Self-points on elliptic curves

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Abstract

Let $E/\mathbb{Q}$ be an elliptic curve of conductor N. We consider trace-compatible towers of modular points in the non-commutative division tower $\mathbb{Q}(E[p^\infty])$. Under weak assumption we can prove that all these points are of infinite order. Furthermore, we use Kolyvagin's construction of derivate classes to find explicit elements in certain Tate-Shafarevich groups.

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1 Introduction

1.1 Definition of self-points

Let $E/\mathbb{Q}$ be an elliptic curve. Write N for its conductor. As proved in [BCDT01], there exists a modular parametrisation

$$\varphi_E \colon X_0(N) \longrightarrow E$$

which is a surjective morphism defined over $\mathbb{Q}$ mapping the cusp ∞ on the modular curve $X_0(N)$ to O. The open subvariety $Y_0(N)$ in $X_0(N)$ is a moduli space for the set of couples (A, C) where A is an elliptic curve and C is a cyclic subgroup in A of order N. More precisely, if $k/\mathbb{Q}$ is a field, then $Y_0(N)(k)$ is in bijection with the set of such couples (A, C) with A and C defined over k, up to isomorphism over the algebraic closure $\bar{k}$.

In particular, we may consider the couple $x_C = (E, C)$ for any given cyclic subgroup C of order N in E as a point in $Y_0(N)(\mathbb{C})$. Its image $P_C = \varphi_E(x_C)$ under the modular parametrisation is called a *self-point* of E. The field of definition of the point P_C on E is the same as the field of definition $\mathbb{Q}(C)$ of C. The compositum of all $\mathbb{Q}(C)$ will be denoted by K_N; it is the smallest field K such that the Galois group $\mathrm{Gal}(\bar{K}/K)$ acts by scalars on $E[N]$.

More generally, for any integer m we define a number field K_m as follows. There is a Galois representation attached to the m-torsion points on E

$$\bar{\rho}_m \colon \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{Aut}(E[m]) \cong \mathrm{GL}_2(\mathbb{Z}/m\mathbb{Z}) \longrightarrow \mathrm{PGL}_2(\mathbb{Z}/m\mathbb{Z}).$$

The field K_m is the field fixed by the kernel of $\bar{\rho}_m$. The Galois group of the extension $K_m/\mathbb{Q}$ can be viewed via $\bar{\rho}_m$ as a subgroup of $\mathrm{PGL}_2(\mathbb{Z}/m\mathbb{Z})$.

We will call *higher self-point* the image under φ_E of any couple (A, C) where A is an elliptic curve which is isogenous to E over $\bar{\mathbb{Q}}$. Though, the most interesting case of higher self-points is the case when the isogeny between E and A is of degree a prime power p^n. In particular this prime p is allowed to divide the conductor N.

This construction imitates the definition of Heegner points, where one uses couples (A, C) with A having complex multiplication. More generally, modular points on elliptic curves were considered earlier by Harris in [Har79] without any restriction on A. This article is a sequel to the previous articles [DW08] and [Wut07] on self-points, where we have emphasised already that the theory of self-points differs from the well-known theory of Heegner points. For instance, there does not seem to be a link between the root numbers and the question of whether the self-points are of infinite order.

We present here not only a generalisation of the previous results on self-points, but also we introduce the construction of derivative classes à la Kolyvagin. Indeed, Kolyvagin [Kol90] was able to find upper bounds on certain Selmer groups by constructing cohomology classes starting from Heegner points. We propose here to do the analogue for self-points. But the situation is radically different as the Galois groups involved are non-commutative and rather than finding upper bounds of Selmer groups over the base field, we will find *lower* bounds on Selmer groups over certain number fields.

1.2 The results for self-points

The main question that arises first is whether we can determine if the self-points are of infinite order in the Mordell-Weil group $E(\mathbb{Q}(C))$. It was shown in [DW08] that the self-points are always of infinite order if the conductor is a prime number. We extend here the method and provide a framework to treat the general case. In theorem 15 we will prove the following.

Theorem 1. *Let $E/\mathbb{Q}$ be a semi-stable elliptic curve of conductor $N \neq 30$ or 210. Then all the self-points are of infinite order*

But the methods are more general and we are able to prove that they are of infinite order in most cases. In fact, we conjecture that this holds whenever E does not admit complex multiplication. In section 6.2 we will give a self-point of finite order on a curve with complex multiplication. In the largest generality, we are able to prove in theorem 5 that there is at least one self-point of infinite order under the assumption that $j(E) \notin \frac{1}{2}\mathbb{Z}$.

Next we address the question of the rank of the group generated by self-points in $E(K_N)$. If N is prime, we saw that the only relation among the self-points is that the sum of all of them is a torsion point in $E(\mathbb{Q})$. For a general conductor, we find that for all proper divisors d of N and all cyclic subgroups B in E of order d, the sum of all self-points P_C with $C \supset B$ is torsion. This is proved in proposition 7 as a consequence of the existence of the degeneracy maps on modular curves. For a lot of semi-stable curves we prove in theorem 17 that these are the only relations among self-points.

Theorem 2. *Let $E/\mathbb{Q}$ be a semi-stable elliptic curve. Suppose that $N \neq 30$ or 210. Suppose that for each prime $p \mid N$ such that $\bar{\rho}_p$ is not surjective, there is a prime $\ell \mid N$ such that the Tamagawa number c_ℓ is not divisible by p. Then the group generated by the self-points is of rank N.*

We conjecture that this holds more generally.

Conjecture 1. *Let $E/\mathbb{Q}$ be an elliptic curve without complex multiplication. Then all the self-points are of infinite order and the only relations among them are produced by the degeneracy maps. In particular, the rank of the group generated by self-points should be equal to*

$$\delta(N) = \prod_{p \mid N} \left\lceil (1 - p^{-2}) \cdot p^{\mathrm{ord}_p(N)} \right\rceil,$$

where $\lceil x \rceil$ denotes the smallest integers larger or equal to x.

The expression $\delta(N)$ in the conjecture is equal to N if and only if N is square-free.

1.3 The results for higher self-points

We are particularly interested in higher self-points that are modular points coming from a couple (E', C') where E' has an isogeny to E of degree a power of a prime p. There are two cases that we treat: when p is a prime of good reduction and when p is a prime of multiplicative reduction.

For simplicity we only sketch the results for the good case here. See section 7 for more details.

Let D be a cyclic subgroup of E of order p^{n+1} and let $E' = E/D$. Given any self-point P_C, we may consider the image C' of C under the isogeny $E \longrightarrow E'$. The higher self-point Q_D is defined to be the image of $(E', C') \in Y_0(N)$ under the modular parametrisation φ_E. It is a point in the Mordell-Weil group of E over the field $\mathbb{Q}(C, D)$, which is contained in $K_{p^{n+1}N}$. In corollary 23, we are able to prove that the higher self-points are all of infinite order in some cases.

Theorem 3. *Let $E/\mathbb{Q}$ be a semi-stable curve of conductor $N \neq 30$, or 210. Suppose that p is a prime such that $p > N$, and such that $\bar{\rho}_p\colon \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{PGL}_2(\mathbb{F}_p)$ is surjective. Let s be the rank of the group generated by self-points in $E(K_N)$. Then the higher self-points in $E(K_{p^{n+1}N})$ generate a group of rank at least $s \cdot (p+1) \cdot p^n$.*

If one assumes that the prime is of ordinary reduction for E, one can weaken the condition on the bad reduction substantially.

Furthermore these higher self-points are trace-compatible in the following sense. Let D be a cyclic subgroup of order p^{n+1} and let a_p be the p^{th} Fourier coefficient of the modular form associated to the isogeny class of E. Then we have

$$\sum_{D' \supset D} Q_{D'} = a_p \cdot Q_D$$

where the sum runs over all cyclic subgroup D' of order p^{n+2} containing D. If the Galois representation $\rho_{K_N,p}\colon \mathrm{Gal}(\bar{K}_N/K_N) \longrightarrow \mathrm{PGL}_2(\mathbb{Z}_p)$ is surjective then we can reformulate this equation, by saying that the trace of $Q_{D'}$ from its field of definition to the field of definition of Q_D is equal to $a_p \cdot Q_D$. This trace compatibility reminds of the definition of an Euler system; but the field $\mathbb{Q}(C, D)$ is not even Galois and F_n/F is not an abelian extension.

The higher self-points are the only known towers of points of infinite order in the division tower $\mathbb{Q}(E[p^\infty])$ of E. But the growth of the rank of the Mordell-Weil group should often be faster than the lower bound $(p+1)p^n$ that we establish here in many cases. This is due to changing signs in the functional equations and the corresponding parity results on the corank of Selmer groups. See [CFKS06] and [MR07]. These results predict, under the assumption of the finiteness of the Tate-Shafarevich group, that there should be more points of infinite order in the division tower not encountered for by higher self-points. Furthermore the higher self-points do not seem to be linked in an obvious way to root numbers. Also it is completely unknown if there is a relation to L-functions (or to non-commutative p-adic L-functions as in [CFK+05]) in analogy to the Gross-Zagier formula for Heegner points.

1.4 Derivatives

In [Kol90], Kolyvagin has used Heegner points of infinite order to construct cohomology classes that obstruct the existence of further points of infinite order. We aim to use a similar construction to build cohomology classes from higher self-points of infinite order.

Let p be a prime of either good ordinary reduction or of multiplicative reduction. If p does not divide the conductor N, define $F_n = K_{p^{n+1}N}$, otherwise let $F_n = K_{p^n N}$. Put $F = F_{-1}$. If we suppose that

$$\rho_{F,p}\colon \mathrm{Gal}(\bar{F}/F) \longrightarrow \mathrm{PGL}_2(\mathbb{Z}_p)$$

is surjective, then $\mathrm{Gal}(F_n/F) = \mathrm{PGL}_2(\mathbb{Z}/_{p^{n+1}\mathbb{Z}})$. We are interested in a particular cyclic subgroup A in $\mathrm{Gal}(F_n/F)$. Choosing a $\mathbb{Z}_p$-basis of the quadratic unramified extension $\mathcal{O}$ of $\mathbb{Z}_p$ gives a map

$$\mathcal{O}^\times \longrightarrow \mathrm{GL}_2(\mathbb{Z}_p) \longrightarrow \mathrm{PGL}_2(\mathbb{Z}_p) \longrightarrow \mathrm{PGL}_2(\mathbb{Z}/_{p^{n+1}\mathbb{Z}}),$$

whose image is a cyclic group A_n of order $(p+1)\cdot p^n$. By a slight abuse of notation we will denote the subfield of F_n fixed by A_n by F_n^A.

The construction of derivatives provides us with a map

$$\partial_n \colon \mathrm{H}^1(A_n, S) \longrightarrow Ш(E/F_n^A).$$

The source is a cohomology group of the saturated higher self-points (see section 8 for the definitions). Although we do not know its exact structure, we can prove that it contains at least p^n elements. It seems plausible to think that the map ∂_n is very often injective, but we do have no means to prove this in a single case. Nevertheless, we are able to show the existence of points of infinite order in $E(F_n^A)$ whenever the map is not injective. Here the final result in theorem 24.

Theorem 4. *Let $E/\mathbb{Q}$ be an elliptic curve. Suppose that E does not have potentially good supersingular reduction for any prime of additive reduction. Let p be a prime of either good ordinary or multiplicative reduction. Assume that $\rho_{F,p}$ is surjective and that K_N contains a self-point of infinite order. Then we have*

$$\#\,\mathrm{Sel}_{p^n}(E/F_n^A) \geqslant p^n.$$

The construction of derivatives relies on a property of modular representation theory. The higher self-points generate in the Mordell-Weil group a copy of the irreducible Steinberg representation. More precisely, if H_n denotes $\mathrm{Gal}(F_n/F)$, there is a certain $\mathbb{Q}[H_n]$-module in $E(F_n)\otimes\mathbb{Q}$ which is irreducible. But this is no longer irreducible over $\mathbb{F}_\ell[H_n]$ when ℓ divides $(p+1)\cdot p^n$. The idea of using modular representation theory to study Selmer groups is developed by Greenberg in [Gre08] and could maybe shed new light on these derivatives.

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2 The fundamental theorem

In this section we prove the following theorem.

Theorem 5. *Let $E/\mathbb{Q}$ be an elliptic curve of conductor N. Suppose that the j-invariant of E is not in $\frac{1}{2}\mathbb{Z}$, then there is at least one self-point P_C of infinite order in $E(K_N)$.*

Proof. Let p be a prime which divides the denominator of the j-invariant of E. If possible, we avoid $p=2$. Note that p^2 may divide N, but we know that E acquires multiplicative reduction over some extension of $\mathbb{Q}$ at p.

First we fix an embedding of $\bar{\mathbb{Q}}$ into $\bar{\mathbb{Q}}_p$. We consider the modular parametrisation over $\bar{\mathbb{Z}}_p$. The modular curve $X_0(N)$ over $\bar{\mathbb{Z}}_p$ has a neighbourhood of the cusp ∞ consisting of couples (A,C) of a Tate curve of the form $A = \bar{\mathbb{Q}}_p^\times/q^{\mathbb{Z}}$ together with a cyclic subgroup C of order N generated by the N^{th} root of unity. The parameter q is a p-adic analytic uniformiser at ∞, so that the $\mathrm{Spf}\,\bar{\mathbb{Z}}_p[\![q]\!]$ is the formal completion of $X_0(N)/\bar{\mathbb{Z}}_p$ at the cusp ∞, see chapter 8 of [KM85].

Let $f_E = \sum a_n q^n$ be the normalised newform associated to E and so $f_E/q \cdot dq$ is the associated differential. Let c_E be the Manin constant (of the not necessarily strong Weil curve E), which by definition is the number such that $\varphi_E^*(\omega_E) = c_E \cdot f_E/q \cdot dq$ where ω_E is the invariant differential on E. The rigid analytic map induced by φ_E on the completion can now be characterised as

$$\log_E(\varphi_E(q)) \;=\; \int_O^{\varphi_E(q)} \omega_E \;=\; c_E \cdot \int_0^q f_E \, \frac{dq}{q} \;=\; c_E \cdot \sum_{n\geqslant 1} \frac{a_n}{n} \cdot q^n \,. \tag{1}$$

Here $\log_E$ denotes the formal logarithm associated to E from the formal group $\hat{E}(\bar{\mathfrak{m}})$ to the maximal ideal $\hat{\mathbb{G}}_a(\bar{\mathfrak{m}}) = \bar{\mathfrak{m}}$ of $\bar{\mathbb{Z}}_p$. We deduce from this description the following lemma that will be useful later. Write $|\cdot|_p$ for the normalised absolute value such that $|p|_p = p^{-1}$.

Lemma 6. *Let (A, C) be a point in $Y_0(N)(\bar{\mathbb{Q}}_p)$ such that A is isomorphic to the Tate curve with parameter $q_0 \neq 0$ and C is isomorphic to $\mu[N]$. If $|q_0|_p < p^{-\frac{1}{p-1}}$, then $\varphi_E(A, C)$ is a point of infinite order on $E(\bar{\mathbb{Q}}_p)$.*

Proof. Under the condition on the absolute value of q_0, we know that the sum on the right hand side of (1) converges. We consider the sum

$$z = c_E \cdot \sum_{n\geqslant 1} \frac{a_n}{n} \cdot q_0^n \,.$$

Since the Manin constant is known to be an integer (see [Edi90]), the absolute value of the right hand side is

$$|z|_p = |c_E|_p \cdot \left| q_0 + \frac{a_p}{p} q_0^p \right|_p$$

as these are the terms of large absolute value. But note that the condition on q_0 implies that the second term on the right hand side is actually slightly smaller that the first, and hence the absolute value of the sum is bounded by $|z|_p = |c_E|_p \cdot |q_0|_p < p^{-\frac{1}{p-1}}$. Therefore the value of z lies in the domain of convergence of the p-adic elliptic exponential $\exp_E$ and we obtain that $\varphi_E(A, C) = \exp_E(z)$. Since we know that $|z|_p \neq 0$, we can deduce that $\exp_E(z)$ is not a torsion point in $E(\bar{\mathbb{Q}}_p)$. ∎

We now proceed to the proof of the theorem. Since E has multiplicative reduction over $\bar{\mathbb{Z}}_p$, there is exactly one of the $x_C = (E, C)$ in the neighbourhood of ∞ on $X_0(N)$ represented by the p-adic Tate parameter q_E associated to E together with the group C isomorphic to $\mu[N]$. If $p \neq 2$, then we know that

$$|q_E|_p = |j(E)|_p^{-1} \leqslant p^{-1} < p^{-\frac{1}{p-1}}$$

and if p had to chosen to be equal to 2 in the beginning then we know that

$$|q_E|_2 = |j(E)|_2^{-1} \leqslant p^{-2} < p^{-\frac{1}{p-1}} \,.$$

Hence in any case, the lemma applies and provides us with a point of infinite order among the self-points. ∎

Note that if the chosen prime p is such that p^2 does not divide N then q_E lies in $p^v\mathbb{Z}_p$, where $v = -\operatorname{ord}_p(j(E))$. Hence the point P_C in the proof will be defined over $\mathbb{Q}_p$.

The restriction at $p = 2$ seems unnecessary. Often one can deduce the result of the theorem by hand for curves whose j-invariant is an odd integer divided by 2. We present here an easy example. For the curve 2450o1 in Cremona's tables [Cre97] with j-invariant $-\frac{189}{2}$, the 2-adic Tate parameter is equal to $2 + 2^2 + 2^4 + \mathbf{O}(2^9)$ and the newform is $f_E = q - q^2 + q^4 + \mathbf{O}(q^8)$. From this one concludes that $\log_E(P_C) = 2^3 + \mathbf{O}(2^5)$. So P_C is of infinite order. Nevertheless we do not see any easy argument to prove that $P_C \neq O$ for a general curve with $j(E) \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$ as it seems that the 2-adic valuation of $\log_E(P_C)$ can be arbitrary large.

2.1 A torsion self-point

This theorem could still be valid if E is a curve with integral j, though not all self-points are of infinite order. We present here a surprisingly easy example of a self-point that is torsion.

The curve 27a2 admits a cyclic isogeny of degree 27 defined over $\mathbb{Q}$ to the curve 27a4 . Let E be any of the two curves. So E has exactly one cyclic subgroup of order 27 defined over $\mathbb{Q}$, i.e. the curve E admits a self-point in $E(\mathbb{Q})$. Since the rank of $E(\mathbb{Q})$ is zero, the self-point has to be of finite order. Note that these curves have complex multiplication. See section 6.2 for more detailed computations on these self-points.

3 Relations

In [DW08] it is shown that the self-points on a curve of prime conductor satisfy exactly one relation. What kind of relations could occur among the self-points for a curve of conductor N? Here is a first part of an answer. But first, we need some more notations. The Galois group $G = G_N = \mathrm{Gal}(K_N/\mathbb{Q})$ was identified with a subgroup of $\mathrm{PGL}_2(\mathbb{Z}/N\mathbb{Z})$. For any divisor d of N, we define the image of G_N under the projection $\mathrm{PGL}_2(\mathbb{Z}/N\mathbb{Z}) \longrightarrow \mathrm{PGL}_2(\mathbb{Z}/d\mathbb{Z})$ as G_d and K_d its fixed field in K_N. In other words K_d is the smallest number field for which the absolute Galois groups acts by scalars on $E[d]$.

Proposition 7. *The sum of all self-points is a torsion point defined over $\mathbb{Q}$. Let $d \neq N$ be a integer dividing N, then there are relations of the form*

$$R_B\colon \qquad \sum_{C \supset B} P_C \text{ is torsion in } E(K_d),$$

where B is any given cyclic subgroup of order d and C runs through all cyclic groups of order N containing B.

Proof. There is a map from $\pi\colon X_0(N) \longrightarrow X_0(d)$ inducing a map $\pi^*\colon J_0(d) \longrightarrow J_0(N)$ on Jacobians. Given a cyclic subgroup subgroup B of order d on E, we may consider the point $x_B = (E, B)$ on $X_0(d)$. The divisor class

$$\pi^*\big[(x_B) - (\infty)\big] = \sum_{C \supset B} \big[(x_C)\big] - \pi^*\big[(\infty)\big]$$

is in the image of π^* in $J_0(N)$ and hence in the kernel of the map $\varphi_E\colon J_0(N) \longrightarrow E$ because N is the exact conductor of E. This gives the relation R_B.

Taking $d = 1$ gives the result that the sum of all self-points is a torsion point. Since this sum is fixed by the Galois group, it has to be a rational point. □

4 The Steinberg representations

The aim is to describe certain irreducible representations that will appear in the study of self-points. Let $N > 1$ be an integer. We are interested in the group $P = \mathrm{PGL}_2(\mathbb{Z}/N\mathbb{Z})$. We will decompose the $\mathbb{Q}[P]$-module V whose basis $\{e_C\}$ as a $\mathbb{Q}$-vector space is in bijection with the projective line $\mathbb{P}^1(\mathbb{Z}/_{N\mathbb{Z}})$ and the action of P is given by the usual permutation on the basis. So it can be written as

$$V = \bigoplus_{C\in\mathbb{P}^1(\mathbb{Z}/_{N\mathbb{Z}})} \mathbb{Q}\, e_C = \mathrm{Ind}_B^P(\mathbb{1}_B)$$

where B is a Borel subgroup of P and $\mathbb{1}_B$ is its trivial representation.

Theorem 8. *The $\mathbb{Q}[\mathrm{PGL}_2(\mathbb{Z}/_{N\mathbb{Z}})]$-module V splits into the sum*

$$V = \bigoplus_{D\mid N} W_D$$

of irreducible $\mathbb{Q}[\mathrm{PGL}_2(\mathbb{Z}/_{N\mathbb{Z}})]$-modules W_D where D runs through all divisors of N. Let $D = \prod_p p^{d_p}$ be the prime decomposition of a divisor D of N. Define

$$\delta_p = p^{d_p} - \lceil p^{d_p-2} \rceil = \lceil p^{d_p} - p^{d_p-2} \rceil \begin{cases} 1 & \text{if } d_p = 0, \\ p & \text{if } d_p = 1 \text{ and} \\ p^{d_p} - p^{d_p-2} & \text{if } d_p > 1. \end{cases}$$

Then $\mathbb{Q}[\mathrm{PGL}_2(\mathbb{Z}/_{N\mathbb{Z}})]$-module W_D has dimension $\delta(D) = \prod_{p\mid D} \delta_p$ as a $\mathbb{Q}$-vector space.

Proof. We split the proof into three parts according to whether N is a prime, a prime power or any integer. The first two cases could also be treated by invoking theorem 3.3 in [Sil70] on page 58, but, since we need the explicit description of W_D later on, we prefer to prove this theorem in details. Since the proof is inductive on N, we will now write P_N for $\mathrm{PGL}_2(\mathbb{Z}/_{N\mathbb{Z}})$ and V_N for its V.

Case when N is prime: Write $p = N$. The claim is simply that the $\mathbb{Q}[P]$-module V_p splits into two irreducible components $W_1 \oplus W_p$. We define W_1 to be the 1-dimensional subspace of V generated by the vector $v_1 = \sum_C e_C$ where the sum runs over all C in $\mathbb{P}^1(\mathbb{F}_p)$. Of course, $W_1 = V_p^P$ is an irreducible $\mathbb{Q}[P]$-submodule of V_p and the space

$$W_p = \left\{ \sum a_C \cdot e_C \;\middle|\; \sum a_C = 0 \right\}$$

is a complement to it. It remains to show that W_p is irreducible. Let g be an element of order p in P, such as the class of $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$. On $V_p \otimes \mathbb{C}$ the element g acts with eigenvalues $\{1, 1, \zeta, \zeta^2, \dots \zeta^{p-1}\}$ where ζ is a primitive p^{th} root of unity. Hence on W_p every p^{th} root of unity appears exactly once as an eigenvalue. So the only possibility for W_p to split up in two $\mathbb{Q}[P]$-submodules would have to involve a 1-dimensional and a $(p-1)$-dimensional submodule.

As we can see from the fact that $\mathrm{PSL}_2(\mathbb{F}_p)$ is a simple group when $p > 3$ and by direct calculations for $p = 2$ and 3, there are only two one-dimensional representations of $\mathrm{PGL}_2(\mathbb{F}_p)$: the trivial representation and the one with kernel $\mathrm{PSL}_2(\mathbb{F}_p)$ of index 2. Since $\mathrm{PSL}_2(\mathbb{F}_p)$ acts transitively on $\mathbb{P}^1(\mathbb{F}_p)$, the one-dimensional subrepresentations of V_p must be contained in $V_p^{\mathrm{PSL}_2(\mathbb{F}_p)} = W_1$.

Case when N is a prime power: We write $N = p^k$ with p being prime. We will prove the statement by induction on k. The case $k = 1$ has been treated already; thus we may assume that $k \geqslant 2$. The claim is that V_{p^k} splits as $\oplus W_{p^m}$ where m runs from 0 to k.

There is a reduction map $\alpha\colon \mathbb{P}^1(\mathbb{Z}/_{p^k\mathbb{Z}}) \longrightarrow \mathbb{P}^1(\mathbb{Z}/_{p^{k-1}\mathbb{Z}})$ which is surjective and any fibre contains p elements. Define

$$V' = \left\{ \sum a_C\, e_C \;\middle|\; a_C = a_{C'} \text{ whenever } \alpha(C) = \alpha(C') \right\}.$$

It is easy to see that V' is isomorphic as a vector space to $V_{p^{k-1}}$ and the action of P_{p^k} factors through the quotient $P_{p^k} \twoheadrightarrow P_{p^{k-1}}$ induced by reduction. By induction V' splits as a $\mathbb{Q}[P_{p^{k-1}}]$-module into the sum

$$V' = \bigoplus_{m=0}^{k-1} W_{p^m}$$

and this is also a decomposition of V' into irreducible $\mathbb{Q}[P_{p^k}]$-modules. As a complement to V', we define

$$W_{p^k} = \Big\{ \sum a_C\, e_C \;\Big|\; \sum_{\alpha(C)=D} a_C = 0 \text{ for all } D \text{ in } \mathbb{P}^1(\mathbb{Z}/_{p^{k-1}\mathbb{Z}}) \Big\}\,.$$

It is clear that W_{p^k} is a $\mathbb{Q}[P_{p^k}]$-submodule of V_{p^k}. If $k > 1$ then its dimension is equal to

$$\dim_{\mathbb{Q}} W_{p^k} = \#\mathbb{P}^1(\mathbb{Z}/_{p^k\mathbb{Z}}) - \#\mathbb{P}^1(\mathbb{Z}/_{p^{k-1}\mathbb{Z}}) = (p+1)\cdot p^{k-1} - (p+1)\cdot p^{k-2} = p^k - p^{k-2}$$

It remains to show that W_{p^k} is irreducible.

Let ∞ be any point in $\mathbb{P}^1(\mathbb{F}_p)$ and write U^∞ for the preimage of ∞ under the reduction map $\mathbb{P}^1(\mathbb{Z}/_{p^k\mathbb{Z}}) \twoheadrightarrow \mathbb{P}^1(\mathbb{F}_p)$. Within V, we define a linear subspace

$$V^\infty = \Big\{ \sum a_C\, e_C \;\Big|\; a_C = 0 \text{ if } C \in U^\infty \Big\}$$

of dimension p^k and let $W^\infty = W_{p^k} \cap V^\infty$ and $V'^\infty = V' \cap V^\infty$. Let g be an element of P_{p^k} of order p^k whose fixed points lie in U^∞. If ∞ is $(0:1)$, then we may take the class of the matrix $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$. The element g acts on $V^\infty \otimes \mathbb{C}$ such that every p^k-th root of unity appears exactly once. The eigenvalues of g on the subspace V'^∞ are all p^{k-1}-st roots of unity. Hence on W^∞ every primitive p^k-th root of unity appears exactly once as an eigenvalue. So W^∞ is an irreducible $\mathbb{Q}[\langle g\rangle]$-module and so, if W_{p^k} splits as a $\mathbb{Q}[P_{p^k}]$-module then W^∞ has to be completely contained in one of the summands. But for any two distinct points ∞ and ∞' in $\mathbb{P}^1(\mathbb{F}_p)$ the spaces W^∞ and $W^{\infty'}$ span the whole of W_{p^k}. Hence W_{p^k} can not be reducible.

General case: The general case follows fairly easily from the previous cases. Let $N = \prod p^{n_p}$ be the prime decomposition of N. We may suppose that N is not a prime power as we have treated this case already. Now the group P_N splits as

$$P_N = \mathrm{PGL}_2(\mathbb{Z}/_{N\mathbb{Z}}) = \prod_{p|N} \mathrm{PGL}_2(\mathbb{Z}/_{p^{n_p}\mathbb{Z}}) = \prod_{p|N} P_{p^{n_p}}$$

by the Chinese remainder theorem. Similarly, we have

$$\mathbb{P}^1(\mathbb{Z}/_{N\mathbb{Z}}) = \prod_{p|N} \mathbb{P}^1(\mathbb{Z}/_{p^{n_p}\mathbb{Z}}) \quad \text{and so} \quad V_N = \bigotimes_{p|N} V_{p^{n_p}}$$

as a $\mathbb{Q}[P_N]$-module. Now we use the previous case to rewrite

$$V_N = \bigotimes_{p|N} \bigoplus_{m=0}^{n_p} W_{p^m}\,.$$

Let D be any divisor of N and $\prod p^{d_p}$ its prime factorisation, then define

$$W_D = \bigotimes_{p|D} W_{p^{d_p}}\,.$$

It is clear from the representation theory of direct products that W_D is irreducible. Rearranging the above decomposition of V_N we find the desired expression $V_N = \oplus_{D|N} W_D$. □

Proposition 9. *Let p be a prime. Let G be a subgroup of a Borel subgroup of $\mathrm{PGL}_2(\mathbb{F}_p)$ acting on $V = \bigoplus \mathbb{Q}e_C$. Suppose that the class of $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ belongs to G. Then V decomposes into irreducible $\mathbb{Q}[G]$-modules as $W_1 \oplus W_1' \oplus W_p'$ where W_p' is an irreducible $\mathbb{Q}[G]$-module of dimension $p-1$.*

Proof. Let C_0 be the element of $\mathbb{P}^1(\mathbb{F}_p)$ which is fixed by the Borel group containing G. By our assumption, we know that C_0 is the only fixed point of G acting on $\mathbb{P}^1(\mathbb{F}_p)$. Hence V contains two linearly independent vectors that are fixed by G, namely e_{C_0} and $v_0 = \sum_{C\neq C_0} e_C$. The $\mathbb{Q}[G]$-submodule

$$W_p' = \left\{ \sum_{C\neq C_0} a_C \cdot e_C \,\middle|\, \sum_{C\neq C_0} a_C = 0 \right\}$$

is a complement to V^G. Now use the class g of the matrix $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ as before to show that W_p' is irreducible as the eigenvalues of g on W_p' are exactly the set of all primitive p-th roots of unity. □

In fact one can show that the theorem 8 holds even as $\mathbb{C}[\mathrm{PGL}_2(\mathbb{Z}/N\mathbb{Z})]$-modules. On the other hand the previous proposition really relies on the fact that we are only considering decompositions as $\mathbb{Q}[G]$-modules. For instance we may well take G to be the cyclic group generated by the matrix $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$; then of course $W_p' \otimes \mathbb{C}$ will split into 1-dimensional representations. But since the p-th roots of unity are not all defined over $\mathbb{Q}$, at least if $p > 2$, this decomposition does not hold in general for W_p'.

We can now reformulate the statement of proposition 7 as follows. There is a G-equivariant map $\iota\colon V_N \longrightarrow E(K_N) \otimes \mathbb{Q}$, defined by sending e_C to P_C. It has a kernel containing all submodules W_d for $d \neq N$ dividing N. So it induces a map

$$\iota\colon W_N \longrightarrow E(K_N) \otimes \mathbb{Q}$$

which is G-equivariant. By the fundamental theorem 5, this morphism is non-trivial if $j \notin \frac{1}{2}\mathbb{Z}$. Hence we can deduce the following corollary.

Corollary 10. *The self-points generate a group of rank at most $\delta(N)$ inside $E(K_N)$. If W_N is an irreducible $\mathbb{Q}[G_N]$-module and the j-invariant is not in $\frac{1}{2}\mathbb{Z}$, then the self-points generate a group of rank $\delta(N)$ and the Galois group acts like the Steinberg representation W_N on it.*

5 Self-points on semi-stable curves

We will suppose in this section that the curve $E/\mathbb{Q}$ is semi-stable. In particular, the j-invariant can not belong to $\frac{1}{2}\mathbb{Z}$ as all primes dividing N must appear in the denominator of $j(E)$ and there is no curve of conductor 2. Hence the fundamental theorem 5 applies to E.

5.1 Some lemmae

In what follows we often have to split up the primes dividing N into two groups. Let s, standing for "surjective", be the product of all primes p dividing N such that the representation $\bar{\rho}_p$ is surjective. Let m, standing for "méchant", be the product of the remaining primes dividing N. Note that there are not many choices for m as described in the following lemma.

Lemma 11. *We have $m \in \{1, 2, 3, 4, 5, 6, 7, 10\}$. If $p \mid m$, then G_p is contained in a Borel group of $\mathrm{PGL}_2(\mathbb{F}_p)$ and hence is either a cyclic or a meta-cyclic[1] group.*

[1] metacyclic : a semi-direct product of cyclic groups

Proof. Let $p \mid m$. By a theorem of Serre in [Ser96], the curve admits a p-isogeny $E \longrightarrow E'$ and either E or E' must have a point of order p defined over $\mathbb{Q}$. Then by Mazur's theorem on torsion points on elliptic curves over $\mathbb{Q}$ in [Maz78], we know now that $p \leqslant 7$ and that $m \leqslant 10$. □

Lemma 12. *Let $E/\mathbb{Q}$ be a semi-stable elliptic curve. Then the largest prime p dividing N is such that the representation $\bar{\rho}_p$ is surjective. Unless N is 30 or 210, we have $p-1 > m$.*

Proof. If N is divisible by a prime $p \geqslant 13$, then the largest prime p divising N cannot divide m and satisfies $p-1 > m$ because $m \leqslant 10$ by the previous lemma. Hence we are left with a finite list of possible N to check. This can be done easily; to illustrate it we show in the table 1 the list of curves of square-free conductors N whose prime divisors are among $\{2,3,5,7\}$. For the full proof, we would need to list also conductors divisible by 11, but then the list will be far too long to be included here. But the only three exceptional isogeny classes can already be seen in the this table.

To each isogeny class, we give the number i of isogenous curves, the maximal degree d of an isogeny among them, the value of m, and the largest $p \mid N$ such that $\bar{\rho}_p$ is surjective. This ends the proof. □

N	14a	15a	21a	**30a**	35a	42a	70a	105a	**210a**	**210b**	210c	210d	210e
i	6	8	6	**8**	3	6	4	4	**8**	**8**	6	4	8
d	18	16	8	**12**	9	8	4	4	**12**	**12**	8	4	16
m	2	1	1	**6**	1	2	2	1	**6**	**6**	2	2	2
p	7	5	7	**5**	7	7	7	7	**7**	**7**	7	7	7

Table 1: Some of the evil curves to be treated separately in lemma 12

Lemma 13. *Let $E/\mathbb{Q}$ be a semistable elliptic curve with $6 \mid N$ and such that the representation $\bar{\rho}_2$ is surjective onto $\mathrm{PGL}_2(\mathbb{F}_2)$. If there exists a prime $p \mid N$ such that $3 \nmid c_p$, then K_2 can not be contained in K_3.*

Proof. We wish to derive a contradiction from the assumption that K_2 is contained in K_3. By assumption, the Galois group $G_2 = \mathrm{Gal}(K_2/\mathbb{Q})$ is $\mathrm{PGL}_2(\mathbb{F}_2)$, which is isomorphic to the symmetric group on three letters $\mathfrak{S}_3$. The Galois group G_3 is contained in $\mathrm{PGL}_3(\mathbb{F}_3) = \mathfrak{S}_4$. Therefore the Galois group $\mathrm{Gal}(K_3/K_2)$ is contained in the Klein group V_4 of $\mathfrak{S}_4$.

Suppose first that the reduction of E at p is split multiplicative. Let q_E be the Tate parameter of E over $\mathbb{Q}_p$. Choose a place υ above p in K_2 and a place w above υ in K_3. Then the completion $K_{3,w}$ is equal to $\mathbb{Q}_p(\zeta_3, \sqrt[3]{q_E})$ and $K_{2,\upsilon}$ is equal to $\mathbb{Q}_p(\sqrt{q_E})$. Since 3 does not divide $c_p \geqslant 1$, we know that q_E can not be a cube. Therefore the degree of $K_{3,w}/K_{2,\upsilon}$ is divisible by 3. But this is impossible as the degree of K_3/K_2 must be a power of 2.

If the reduction is non-split multiplicative at p, then one can do the same argument but transposed to the extension L of $\mathbb{Q}_p$ over which E acquires split multiplicative reduction. As $L/\mathbb{Q}_p$ is of degree 2, we still find that the degree of $K_{3,w}/K_{2,\upsilon}$ must be a multiple of 3. □

Lemma 14. *Let $E/\mathbb{Q}$ be a semi-stable elliptic curve. For the second and third point below, we assume that, if $2 \mid N$ and $3 \mid N$ then there is a prime $p \mid N$ such that $3 \nmid c_p$.*

i). Then G_s acts transitively on the set $\mathbb{P}^1(\mathbb{Z}/_{s\mathbb{Z}})$ of cyclic subgroup of order s in E.

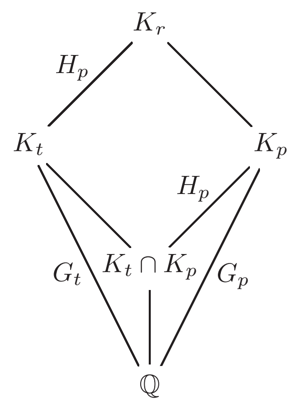
ii). The Steinberg representation W_s is irreducible as a $\mathbb{Q}[G_s]$-module.

iii). Let $U_1 \oplus \cdots \oplus U_k$ be the decomposition of W_m into irreducible $\mathbb{Q}[G_m]$-modules then we have the decomposition of W_N into irreducible $\mathbb{Q}[G_N]$-modules as follows

$$W_n = \bigoplus_{i=1}^{k} \big(U_i \otimes W_s\big)\,.$$

Proof. We will first prove by induction the statement in ii) with s replaced by any of its divisors r, assuming the additional hypothesis. If $r = p$ is prime then $G_p = \mathrm{PGL}_2(\mathbb{F}_p)$ and theorem 8 shows that W_p is irreducible as a $\mathbb{Q}[G_p]$-module. Let p be the largest prime factor of r. We may suppose that r is composite and so $p > 2$. Put $t = \frac{r}{p} \geqslant 2$. We assume that W_t is an irreducible $\mathbb{Q}[G_t]$-module. We wish to prove that W_r is an irreducible $\mathbb{Q}[G_r]$-module.

The Galois group $H_p = \mathrm{Gal}(K_r/K_t)$ is isomorphic to the Galois group of the extension $K_p/K_t \cap K_p$. Hence H_p is a normal subgroup of $G_p = \mathrm{PGL}_2(\mathbb{F}_p)$. We use the fact that $\mathrm{PSL}_2(\mathbb{F}_p)$ is simple for $p > 3$. So H_p is either all of G_p, $\mathrm{PSL}_2(\mathbb{F}_p)$, the trivial group or, in the case $p = 3$, the Klein group V_4 in $\mathrm{PGL}_2(\mathbb{F}_3) = \mathfrak{S}_4$. Treating the four cases separately, we will prove that W_p is an irreducible $\mathbb{Q}[H_p]$-module.



First, if H_p is all of G_p then W_p is irreducible as a $\mathbb{Q}[H_p]$-module by theorem 8. If H_p is equal to $\mathrm{PSL}_2(\mathbb{F}_p)$, then W_p could split at most into two subspace of equal dimension as $\mathrm{PSL}_2(\mathbb{F}_p)$ has index 2 in $\mathrm{PGL}_2(\mathbb{F}_p)$. But the dimension of W_p is odd, unless $p = 2$ which we excluded. Hence W_p is irreducible.

Next, we will exclude the case when H_p is trivial. If it were so, then there is a surjective map from G_t onto $G_p = \mathrm{PGL}_2(\mathbb{F}_p)$. The group G_t is contained in $\mathrm{PGL}_2(\mathbb{Z}/t\mathbb{Z})$ whose order is

$$\prod_{\ell \mid t} \ell \cdot (\ell + 1) \cdot (\ell - 1)\,.$$

So the order of G_t can not be divisible by p as p is larger than any of the ℓ, unless $p = 3$ and $t = 2$. But it is also impossible that there is a surjective map from $\mathrm{PGL}_2(\mathbb{F}_2)$ onto $\mathrm{PGL}_2(\mathbb{F}_3)$. So H_p is not trivial.

Finally, we treat the case when H_p is the Klein group in $\mathrm{PGL}_2(\mathbb{F}_3) = \mathfrak{S}_4$. Since $p = 3$, we have $t = 2$. As $G_2 = \mathrm{PGL}_2(\mathbb{F}_2) = \mathfrak{S}_3$, the only possibility for this case is when K_2 is contained in K_3. But it was shown in lemma 13 that this is not possible under our additional hypothesis.

Let X be a sub-$\mathbb{Q}[G_r]$-module of $W_r = W_p \otimes W_t$. As H_p acts trivially on W_t, we deduce that there is a subspace Z of W_t such that $X = W_p \otimes Z$. By induction hypothesis, we know that W_t is irreducible as a $\mathbb{Q}[G_t]$-module. Hence $Z = W_t$ and we have shown that W_r is $\mathbb{Q}[G_r]$-irreducible.

Now we will prove i). If the additional hypothesis is verified then W_s is an irreducible $\mathbb{Q}[G_s]$-module by ii), hence G_s acts transitively on $\mathbb{P}^1(\mathbb{Z}/s\mathbb{Z})$. But the only place where we used the additional hypothesis in the proof of ii) is when we excluded the possibility that H_p is the Klein group in $\mathrm{PGL}_2(\mathbb{F}_3)$. But since the Klein group acts transitively on $\mathbb{P}^1(\mathbb{F}_3)$, we can prove directly the truth of i) in general.

Finally we must prove iii). We follow once again the same lines as the proof of ii). Of course, we may assume that $m > 1$. Let $1 \leqslant i \leqslant k$ and let $r \mid s$. We will prove by induction that $U_i \otimes W_r$ is an irreducible $\mathbb{Q}[G_{rm}]$-module. Let p be the largest prime dividing r and let $t = \frac{r}{p}$. By induction, we may suppose that $U_i \otimes W_t$ is G_{tm}-irreducible. Let $H_p = \mathrm{Gal}(K_{rm}/K_{tm}) \subset \mathrm{PGL}_2(\mathbb{F}_p)$. As before, if we can prove that W_p is an irreducible $\mathbb{Q}[H_p]$-module then we know that $U_i \otimes W_r = U_i \otimes W_t \otimes W_p$ is G_{rm}-irreducible. Once again we must exclude only the possibility that H_p is trivial or equal to the Klein group V_4 in $\mathrm{PGL}_2(\mathbb{F}_3)$.

Suppose first that $p = 2$. By maximality of p, we must have $t = 1$. If H_p is trivial, then there is a surjective map from G_m to $\mathrm{PGL}_2(\mathbb{F}_2)$. Running through all the possible odd m in lemma 11, we find that only $m = 3$ can be possible. Moreover in this case we must have $K_2 = K_3$. Again we use the previous lemma 13 to exclude this possibility.

We treat now the case that $p = 3$. Then $t = 1$ or $t = 2$. Suppose that H_p is trivial. There must be a surjective map from G_{tm} to $\mathrm{PGL}_2(\mathbb{F}_3) \cong \mathfrak{S}_4$. We can check that if $t = 1$ then we must have $m = 7$ as otherwise $\#G_m$ will not be a multiple of 3. But $\#G_7$ is not divisible by 24. If $t = 2$, then m can only be 5 or 7. Again it can not be 7. So we must have $G_{tm} \subset \mathfrak{S}_3 \times (\mathbb{Z}/4\mathbb{Z} \ltimes \mathbb{Z}/5\mathbb{Z})$ and it is easy to check that the latter group does not have a subquotient isomorphic to $\mathfrak{S}_4$.

Continuing with the case $p = 3$, we suppose now that H_p is the Klein group in $\mathrm{PGL}_2(\mathbb{F}_3)$. This time we have a surjection of G_{tm} onto $\mathfrak{S}_3$. If $t = 1$ then we can again check that there is no possibility for G_m. So suppose that $t = 2$. Then G_{tm} is contained in $\mathfrak{S}_3 \times G_m$. Then the only possibility for the surjection is that G_m lies in its kernel and $\mathrm{PGL}_2(\mathbb{F}_2)$ maps isomorphically onto $\mathfrak{S}_3$. In this case we would have that K_2 is contained in K_3. Once again the lemma 13 excludes this.

The very last step is to assume that $p > 3$ and that H_p is trivial. Then there is a surjective map from G_{tm} to $\mathrm{PGL}_2(\mathbb{F}_p)$. By the maximality of p, we know that $\#\,\mathrm{PGL}_2(\mathbb{Z}/t\mathbb{Z})$ is not divisible by p. Therefore $p \neq m$ must divide $\#G_m$. Running through the list of possible groups in lemma 11, we find that this is not possible. □

5.2 Results for semi-stable curves

Theorem 15. *Let $E/\mathbb{Q}$ be a semi-stable elliptic curve of conductor N with $N \neq 30$ or 210. Then all the self-points P_C are of infinite order in $E(\mathbb{Q}(C))$.*

Proof. By lemma 12, we may choose a prime p dividing N such that $\bar{\rho}_p$ is surjective and such that $p - 1 > m$.

Any cyclic subgroup C of order N may be written as $C = A \oplus B$ with A of order m and B of order $s = \frac{N}{m}$. Now we use the previous lemma. For any fixed A, the group G_N acts transitively on the set $\{A \oplus B\}_B$ as B runs over all cyclic subgroups of order s in E. Hence all self-points $\{P_C\}$ with the m-part A fixed are conjugate in $E(K_N)$. In particular, if $m = 1$ then all self-points are conjugate and the fundamental theorem 5 proves the theorem. So suppose now that $m > 1$.

Now we use the p-adic proof of the fundamental theorem 5. We identify the curve $E/\bar{\mathbb{Q}}_p$ with the Tate curve $\bar{\mathbb{Q}}_p^{\times}/q_E^{\mathbb{Z}}$. Fix a cyclic subgroup A of order m in E and let $B = \mu[s]$ and $C = A \oplus B$. Since any self-point is conjugate to such a point, it is sufficient to prove that P_C is of infinite order.

For each $\ell \mid m$, let A_ℓ be the ℓ-torsion part of A. Write A'' for the direct sum of all A_ℓ such that A_ℓ is generated by the ℓ-th roots of unities $\mu[\ell]$ in $E(\bar{\mathbb{Q}}_p)$. Write A' for the sum of all other A_ℓ. So $A = A' \oplus A''$. Denote the order of A' by m' and, likewise, the order of A'' by m''. Now we consider the isogeny ψ with kernel A'

$$0 \longrightarrow A' \longrightarrow E \xrightarrow{\psi} E' \longrightarrow 0\,.$$

If $\hat{A}'$ is the kernel of the dual isogeny $\hat{\psi}\colon E' \longrightarrow E$, then we may consider the point

$$x'_C = \bigl(E', \hat{A}' \oplus \psi(A'') \oplus \psi(B)\bigr) \in X_0(N)(\bar{\mathbb{Q}}_p)$$

which is nothing else but the Atkin-Lehner involution $w_{m'}$ applied to the point $x_C = (E, C)$. We know already that $\psi(B) = \mu[k]$ and $\psi(A'') = \mu[m'']$, but we also see that the group $\hat{A}'$ is isomorphic to $\mu[m']$. Hence the point x'_C lies now close to the cusp ∞ and its Tate-parameter will be a certain m'-th root u of q_E. Since

$$|u|_p = \Bigl(|q_E|_p\Bigr)^{\frac{1}{m'}} = p^{-\frac{c_p}{m'}} < p^{-\frac{1}{p-1}}$$

as $m' \leqslant m < p-1$, we can apply lemma 6 to show that $\varphi_E(x'_C)$ is of infinite order. But we also know that the Atkin-Lehner involutions w_ℓ act like multiplication by $-a_\ell \in \{\pm 1\}$ for all primes ℓ dividing N as shown in [AL70]. So $P_C = \varphi_E(x_C) = \pm\varphi_E(x'_C) + T$ where T is a point of finite order, and hence P_C is of infinite order. □

As remarked earlier we have a G_N-equivariant map

$$\iota\colon W_N \longrightarrow E(K_N)\otimes\mathbb{Q}$$

The second point of lemma 14 shows the following

Theorem 16. *Let $E/\mathbb{Q}$ be a semi-stable elliptic curve with $N \neq 30,\, 210$ and suppose that all the representations $\bar\rho_p$ for all primes $p \mid N$ are surjective, then the group generated by the self-points is of rank N and the Galois groups acts like the irreducible Steinberg representation W_N on it.*

We prove now an extension of this theorem to the case when $m \neq 1$. In particular W_N might not be irreducible anymore. Unfortunately we can not prove that the rank is N in general for a semistable curve as we have to exclude the possibility that the curve has two distinct isogenies of the same degree defined over $\mathbb{Q}$. For, if the curve has two isogenies of degree p over $\mathbb{Q}$, then in the decomposition of W_N into irreducible $\mathbb{Q}[G]$-modules, there will be a representation that appears with multiplicity 2. The second hypothesis in the following theorem excludes this possibility, but it is also needed elsewhere to be able to apply the lemmae from the previous section.

Theorem 17. *Let $E/\mathbb{Q}$ be a semi-stable elliptic curve. Suppose that $N \neq 30$ or 210. Suppose that for each prime $p \mid N$ such that $\bar\rho_p$ is not surjective, there is a prime $\ell \mid N$ such that the Tamagawa number c_ℓ is not divisible by p. Then the group generated by the self-points is of rank N.*

Proof. As a consequence of the second hypothesis, we know that for each $p \mid N$ there is an element of order p in G_p. See the appendix of [Ser68]. Since either G_p is all of $\mathrm{PGL}_2(\mathbb{F}_p)$ or it is contained in the Borel subgroup, we conclude that, either G_p acts transitively on $\mathbb{P}^1(\mathbb{F}_p)$ or it has one single fixed point, which we will call $C_p \in \mathbb{P}^1(\mathbb{F}_p)$.

Let $p \mid m$. Then by proposition 9, the $\mathbb{Q}[G_p]$-module W_p decomposes as the sum of the trivial part W'_1 and an irreducible part W'_p of dimension $p-1$. If m is not prime it can only be either $2\cdot 3$ or $2\cdot 5$ by Mazur's theorem. If $m = 6$ then W_6 decomposes as $W'_1 \oplus W'_2 \oplus W'_3 \oplus W'_6$ where $W'_6 = W'_2 \otimes W'_3$. To see that the latter is also irreducible one needs only to note that the dimension of W'_2 is 1. In the same way, for $m = 10$, we have an irreducible component W'_{10}.

Using lemma 14, we know now that W_N decomposes as

$$W_N = \bigoplus_{d\mid m} (W'_d \otimes W_s)$$

into irreducible $\mathbb{Q}[G_N]$-modules. We must now prove that none of the components belongs to the kernel of the map $\iota\colon W_N \longrightarrow E(K)\otimes\mathbb{Q}$.

First recall the definition of $W'_d \otimes W_s$. It contains all elements

$$\sum_{C\in\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})} a_C\, e_C \;\in\; \bigoplus_{C\in\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})} \mathbb{Q}\, e_C$$

subject to the following three conditions.

- For all $N \neq b \mid N$ and all cyclic subgroups B of order b, the sum $\sum_{C \supset B} a_C$ vanishes.
- For all primes $p \mid d$ and all $C \supset C_p$, we have $a_C = 0$.
- For all primes $p \mid \frac{m}{d}$ and all $C \not\supset C_p$, we have $a_C = 0$.

Let $d \mid m$. Define A to be the direct sum of C_p for all $p \mid \frac{m}{d}$. So A is a cyclic group of order $\frac{m}{d}$. The map ι on $W'_d \otimes W_s$ is induced from the map

$$\iota_d \colon \bigoplus_{D} \mathbb{Q}\, e_{A \oplus D} \longrightarrow E(K) \otimes \mathbb{Q}$$

where D runs through all the cyclic subgroups D in E of order $d \cdot s$ such that D does not contain any of the C_p with $p \mid d$. As this map sends $e_{A\oplus D}$ to the self-point $P_{A \oplus D}$, it follows from theorem 15 that the map ι_d is not trivial.

Now we use the relations in proposition 7 to see that, for all $b \mid ds$ and all cyclic groups B of order b, not containing any of the C_p, we have

$$\sum_{D \supset B} e_{A\oplus D} \in \ker \iota_d\,.$$

Hence the only irreducible part of the domain of ι_d which does not lie in the kernel is $W'_d \otimes W_s$ Hence ι_d induces an injection $W'_d \otimes W_s \rightarrowtail E(K) \otimes \mathbb{Q}$. □

The hypothesis in this last theorem is fulfilled for the very large part of semi-stable curves. We could not find an strong Weil curve with $N < 10'000$ for which the theorem would not apply. The first curve which does not satisfy the hypothesis with $p = 3$ is 651e2 as it has $G_3 = \mathbb{Z}/2\mathbb{Z}$ and the Tamagawa numbers are $c_3 = 3$, $c_7 = 3$, and $c_{31} = 3$. For $p = 2$, the examples that do not satisfy the hypothesis are exactly those which have all 2-torsion points defined over $\mathbb{Q}$, like for instance 30a2.

6 Examples

The following table 2 shows some computations done for the optimal curves (with one exception) of smallest conductor. We do not give the complete explanation of how one obtains these results. For more detail, we refer the reader to [DW08] and [Wut07]. But we will consider two curves in more detail later.

The curves in table 2 are labelled as in Cremona's tables [Cre97]. The first line shows the structure of the torsion group over $\mathbb{Q}$, e.g. $2 \cdot 4$ means that $E(\mathbb{Q})_{\text{tors}} = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$. The next line indicates the largest degree of a cyclic isogeny defined over $\mathbb{Q}$ on E. The last two lines are those containing information about self-points, first we counted the number of irreducible $\mathbb{Q}[G_N]$-modules in W_N and finally, we computed the rank of the group generated by self-points in $E(K_N)$. The two values in bold face are lower than the usual conjectured rank, which is no surprise since these two curves have complex multiplication. When there is no $*$ sign next to the rank, the value is proven using the results in the previous section. The sign $*$ indicates that we have only empirically computed the rank using the following method.

Using high precision computation we may find a very good approximation to the values of

$$z_C = \int_{x_C}^{\infty} f_E(q) \frac{dq}{q}$$

as elements of $\mathbb{C}$, where C runs over all cyclic subgroups of order N in E. Hence z_C maps to P_C under $\mathbb{C} \longrightarrow \mathbb{C}/\Lambda_E \longrightarrow E(\mathbb{C})$ where $\Lambda_E = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ is the period lattice of E. Let t be the

N	11a1	14a1	15a1	17a1	19a1	20a1	21a1	24a1	26a1
tors.	5	$2\cdot 3$	$2\cdot 4$	4	3	$2\cdot 3$	$2\cdot 4$	$2\cdot 4$	3
isog.	25	18	16	4	9	6	8	8	9
W_N	1	2	1	1	1	2	1	4	1
rank	11	14	15	17	19	15	21	18^*	26

N	26b1	27a**2**	30a1	32a1	33a1	34a1	35a1	37a1	38a1
tors.	7	3	$2\cdot 3$	4	$2\cdot 2$	$2\cdot 3$	3	1	3
isog.	7	27	12	4	4	6	9	1	9
W_N	1	5	4	?	1	2	1	1	1
rank	26	**20**	30^*	**12***	33	34	35	37	38

Table 2: The ranks of the group generated by self-points for some curves

order of the torsion subgroup of E over $\mathbb{Q}$. Consider the abelian group spanned by $\frac{1}{t}\omega_1$, $\frac{1}{t}\omega_2$ and all the z_C in a complex vector space of dimension $2 + \#\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$. Using the LLL-algorithm, we find small vectors in this lattice. These are likely to give relations

$$b_1\omega_1 + b_2\omega_2 + \sum_C a_C z_C = 0$$

with b_1, b_2, and a_C all integers. This yields a probable relation among the self-points. Unfortunately we might not catch those relations involving torsion points on E not defined over $\mathbb{Q}$. So to increase the likelihood of finding all relations we multiply t by a product of small primes. For all cases for which we were able to determine the rank, this empirical computation gave the same answer. In principle these computations could be made rigorous by considering exact estimates for the error terms.

6.1 Conductor 24

We present here an example of a curve where we are unable to determine the rank of the group generated by self-points. The Mordell-Weil group of the curve 24a1, given by the equation

$$E: \quad y^2 = x^3 - x^2 - 4\cdot x + 4$$

is $E(\mathbb{Q}) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$. The situation is rather complicated and we do not explain all computations here. The field K_4 turns out to be $\mathbb{Q}(i,\sqrt{3})$, which happens to be equal to $\mathbb{Q}(E[4])$. There is are two non-trivial Galois-orbits of 4-torsion points, one over $\mathbb{Q}(\sqrt{3})$ and the other over $\mathbb{Q}(\sqrt{-3})$. Hence the representation V_4 splits as

$$V_4 = \mathbb{1} \oplus \mathbb{1} \oplus \mathbb{1} \oplus \mathbb{1} \oplus \mathbb{1}(\sqrt{3}) \oplus \mathbb{1}(\sqrt{-3})\,,$$

where $\mathbb{1}(\sqrt{d})$ is the one-dimensional representation corresponding to the Dirichlet character associated to $\mathbb{Q}(\sqrt{d})$. Now, the field K_8 can be computed, too. In turns out that it coincides with $\mathbb{Q}(E[8])$ in this case. It is a degree 16 extension of discriminant $2^{36}\cdot 3^{12}$. It contains the extension $\mathbb{Q}(i,\sqrt{2},\sqrt{3})$. The sub-extension K_4 is fixed by the centre of the Galois group G_8. The group G_8 admits two irreducible 2-dimensional representations, one of which we call Z_2. Then the representation V_8 splits in many components and we find that

$$W_8 = \mathbb{1}(\sqrt{2}) \oplus \mathbb{1}(\sqrt{-2}) \oplus Z_2 \oplus Z_2\,.$$

The first two factors correspond to two couples of lines in $E[8]$ defined over $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{-2})$ respectively. The other lines are defined over fields of degree 4.

Using that the field K_3 intersects K_8 in $\mathbb{Q}(\sqrt{-3})$, we find that W_{24} splits into 4 irreducible factors $W_{24} = W_3(\sqrt{2}) \oplus W_3(\sqrt{-2}) \oplus Z_6 \oplus Z_6$. Here $Z_6 = W_3 \otimes Z_2$ is an irreducible representation of dimension 6. In particular, this representation appears with multiplicity 2. So the usual proof that there are no further relations among self-points will not work.

The cyclic subgroup of order 8 in E which corresponds to $\mu[8]$ over $\mathbb{Q}_3$ contains the rational 4-torsion point. So one of the two factors of dimension 3 in W_{24} certainly appears in $E(K_N)\otimes\mathbb{Q}$. But we are unable to show that any other self-points are of infinite order with the means of theorem 15.

So we can only conclude that the rank r of the group generated by the self-points satisfies $3 \leqslant r \leqslant 18$. But we strongly believe that $r = 18$ as suggested by the empirical computations.

6.2 Conductor 27

There are four curves of conductor 27 forming the following isogeny graph

$$27\mathrm{a}2 \longleftarrow 27\mathrm{a}1 \longleftarrow 27\mathrm{a}3 \longleftarrow 27\mathrm{a}4$$

The isogenies $\longleftarrow$ are all of degree 3 and, in the sense that they are drawn here, the kernels are $\mathbb{Z}/3\mathbb{Z}$ while the dual isogenies have kernel $\mu[3]$. Over the field $F = \mathbb{Q}(\sqrt{-3}) = \mathbb{Q}(\zeta)$ with ζ a third root of unity, the curves 27a1 and 27a3 become isomorphic, the same holds for the curves 27a2 and 27a4. The first couple has complex multiplication by the maximal order $\mathbb{Z}[\zeta]$, while the second couple has cm by $\mathbb{Z}[3\zeta]$.

Let E be the curve 27a2 defined by

$$y^2 \; + \; y \; = \; x^3 \; - \; 270 \cdot x \; - \; 1708\,.$$

Theorem 18. *The self-points on the curve 27a2 generate a group of rank* 20 *in* $E(K_{27})$. *There are exactly two linearly independent self-points defined over* $K_3 = \mathbb{Q}(\sqrt[6]{-3})$ *and they generate a subgroup of finite index in* $E(K_3)$.

The proof is contained in the following explanations. But we do omit certain computations from the presentation here.

The field K_3 is equal to $\mathbb{Q}(\sqrt[6]{-3})$ and the Galois group G_3 is a dihedral group of order 6. In fact some 3-torsion points are defined over $F = \mathbb{Q}(\sqrt{-3})$ and some others are over $\mathbb{Q}(\sqrt[3]{-3})$ and we have $V_3 = \mathbb{1} \oplus \mathbb{1}(\sqrt{-3}) \oplus Z_2$ where Z_2 is the unique irreducible 2-dimensional representation of G_3.

In order to determine the structure of V_{27}, we need to use the theory of complex multiplication. Let H_{27} be the subgroup $\mathrm{Gal}(K_{27}/F)$ inside G_{27}. We know that the representation $\bar\rho_{27,F}$ now maps to

$$\bar\rho_{27,F}\colon H_{27} \rightarrowtail \frac{\mathrm{Aut}_{\mathcal{O}/27\mathcal{O}}(E[27])}{(\mathbb{Z}/27\mathbb{Z})^\times} = \frac{(\mathcal{O}/27\mathcal{O})^\times}{(\mathbb{Z}/27\mathbb{Z})^\times} = \{ \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{PGL}_2(\mathbb{Z}/27\mathbb{Z})\} \cong \mathbb{Z}/27\mathbb{Z}$$

where $\mathcal{O} = \mathbb{Z}[3\,\zeta]$ is the ring of endomorphisms of E/F. It is possible to verify that H_{27} is equal to this group and hence G_{27} is a dihedral group of order 54 generated by $h = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $s = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. The computation of V_{27} is now easy and one finds

$$W_{27} = \mathbb{1} \oplus \mathbb{1}(\sqrt{-3}) \oplus Z_2 \oplus Z_2 \oplus Z_{18}.$$

Here Z_2 is the unique 2-dimensional irreducible $\mathbb{Q}[G_{27}]$-module (the action of h has trace -1) and Z_{18} is the unique irreducible 18-dimensional $\mathbb{Q}[G_{27}]$-module (it splits over $\mathbb{C}$ into six 2-dimensional representations). As the curve 27a2 is not the strong Weil curve in the isogeny class, the modular parametrisation φ_E from the elliptic curve $X_0(27)$ to E is not an isomorphism but an isogeny of degree 3. The curve $X_0(27)$ has six cusps represented by the classes $\{\infty, 0, \frac{1}{3}, \frac{2}{3}, \frac{2}{9}, \frac{4}{9}\}$. The group $X_0(27)(\mathbb{Q})$ contains the cusps ∞ and 0 and the self-point obtained from the isogeny 27a2 $\longrightarrow$ 27a4. They form exactly the kernel of φ_E. The other cusps are mapped to the 3-torsion points defined over F on E. In fact $E(F) = \mathbb{Z}/3\mathbb{Z}$ and $E(K_3)_{\text{tors}} = \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$. A two-descent over K_3 shows that the 2-Selmer group of E/K_3 has two copies of $\mathbb{Z}/2\mathbb{Z}$ in it.

The trivial factor in W_{27} corresponds to the self-point obtained from the 27-isogeny defined over $\mathbb{Q}$ on 27a2. We know that it is the point O in $E(\mathbb{Q})$. The factor $\mathbb{1}(\sqrt{-3})$ in W_{27} must also belong to the kernel of $\iota\colon W_{27} \longrightarrow E(K_{27}) \otimes \mathbb{Q}$ as the Mordell-Weil group $E(F)$ is of rank 0. From the factors Z_2 at least one of them must be in the kernel as the rank of $E(K_3)$ is bounded by 2 from above. It is not hard to check by looking at traces of Frobenii that the torsion subgroup of $E(K_{27})$ only contains nine 3-torsion points. Since the degree of φ_E is 3, there are at most 27 points in $X_0(27)(K_{27})$ which map to torsion points in $E(K_{27})$ under φ_E. As there are 36 points x_C, we conclude that at least 9 self-points are of infinite order. By looking at the decomposition of W_{27} we see that Z_{18} can not belong to the kernel of ι.

Finally we have to show that there is a self-point of infinite order in $E(K_3)$. This will show that the second copy of Z_2 does not belong to the kernel of ι. This can be done numerically. The point $\tau_C = \frac{1}{6} \cdot (-1 + \sqrt{-3})$ in the upper half plane corresponds to a point x_C in $X_0(27)$. We find that

$$-\frac{1}{8}\big(36 \cdot s^5 + 15 \cdot s^4 - 45 \cdot s^3 - 18 \cdot s^2 + 69 \cdot s + 99\big) \qquad \text{with } s = \sqrt[6]{-3}$$

is the x-coordinate of the self-point P_C in $E(K_3)$. Its canonical height is 1.5191 and hence P_C is of infinite order. This point P_C and its conjugates over F will generate a group of rank 2 in $E(K_3)$. Since we have computed the 2-Selmer group earlier, we conclude that the rank of $E(K_3)$ is as claimed equal to 2.

It seems plausible that this point P_C can also be constructed as an “exotic Heegner point” using the construction of Bertonlini, Darmon and Prasanna in [BDP07]. But the authors exclude there explicitly the case of conductor $N = 27$.

7 Higher self-points

In this section, we investigate on three particular cases of higher self-points. Let $E/\mathbb{Q}$ be an elliptic curve of conductor N. For any cyclic subgroup D in E we may consider the isogenous curve E/D with a suitable choice of a cyclic subgroup of order N in it. In the first case, we use subgroups D defined over $\mathbb{Q}$ to construct new points and for the two other cases we use subgroups D of prime-power order p^n, first when p divides the conductor and then when it does not divide the conductor.

7.1 Self-points via rational isogenies

Let D be a cyclic subgroup in E defined over $\mathbb{Q}$. Suppose for simplicity that the order of D is prime to N. Then for any cyclic subgroup C of order N on E,

$$Q_D = \varphi_E\big(E/D, (C + D)/D\big)$$

is a higher self-point defined over the same field as P_C. It would be interesting to know in general when P_C and Q_D are linearly independent. For instance this can be shown on the curves of conductor 11 : There are 3 curves in the isogeny class and hence we find, for any fixed C, one self-point and two higher self-points on E defined over $\mathbb{Q}(C)$. Using the canonical height pairing, we can prove the linear independence of these three points computed explicitly on E. So the rank of $E(\mathbb{Q}(C))$ will have to be at least 3. See [DW08] and [Wut07] for more details on this example.

In some cases the method of the proof of theorem 15 can be used to show that Q_D is also of infinite order. But the methods of the proof of theorem 17 will not be sufficient to prove the independence of P_C and Q_D.

7.2 The multiplicative case

Let now p be a prime dividing N exactly once, i.e. E has multiplicative reduction at p. Let M be such that $N = p \cdot M$. As a base-field we will consider here the number field $F = K_M$, the smallest field such that its absolute Galois group acts as scalars on $E[M]$. In the particular situation when $N = p$ is prime then $F = \mathbb{Q}$; the same is true for instance if E is a curve of conductor 14 and $p = 7$.

For any $n \geqslant 0$, we define now F_n to be the field $K_{p^n N}$ and H_n to be the Galois group of F_n/F. Via the Galois representation

$$\rho_{F,p}\colon\ \mathrm{Gal}(\bar{F}/F) \longrightarrow \mathrm{Aut}(T_pE) \cong \mathrm{GL}_2(\mathbb{Z}_p) \longrightarrow \mathrm{PGL}_2(\mathbb{Z}_p)$$

the group H_n identifies with a subgroup of $\mathrm{PGL}_2(\mathbb{Z}/p^{n+1}\mathbb{Z})$.

Fix a subgroup B of order M in E. Let $n \geqslant 0$ and let D be a cyclic subgroup of order p^{n+1} in E. Let $A = D[p]$ and $C = A \oplus B$, which is a cyclic subgroup of order N. Write ψ for the isogeny $E \longrightarrow E'$ of kernel D and $\hat{\psi}$ for its dual. Define

$$C' = \ker(\hat{\psi})[p] \oplus \psi(B)\,,$$

which is a cyclic subgroup of E' of order $M \cdot p = N$. The image of the point $y_D = (E', C') \in Y_0(N)$ through the map φ_E will be denoted by Q_D. It is by definition a higher self-point. We will say that "Q_D lies over P_C" or "over B".

In particular, if $n = 0$, then $D = A$ is a cyclic subgroup of order p. From the construction above, we see that the point y_D is nothing but $w_p(x_C)$ where w_p is the Atkin-Lehner involution on $X_0(N)$. Hence we have that $Q_D = -a_p \cdot P_C + T$ for some 2-torsion point T defined over $\mathbb{Q}$. Here $a_p = \pm 1$ is, as before, the Hecke eigenvalue of the newform f_E attached to the isogeny class of E.

Let D be a cyclic subgroup of E of order p^{n+1}. By the definition of the Hecke operator T_p on $J_0(N)$, we have that

$$T_p\bigl((y_D) - (\infty)\bigr) = \sum_{D' \supset D} \bigl((y_{D'}) - (\infty)\bigr)$$

where the sum runs over all cyclic subgroups D' in E of order p^{n+2} containing D. This gives us the relation

$$a_p \cdot Q_D = \sum_{D' \supset D} Q_{D'}\,. \tag{2}$$

Hence by induction, we know that Q_D is of infinite order if the self-point P_C is.

Lemma 19. *Let B be a fixed subgroup of order M in E and let $n \geqslant 0$. Then $\sum_D Q_D$ is a torsion point in $E(F)$, where the sum is over all cyclic subgroups D of E of order p^{n+1}.*

Proof. Suppose first that $n = 0$. Then we sum over all cyclic subgroups $D = A$ of order p which gives

$$\sum_D Q_D = \sum_{C \supset B} (-a_p P_C + T) = (p+1) \cdot T - a_p \sum_{C \supset B} P_C.$$

The first term on the right hand side is clearly torsion and the second term contains exactly one of the relations from proposition 7. Now by induction, we assume that the statement holds for n. But then $\sum_{D'} Q_{D'}$, with the sum running over all cyclic subgroups D' of order p^{n+2}, is, by (2), equal to $a_p \cdot \sum_D Q_D$, with the sum now running over cyclic subgroups of order p^{n+1}. $\square$

The $\mathbb{Q}$-vector space with basis $\{e_D\}_D$ in bijection with $\mathbb{P}^1(\mathbb{Z}/p^{n+1}\mathbb{Z})$ is a natural $\mathbb{Q}[H_n]$-module. Define

$$V'_{(n)} = \frac{\bigoplus_A \mathbb{Q}\, e_D}{\mathbb{Q}(\sum_D e_D)}$$

which is a vector space of dimension $p^{n+1} + p^n - 1$.

Fix a cyclic subgroup B of order M in E. By the previous lemma, there is a morphism of $\mathbb{Q}[H_n]$-modules given by

$$\iota_n = \iota_{B,n} \colon \quad V'_{(n)} \longrightarrow E(F_n) \otimes \mathbb{Q}$$
$$e_D \longmapsto Q_D$$

We assume that the representation $\rho_{F,p}$ is surjective onto $\mathrm{PGL}_2(\mathbb{Z}_p)$. So $H_n \cong \mathrm{PGL}_2(\mathbb{Z}/p^{n+1}\mathbb{Z})$ and the $\mathbb{Q}[H_n]$-module $V'_{(n)}$ is the Steinberg representation, which was denoted by V_{p^n}/W_1 earlier in section 4.

Theorem 20. *Suppose $E/\mathbb{Q}$ is an elliptic curve and p a prime of multiplicative reduction. Suppose that $\rho_{F,p}$ is surjective and that there is a self-point P_C of infinite order in $E(F_0)$. Then for all $n \geqslant 0$ and all cyclic subgroups D of order p^{n+1} with $D[p] \subset C$ the point Q_D is of infinite order. They generate in $E(F_n) \otimes \mathbb{Q}$ a $\mathbb{Q}[H_n]$-module isomorphic to the representation $V'_{(n)}$ of dimension $p^{n+1} + p^n - 1$.*

As a special case, we recover Theorem 8 in [DW08] in the case when $N = p$ is prime and $F = \mathbb{Q}$.

Proof. We only have to show that ι_n is injective. Suppose $n \geqslant 0$ is the smallest value such that ι_n is not injective. Since $V'_{(n)} = W_{p^{n+1}} \oplus V'_{(n-1)}$ if $n > 0$ and $V'_{(0)} = W_p$, this means that ι_n induced on $W_{p^{n+1}}$ is not injective. Since this is an irreducible $\mathbb{Q}[H_n]$-module when $\rho_{F,p}$ is surjective, this means that ι_n is trivial on $W_{p^{n+1}}$. This is impossible since we have shown that all Q_D above P_C are of infinite order. $\square$

7.3 The good case

Let p be a prime not dividing N, i.e. of good reduction for E. Let F be a number field such that $E(F)$ contains a self-point P_C of infinite order. We fix the corresponding cyclic subgroup C of order N in E.

For any $n \geqslant 0$, let F_n be the smallest Galois extension of F such that the absolute Galois group $\mathrm{Gal}(\bar{F}/F)$ acts via scalars on $E[p^{n+1}]$, hence $F_n = F \cdot K_{p^{n+1}}$. Define H_n to be the Galois group $\mathrm{Gal}(F_n/F)$, which will be considered as a subgroup of $\mathrm{PGL}_2(\mathbb{Z}/p^{n+1}\mathbb{Z})$.

For any $n \geqslant 0$ and any cyclic subgroup D of order p^{n+1} we construct a higher self-point Q_D in $E(F_n)$ as follows. Let $\psi\colon E \longrightarrow E/D$ be the isogeny associated to D. Put $y_D = (E/D, \psi(C)) \in Y_0(N)$ and $Q_D = \varphi_E(y_D)$. This is a higher self-point "above P_C".

Again we may use the definition of the Hecke operator T_p to prove that, for all $n \geqslant 0$ and D as before

$$a_p \cdot Q_D = \sum_{D' \supset D} Q_{D'}, \tag{3}$$

where the sum runs over all cyclic subgroups D' of order p^{n+2} in E containing D. Furthermore we have

$$a_p \cdot P_{C} = \sum_{D} Q_{D} \tag{4}$$

with the sum running over all cyclic subgroups D of order p in E.

Let $V_{(n)} = V_{p^{n+1}}$ be the $\mathbb{Q}[H_n]$-module whose basis $\{e_D\}_D$ as a vector space over $\mathbb{Q}$ is in bijection with $\mathbb{P}^1(\mathbb{Z}/_{p^{n+1}\mathbb{Z}})$. We have a H_n-morphism defined by

$$\iota_n = \iota_{C,n}\colon \quad V_{(n)} \longrightarrow E(F_n) \otimes \mathbb{Q}$$
$$e_D \longmapsto Q_D$$

Theorem 21. *Let $E/\mathbb{Q}$ be an elliptic curve of conductor N. Let p be a prime of good and ordinary reduction for E. Let F be a number field such that $E(F)$ contains a self-point P_C of infinite order. Suppose that the representation $\rho_{F,p}$ is surjective. Then all higher self-points Q_D constructed above are of infinite order and they generate a group of rank $p^n \cdot (p+1)$.*

Proof. By induction on n using the formulae (3) and (4) and the hypothesis that p is ordinary to guarantee that $a_p \neq 0$. □

The above easy proof of the theorem breaks down if E has supersingular reduction at p, for a_p is then almost always equal to 0.

Theorem 22. *Let $E/\mathbb{Q}$ be a a semi-stable elliptic curve of conductor $N \neq 30$ or 210. Let $p > N$ be a supersingular prime for E. Let $F = K_N$. Suppose that the representation $\rho_{F,p}$ is surjective. Then all higher self-points Q_D above a given self-point P_C are of infinite order and they generate a group of rank $p^n \cdot (p+1)$.*

Proof. We follow the proof of theorem 15. Let $\ell > 2$ be a prime dividing N. We proved that the self-points are of infinite order by showing that when a certain Atkin-Lehner involution is applied to one of the conjugates of x_C one obtains a point ℓ-adically close to the cusp ∞ on $X_0(N)(\bar{\mathbb{Q}}_\ell)$.

Let Q_D be a higher self-point above the self-point P_C. Since $\rho_{F,p}$ is surjective, the point Q_D will be conjugate over K_N to a all other higher self-point above the same self-point. Therefore without loss of generality we may assume that the cyclic subgroup D on E corresponds to $\mu[p^{n+1}]$ in $E(\bar{\mathbb{Q}}_\ell)$. Then the point $y_D = (E', C')$ is represented by a Tate curve over $\bar{\mathbb{Q}}_\ell$ with parameter $q_{E'}$ equal to the p^{n+1}-st power of q_E.

Let r be a divisor of N such that $w_r(y_D)$ is the couple $(E'', \mu[N])$ with E'' the Tate curve with parameter $q_{E'}^{1/r}$. Using the fact that $p > N \geqslant r$, we find that

$$\left| q_{E'}^{1/r} \right|_\ell = |q_E|_\ell^{\frac{p^{n+1}}{r}} \leqslant \ell^{-\frac{p}{r}\cdot p^n} \leqslant \ell^{-1} < \ell^{-\frac{1}{\ell-1}}$$

and hence, the lemma 6 shows that $\varphi_E(E'', \mu[N])$ is of infinite order. Then as usual Q_D differs from $\pm\varphi_E(w_r(y_D))$ by a torsion point. So Q_D is of infinite order.

Since the representation W_{p^n} is irreducible for $\mathrm{PGL}_2(\mathbb{Z}/_{p^{n+1}\mathbb{Z}})$, we can show by induction that the rank of the group generated by higher self-points is $\dim(V_{(n)}) = p^n \cdot (p+1)$. □

Putting the previous two results together, we are able to show a corollary which hold for all but finitely many primes p.

Corollary 23. *Let $E/\mathbb{Q}$ be a semi-stable curve of conductor $N \neq 30$, or 210. Suppose that p is a prime such that $p > N$, (so it is of good reduction), and such that $\bar{\rho}_p\colon \operatorname{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \operatorname{PGL}_2(\mathbb{F}_p)$ is surjective. Let s be the rank of the group generated by self-points in $E(K_N)$. Then the higher self-points in $E(K_{p^{n+1}N})$ generate a group of rank at least $s \cdot (p+1) \cdot p^n$.*

Proof. Take $F = K_N$ in the previous theorems. We only have to show the condition that $\rho_{F,p}$ is surjective. Note that it is enough to show that $\bar{\rho}_{F,p}\colon \operatorname{Gal}(\bar{F}/F) \longrightarrow \operatorname{PGL}_2(\mathbb{F}_p)$ has all of $\operatorname{PSL}_2(\mathbb{F}_p)$ in its image, since the representation V_{p^n} will still have the same decomposition.

Let H_p be the group $\operatorname{Gal}(K_{Np}/K_N)$, i.e. the image of $\bar{\rho}_{F,p}$. It is equal to the normal subgroup in $\operatorname{Gal}(K_p/\mathbb{Q}) \cong \operatorname{PGL}_2(\mathbb{F}_p)$ corresponding to the subextension $K_p/K_N \cap K_p$. Since $p > 11$ when $p > N$, we have that $\operatorname{PGL}_2(\mathbb{F}_p)$ has only three normal subgroups, namely itself, $\operatorname{PSL}_2(\mathbb{F}_p)$ and $\{1\}$. By the remark above, we only have to exclude that H_p is not trivial.

If H_p was trivial, then p, dividing the order of $\operatorname{PGL}_2(\mathbb{F}_p)$, would have to divide the order of G_N, which is a subgroup of $\operatorname{PGL}_2(\mathbb{Z}/N\mathbb{Z})$. But when $p > N$, then p cannot divide the order of $\operatorname{PGL}_2(\mathbb{Z}/N\mathbb{Z})$, except when $p = 3$ and $N = 2$, which cannot occur as a conductor. □

8 Derivatives

Let $E/\mathbb{Q}$ be an elliptic curve of conductor N. Let p be an *odd* prime of *ordinary*, either good or multiplicative, reduction. In order to treat the cases of higher self-points discussed in the sections 7.2 and 7.3 simultaneously, we choose now a base field F. If E has good ordinary reduction at p, then F is any number field such that $E(F)$ contains a self-point P_C of infinite order. If p divides N, then F is a number field such that the absolute Galois group of F acts by scalars on $E[\frac{N}{p}]$.

We will suppose from now on that

$$\rho_{F,p}\colon \operatorname{Gal}(\bar{F}/F) \longrightarrow \operatorname{PGL}_2(\mathbb{Z}_p)$$

is surjective.

We let F_n be the smallest extension of F such that $H_n = \operatorname{Gal}(F_n/F)$ acts by scalars on $E[p^{n+1}]$. By assumption the map $\rho_{F,p}$ induces an isomorphism from H_n to $\operatorname{PGL}_2(\mathbb{Z}/p^{n+1}\mathbb{Z})$. Also, this implies that $E(F_n)$ has no p-torsion elements.

Let $\mathcal{O}$ be the ring of integers in the unramified quadratic extension of $\mathbb{Q}_p$. Choosing a basis of $\mathcal{O}$ over $\mathbb{Z}_p$, we get a homomorphism

$$\Psi\colon \mathcal{O}^\times \longrightarrow \operatorname{GL}_2(\mathbb{Z}_p) \longrightarrow \operatorname{PGL}_2(\mathbb{Z}_p)\,,$$

whose kernel is $\mathbb{Z}_p^\times$. The image of the composition

$$\mathcal{O}^\times \xrightarrow{\ \Psi\ } \operatorname{PGL}_2(\mathbb{Z}_p) \longrightarrow \operatorname{PGL}_2(\mathbb{Z}/p^{n+1}\mathbb{Z}) \longrightarrow H_n$$

will be denoted by A_n. This is a cyclic group of order $(p+1)\cdot p^n = \#\mathbb{P}^1(\mathbb{Z}/p^{n+1}\mathbb{Z})$; it is the projective version of the non-split Cartan group in $\operatorname{GL}_2(\mathbb{Z}/p^{n+1}\mathbb{Z})$. To simplify the notations we will write F_n^A for the subfield of F_n fixed by A_n.

Theorem 24. *Let $E/\mathbb{Q}$ be an elliptic curve. Suppose that E does not have potentially good supersingular reduction for any prime of additive reduction. Let p be a prime of either good ordinary or multiplicative reduction. Let F be the number field as above and assume that $\rho_{F,p}$ is surjective. Then we have*

$$\#\operatorname{Sel}_{p^n}(E/F_n^A) \geqslant p^n$$

where A is any non-split Cartan group in $\operatorname{PGL}_2(\mathbb{Z}_p)$.

The proof of this theorem will be completed in section 8.3.

Since there are no p-torsion points in $E(F_n)$, as $\rho_{F,p}$ is assumed to be surjective, there is an isomorphism

$$\mathrm{H}^1(F_n^A, E[p^k]) \longrightarrow \mathrm{H}^1(F_n, E[p^k])^{A_n}$$

induced by the restriction map. This implies that the map

$$\mathrm{Sel}_{p^n}(E/F_n^A) \longrightarrow \mathrm{Sel}_{p^n}(E/F_n)^{A_n}$$

is injective. We conjecture that the elements in the Selmer group constructed above do not lie in the image of the Kummer map, but represent non-trivial elements in the Tate-Shafarevich group $Ш(E/F_n^A)[p^n]$. If so, these classes in the Tate-Shafarevich group will capitulate in the extension F_n/F_n^A, since the elements of the Selmer group in the theorem restrict to elements in the image of the higher self-points inside $\mathrm{Sel}_{p^n}(E/F_n)$.

8.1 The field extension

Lemma 25. *The cyclic group A_n intersects trivially any Borel subgroup in H_n.*

Proof. We prove the statement that the image of Ψ in $\mathrm{PGL}_2(\mathbb{Z}_p)$ intersects trivially any of its Borel subgroups B. Let L be the $\mathbb{Z}_p$-line $\mathcal{O}$ such that B is the stabiliser under the action of $\mathrm{PGL}_2(\mathbb{Z}_p)$ on $\mathbb{P}^1(\mathbb{Z}_p)$ viewed as the set of $\mathbb{Z}_p$-modules in $\mathcal{O}$ generated by a unit. Let $\alpha \in \mathcal{O}^\times$ be any element with a non-trivial image under Ψ, then $\alpha \notin \mathbb{Z}_p^\times$ can not fix L. □

This implies in particular that any generator α_n of A_n acts simply transitively on the set $\mathbb{P}^1(\mathbb{Z}/p^{n+1}\mathbb{Z})$.

Lemma 26. *Let υ be either a place of ordinary reduction above p or a infinite place or a place of potentially multiplicative reduction. Then the image of*

$$\bar{\rho}_{F_\upsilon,p} \colon \mathrm{Gal}(\bar{F_\upsilon}/F_\upsilon) \longrightarrow \mathrm{PGL}_2(\mathbb{Z}/p^{n+1}\mathbb{Z})$$

lies in a Borel subgroup of $\mathrm{PGL}_2(\mathbb{Z}/p^{n+1}\mathbb{Z})$.

Proof. First suppose that υ divides p. As E is of ordinary reduction at υ, there is a cyclic subgroup of $E[p^{n+1}]$ of order p^{n+1} which is fixed by the Galois group $\mathrm{Gal}(\bar{F_\upsilon}/F_\upsilon)$. This subgroup consists of all elements of $E[p^{n+1}]$ with trivial reduction over $\bar{F_\upsilon}$. Therefore the image of $\bar{\rho}_{F_\upsilon,p}$ is contained in the stabiliser of this point in $\mathbb{P}^1(\mathbb{Z}/p^{n+1}\mathbb{Z})$, which is a Borel subgroup.

Now, let υ be a place of split multiplicative reduction for E. From the description of E as a Tate curve over F_υ, we see that there is subgroup isomorphic to $\mu[p^{n+1}]$ inside $E[p^{n+1}]$. As before $\mathrm{Gal}(\overline{F_\upsilon}/F_\upsilon)$ will fix this subgroup and hence the image of $\bar{\rho}_{F_\upsilon,p}$ is contained in a Borel subgroup.

Next, we suppose that υ is a place of bad reduction, but not of split multiplicative type. Then by hypothesis, E has either non-split multiplicative or additive and potentially multiplicative reduction. In both cases there exists a quadratic extension L of F_υ, unramified in the first case and ramified in the second, such that E has split multiplicative reduction over L, see page 312 in [Ser72]. Hence $E[p^{n+1}]$ can be described as the set of $\zeta^i \cdot a^j$ with ζ a primitive p^{n+1}-st root of unity, a a p^{n+1}-st root of the Tate-parameter q and $0 \leqslant i,j < p^{n+1}$; but the action of $\sigma \in \mathrm{Gal}(\overline{F_\upsilon}/F_\upsilon)$ is given by $\sigma * (\zeta^i \cdot a^j) = \chi_L(\sigma) \cdot \sigma(\zeta)^i \cdot \sigma(a)^j$ where χ_L is the quadratic character associated to L/F_υ. Therefore the subgroup generated by ζ is still fixed under $\mathrm{Gal}(\overline{F_\upsilon}/F_\upsilon)$.

Finally, we have to treat the case when υ is an infinite place. But for any p, there is a cyclic subgroup of order p^{n+1} in $E(\mathbb{R})$, hence the image is contained in a Borel subgroup. □

Remark: We used here in a crucial way the assumption that p is a prime of ordinary reduction. Certainly it will not hold for places of additive reduction that are potentially supersingular.

Proposition 27. *Suppose that none of the primes of additive reduction for E are potentially good supersingular. Then then extension F_n/F_n^A is nowhere ramified. Moreover all places above ∞, p, and N split completely in this extension.*

Proof. Since F_n is a subfield of $F(E[p^\infty])$ it is clear that it is unramified outside ∞, p and N. By the previous lemma, we know that the decomposition group of a place υ dividing $\infty \cdot p \cdot N$ in F inside H_n is contained in a Borel. Since any Borel intersects $A_n = \mathrm{Gal}(F_n/F_n^A)$ trivially by lemma 25, we have that the places above $\infty \cdot p \cdot N$ in F_n^A split completely. □

8.2 The A-cohomology of the Steinberg representation

Let

$$V_n' = \Bigl\{ f \colon \mathbb{P}^1(\mathbb{Z}/_{p^{n+1}\mathbb{Z}}) \longrightarrow \mathbb{Q} \Bigm| \sum_D f(D) = 0 \Bigr\}$$

be the $\mathbb{Q}[H_n]$-module considered earlier in section 7.2. It is a $\mathbb{Q}$-vector space of dimension $m-1$ with $m = (p+1)\cdot p^n$. There is a natural lattice T_n' in V_n' which is fixed by H_n, defined by

$$T_n' = \Bigl\{ f \colon \mathbb{P}^1(\mathbb{Z}/_{p^{n+1}\mathbb{Z}}) \longrightarrow \mathbb{Z} \Bigm| \sum_D f(D) = 0 \Bigr\}.$$

Lemma 28. *We have*

$$\mathrm{H}^1(A_n, T_n') = \mathbb{Z}/_{m\mathbb{Z}}\,.$$

Proof. Note first that the A_n-fixed part of V_n' is trivial since A_n act transitively on $\mathbb{P}^1(\mathbb{Z}/_{p^{n+1}\mathbb{Z}})$, for a function $f \colon \mathbb{P}^1(\mathbb{Z}/_{p^{n+1}\mathbb{Z}}) \longrightarrow \mathbb{Q}$ that is fixed by A_n would necessarily be constant, but then $\sum_D f(D) = 0$ implies that $f = 0$. Consider now the exact sequence of H_n-modules

$$0 \longrightarrow T_n' \longrightarrow V_n' \longrightarrow T_N'/V_n' \longrightarrow 0\,,$$

which induces an isomorphism

$$\bigl(T_n'/V_n'\bigr)^{A_n} \longrightarrow \mathrm{H}^1(A_n, T_n')$$

since $\mathrm{H}^1(H_n, V_n') = 0$ as V_n' is divisible. So we are looking to determine the A_n-fixed functions in

$$T_n'/V_n' = \Bigl\{ f \colon \mathbb{P}^1(\mathbb{Z}/_{p^{n+1}\mathbb{Z}}) \longrightarrow \mathbb{Q}/_{\mathbb{Z}} \Bigm| \sum_D f(D) = 0 \Bigr\}.$$

Such a function must be constant, since A_n acts transitively, say $f(D) = f_0$. Then $m \cdot f_0 = 0$, so $f_0 \in \frac{1}{m}\mathbb{Z}$ gives the result. □

Proposition 29. *Let U be any lattice in V_n' which is fixed by H_n, then*

$$\#\,\mathrm{H}^1(A_n, U) = m\,.$$

Proof. The lattice U is contained in a scaled version of T'_n with finite index, say

$$0 \longrightarrow U \longrightarrow T'_n \longrightarrow Z \longrightarrow 0\,.$$

Since the Herbrand quotient[2] satisfies $h(A_n, Z) = 1$ for the finite A_n-module Z, we have

$$\#\,\mathrm{H}^1(A_n, U) = h(A_n, U) = h(A_n, T'_n) = \#\,\mathrm{H}^1(A_n, T'_n) = m\,. \qquad \square$$

It is not true in general that $\mathrm{H}^1(A_n, U)$ is cyclic. For $n = 0$, it can have up to three cyclic factors.

8.3 Proof of Theorem 24

We have an injection

$$\iota\colon V'_n \rightarrowtail E(F_n) \otimes \mathbb{Q}$$
$$f \longmapsto \sum_D f(D) \cdot Q_{D}\,.$$

Where Q_D is the higher self-point constructed in section 7.2 and section 7.3. Let S_n be the saturated group generated by the higher self-points in $E(F_n)$, that is

$$S_n = \Big\{ P \in E(F_n) \;\Big|\; \text{there is a } k > 0 \text{ such that } k \cdot P \in \mathbb{Z}[H_n] \cdot Q_D \Big\}\,.$$

By definition all torsion points in $E(F_n)$ belong to S_n, moreover we have

$$0 \longrightarrow E(F_n)_{\mathrm{tors}} \longrightarrow S_n \longrightarrow U_n \longrightarrow 0$$

where U_n can be identified as a H_n-stable lattice in the image of ι. Because there are no A_n-fixed elements in U_n, we find

$$0 \longrightarrow \mathrm{H}^1(A_n, E(F_n)_{\mathrm{tors}}) \to \mathrm{H}^1(A_n, S_n) \to \mathrm{H}^1(A_n, U_n) \longrightarrow$$
$$\longrightarrow \mathrm{H}^2(A_n, E(F_n)_{\mathrm{tors}}) \to \mathrm{H}^2(A_n, S_n) \longrightarrow 0\,.$$

Since the Herbrand quotient $h(A_n, E(F_n)_{\mathrm{tors}})$ is trivial, we find

$$\#\,\mathrm{H}^1(A_n, S_n) = \#\,\mathrm{H}^1(A_n, U'_n) \cdot \#\,\mathrm{H}^1(A_n, S_n) \geqslant \#\,\mathrm{H}^1(A_n, U'_n) = m = (p+1) \cdot p^n$$

by proposition 29. Note also that since $E(F_n)$ has no p-torsion points, we know that

$$\#\,\mathrm{H}^1(A_n, S_n)[p^n] = \#\,\mathrm{H}^1(A_n, U_n)[p^n] = p^n\,.$$

Consider now the natural inclusion of S_n into $E(F_n)$. The cokernel of this inclusion Y_n is a free $\mathbb{Z}$-module. The long exact sequence

$$0 \longrightarrow E(F_n^A)_{\mathrm{tors}} \longrightarrow E(F_n^A) \longrightarrow {Y_n}^{A_n} \longrightarrow \mathrm{H}^1(A_n, S_n) \longrightarrow \mathrm{H}^1(A_n, E(F_n)) \qquad (5)$$

shows that ${Y_n}^{A_n}$ has the same rank as $E(F_n^A)$.

[2] For a finite cyclic group G acting on a G-module A, we define $h(G, A) = \#\,\mathrm{H}^1(G, A)/\#\,\mathrm{H}^2(G, A)$.

Composing the last map in the above sequence with the inflation map will be called the *derivation map*

$$\partial_n\colon\ \mathrm{H}^1(A_n, S_n) \longrightarrow \mathrm{H}^1(A_n, E(F_n)) \xrightarrow{\text{inf}} \mathrm{H}^1(F_n^A, E)\,.$$

Since S_n has no p-torsion elements, we can identify the p^n-torsion part of the source with

$$\Big(\frac{S_n}{p^n\, S_n}\Big)^{A_n} \xrightarrow{\cong} \mathrm{H}^1(A_n, S_n)[p^n]$$

and therefore we call the image of ∂_n the *derived classes* of higher self-points.

Lemma 30. *The image of ∂_n is contained in $Ш(E/F_n^A)$.*

Proof. Let κ be the lift of an element in the image of ∂_n under the map

$$\mathrm{H}^1(F_n^A, E[m']) \longrightarrow \mathrm{H}^1(F_n^A, E)[m']$$

for a sufficiently large m'. Since the extension F_n/F_n^A is non-ramified at a place υ outside the set Σ of places in F_n^A above p, N or ∞, the restriction of κ to $\mathrm{H}^1(F_{n,\upsilon}^A, E[m'])$ will lie in $\mathrm{H}^1_f(F_n^A, E[m'])$. Now for any place υ in Σ, the place υ splits completely in extension F_n/F_n^A by proposition 27, so the restriction of κ to $\mathrm{H}^1(F_{n,\upsilon}^A, E)[m']$ is trivial as it comes from the inflation $\mathrm{H}^1(F_n/F_n^A, E(F_n)) \xrightarrow{\text{inf}} \mathrm{H}^1(F_n^A, E)$. Hence κ belongs to the Selmer group within $\mathrm{H}^1(F_n^A, E[m'])$. □

We can now end the proof of theorem 24. Denote by s the minimal number of generators of the kernel of ∂_n. From the long exact sequence (5), we see that the rank of $Y_n^{A_n}$ is at least s. So, if ∂_n is not injective, then $\mathrm{rank}(E(F_n^A))$ is positive. So either the image of ∂, lifted to the Selmer group, will contribute p^n elements or else $E(F_n^A)$ will give rise to a copy of ${}^{\mathbb{Z}}\!/_{p^n\mathbb{Z}}$ in $\mathrm{Sel}_{p^n}(E/F_n^A)$. □

We add here a comment on the case when E has supersingular reduction at p. It turns out that construction of derivative classes in $\mathrm{H}^1(F_n^A, E)$ using higher self-points works the same, provided that the higher self-points are of infinite order. The main difference is that the cohomology classes do not belong to the Tate-Shafarevich group. In fact, under the assumption that the derivative map is not trivial, they will provide classes that are orthogonal to elements from the Selmer group and could be used to bound the Selmer group from above; just like Kolyvagin's classes built from Heegner points. Unfortunately we do not know a way of proving the assumption and hence these derivative classes can not be used to say something about the Selmer group.

8.4 Derivative of self-points

Rather than constructing derivative classes of higher self-point, we can also produce cohomology classes from self-points. We only sketch here the results whose proofs are in the similar to the previous sections.

Let $E/\mathbb{Q}$ be an elliptic curve of conductor N. Assume for simplicity that $N = p$ is prime. Put $K = K_p$. It is known that ρ_p is surjective, see [DW08] for more details. So the Galois group $G = \mathrm{Gal}(K/\mathbb{Q})$ is isomorphic to $\mathrm{PGL}_2(\mathbb{F}_p)$. Let A be any cyclic subgroup of order $p+1$ in G.

Theorem 31. *There is map ∂ to the Tate-Shafarevich $Ш(E/K^A)$ from a group of order at least $p+1$. If this map is not injective, then there are points of infinite order defined over K^A that only become divisible in $E(K)$. If r is the difference of the rank of $E(\mathbb{Q}(C))$ and $E(\mathbb{Q})$, then*

$$\mathrm{Sel}_{p+1}(E/K^A) \geqslant (p+1)^r \cdot \#E(\mathbb{Q})[p+1]\,.$$

As before we consider the saturation of the self-points S in $E(K)$. We know that S modulo its torsion-part is a lattice U in the Steinberg representation of $\mathrm{PGL}_2(\mathbb{F}_p)$. As we have seen in section 8.2, the cohomology group $\mathrm{H}^1(A,U)$ will have $p{+}1$ elements. In section 4 of [DW08], we have computed the torsion subgroup of $E(K)$. Using this we can compute that $E(K^A)_{\mathrm{tors}} = E(\mathbb{Q})_{\mathrm{tors}}$ and that

$$\mathrm{H}^1(A, E(K)_{\mathrm{tors}}) = \mathrm{H}^2(A, E(K)_{\mathrm{tors}}) = \begin{cases} \mathbb{Z}/2\mathbb{Z} \\ 0 \end{cases}$$

with the non-trivial case exactly when E is one of the curves 17a2, 17a3, 17a4 or any Neumann-Setzer curve. As before this shows that $\mathrm{H}^1(A,S)$ has either $p+1$ or $2\,(p+1)$ elements. The derivative map is again

$$\partial\colon\ \mathrm{H}^1(A,S) \longrightarrow \mathrm{H}^1(A,E(K)) \longrightarrow \mathrm{H}^1(K^A,E)$$

and its image is in the Tate-Shafarevich group $Ш(E/K^A)$.

We should add here that the control theorem for the Selmer group is not necessarily perfect; the kernel of

$$\mathrm{Sel}_{p+1}(E/K^A) \longrightarrow \mathrm{Sel}_{p+1}(E/K)$$

can be of order 1 or 2.

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