The sub-leading coefficient of the L-function of an elliptic curve

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Abstract

We show that there is a relation between the leading term at $s = 1$ of an L-function of an elliptic curve defined over an number field and the term that follows.

Let E be an elliptic curve defined over a number field K. We will assume that the L-function $L(E, s)$ admits an analytic continuation to $s = 1$ and that it satisfies the functional equation. By modularity [1], we know that this holds when $K = \mathbb{Q}$. The conjecture of Birch and Swinnerton-Dyer predicts that the behaviour at $s = 1$ is linked to arithmetic information. More precisely, if

$$L(E, s) = a_r\,(s-1)^r + a_{r+1}\,(s-1)^{r+1} + \cdots$$

is the Taylor expansion at $s = 1$ with $a_r \neq 0$, then r should be the rank of the Mordell-Weil group $E(K)$ and the leading term a_r is equal to a precise formula involving the Tate-Shafarevich group of E. It seems to have passed unnoticed that the sub-leading coefficient a_{r+1} is also determined by the following formula.

Theorem 1. *With the above assumption, we have the equality*

$$a_{r+1} = \Bigl([K:\mathbb{Q}]\cdot(\gamma + \log(2\pi)) - \tfrac{1}{2}\log(N) - \log|\Delta_K|\Bigr)\cdot a_r \tag{1}$$

where $\gamma = 0.577216\ldots$ is Euler's constant, N is the absolute norm of the conductor ideal of E/K and Δ_K is the absolute discriminant of $K/\mathbb{Q}$.

In particular, the conjecture of Birch and Swinnerton-Dyer also predicts completely what the sub-leading coefficient a_{r+1} should be. One consequence for $K = \mathbb{Q}$ is that for all curves with conductor $N > 125$, and this is all but 404 isomorphism classes of curves, the sign of a_{r+1} is the opposite of a_r. Of course, it is believed that a_r is positive for all $E/\mathbb{Q}$.

Proof. Set $f(s) = B^s \cdot \Gamma(s)^n$ with $n = [K:\mathbb{Q}]$ and $B = \sqrt{N}\cdot|\Delta_K|/(2\pi)^n$. Then $\Lambda(s) = f(s)\cdot L(E,s)$ is the completed L-function, which satisfies the functional equation $\Lambda(s) = (-1)^r\cdot\Lambda(2-s)$, see [3]. For $i \equiv r+1 \pmod 2$ it follows that $\frac{d^i}{ds^i}\Lambda(s)\big|_{s=1} = 0$. Hence for $i = r+1$, we obtain that

$$(r+1)\cdot f'(s)\cdot\frac{d^r}{ds^r}L(E,s) + f(s)\cdot\frac{d^{r+1}}{ds^{r+1}}L(E,s)$$

is zero at $s = 1$. Therefore $(r+1)\,f'(1)\,r!\,a_r + f(1)\,(r+1)!\,a_{r+1} = 0$. It remains to note that $f(1) = B$ and $f'(1) = B\cdot\bigl(\log(B) + n\cdot\Gamma'(1)\bigr)$ together with $\Gamma'(1) = -\gamma$. □

Obviously a similar formula holds for the L-function of a modular form of weight 2 for $\Gamma_0(N)$. More generally, for any L-function with a functional equation there is a relation between the leading and the sub-leading coefficient of the Taylor expansion of the L-function at the central point.

Sub-leading coefficients of Dirichlet L-functions have been investigated; for instance Colmez [2] makes a conjecture, which is partially known. However these concern the much harder case when s is not at the centre but the boundary of the critical strip of the L-function.

References

[1] Christophe Breuil, Brian Conrad, Fred Diamond, and Richard Taylor, *On the modularity of elliptic curves over* $\mathbf{Q}$*: wild 3-adic exercises*, J. Amer. Math. Soc. **14** (2001), no. 4, 843–939.

[2] Pierre Colmez, *Périodes des variétés abéliennes à multiplication complexe*, Ann. of Math. (2) **138** (1993), no. 3, 625–683.

[3] Dale Husemöller, *Elliptic curves*, second ed., Graduate Texts in Mathematics, vol. 111, Springer-Verlag, New York, 2004, With appendices by Otto Forster, Ruth Lawrence and Stefan Theisen.