Exceptional holonomy 8-manifold with SU(3)xSU(2)xU(1) isometry

Background:

Exceptional holonomy 7-manifolds are of importance for compactifications of M-theory. They are closely related to exceptional holonomy 8-manifolds

Both are also of importance for mathematics, as the last entries in Berger's classification list to be understood. Also of importance for Donaldson's programme to construct invariants of 8-manifolds by repeating the success of the 4D story.

Closed examples are difficult to construct (metrics are not explicit), similar to how closed hyper Kahler metrics in 4D are never explicit.

But cohomogeneity one examples (complete but not closed) are understood, with the first examples coming from work by Bryant and Salamon, and then many more examples constructed by physicists.

For our purposes, we are interested in 8D examples. These are of two types:

Asymptotically conical: $ds^2 = dr^2 + r^2 g^7_{\text{base}}$

Circle of finite size at infinity

Asymptotically locally conical: $ds^2 = dr^2 + r^2 g^6_{\text{base}} + d\phi^2$

In both cases, a cohomogeneity one metric is characterised by the geometry of its 7D base, which is a homogeneous group space G/H

Before constructing concrete examples, useful to recall basic facts

Cayley form $\Phi \in \Lambda^4(M)$ is a differential form of a special algebraic type. It defines on M a Riemannian metric and a triple cross product.

Alternatively, it defines on M a Riemannian metric g_Φ and a unit spinor Ψ_Φ

Proposition: The Riemannian metric g_Φ has holonomy contained in Spin(7) if and only if $d\Phi = 0$

Alternatively, when the spinor Ψ_Φ is parallel $\nabla_{g_\Phi} \Psi_\Phi = 0$

Similar statements in 7D

A generic 3-form $\Omega \in \Lambda^3(M)$ of a positive type defines on M a Riemannian metric as well as a cross-product.

Alternatively, it defines Riemannian metric g_Ω as well as a unit spinor Ψ_Ω

Proposition: The Riemannian metric g_Ω has holonomy contained in G2 if and only if $d\Omega = 0, \quad d^*\Omega = 0$

Alternatively, when the spinor Ψ_Ω is parallel $\nabla_{g_\Omega} \Psi_\Omega = 0$

Remark: Can replace the parallel spinor condition with the Killing spinor condition $(X\nabla_{g_\Omega})\Psi_\Omega = X \cdot \Psi_\Omega$

Equivalent to $d\Omega = {}^*\Omega$ One gets so-called nearly parallel G2 structures

Proposition: The cone metric $ds^2 = dr^2 + r^2 g^7_{\text{base}}$

has holonomy contained in Spin(7) if and only if the base carries the geometry of a nearly parallel G2 structure

Remark: Similar statement for cones on 6D bases requires the base to carry the so-called nearly Kahler geometry

The conical exceptional holonomy 8D manifolds are classified by the number of the Killing spinors on the cone base

3 Killing spinors: Base is 3-Sasakian, and cone is hyper Kahler

2 Killing spinors: Base is Sasakian, and cone is Calabi-Yau, that is holonomy SU(4)

1 Killing spinor: Cone has holonomy Spin(7)

From paper: Nearly parallel G2 structures, Friedrich, Kath, Moroianu, Semmelmann

For purposes envisaged (isometry GSM), we are interested in the case with two Killing spinors

In this case, the 7D base is actually a circle bundle over a 6D Kahler-Einstein manifold

This fits with the known construction by Calabi of a SU(n) holonomy metric in the total space of a complex line bundle over a KE (n-1)-manifold

For the case of interest, the 6D KE space is just the product $X^6 = \mathbb{CP}^2 \times S^2$

Metric explicitly (from Duff, Nilsson "Kaluza-Klein supergravity"), Section 9.2

$$ds^2_{\mathbb{CP}^2} = \left(\frac{6}{\Lambda_4}\right)[d\mu^2 + \frac{1}{4}\sin^2\mu(\sigma_1^2 + \sigma_2^2 + \cos^2\mu\sigma_3^2)]$$

$$ds^2_{S^2} = \left(\frac{1}{\Lambda_2}\right)[d\theta^2 + \sin^2\theta d\phi^2]$$

Both are Kahler-Einstein

Note that their product is only Einstein if $\Lambda_2 = \Lambda_4$ It is then also Kahler-Einstein

These metrics are then put together into a U(1) bundle over $\mathbb{CP}^2 \times S^2$

$$ds^2 = c^2(d\tau + m\sin^2\mu\,\sigma_3 + n\cos\theta d\phi)^2 + ds^2_{\mathbb{CP}^2} + ds^2_{S^2}$$

Here m,n are integers characterising the U(1) bundle.

The exterior derivative of these 1-forms gives the Kahler forms on the two spaces

Connections on non-trivial U(1) bundles over $\mathbb{CP}^2, S^2$

The Ricci tensor is

$$R_{00} = c^2(\frac{4}{9}m^2\Lambda_4^2 + \frac{1}{2}n^2\Lambda_2^2),$$

$$R_{ij} = \Lambda_4(1 - \frac{2}{9}c^2m^2\Lambda_4)\delta_{ij}, \quad i,j = 1,2,3,4$$

$$R_{ij} = R_{ij} = \Lambda_2(1 - \frac{1}{2}c^2n^2\Lambda_2)\delta_{ij}, \quad i,j = 5,6$$

Einstein condition is then an algebraic problem $x := \frac{\Lambda_4}{\Lambda_2}$

$$\Lambda_4 = \frac{4x}{1+2x}\Lambda \qquad \Lambda_2 = \frac{4}{1+2x}\Lambda$$

$$c^{-2} = \frac{4(4m^2x^2 + 9n^2)}{9(1+2x)}\Lambda$$

$$\left(\frac{m}{n}\right)^2 = \frac{9(2x-1)}{4x^2(3-2x)}$$

Given m/n, we get a cubic equation for x, which has exactly one real root $1/2 \leq x \leq 3/2$

The case $x = 1/2$ gives $\mathbb{CP}^2 \times S^3$ Isometry SU(3) x SO(4)

The case $x = 3/2$ gives $S^5 \times S^2$ Isometry SO(6) x SO(3)

All other cases have isometry SU(3) x SU(2) x U(1)

We are interested in the remaining special case $x = 1$ or $\Lambda_4 = \Lambda_2$ or $m = 3, n = 2$

This is the case when is Kahler-Einstein

The arising 7D metric is the special case of the Calabi construction, which gives a SU(4) holonomy metric in the total space of the complex line bundle over a KE 6-manifold

Explicitly, if set $\Lambda = 3/4$

$$ds^2 = \frac{1}{8}\left(d\tau + 3\sin^2\mu\,\sigma_3 + 2\cos\theta\,d\phi\right)^2$$
$$+6(d\mu^2 + \frac{1}{4}\sin^2\mu(\sigma_1^2 + \sigma_2^2 + \cos^2\mu\sigma_3^2) + (d\theta^2 + \cos^2\theta d\phi^2)$$

The cone over this has holonomy SU(4) and isometry SU(3)xSU(2)xU(1)

The resolution of this cone is provided by the Calabi metric in the total space of the complex line bundle over $\mathbb{CP}^2 \times S^2$

Calabi construction

On a Kahler manifold we have ω $d\omega = 0$ Closed symplectic form

Ω Top holomorphic form $d\Omega + ia\Omega = 0$

This defines a complex structure by declaring it to be (n,0) form. This complex structure is integrable as a consequence of this equation

These data satisfy $\Omega\Omega = 0, \quad \Omega\omega = 0$

$\Omega\bar{\Omega} \sim \omega^n$ where the proportionality coefficient is a constant

In addition, in terms of these data the Einstein condition becomes the statement that $da = s\omega$ where s is a multiple of scalar curvature

Example of the 2-sphere

$$\omega = \frac{4dz \wedge d\bar{z}}{(1+|z|^2)^2} \qquad \Omega = \frac{2dz}{1+|z|^2} \qquad a = \frac{1}{i}\frac{zd\bar{z} - \bar{z}dz}{1+|z|^2} \qquad da = \frac{1}{2}\omega$$

Calabi considers the total space of the $\Lambda^{n,0}$ line bundle

Let t be the complex coordinate along the fibre

Introduce $\theta := dt - iat$ Then $d|t|^2 = \bar{t}dt + td\bar{t} = \bar{t}(\theta + iat) + t(\bar{\theta} - ia\bar{t}) = \bar{t}\theta + t\bar{\theta}$

This will be the (1,0) form along the fibres

Consider $\tilde{\omega} = u\,\omega + \frac{1}{is}u'\theta \wedge \bar{\theta}$ Here u is a function of $|t|^2$ and prime denotes the derivative

Proposition: $\tilde{\omega}$ is closed

Proof: $d\theta = -is\omega t + ia\theta$

$$d\tilde{\omega} = u'(\bar{t}\theta + t\bar{\theta})\omega + \frac{1}{is}u'(-is\omega t + ia\theta)\bar{\theta} - \frac{1}{is}u'\theta(is\omega\bar{t} - ia\bar{\theta})$$

The $a\theta\bar{\theta}$ terms cancel, as do the other terms as well

Proposition: The form $\theta \wedge \Omega = d(t\Omega)$ is exact, and so closed

Proof:

$$d(t\Omega) = dt \wedge \Omega - ita\Omega = \theta \wedge \Omega$$

Proposition: The algebraic constraints are satisfies $\tilde{\omega} \wedge \theta \wedge \Omega = 0$

These data thus define an integrable complex structure in which $\theta \wedge \Omega$ is the top holomorphic form, and $\tilde{\omega}$ is a (1,1) Kahler form

The metric defined by these data will be Einstein if $\tilde{\omega}^{n+1} \sim \theta \wedge \bar{\theta} \wedge \Omega \wedge \bar{\Omega}$

This gives the equation $\frac{1}{s}u^n u' = const$ Whose solution is $u^{n+1} = const(s|t|^2 + K)$

Has holonomy contained in SU(n+1)

This gives the following Calabi-Yau metric

$$ds^2 = (s|t|^2 + K)^{\frac{1}{n+1}} ds^2_{KE} + \frac{1}{n+1}(s|t|^2 + K)^{-\frac{n}{n+1}}|dt - iat|^2$$

Example: Applying this construction to S2 we get Eguchi-Hanson instanton

Proposition: This metric is asymptotically conical, with the circle bundle over the KE metric as the base of the cone

Conclusion: The regular, complete metric that has holonomy SU(4) and isometry SU(3) x SU(2) x U(1) is obtained by applying Calabi's construction to $\mathbb{CP}^2 \times S^2$

Asymptotically, this is the conical metric explicitly described before