4 L-functions

In this chapter, we will use analytic tools to study prime numbers. We first start by giving a new proof that there are infinitely many primes and explain what one can say about exactly "how many primes there are". Then we will use the same tools to proof Dirichlet's theorem on primes in arithmetic progressions.

4.1 Riemann's zeta-function and its behaviour at $s = 1$

Definition. The **Riemann zeta function** $\zeta(s)$ is defined by

$$\zeta(s) = \sum_{n\geqslant 1} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots .$$

The series converges (absolutely) for all $s > 1$.

Remark. The series also converges for complex s provided that $\mathrm{Re}(s) > 1$.

Example. From the last chapter, $\zeta(2) = \pi^2/6$, $\zeta(4) = \pi^4/90$. But $\zeta(1)$ is not defined since $\sum \frac{1}{n}$ diverges. However we can still describe the behaviour of $\zeta(s)$ as $s \searrow 1$ (which is my notation for $s \to 1^+$, i.e. $s > 1$ and $s \to 1$).

Proposition 4.1.

$$\lim_{s\searrow 1} (s-1)\cdot\zeta(s) = 1.$$

Proof. Summing the inequality $(n+1)^{-s} < \int_n^{n+1} \frac{1}{x^s}dx < n^{-s}$ for $n \geqslant 1$ gives

$$\zeta(s) - 1 < \int_1^{\infty} \frac{1}{x^s}dx = \frac{1}{s-1} < \zeta(s)$$

and hence $1 < (s-1)\zeta(s) < s$; now let $s \searrow 1$. □

Corollary 4.2.

$$\lim_{s\searrow 1} \frac{\log\zeta(s)}{\log\left(\frac{1}{s-1}\right)} = 1.$$

Proof. Write

$$\frac{\log\zeta(s)}{\log\left(\frac{1}{s-1}\right)} = \frac{\log(s-1)\zeta(s)}{\log\left(\frac{1}{s-1}\right)} + 1$$

and let $s \searrow 1$. □

Remark (This can be ignored by those who did not take G12COF). It is possible to extend the Riemann zeta function $\zeta(s)$ to an entire function on the whole of $\mathbb{C}\setminus\{1\}$, which has a simple pole at $s = 1$ with residue 1 (as can be seen from proposition 4.1). Although the series $\sum_{n\geqslant 1} n^{-s}$ we used originally to define $\zeta(s)$ only converges when $\mathrm{Re}(s) > 1$, there are alternative formulas which agree with this when $\mathrm{Re}(s) > 1$ but also make sense for all complex s (except $s = 1$); and then $\zeta(s) - \frac{1}{s-1}$ is entire.

4.2 Euler Product expansion for $\zeta(s)$

Notation. Let $\mathcal{P}$ denote the set of all primes, so $\mathcal{P} = \{2, 3, 5, 7, \dots\}$.

Proposition 4.3 (Euler Product expansion of $\zeta(s)$)**.** *For $s > 1$,*

$$\zeta(s) = \prod_{p \in \mathcal{P}} \frac{1}{1 - p^{-s}}\,.$$

Proof. For each prime p we have

$$\frac{1}{1-p^{-s}} = 1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots$$

multiplying over all primes gives the sum of terms n^{-s} with each $n \in \mathbb{N}$ appearing exactly once because of unique factorisation. □

Remark. As a formal identity this result encodes the unique factorisation of positive integers into a single formula; but the product also converges properly, so this is a genuine equality of analytic functions.

Lemma 4.4. *There is a bound b such that*

$$\left| \log \zeta(s) - \sum_{p \in \mathcal{P}} \frac{1}{p^s} \right| < b$$

for all $s > 1$.

Proof. Using the series expansion for $\log(1+x)$ with $x = -p^{-s}$ gives

$$\log\left(\frac{1}{1-p^{-s}}\right) = \sum_{m \geqslant 1} \frac{1}{m} \cdot \frac{1}{p^{ms}}\,;$$

using the Euler Product formula for $\zeta(s)$ then gives

$$\log \zeta(s) = \sum_{p \in \mathcal{P}} \sum_{m \geqslant 1} \frac{1}{m} \cdot \frac{1}{p^{ms}} = \sum_{p \in \mathcal{P}} \frac{1}{p^s} + R(s)$$

where $R(s) = \sum_{m \geqslant 2} \sum_{p \in \mathcal{P}} m^{-1} \cdot p^{-ms}$; finally, $R(s)$ is bounded. □

Theorem 4.5. *The sum $\sum \frac{1}{p}$, where p runs over all primes p, diverges.*

Proof. By proposition 4.1 $\zeta(s)$ has a pole at $s = 1$, so has $\log \zeta(s)$. By lemma 4.4, this means that $\sum_{p \in \mathcal{P}} p^{-s}$ diverges as $s \searrow 1$. □

This divergence is very slow:

x	10	100	10^3	10^4	10^5	10^6
$\sum_{p \leqslant x} \frac{1}{p}$	1.176	1.803	2.198	2.483	2.705	2.887

In fact we have $\sum_{p \leqslant x} \frac{1}{p} \sim \log(\log(x))$. This follows from the following BIG theorem. Define $\pi(x)$ to be the number of primes $p \leqslant x$.

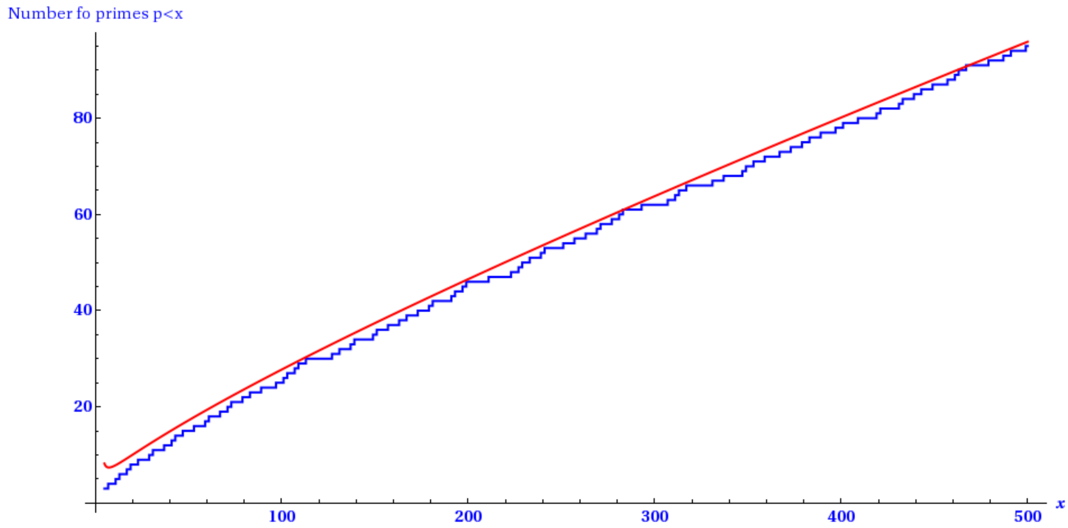

Figure 1: The prime number theorem

Theorem 4.6 (Prime Number Theorem)**.** *We have*

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\log(x)} = 1\,.$$

The proof uses complex function theory and the fact that $\zeta(1+it) \neq 0$ for $0 \neq t \in \mathbb{R}$.

x	10	100	10^3	10^4	10^5	10^6	10^7	10^8
$\pi(x)$	4	25	168	1229	9592	78498	$6.6\cdot 10^5$	$5.8\cdot 10^6$
$\pi(x)/(x/\log(x))$	0.921	1.151	1.161	1.132	1.104	1.084	1.071	1.061
$\pi(x) - x/\log(x)$	-0.3	3.3	23	143	906	6116	$4.4\cdot 10^4$	$3.3\cdot 10^5$

x	10^9	10^{10}	10^{11}	10^{12}	10^{13}	...	10^{23}
$\pi(x)$	$5.1\cdot 10^7$	$4.6\cdot 10^8$	$4.1\cdot 10^9$	$3.8\cdot 10^{10}$	$3.5\cdot 10^{11}$	...	$1.9\cdot 10^{21}$
$\pi(x)/(x/\log(x))$	1.054	1.048	1.043	1.039	1.034	...	1.020
$\pi(x) - x/\log(x)$	$2.6\cdot 10^6$	$2.1\cdot 10^7$	$1.7\cdot 10^8$	$1.4\cdot 10^9$	$1.2\cdot 10^{10}$	...	$3.7\cdot 10^{19}$

4.3 Dirichlet densities of sets of primes

We have now found twice the fact that there are infinitely many primes. We also know that there are infinitely many primes of the form $4k+1$. Can we generalise this? So the question is:

> In an arithmetic progression $a, a+m, a+2m, a+3m, \ldots$, with $\gcd(a,m) = 1$, do infinitely many primes occur?

Obviously if a and m are not coprime then there are only finitely many primes in the arithmetic progression.

Examples.

- $a=1$, $m=1$: $1,2,3,4,\ldots$ –yes (all primes).
- $a=1$, $m=2$: $1,3,5,7,\ldots$ –yes (all odd primes).
- $a=1$, $m=4$: $1,5,9,13,17,\ldots$ –yes (all primes of the form $4k+1$: see chapter II).
- $a=3$, $m=4$: $3,7,11,15,\ldots$ –how many primes have the form $4k+3$???

We actually want to say a bit more. Are there more primes of the form $4k+1$ than primes of the form $4k+3$? But in order to answer such question, we first need a good way of measuring how big a set of primes is, as a proportion of the set of all primes? In other words, if X is some set of primes, can we define its **density** $\operatorname{dens}(X)$?

Definition. Let $X \subset \mathcal{P}$. We say that X has natural density δ if the limit

$$\lim_{N\to\infty} \frac{\#\{p \in X \mid p \leqslant N\}}{\#\{p \in \mathcal{P} \mid p \leqslant N\}}$$

exists and is equal to δ.

Example. Let X be the set of primes of the form $4k+1$. Then

N	100	10^3	10^4	10^5	10^6
$\frac{\#\{p\in X \mid p\leqslant N\}}{\#\{p\in\mathcal{P} \mid p\leqslant N\}}$	0.44	0.47	0.495	0.4986	0.49906

So it looks like the natural density of X is $\frac{1}{2}$. This is true but too hard to prove. We will replace the natural density by another notion of density.

We'll consider the quotient

$$\sum_{p\in X} \frac{1}{p^s} \Big/ \log\Bigl(\frac{1}{s-1}\Bigr)$$

as a function of s, and study its behaviour as $s \searrow 1$. When $X = \mathcal{P}$, lemma 4.4 and corollary 4.2 imply that

$$\lim_{s\searrow 1} \frac{\sum_{p\in\mathcal{P}} p^{-s}}{\log \frac{1}{s-1}} = 1\,. \qquad (☺)$$

Definition. Let $X \subset \mathcal{P}$ be any set of prime numbers. Define the **Dirichlet density** of X to be the value of the limit

$$\operatorname{dens}(X) = \lim_{s\searrow 1} \frac{\sum_{p\in X} \frac{1}{p^s}}{\log \frac{1}{s-1}},$$

provided that the limit exists.

Proposition 4.7 (Properties of Dirichlet Density)**.**

(i). $0 \leqslant \operatorname{dens}(X) \leqslant 1$, *if it exists.*

(ii). $\operatorname{dens}(\mathcal{P}) = 1$.

(iii). If X is finite or empty then $\operatorname{dens}(X) = 0$.

(iv). If $X_1 \cap X_2 = \emptyset$ *then* $\operatorname{dens}(X_1 \cup X_2) = \operatorname{dens}(X_1) + \operatorname{dens}(X_2)$.

Proof. The second statement was shown before. The first comes from

$$0 \leqslant \sum_{p\in X} \frac{1}{p^s} \leqslant \sum_{p\in\mathcal{P}} \frac{1}{p^s} \qquad \text{for all } s > 1,$$

the third from the fact that $\sum_{p\in X} p^{-s}$ is bounded if X is finite or empty, and the last one follows from

$$\sum_{p\in X_1\cup X_2} \frac{1}{p^s} = \sum_{p\in X_1} \frac{1}{p^s} + \sum_{p\in X_2} \frac{1}{p^s}\,.$$

Each time we may divide by $\log \frac{1}{s-1}$ and then letting $s \searrow 1$. □

How will we use this? As a corollary we can say the following: If $\operatorname{dens}(X) > 0$ then X is infinite. Or even that if $\operatorname{dens}(X) > 0$ then $\sum_{p\in X} \frac{1}{p}$ diverges.
We'll use this to answer the above question: it turns out that the density of the set of primes $p \equiv a \pmod{m}$ is positive when $\gcd(a,m) = 1$, so these sets are infinite.

4.4 A special case of Dirichlet's theorem

Recall the Riemann zeta function $\zeta(s) = \sum_{n\geqslant 1} \frac{1}{n^s}$. We used this to find the density of the set of all primes. It is the simplest example of an "L-function", which is a function given by a series of the form $\sum_{n\geqslant 1} \frac{f(n)}{n^s}$ for some arithmetic function f.
Here we will consider some of these, which are related to the problem of whether there are infinitely many primes in arithmetic progressions, by studying the density of sets of primes.

Notation. For all $a, m \in \mathbb{Z}$ with $m > 0$ and $\gcd(a, m) = 1$ we set

$$\mathcal{P}(a, m) = \{p \in \mathcal{P} \mid p \equiv a \pmod{m}\} = \{\text{primes of the form } a + mk \text{ for some } k \in \mathbb{Z}\}\,.$$

Our aim is to prove that all these sets are infinite by finding their densities. To start with we'll take $m = 4$ and study the densities of

$$\mathcal{P}(1, 4) = \{p \in \mathcal{P} \mid p \equiv 1 \pmod{4}\} \qquad \text{and} \qquad \mathcal{P}(3, 4) = \{p \in \mathcal{P} \mid p \equiv 3 \pmod{4}\}.$$

Definition. Define the arithmetic function $\chi_1 \colon \mathbb{Z} \to \{-1, 0, 1\}$ by

$$\chi_1(n) = \begin{cases} +1 & \text{if } n \equiv 1 \pmod{4}; \\ 0 & \text{if } n \text{ is even}; \\ -1 & \text{if } n \equiv 3 \pmod{4}. \end{cases}$$

In the problem sheet, we will show that $\chi_1(mn) = \chi_1(m)\chi_1(n)$ for all $m, n \in \mathbb{Z}$.

Definition. The function $L(s, \chi_1)$ is defined for $s > 1$ by the series

$$L(s, \chi_1) = \sum_{n\geqslant 1} \frac{\chi_1(n)}{n^s} = \frac{1}{1^s} - \frac{1}{3^s} + \frac{1}{5^s} - \frac{1}{7^s} + - + \cdots$$

Since $|\chi_1(n)n^{-s}| \leqslant |n^{-s}|$, convergence of $L(s, \chi_1)$ for $s > 1$ follows by comparison with the series for $\zeta(s)$. This is our first example of a **Dirichlet L-function**. Note that the coefficients $\chi_1(n)$ depend on n modulo 4; other Dirichlet L-functions will have coefficients defined using other moduli.

Proposition 4.8. *$L(s, \chi_1)$ has an Euler Product expansion*

$$L(s, \chi_1) = \prod_{p\in\mathcal{P}} \frac{1}{1 - \frac{\chi_1(p)}{p^s}}\,.$$

Proof. Similar to proposition 4.3: see problem sheet 4. □

Lemma 4.9. *There is a bound* $\left|\log L(s, \chi_1) - \sum_{p\in\mathcal{P}} \frac{\chi_1(p)}{p^s}\right| < b_1$ *for all* $s > 1$.

Proof. As in lemma 4.4. □

Lemma 4.10. $\log L(s, \chi_1)$ *is bounded as* $s \searrow 1$*; in fact* $L(1, \chi_1) = \frac{\pi}{4}$.

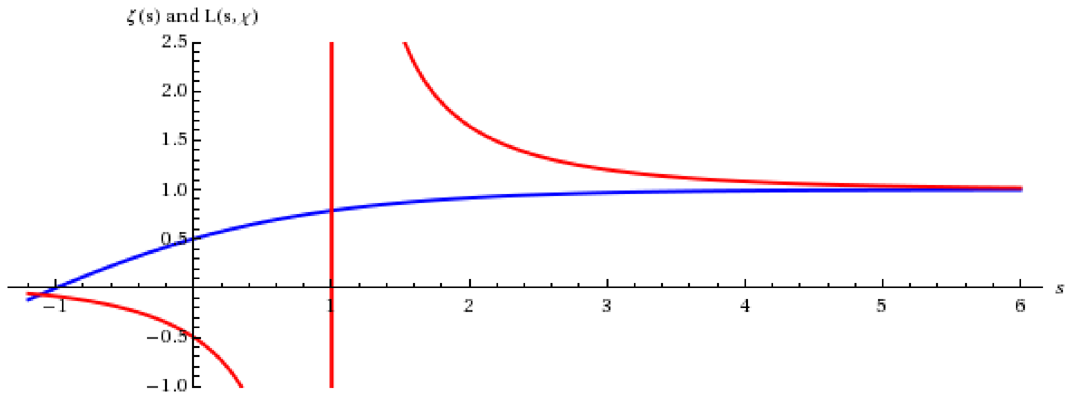

Figure 2: The graphs of $\zeta(s)$, $L(s,\chi_1)$

Proof. On the one hand, for all $s > 1$ we have

$$L(s,\chi_1) = (1-\frac{1}{3^s}) + (\frac{1}{5^s}-\frac{1}{7^s}) + (\frac{1}{9^s}-\frac{1}{11^s}) + \cdots > 1-\frac{1}{3^s} > \frac{2}{3}.$$

On the other hand,

$$L(s,\chi_1) = 1-(\frac{1}{3^s}-\frac{1}{5^s})-(\frac{1}{7^s}-\frac{1}{9^s})-\cdots < 1.$$

So $\frac{2}{3} < L(s,\chi_1) < 1$ for all $s > 1$. For the last part, the formula

$$L(1,\chi_1) = 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots = \frac{\pi}{4}$$

is Leibnitz formula for π. It can be found by putting $x = 1$ in

$$\arctan(x) = x - \frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\frac{x^9}{9}-+\cdots.$$

□

Theorem 4.11. *Both $\mathcal{P}(1,4)$ and $\mathcal{P}(3,4)$ have density $\frac{1}{2}$; in particular, both sets are infinite.*

Proof. By the previous two lemmas,

$$\sum_{p\equiv 1 \pmod 4}\frac{1}{p^s} - \sum_{p\equiv 3 \pmod 4}\frac{1}{p^s} = \sum_{p\in\mathcal{P}}\frac{\chi_1(p)}{p^s}$$

is bounded as $s \searrow 1$. Dividing by $\log(1/(s-1))$ and letting $s\searrow 1$ gives

$$\operatorname{dens}(\mathcal{P}(1,4)) - \operatorname{dens}(\mathcal{P}(3,4)) = 0.$$

But $\operatorname{dens}(\mathcal{P}(1,4)) + \operatorname{dens}(\mathcal{P}(3,4)) = 1$ since $\mathcal{P}(1,4)\cup\mathcal{P}(3,4)$ is the set of all primes except $p = 2$. Hence the result. □

4.5 Dirichlet characters

Definition. Let $m > 1$ be an integer. A **Dirichlet character modulo m** is an arithmetic function $\chi\colon \mathbb{Z}\to\mathbb{C}$ such that

(i). $\chi(a) = 0$ if and only if $\gcd(a,m) > 1$, (ii). $\chi(a+m) = \chi(a)$ for all a, and

(iii). $\chi(ab) = \chi(a) \cdot \chi(b)$ for all a and b.

[G13GTH-people] In other words χ factors as $\mathbb{Z} \to \mathbb{Z}/_{m\mathbb{Z}} \to \mathbb{C}$ such that restricted to $\left(\mathbb{Z}/_{m\mathbb{Z}}\right)^{\times}$ it is a group homomorphism to $\mathbb{C}^{\times}$. We consider the group $\left(\mathbb{Z}/_{m\mathbb{Z}}\right)^{\times}$ of all invertible residue classes modulo m. If $m = p$ is prime then $\left(\mathbb{Z}/_{p\mathbb{Z}}\right)^{\times}$ is cyclic of order $p-1$. The order of $\left(\mathbb{Z}/_{m\mathbb{Z}}\right)^{\times}$ is, by definition, equal to $\varphi(m)$, but the group need not be cyclic as the example $m = 12$ shows : $\left(\mathbb{Z}/_{12\mathbb{Z}}\right)^{\times} = \{\bar{1}, \bar{5}, \bar{7}, \overline{11}\}$; since $1^2 \equiv 7^2 \equiv 5^2 \equiv 11^2 \equiv 1 \pmod{12}$, this group is not cyclic of order 4, but rather of the form $\mathbb{Z}/_{2\mathbb{Z}} \times \mathbb{Z}/_{2\mathbb{Z}}$.

Proposition 4.12 (Properties of Dirichlet characters)**.** *(i). $\chi(1) = 1$.*

(ii). If a is coprime to m, then $\chi(a)$ is a $\varphi(m)^{th}$ root of unity in $\mathbb{C}$.

(iii). If a is coprime to m, then $|\chi(a)| = 1$.

Proof. $\chi(1) = \chi(1 \cdot 1) = \chi(1)^2$, but $\chi(1) \neq 0$. Then by Euler's theorem $\chi(a)^{\varphi(m)} = \chi(a^{\varphi(m)}) = \chi(1) = 1$. So $\chi(a)$ is of the form $e^{2\pi i t/\varphi(m)}$ for some t. □

Examples of Dirichlet characters.

- The **trivial Dirichlet character**. If $m > 1$ is any integer, then the function

$$\chi_{\circ}(a) = \begin{cases} 0 & \text{if } \gcd(a,m) > 1 \text{ and} \\ 1 & \text{if } \gcd(a,m) = 1 \end{cases}$$

  is a Dirichlet charachter.

- For $m = 4$ then χ_1 used in the previous section is a Dirichlet character.

- For any odd prime p, the map $\chi(a) = (\frac{a}{p})$ is a Dirichlet character.

- Let p be a prime and let $0 \leqslant t < p-1$. Choose a primitive root g modulo p. Put $\omega = e^{2\pi i/(p-1)}$. We define a Dirichlet character χ_t as follows. If $p \mid a$, then $\chi_t(a) = 0$. If $p \nmid a$, then $a \equiv g^k \pmod{p}$ for some $0 \leqslant k < p-1$. Define $\chi_t(a) = \omega^{tk}$.

Lemma 4.13. *(i). For $m = 4$ there are only two Dirichlet characters modulo 4. Namely $\chi_{\circ}$ and χ_1.*

(ii). If $m = p$ is prime, then the $p-1$ Dirichlet charachter χ_t for $0 \leqslant t < p-1$ are all Dirichlet characters modulo p.

Proof. A Dirichlet character is determined by its values on the integers $1 \leqslant a < m$ that are prime to m. For $m = 4$, a Dirichlet character χ must satisfy $\chi(1) = 1$ and $\chi(3) \in \{\pm 1\}$.
Let χ be a Dirichlet character modulo a prime $m = p$. Then $\chi(g)$, for a primitive element g, must be a power of ω, say ω^t. But then $\chi(g^k) = \omega^{tk}$ for all u, so $\chi = \chi_t$. □

Lemma 4.14. *Let $m > 1$ be an integer. Then*

$$\sum_{\text{all } \chi} \chi(a) = \begin{cases} \varphi(m) & \text{if } a \equiv 1 \pmod{m} \\ 0 & \text{otherwise,} \end{cases}$$

where the sum runs over all distinct Dirichlet characters modulo m.

Proof only when $m = p$ is prime. First, if a is not coprime to p, then the sum is obviously 0. Then if $a \equiv 1 \pmod{p}$, then we are summing $\varphi(p)$ terms of value $\chi(a) = \chi(1) = 1$.
Suppose now that a is coprime to p and that $a \not\equiv 1 \pmod{p}$. Then there exists $1 \leqslant k < p-1$ such that $a \equiv g^k \pmod{p}$ where g is a primitive root modulo m. Now we have

$$\sum_{\text{all } \chi} \chi(a) = \sum_{t=0}^{p-2} \chi_t(a) = \sum_{t=0}^{p-2} \chi_t(g^k) = \sum_{t=0}^{p-2} \omega^{tk} = \frac{(\omega^k)^{(p-1)} - 1}{\omega^k - 1} = \frac{1-1}{\omega^k - 1} = 0\,. \qquad \square$$

4.6 Dirichlet's L-functions

Let $m > 1$ be an integer. For each character χ modulo m, we form the L-function

$$L(s,\chi) = \sum_{n \geqslant 1} \frac{\chi(n)}{n^s}\,.$$

Since $|\chi(n)\, n^{-s}| \leqslant n^{-s}$, this converges absolutely for $s > 1$. Note that it is now a function with values in $\mathbb{C}$. It still admits an Euler product

$$L(s,\chi) = \prod_{p \in \mathcal{P}} \frac{1}{1 - \frac{\chi(p)}{p^s}}$$

since χ is strictly multiplicative (see problem sheet).
For the trivial charachter $\chi_\circ$, we find the following

$$L(s,\chi_\circ) = \prod_{p \in \mathcal{P}} \frac{1}{1 - \frac{\chi_\circ(p)}{p^s}} = \prod_{p \nmid m} \frac{1}{1 - \frac{1}{p^s}} = \zeta(s) \cdot \prod_{p \mid m} \Bigl(1 - \frac{1}{p^s}\Bigr)$$

So the behaviour of $L(s,\chi_\circ)$ as $s \searrow 1$ is the same as the behaviour of $\zeta(s)$. But for all other charachters the situation is different.

Proposition 4.15. *If χ is a non-trivial Dirichlet character modulo m, then $L(s,\chi)$ stays bounded as $s \searrow 1$ and $L(1,\chi) \neq 0$.*

The proof is omitted as it uses the theory of complex functions.
We saw that for $m = 4$, we have $L(1,\chi_1) = \pi/4$. For $m = 3$ and the unique non-trivial character χ_1, we have $L(1,\chi_1) = \frac{\pi}{3\sqrt{3}}$. For $m = 5$, we find

$$L(1,\chi_1) = \overline{L(1,\chi_3)} = \frac{\pi}{5\sqrt{5}} \cdot \Bigl(\sqrt{5 + 2\sqrt{5}} + i\sqrt{5 - 2\sqrt{5}}\Bigr) \qquad \text{and} \qquad L(1,\chi_2) = \frac{1}{\sqrt{5}} \cdot \log\Bigl(\frac{3+\sqrt{5}}{2}\Bigr)\,.$$

4.7 Dirichlet's theorem on primes in arithmetic progressions

Theorem 4.16 (Dirichlet's Theorem on Primes in Arithmetic Progressions)**.**
If $\gcd(a,m) = 1$ *then*

$$\operatorname{dens}(\mathcal{P}(a,m)) = \frac{1}{\varphi(m)}.$$

In particular, there are infinitely many primes $p \equiv a \pmod{m}$.

Remark. For fixed m, all primes except for the finitely many prime divisors of m lie in exactly *one* of the $\varphi(m)$ disjoint sets $\mathcal{P}(a,m)$ with $1 \leqslant a < m$ and $\gcd(a,m) = 1$. Dirichlet's theorem says that these sets all have the same density.

Very sketchy proof. Let a be an integer prime to m. Let b be an integer such that $ab \equiv 1 \pmod{m}$. For each character χ, we find that

$$\left|\log L(s,\chi) - \sum_{p\in\mathcal{P}} \frac{\chi(p)}{p^s}\right| < b_\chi$$

for some bound b_χ valid for $s > 1$ by a result similar to lemma 4.9. (Here I omit the discussion of what "log" we take.) So on the one hand we have that

$$\left|\sum_{\text{all }\chi} \chi(b)\cdot\log L(s,\chi) - \sum_{\text{all }\chi}\chi(b)\cdot\sum_{p\in\mathcal{P}}\frac{\chi(p)}{p^s}\right| \;<\; \sum_{\text{all }\chi} |\chi(b)|\cdot b_\chi = B$$

On the other hand, note that the expression $\log L(s,\chi)$ is bounded itself by proposition 4.15, except if $\chi = \chi_\circ$. Hence

$$\sum_{\text{all }\chi}\chi(b)\cdot\log L(s,\chi)$$

differs from $\chi_\circ(b)\cdot\log\zeta(s) = \log\zeta(s)$ by a quantity that remains smaller than a certain bound B' as $s \searrow 1$. Now we can compute

$$\begin{aligned}\sum_{\text{all }\chi}\chi(b)\cdot\sum_{p\in\mathcal{P}}\frac{\chi(p)}{p^s} &= \sum_{p\in\mathcal{P}}\sum_{\text{all }\chi}\frac{\chi(b)\cdot\chi(p)}{p^s}\\ &= \sum_{p\in\mathcal{P}}\frac{\sum_\chi \chi(b\cdot p)}{p^s}\end{aligned}$$

Now we use lemma 4.14

$$\begin{aligned}&= \sum_{bp\equiv 1 \pmod{m}} \frac{\varphi(m)}{p^s}\\ &= \varphi(m)\cdot\sum_{p\in\mathcal{P}(a,m)}\frac{1}{p^s}\end{aligned}$$

since $bp \equiv 1 \pmod{m}$ if and only if $p \equiv a \pmod{m}$. Dividing through by $\log\frac{1}{s-1}$ and letting $s \searrow 1$, we obtain

$$1 = \lim_{s\searrow 1}\frac{\log\zeta(s)}{\log\frac{1}{s-1}} = \varphi(m)\cdot\lim_{s\searrow 1}\frac{\sum_{p\in\mathcal{P}(a,m)}\frac{1}{p^s}}{\log\frac{1}{s-1}} = \varphi(m)\cdot\operatorname{dens}(\mathcal{P}(a,m))\,. \qquad \square$$