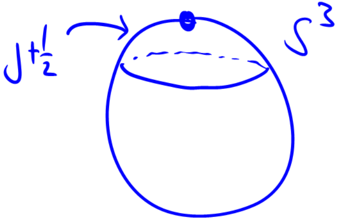
Examples in 3d

Turaev–Viro $\quad f(\ell) = \delta(\ell_1 - j)$



$$Z_{TV}(S^3, f) = (\dim_q j)^2 = \frac{\sin^2 \frac{\pi}{r}(2j+1)}{\sin^2 \pi/r}$$

$\propto$ prob 2 points on S^3 (radius $r/2\pi$) are distance $j+\frac{1}{2}$ apart

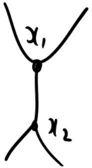


Matter couplings

In 3d, TV, Γ = planar graph with $\delta(l_i - j_i)$ on each edge

$$Z_{TV}(M, f) = \text{Feynman diagram measure}$$

$$\int_{x_i \in S^3} \prod dx_i = \int \prod_{ij} d\|x_i - x_j\| \; J$$



$$Z_{TV} = J \quad \text{at discrete values!}$$

Problem : Investigate 4d version further.

Exercises

1. In a theory based on $\mathrm{SU}(2)$ spin networks, explain why the dimension of $H(X_j, X_k, X_l, X_m) = 0$ unless $j \leq k + l + m$. Give a geometric interpretation of this inequality in terms of the areas of the four faces of a Euclidean tetrahedron.

2. Find four Euclidean triangles that glue together to form the boundary of a tetrahedron that does not embed metrically in Euclidean 3-space. What other space does it embed in?

4d Spin Foam References

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- Crane, Kauffman, Yetter: Evaluating the Crane-Yetter Invariant
- Engle, Livine, Pereira, Rovelli: LQG vertex with finite Immirzi parameter
- Barrett, Dowdall, Fairbairn, Hellmann, Pereira: Lorentzian spin foam amplitudes: Graphical calculus and asymptotics.
- Barrett, Fairbairn, Hellmann: Quantum gravity asymptotics from the SU(2) 15j-symbol.

3d asymptotics References

- Ponzano and Regge: Semiclassical limit of Racah coefficients
- Gurau: The Ponzano-Regge asymptotic of the 6j symbol: an elementary proof
- Dowdall, Gomes, Hellmann: Asymptotic analysis of the Ponzano-Regge model for handlebodies.
- Taylor and Woodward: 6j symbols for $U_q(sl_2)$ and non-Euclidean tetrahedra

Observables references

- Barrett: Geometrical measurements in three-dimensional quantum gravity
- Barrett, Garcia-Islas, Faria Martins: Observables in the Turaev-Viro and Crane-Yetter models
- Freidel and Livine: Ponzano-Regge model revisited III: Feynman diagrams and effective field theory
- Barrett: Feynman diagrams coupled to three-dimensional quantum gravity
- Barrett, Kerr, Louko: A topological state sum model for fermions on the circle