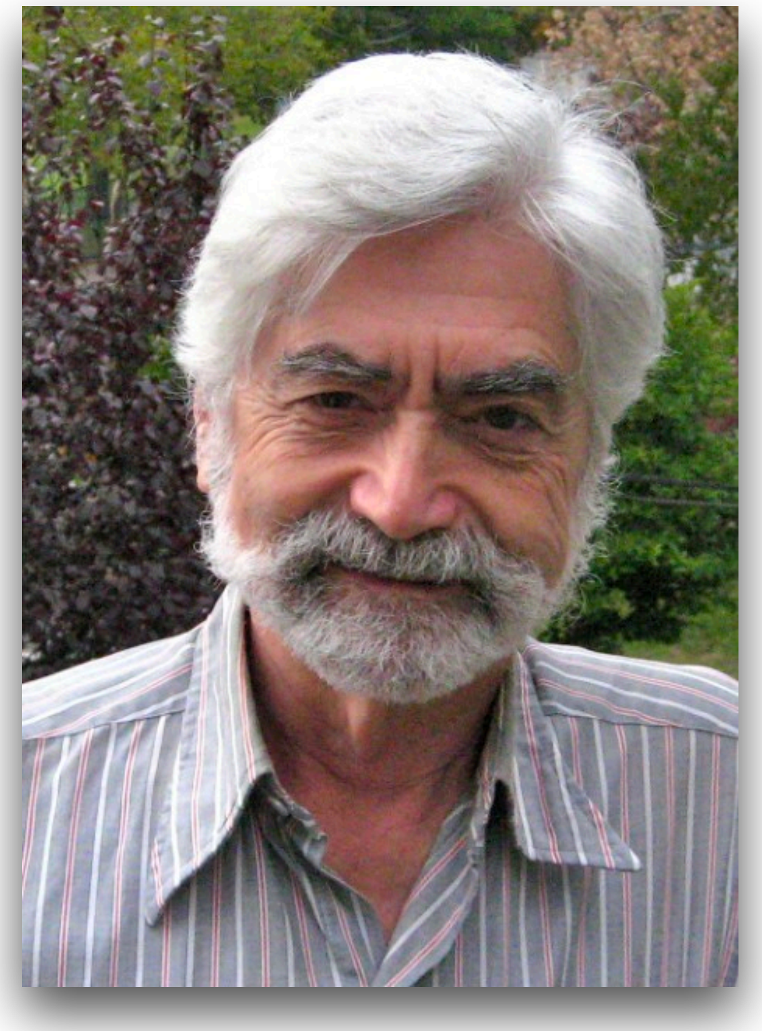
Hidden patterns in the Standard Model of particle physics:

The geometry of SO(10) unification

Kirill Krasnov (Nottingham)

Standard Model Timeline

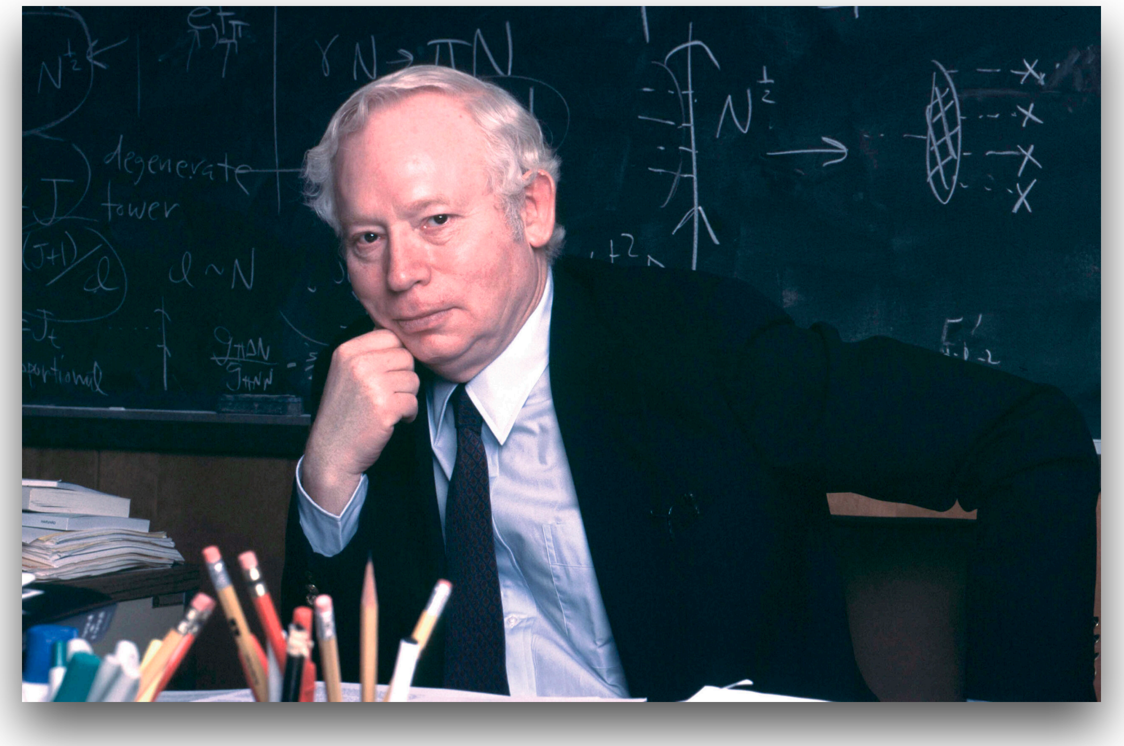
Murray Gell-Mann



George Zweig

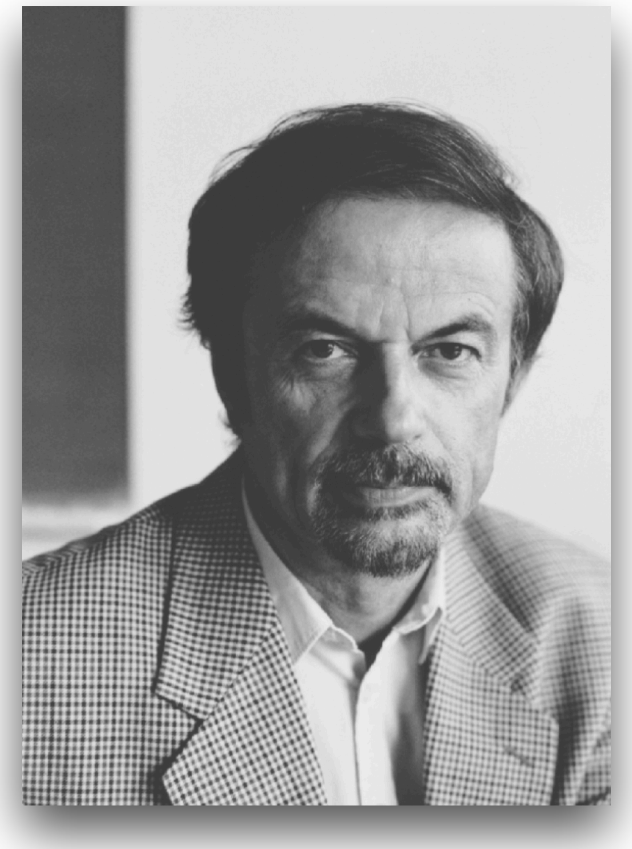
1964: Quark model (with three quarks: up, down, strange) was proposed to explain proliferation of new particles

1967: Electroweak SU(2)xU(1) theory proposed



Steven Weinberg

Abdus Salam



Harald Fritzsch

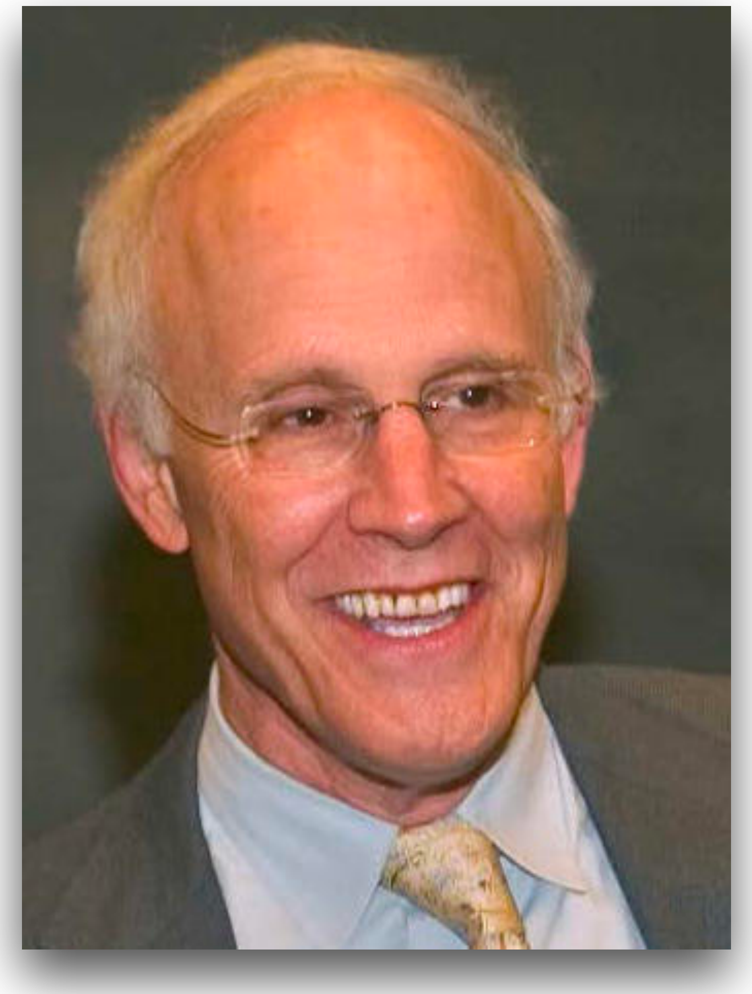
Murray Gell-Mann

1973: SU(3) gauge theory of quarks and gluons proposed - QCD

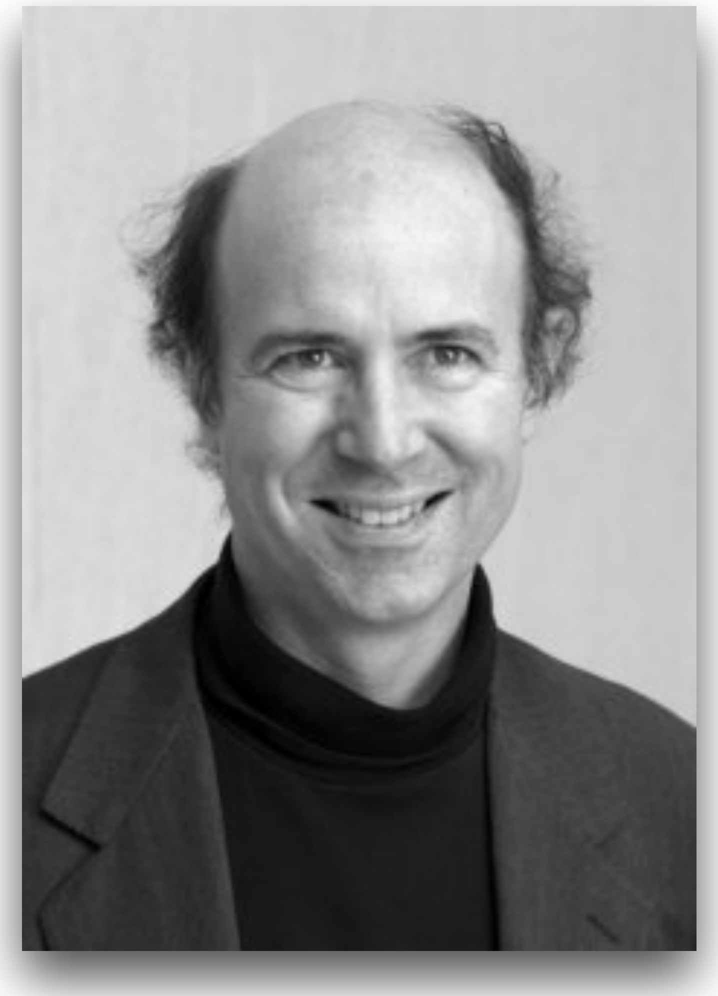
50 Years of the SM in 2023

1973: Discovery of asymptotic freedom in QCD

David Politzer



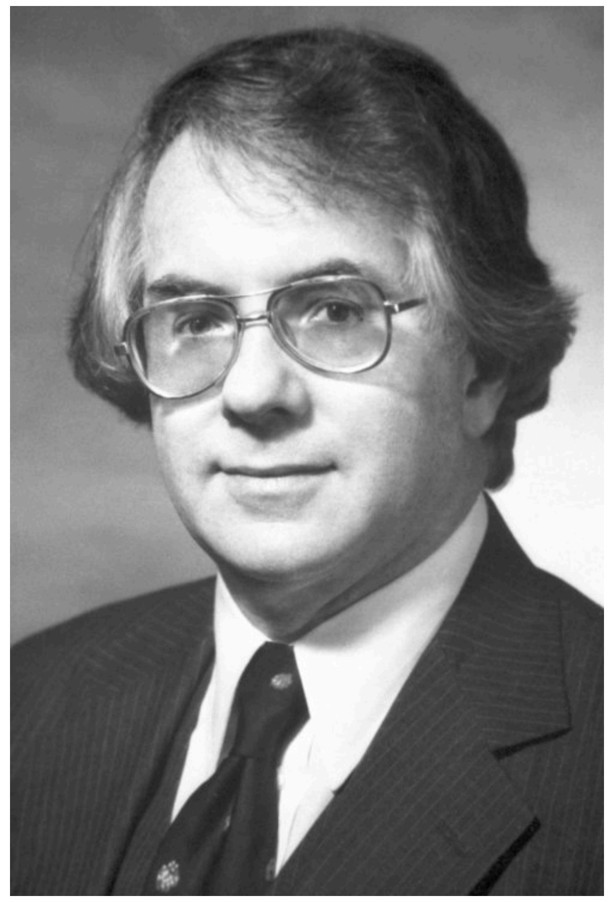
David Gross



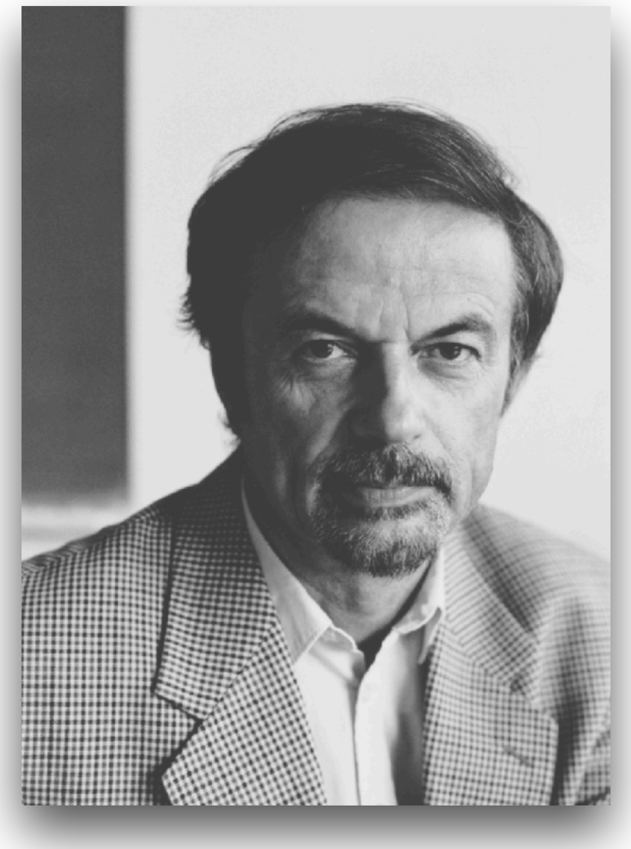
Frank Wilczek

Beyond the Standard Model

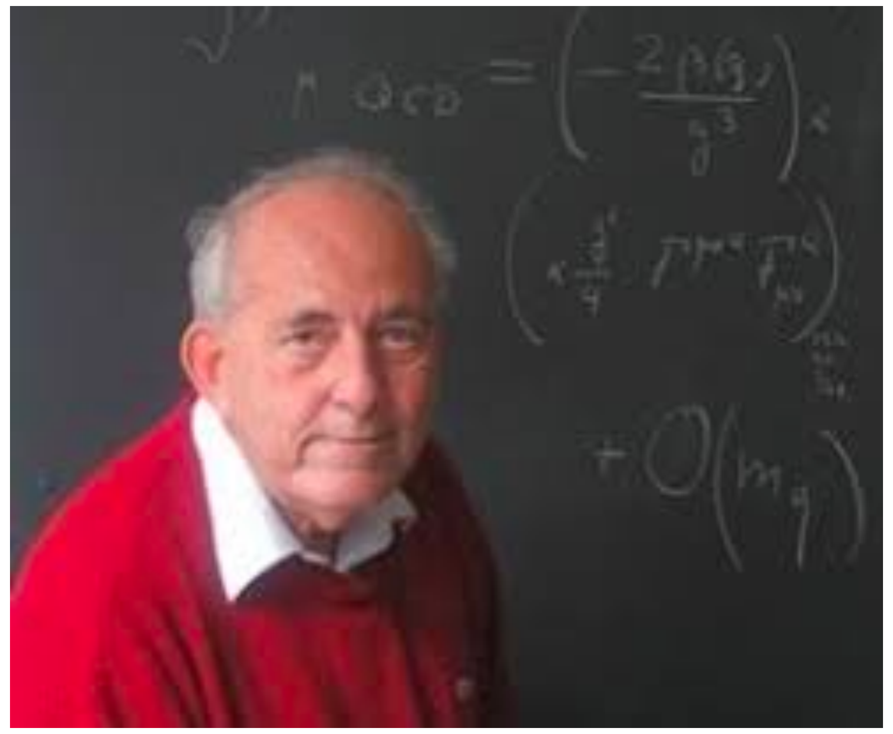
Howard Georgi



Sheldon Glashow



Harald Fritzsch



Peter Minkowski

1973-4: SO(4)xSO(6), SO(10) and SU(5) Grand Unified Theories discovered

Intriguingly, Georgi discovers SO(10) GUT before SU(5), and pursues the latter with Glashow because it is simpler

See e.g. Howard Georgi, “The future of grand unification”, 2007

Current status of GUT

The minimal SU(5) GUT is ruled out experimentally (proton decay)

SUSY GUT models have now less appeal, after no low energy SUSY was seen by the LHC. No SUSY in this talk

Many GUT models can be constructed - the choice is in the Higgs field content, renormalizable vs. higher dimension operators

Models are complicated (need several different Higgs fields)

Many (but not all) simply predict no new physics till very high energies - and so not testable

After no convincing progress in this direction over the last 50 years, there is certain fatigue and loss of interest

As the result, what was universally known by the community in the 70's and 80's is no longer transferred to the younger generation of researchers. One of the goals of this talk is an attempt to rectify this - here in PI.

Aims of the talk

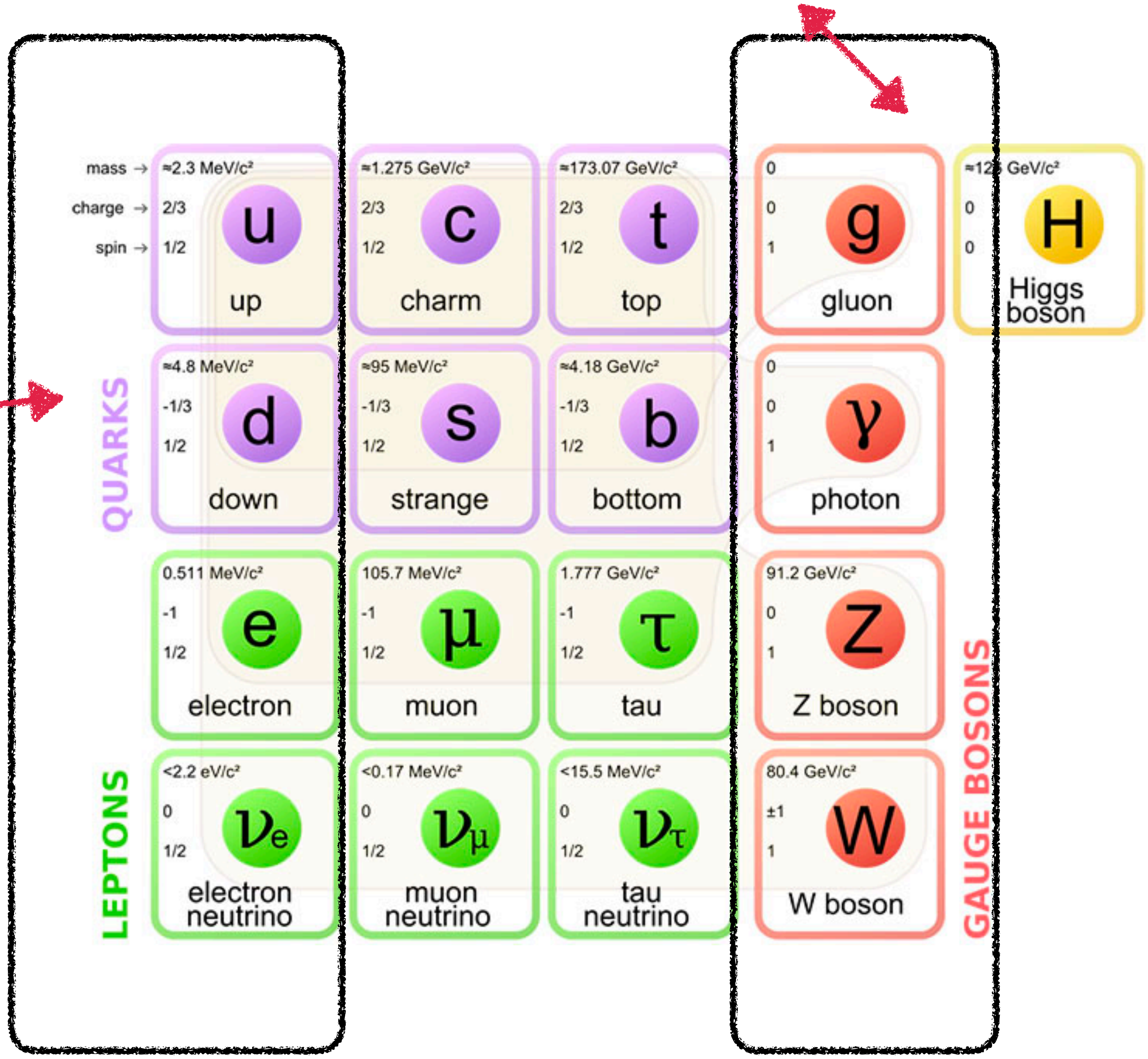
- Describe the basics of SO(10) GUT, concentrating on explaining known facts in as simple terms as possible
- This is not a phenomenology talk - I will concentrate on math (geometry) rather than physics
- Factually, very little if anything is new in my presentation
- The point of view is not the standard one. Symmetry breaking in geometric terms rather than in terms of Higgs fields.

Geometry of SO(10) symmetry breaking

Standard Model fermions

The goal of the talk is to explain what SM fermions are and how they are described. I will ignore the part of the SM describing the dynamics of the gauge fields.

I will mostly concentrate on one (first) generation



Dirac Fermions - Lorentz spinors

Every fermionic particle in the table is a Dirac fermion = Lorentz spinor, which is described by the Dirac Lagrangian

The Standard Model gauge group

$$G_{\mathrm{SM}} = \mathrm{SU}(3) \times \underbrace{\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y}$$

Strong force

Electroweak force

Every particle transforms in a representation of this gauge group

The complication is that each particle has two components (left-handed and right-handed) and these transform as different representations - one says that SM is chiral

To understand this, we need to describe a Dirac fermion in more detail

Spinors - first encounter

Gamma-matrices $$\gamma_0 = \begin{pmatrix} 0 & -\mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}, \qquad \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \qquad i = 1, 2, 3$$

$$\mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$ Pauli matrices

Satisfy the Clifford algebra relations $\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\eta_{\mu\nu}$ $\qquad \eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$

Minkowski metric

Dirac spinor $\Psi = \begin{pmatrix} \xi \\ \tilde{\eta} \end{pmatrix}$ $\qquad \xi, \tilde{\eta} \in \mathbb{C}^2$ two component columns with complex entries

We will say that

$\xi \in S_+$ is a left-handed 2-component spinor

$\tilde{\eta} \in S_-$ is a right-handed 2-component spinor

Lie algebra of the Lorenz group $\mathfrak{so}(1,3)$ is generated by the products of distinct gamma-matrices

$$X(A) := \frac{1}{4} A^{\mu\nu} \gamma_\mu \gamma_\nu \qquad [X(A), X(B)] = X_{[A,B]} \qquad ([A, B])^{\mu\nu} := A^{\mu\rho} \eta_{\rho\sigma} B^{\sigma\nu} - B^{\mu\rho} \eta_{\rho\sigma} A^{\sigma\nu}$$

Gamma-matrices are off-diagonal $\gamma : S_+ \to S_-$ $\gamma : S_- \to S_+$

Products of an even number of gamma-matrices preserve the spaces $S_\pm$

In particular the Lie algebra $\mathfrak{so}(1,3)$ preserves $S_\pm$

We will say that the 2-component spinors in $S_\pm$ are Weyl spinors (to distinguish them from 4-component Dirac spinors)

Weyl spinors are irreducible representations of the Lorentz group. The Dirac spinor is a reducible representation

Inner product: (Lorentz) invariant inner product on $S_\pm$ is anti-symmetric

$$\langle \xi_1, \xi_2 \rangle = \xi_1^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \xi_2, \qquad \xi_{1,2} \in S_+$$

Similarly on S_-

Charge conjugation:

There is an invariant anti-linear (i.e. involving complex conjugation) map $* : S_+ \to S_-$

$$S_- \ni \xi^* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \bar{\xi}, \qquad \xi \in S_+$$

Complex-conjugate 2-component spinor

Remark: the notions of invariant inner product and charge conjugation have analogs in any dimension and in any signature

Weyl Lagrangian

The Dirac Lagrangian is the kinetic term for a Dirac fermion

A Dirac spinor is a pair of Weyl spinors $\Psi = \begin{pmatrix} \xi \\ \tilde{\eta} \end{pmatrix}$ $\xi \in S_+, \tilde{\eta} \in S_-$

Given that we have the charge conjugation operation $* : S_+ \to S_-$

It is very convenient to parametrise all right-handed spinors as charge conjugates of left-handed ones

can always parametrise $\tilde{\eta} = \eta^*$ so that $\Psi = \begin{pmatrix} \xi \\ \eta^* \end{pmatrix}$, $\xi, \eta \in S_+$

The Dirac Lagrangian then splits as the sum of two kinetic terms for the Weyl spinors $\xi, \eta \in S_+$

Define $\not{\partial} : S_+ \to S_-$ $\not{\partial} := -\mathbb{I}\partial_t + \sigma_i \partial_i \equiv \sigma^\mu \partial_\mu$

$$S[\xi] := i \int_{\mathbb{R}^{1,3}} \langle \xi^*, \not{\partial} \xi \rangle$$

Lorentz and translation invariant. Can be seen to be real by the integration by parts argument

When the spinor also transforms in some representation of some gauge group, we make the Lagrangian gauge-invariant by extending the derivative to the covariant derivative

$$S[\xi, A] := i \int_{\mathbb{R}^{1,3}} \langle \xi^*, \sigma^\mu (\partial_\mu + A_\mu) \xi \rangle$$

This describes how fermions interact with gauge fields

Particles of the SM

We now describe the particle content of one (first) generation of the SM

We describe everything in terms of 2-component (Weyl) spinors, and use left-handed spinors to parametrise all particles

Representations of $G_{\text{SM}} = \text{SU}(3) \times \text{SU}(2)_L \times \text{U}(1)_Y$ needed to describe one generation

Particles	SU(3)	SU(2)	Y	T^3	$Q = T^3 + Y$
$Q = \begin{pmatrix} u \\ d \end{pmatrix}$	triplet	doublet	$1/6$	$1/2$ $-1/2$	$2/3$ $-1/3$
$\bar{u}$	anti-triplet	singlet	$-2/3$	0	$-2/3$
$\bar{d}$			$1/3$		$1/3$
$L = \begin{pmatrix} \nu \\ e \end{pmatrix}$	singlet	doublet	-1/2	$1/2$ $-1/2$	0 -1
$\bar{e}$	singlet	singlet	1	0	1

This is impossible to remember unless you work with it every day. SO(10) GUT provides the organising principle, from which this table can be derived

15 particles here, counting those of different colour separately

Remark: Here bar is just part of the name of the 2-component Weyl spinor. It is not complex conjugation. Its charge conjugate is the right-handed 2-component spinor that is part of the Dirac spinor needed to describe this particle

Spinors in higher D and Clifford Algebras

Clifford algebras are algebras generated by the higher D analogs of the already encountered gamma-matrices

Spinors are “columns” on which gamma-matrices act

Definition: Given a (real) vector space V, with a metric $(\cdot, \cdot)$ on it, the Clifford algebra Cl(V) is the algebra generated by vectors from V subject to the relation $uv + vu = 2(u, v)$

Or, more concretely, assuming the metric on V is positive definite and choosing an orthonormal basis, the Clifford algebra is the one generated by the gamma-matrices satisfying $\gamma_i \gamma_j + \gamma_j \gamma_i = \delta_{ij} \mathbb{I}$

As a vector space, Clifford algebra is spanned by products of distinct gamma-matrices, and has dimension 2^n where n is the dimension of V

The deep classification result states that Clifford algebras are matrix algebras over $\mathbb{R}, \mathbb{C}, \mathbb{H}$ or sometimes direct sum of two such algebras, depending on dimension and signature

Spinors are “columns” on which these matrix algebras act, or irreducible representations of Cl

Group Spin(V) is the group generated by the products of an even number of Clifford elements of squared norm one

It is the double cover of the special orthogonal group SO(V)

Spinors of Spin(2n)

The goal now is to present a concrete and efficient model for Cl(2n) and the Spin group in Euclidean space $\mathbb{R}^{2n}$ of an arbitrary (even) dimension.

Spin(2n) is the double cover of SO(2n) $\quad \mathrm{SO}(2n) = \mathrm{Spin}(2n)/\mathbb{Z}_2$

This model arises if one chooses a complex structure in $\mathbb{R}^{2n}$

Definition: An (orthogonal) complex structure in $\mathbb{R}^{2n}$ is a map $J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that $J^2 = -\mathbb{I}$ and $(Ju, Jv) = (u, v), \forall u, v \in \mathbb{R}^{2n}$

Example: $\mathbb{R}^2$ $\quad J\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$

There are no real eigenvectors. But any real vector can be split into a complex eigenvector, plus its complex conjugate

$$\mathbb{R}^{2n} = \mathbb{C}^n \oplus \overline{\mathbb{C}^n}$$

The +i eigenspace of J. Can be seen to be totally null

Example: $\mathbb{R}^2$ $\quad$ The +i eigenspace of J is spanned by $\begin{pmatrix} 1 \\ -i \end{pmatrix}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{x+iy}{2}\begin{pmatrix} 1 \\ -i \end{pmatrix} + \frac{x-iy}{2}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

Complex structures are very important because they provide a real viewpoint on various complex Lie groups

Proposition: The subgroup of $\mathrm{GL}(2n, \mathbb{R})$ that commutes with a fixed complex structure on $\mathbb{R}^{2n}$ is $\mathrm{GL}(n, \mathbb{C})$

The subgroup of $\mathrm{SO}(2n, \mathbb{R})$ that commutes with a fixed orthogonal complex structure on $\mathbb{R}^{2n}$ is $\mathrm{U}(n)$

Rephrasing this, the choice of U(n) subgroup of SO(2n) is the choice of an orthogonal complex structure on $\mathbb{R}^{2n}$

Fundamental fact about spinors:

U(n) is also a subgroup of Spin(2n), and the restriction of the spinor representation S of Spin(2n) to U(n) is

$$S\Big|_{\mathrm{U}(n)} = \Lambda\mathbb{C}^n$$

All anti-symmetric complex tensors in n dimensions, or all complex differential forms

Rephrasing, restricting to U(n), we get a very concrete and powerful model of Spin(2n), Cl(2n) and spinors

Spinors concretely

$$\dim(S) = 2^n$$

Spinor is a general “inhomogeneous” differential form

$$S \equiv \Lambda\mathbb{C}^n \ni \psi = a + \sum_{i=1}^{n} a_i dz^i + \sum_{i<j} a_{ij} dz^i \wedge dz^j + \ldots + a_{1\ldots n} dz^1 \wedge \ldots \wedge dz^n$$

where all the coefficients are complex

We introduce the creation/annihilation operators

$$a_i^\dagger \psi := dz^i \wedge \psi$$

$$a_i \psi := (d/dz^i) \lrcorner \psi$$

These satisfy the fermionic algebra

$$a_i^\dagger a_j + a_j a_i^\dagger = \delta_{ij} \qquad i, j = 1, \ldots, n$$

We now define $\gamma_i = i(a_i - a_i^\dagger), \qquad \gamma_{i+n} = a_i + a_i^\dagger$

Can easily check that satisfy the Clifford defining relations $\gamma_I \gamma_J + \gamma_J \gamma_I = 2\delta_{IJ} \qquad I, J = 1, \ldots, 2n$

And so we get a concrete model of the Clifford algebra, the space of spinors, and the Spin group

Additional information about spinors

Differential forms split into even and odd degree ones

$$S = \Lambda\mathbb{C}^n = \Lambda^{even} \oplus \Lambda^{odd} \equiv S_+ \oplus S_-$$

Creation/annihilation operators and thus gamma-matrices map even into odd and vice versa

But elements of Spin(2n) preserve the "chirality"

$S_\pm$ are irreducible representations of Spin(2n). Known as chiral, or Weyl spinors

Proposition: $\langle \psi_1, \psi_2 \rangle := \tilde{\psi}_1 \wedge \psi_2 \Big|_{top}$

is the Spin(2n) invariant bilinear inner product on S

Here tilde is the operation that reverses the order of all elementary 1-forms dz^i

To get a number one restricts to the top form

Proposition: The product of n gamma-matrices containing the imaginary unit, followed by the complex conjugation, or the product of all gamma-matrices not containing i, again followed by complex conjugation, are invariant anti-linear maps on S. They agree on $S_\pm$ modulo sign.

Rephrasing, there exists "charge conjugation", an anti-linear map that either preserves $S_\pm$ or sends one to the other, depending on the dimension. When n=0 mod(4) there exist Majorana-Weyl spinors.

Spinors of Spin(10)

Particles of one generation of the SM can be fit into a single Weyl spinor representation of Spin(10)

For concreteness, we will work with that of odd degree differential forms. Bit easier to see how particles fit here

$$S_- = \Lambda^1\mathbb{C}^5 + \Lambda^3\mathbb{C}^5 + \Lambda^5\mathbb{C}^5$$

$\dim_{\mathbb{C}}(S_-) = 16$ $\quad 5 \quad 10 \quad 1$

SM particles will fit here

This is where the right-handed neutrino lives

$\mathrm{U}(5) \subset \mathrm{Spin}(10)$ preserves this decomposition

The subgroup of U(5) that also fixes a given vector in $\Lambda^5\mathbb{C}^5$ is $\mathrm{SU}(5)$

Rephrasing, we can break Spin(10) to Georgi-Glashow SU(5) by taking the Higgs field $\mathbf{16}_H$

and selecting it to point in the direction of the right-handed neutrino

Alternatively, the same is achieved by choosing a complex structure on $\mathbb{R}^{10}$

as well as a top form in $\mathbb{C}^5$

There is a connection to so-called pure spinors here that I don't have time to discuss

SM gauge group as subgroup of SU(5)

The key observation is that $G_{\rm SM} = {\rm SU}(3) \times {\rm SU}(2) \times {\rm U}(1)$

is precisely the subgroup of SU(5) that preserves the split $\mathbb{C}^5 = \mathbb{C}^3 \oplus \mathbb{C}^2$

Indeed, having made a choice of such a split, the subgroup that preserves it consists of matrices

$$ {\rm SU}(3) \times {\rm SU}(2) \times {\rm U}(1) \ni (g_3, g_2, e^{i\phi}) \to \begin{pmatrix} e^{-i\phi/3} g_3 & 0 \\ 0 & e^{i\phi/2} g_2 \end{pmatrix} \in {\rm SU}(5) $$

The only choice made here was that of the overall factor in front of ϕ on the right-hand-side

With this choice vectors in $\mathbb{C}^3$ will have fractional 1/3 charges, which is correct for quarks

and those in $\mathbb{C}^2$ will have half-integer Y charges, which is again correct

Now let's try to match components of the spinor with particles. We start with $\Lambda^1 \mathbb{C}^5 = \Lambda^1 \mathbb{C}^3 \oplus \Lambda^1 \mathbb{C}^2$

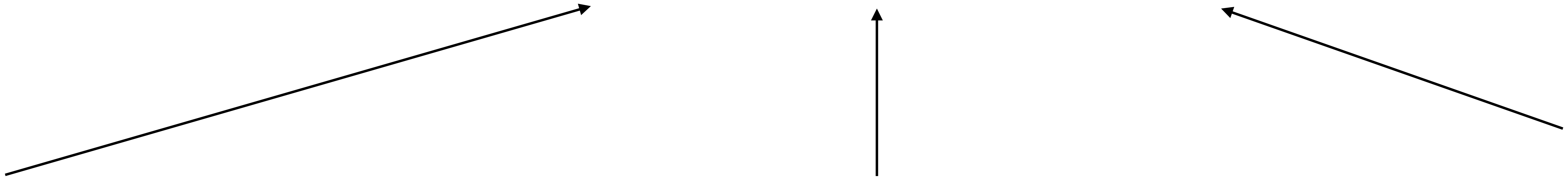
The only particles that can be identified with the SU(2) doublet are $L = \begin{pmatrix} \nu \\ e \end{pmatrix}$ of Y charge -1/2

The SU(3) triplet will then have the Y charge of 1/3, and so must be identified with $\bar{d}$

But it is anti-triplet, and so we must correct the identification $\Lambda^1 \overline{\mathbb{C}}^5 = \Lambda^1 \overline{\mathbb{C}}^3 \oplus \Lambda^1 \overline{\mathbb{C}}^2 = \bar{d} \oplus L$

It is then an exercise to compute the decomposition

$$\Lambda^3(\overline{\mathbb{C}}^3 \oplus \overline{\mathbb{C}}^2) = \Lambda^3\overline{\mathbb{C}}^3 \oplus (\Lambda^2\overline{\mathbb{C}}^3 \otimes \overline{\mathbb{C}}^2) \oplus (\Lambda^1\overline{\mathbb{C}}^3 \otimes \Lambda^2\overline{\mathbb{C}}^2)$$



SU(3) singlet of Y charge +1

SU(2) doublet, SU(3) triplet of Y charge 2/3-1/2=1/6

SU(3) anti-triplet of Y charge 1/3-1=-2/3

$\bar{e}$

Q

$\bar{u}$

To summarise, the subgroup of SU(5) that preserves the 3+2 split is $G_{\text{SM}} = \text{SU}(3) \times \text{SU}(2) \times \text{U}(1)$

And the Weyl spinor of Spin(10) splits as

$$\Lambda^{odd}\overline{\mathbb{C}}^5 = \Lambda^1(\overline{\mathbb{C}}^3 \oplus \overline{\mathbb{C}}^2) \oplus \Lambda^3(\overline{\mathbb{C}}^3 \oplus \overline{\mathbb{C}}^2) \oplus \Lambda^5(\overline{\mathbb{C}}^3 \oplus \overline{\mathbb{C}}^2) = (\bar{d} \oplus L) \oplus (\bar{e} \oplus Q \oplus \bar{u}) \oplus \bar{\nu}$$

All particles fit perfectly, and after fitting the first pair the Y-charges of the rest are correctly predicted!

Surely, the mother Nature is telling us that we are on the right track here!

All this was known already 50 years ago, to Howard Georgi in particular

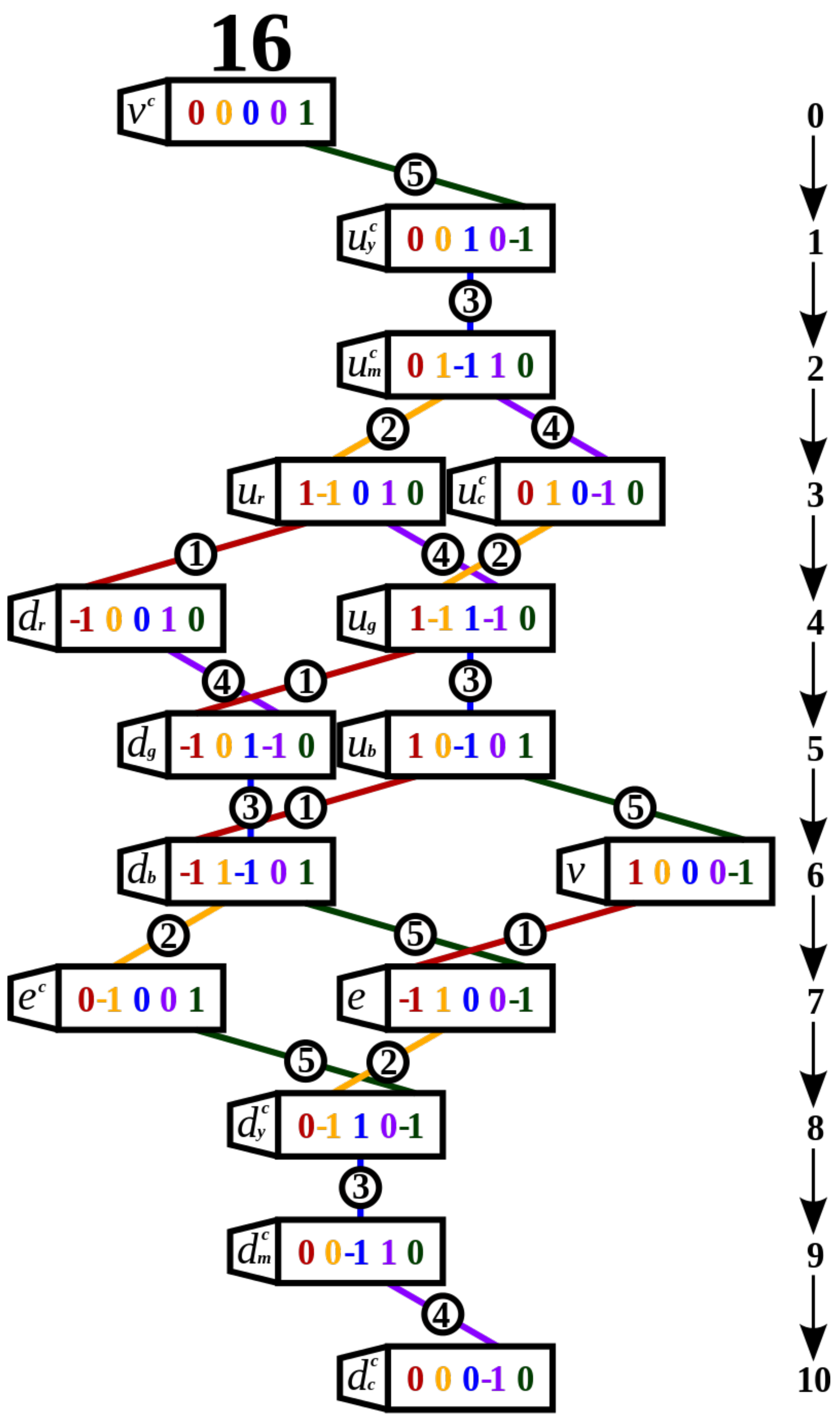
Except that he thought about spinors of Spin(10) differently - weights

The machinery of roots and weights is very powerful, and allows to talk about any representation of any simple Lie algebra

But one needs much more preparatory steps to see how the particles fit into a single Weyl spinor of Spin(10)

In contrast, with our method that describes only simplest representations - spinor, vector, adjoint - we derived the fit by completely elementary means

We don’t need to remember the particle content of the SM - it can be derived



One only needs to remember that the SU(2) doublet L of leptons has Y charge of -1/2

And this is easy to remember because $Q = Y + T_3$ eigenvalues of $T_3 = \pm 1/2$

and we want the electric charges to be $Q_\nu = 0, Q_e = -1$ Y charges of all other particles arising are fixed.

Symmetry breaking

The key elements of the symmetry breaking that led to $G_{\rm SM} = {\rm SU}(3) \times {\rm SU}(2) \times {\rm U}(1)$ were

1) Choice of a spinor pointing in the direction of the right-handed neutrino, to break to SU(5)

1) Choice of $\mathbb{C}^5 = \mathbb{C}^3 \oplus \mathbb{C}^2$ split

In model building with smallest Higgs representations one chooses

$\mathbf{16}_H$ to effect the first step of this symmetry breaking

$\mathbf{Adj} \equiv \mathbf{45}_H$ for the second step. These two must be appropriately aligned, which is dynamically non-trivial

Finally, one usually takes the SM Higgs to reside in $\mathbf{10}_H$

This minimal model is not phenomenologically viable, for it cannot give the right Yukawa couplings

Options are bigger Higgs representations and/or non-renormalizable, higher dimension operators

This is why no convincing Spin(10) GUT model emerges

New observation

From paper arXiv:2209.05088

The choice of a split $\mathbb{C}^5 = \mathbb{C}^3 \oplus \mathbb{C}^2$ is nothing else but a choice of a second complex structure $\tilde{J}$ on $\mathbb{R}^{10}$ that commutes with J that gives U(5)

To see this, we need to introduce the notion of (an orthogonal) product structure

Definition: An orthogonal product structure K on $\mathbb{R}^{2n}$ is a linear map $K : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that $K^2 = \mathbb{I}$ and $(Ku, Kv) = (u, v)$

Proposition: An orthogonal product structure K on splits $\mathbb{R}^{2n}$ orthogonally into the two eigenspaces of K

$$\mathbb{R}^{2n} = V^+ \oplus V^- \quad \text{where} \quad KV^\pm = \pm V^\pm \quad \text{and} \quad (V^+, V^-) = 0$$

Proposition: The subgroup of Spin(2n) that commutes with an orthogonal complex structure K is $\mathrm{Spin}(k) \times \mathrm{Spin}(2n-k)$ with any $k = 0, \ldots, 2n$ possible

Definition: We call an orthogonal product structure K on $\mathbb{R}^{2n}$ and an orthogonal complex structure J on $\mathbb{R}^{2n}$ compatible if they commute $[K, J] = 0$

Proposition: Given an orthogonal complex structure J on $\mathbb{R}^{2n}$ and a compatible with it (commuting) orthogonal product structure K, defines the split $\mathbb{C}^n = \mathbb{C}^k \oplus \mathbb{C}^{n-k}, \quad k = 0, \ldots, n$

Proof: Indeed, both operators can be simultaneously diagonalised, and so we have

$$J\mathbb{C}^n = +i\mathbb{C}^n \quad \text{and} \quad K\mathbb{C}^k = \mathbb{C}^k, \quad K\mathbb{C}^{n-k} = -\mathbb{C}^{n-k}$$

Proposition: The subgroup of Spin(2n) that commutes with a commuting pair J,K is the intersection of

$$\mathrm{U}(n) \quad \text{and} \quad \mathrm{Spin}(2k) \times \mathrm{Spin}(2n-2k)$$

Proposition: Given a pair of commuting (orthogonal) complex structures $J, \tilde{J}$ the product $K = J\tilde{J}$ is an orthogonal product structure compatible with both $J, \tilde{J}$

In the opposite direction, given a pair commuting complex and product structures J, K their product $\tilde{J} = JK$ is another complex structure that commutes with J

Compatible complex, product structure ⇔ Two commuting complex structures

Application to Spin(10)

Thus, a pair of commuting complex structures on $\mathbb{R}^{10}$ gives either

$$\mathbb{C}^5 = \mathbb{C}^3 \oplus \mathbb{C}^2 \quad \text{or} \quad \mathbb{C}^5 = \mathbb{C}^4 \oplus \mathbb{C}^1$$

The option relevant for the SM

Not completely irrelevant, comments below

Theorem: The Standard Model gauge group $G_{\rm SM} = {\rm SU}(3) \times {\rm SU}(2) \times {\rm U}(1)$ is the subgroup of Spin(10) that is the intersection of SU(5) that fixes a complex structure J and its holomorphic top form, and another U(5) that fixes $\tilde{J} : [J, \tilde{J}] = 0$ with this second complex structure such that the resulting split is of the type $\mathbb{C}^5 = \mathbb{C}^3 \oplus \mathbb{C}^2$

Remark: The only other option with such a pair of commuting complex structures is that they define the split of the type $\mathbb{C}^5 = \mathbb{C}^4 \oplus \mathbb{C}^1$ In this case, the common stabiliser of all the data is ${\rm SU}(4) \times {\rm U}(1)$

Remark: Complex structure together with the top holomorphic form for it is encoded in a spinor of a special algebraic type - pure spinor. Thus, the SM gauge group arises as one stabilising a pure spinor and another pure spinor projectively. The condition on this pair of pure spinors that guarantees that what arises is $\mathbb{C}^5 = \mathbb{C}^3 \oplus \mathbb{C}^2$ can be stated as conditions minimising certain potential, see the paper cited.

Final remarks

When both structures $\mathbb{C}^5 = \mathbb{C}^3 \oplus \mathbb{C}^2$ and $\mathbb{C}^5 = \mathbb{C}^4 \oplus \mathbb{C}^1$ are present

we get the split $\mathbb{C}^5 = \mathbb{C}^3 \oplus \mathbb{C}^1 \oplus \mathbb{C}^1$ which is the structure relevant to the SM after the EW symmetry breaking

So, we can effect all of the symmetry breaking from Spin(10) to SU(3)xU(1) by a collection of commuting complex structures with their holomorphic top forms, or just complex structures

Complex structures with their holomorphic top forms are described by pure spinors. Complex structures are described by projective pure spinors. Conditions that associated complex structures commute are easy to impose as potential minimising conditions on the pure spinors. Even the types of arising splitting $\mathbb{C}^5 = \mathbb{C}^3 \oplus \mathbb{C}^2$ or $\mathbb{C}^5 = \mathbb{C}^4 \oplus \mathbb{C}^1$ can be controlled by the potentials. See the paper cited.

Conclusion: There exists a Spin(10) GUT model whose Higgs fields is a collection of $\mathbf{16}_H$

Two is not enough, three is sufficient, but probably the nicest model arises when one takes four.

Summary

- SM was proposed exactly 50 years ago, and almost immediately all GUT's were discovered.
- Spin(10) GUT is provides ultimate unification, where both forces and particles (of one generation) are unified. Also predicts the right-handed neutrino.
- There is not yet a convincing concrete Spin(10) GUT. But the representation theory provides a welcome organising principle. One needs to know very little to derive the SM particle content with all the charges.
- The only thing to remember is that one needs to choose SU(5) and then $\mathbb{C}^5 = \mathbb{C}^3 \oplus \mathbb{C}^2$

Outlook

- I argued that the best way to think about the symmetry breaking is in terms of commuting complex structures.
- One needs two commuting complex structures to see $G_{\text{SM}} = \text{SU}(3) \times \text{SU}(2) \times \text{U}(1)$
- Geometry with two commuting complex structures is known under the name of bi-Hermitian geometry.
- Complex structures are naturally parametrised by spinors of special algebraic type - pure spinors.
- Where all this points to: An overlooked Spin(10) GUT that only has Higgs fields in $\mathbf{16}_H$