Exemples of computations with `shark`

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A good ordinary example

Let E be the following curve

```
sage : e = EllipticCurve('446d1'); p=5; show(e)
```

$$y^2 + xy = x^3 - x^2 - 4x + 4$$

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$$y^2 + xy = x^3 - x^2 - 4x + 4$$

It has rank 2 and good ordinary reduction at $p = 5$.

```
sage : e.rank()
```

$$2$$

```
sage : e.is_ordinary(p)
```

True

But it has anomalous reduction

```
sage : e.Np(p)
```

$$10$$

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with Tamagawa numbers

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sage : e.tamagawa_numbers()
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$$[2, 1]$$

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with Tamagawa numbers

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$$[2, 1]$$

and no torsion point in $E(\mathbb{Q})$.

```
sage : tors= e.torsion_order();tors
```

$$1$$

The p-adic L-function is approximated by

```
sage : lp = e.padic_lseries(p);lps =
lp.series(5,prec=7);lps
```

$$O(5^7) + O(5^4)\cdot T + (5 + 5^2 + 3\cdot 5^3 + O(5^4))\cdot T^2$$
$$+(2\cdot 5 + 3\cdot 5^2 + 3\cdot 5^3 + O(5^4))\cdot T^3$$
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\end{gathered}
$$

- Here we used a Riemann sum on $\mathbb{Z}_p^\times = \bigsqcup (a + p^5\mathbb{Z}_p)$.

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- We have $\mathrm{ord}_{T=0}\,\mathcal{L}_p(E,T) \leqslant 2$.

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- The leading term has valuation 1.
- The sixth coefficient is a unit.

The p-adic regulator

$$\mathrm{Reg}_p(E/\mathbb{Q})/\log(\gamma)^2$$

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sage : reg = e.padic_regulator(p);R =
Qp(p,10);lg = log(R(1+p));reg = R(reg)/lg^2;reg
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$$2 \cdot 5^{-1} + 4 + 3 \cdot 5 + 2 \cdot 5^2 + 5^4 + 5^5 + 2 \cdot 5^6 + 3 \cdot 5^7 + O(5^8)$$

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$$2\cdot 5^{-1}+4+3\cdot 5+2\cdot 5^2+5^4+5^5+2\cdot 5^6+3\cdot 5^7+O(5^8)$$

Its valuation is -1; that is minimal for anomalous primes.

Putting things together

```
sage : eps = (1-1/lp.alpha())^2;
lpstar/eps/reg/e.tamagawa_product()*tors^2
```

$$1 + O(5^3)$$

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- The p-adic BSD predicts $\#Ш(E/\mathbb{Q}) \equiv 1 \pmod{125}$.
- The main conjecture holds for E at p.

Actually we have

$$\mathcal{L}_p(E,T) = T \cdot \big((T+1)^5 - 1\big) \cdot u$$

for some unit $u \in \Lambda^\times$.

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- $\mathrm{rank}(E(\mathbb{Q}_\infty)) = 2 + 4 = 6$ and that

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for some unit $u \in \Lambda^\times$. This tells us immediately that

- $\mathrm{rank}(E(\mathbb{Q}_\infty)) = 2 + 4 = 6$ and that
- $\text{Ш}(E/\mathbb{Q}_\infty)(p)$ is finite.

Further examples

389a1	5	$1 + O(5^4)$
389a1	7	$1 + O(7^3)$
389a1	11	$1 + O(11^2)$
389a1	13	$1 + O(13^2)$
389a1	17	$1 + O(17^2)$
389a1	19	$1 + O(19^2)$
433a1	5	$4 + 4 \cdot 5 + 4 \cdot 5^2 + 4 \cdot 5^3 + O(5^4)$
433a1	7	$6 + 6 \cdot 7 + 6 \cdot 7^2 + O(7^3)$
433a1	11	$10 + 10 \cdot 11 + O(11^2)$
433a1	13	$12 + O(13)$
433a1	17	$16 + 16 \cdot 17 + O(17^2)$
433a1	19	$18 + 18 \cdot 19 + O(19^2)$
446d1	5	$1 + O(5^3)$
446d1	7	$1 + O(7^2)$
446d1	11	$1 + O(11^2)$
446d1	13	$1 + O(13^2)$
446d1	17	$1 + O(17^2)$

A split multiplicative example

A more complicated example is given by the curve

```
sage : e= EllipticCurve('817a1'); p=43; show(e)
```

$$y^2 + y = x^3 + x^2 + x + 6$$

A split multiplicative example

A more complicated example is given by the curve

```
sage : e= EllipticCurve('817a1'); p=43; show(e)
```

$$y^2 + y = x^3 + x^2 + x + 6$$

which has split multiplicative reduction at $p = 43$.

```
sage : eq = e.tate_curve(p);eq.is_split()
```

True

The p-adic regulator is

```
sage : reg = eq.padic_regulator(4);reg
```

$$23 \cdot 43^2 + 41 \cdot 43^3 + 36 \cdot 43^4 + 42 \cdot 43^5 + 26 \cdot 43^6 + O(43^7)$$

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The very first non-trivial appoximation to the p-adic L-series is

```
sage : lp = e.padic_lseries(p);lps =
lp.series(2,prec=4);lps
```

$$O(43^4) + O(43^1) \cdot T + O(43^1) \cdot T^2 + (27 + O(43)) \cdot T^3 + O(T^4)$$

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```

$$O(43^4) + O(43^1) \cdot T + O(43^1) \cdot T^2 + (27 + O(43)) \cdot T^3 + O(T^4)$$

The L-invariant

```
sage : Li = eq.L_invariant(4); Li
```

$$34 \cdot 43 + 40 \cdot 43^2 + 30 \cdot 43^3 + O(43^4)$$

We find

```
sage : lpstar=lps(3); R=Qp(p,10);
lg=log(R(1+p)); tors=e.torsion_order();
tam=e.tamagawa_product()
-lpstar/reg*lg^3/Li*tors^2/tam
```

$$1 + O(43)$$

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Hence $Ш(E/\mathbb{Q})(43)$ of this rank 2 curve is trivial and

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$$1 + O(43)$$

Hence $\text{Ш}(E/\mathbb{Q})(43)$ of this rank 2 curve is trivial and

$$\#\text{Ш}(E/\mathbb{Q}) \equiv 1 \pmod{43}.$$