Colour/Kinematics Duality and the Drinfeld Double of the Lie algebra of Diffeomorphisms

Based on 1603.02033 with Chih-Hao Fu

Kirill Krasnov
(Nottingham)

erc

3 Single Cask Whiskies: Morning After

Kirill Krasnov

Take home message

"Why" C/K works at 4 points:

There is a non-trivial Lie-algebraic structure encoded by the YM Feynman rules

C/K at 4 points is just the Jacobi identity

Directly related to the diffeomorphisms!

Certain natural structure built from the algebra of vector fields with its Lie bracket and the metric

Why I got interested in the subject

I am interested in Lagrangian formulations of GR and YM that "simplify things"

Colour-Kinematics, while an on-shell statement, suggests that there is a cubic formulation of YM with some remarkable properties

So this project resulted from the desire to understand colour/kinematics off-shell

Difference with the "normal" viewpoint

Strong on-shell viewpoint in this community: this is what simplifies computations

On-shell kinematic numerators are more or less equivalent to colour-ordered amplitudes: both provide the (n-3)! basis in which all amplitudes can be decomposed

But there exists a hint of off-shell Lie-algebraic structure: the self-dual sector story of Monteiro and O'Connell

What we don't have

Usual Feynman rules do not lead to C/K dual numerators beyond 4 points

The structure that we observe at 4 points suggests that one must modify the Feynman rules

Higher point vertices required, but this time with clear Lie algebraic interpretation

Certain problem at 4 points prevents us from making the next step

Plan

- A Lie-algebraic viewpoint on YM Feynman rules
- Drinfeld Double
- 5 points and modification of Feynman rules
- Prospects and conclusions

Part I: YM Feynman rules

$$L = \frac{1}{4}(F^a_{\mu\nu})^2 \qquad \Rightarrow$$

$$L^2 = \frac{1}{2}(\partial_\mu A^a_\nu)^2 - \frac{1}{2}(\partial^\mu A^a_\mu)^2$$

$$L^3 = f^{abc}\partial^\mu A^{\nu a} A^b_\mu A^c_\nu$$

$$L^4 = \frac{1}{4} f^{abc} f^{cef} A^{\mu a} A^{\nu b} A^e_\mu A^f_\nu$$

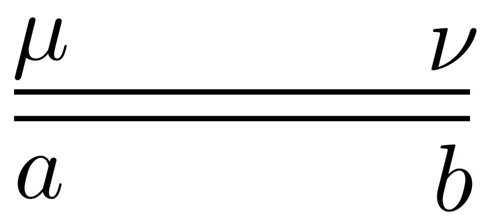
Gauge-fixing

$$L_{g.f.} = \frac{1}{2}(\partial^\mu A_\mu^a)^2$$

Ghosts not relevant as consider only tree level

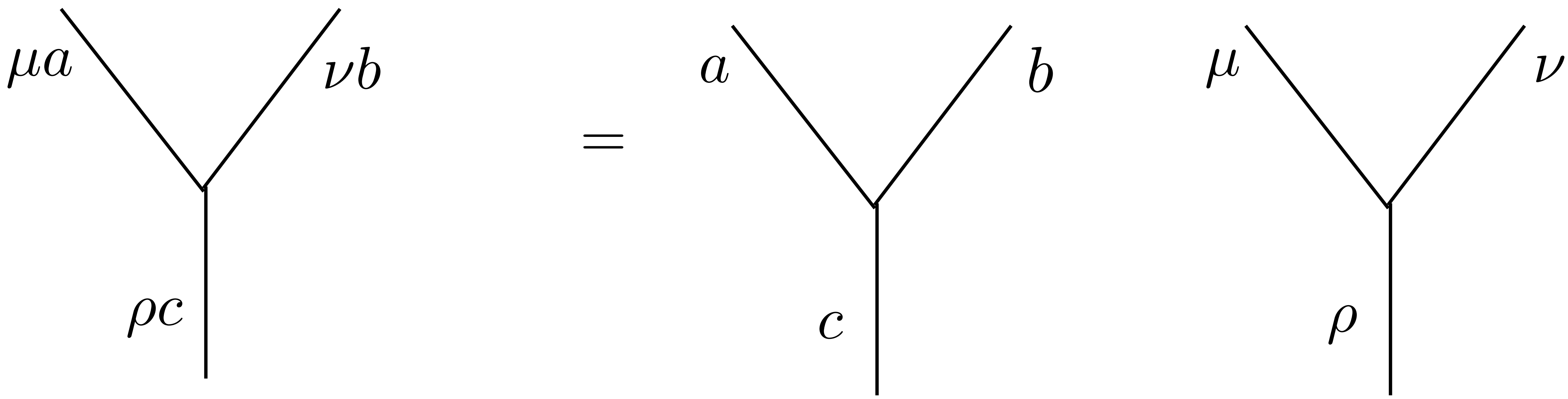
Propagator

$$\langle A_\mu^a A_\nu^b \rangle \ = \frac{i}{k^2}$$



Cubic vertex factor

$$\langle A_\mu^a(k_1) A_\nu^b(k_2) A_\rho^c(k_3)\rangle = f^{abc} v_{\mu\nu\rho}(k_1, k_2, k_3)$$



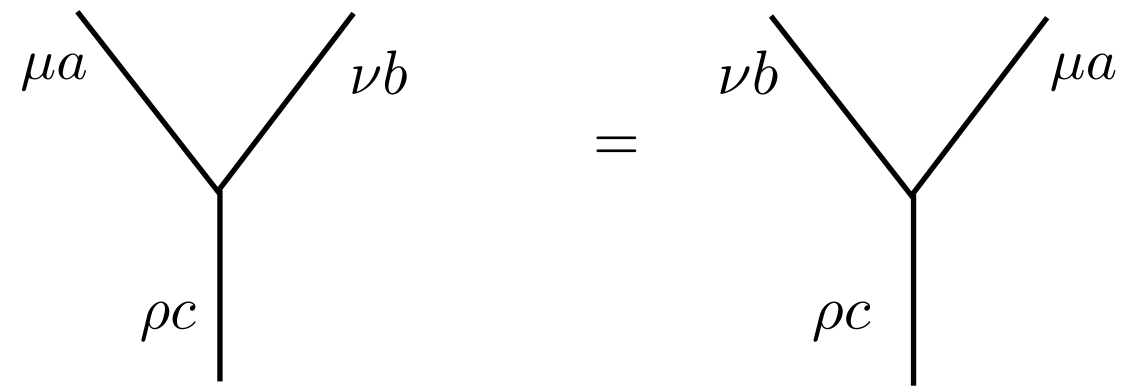
Convenient to use placeholders - vector fields

$$(\xi_1 \xi_2) \equiv \eta_{\mu\nu} \xi_1^\mu \xi_2^\nu \qquad v_{\mu\nu\rho}(k_1, k_2, k_3) \xi_1^\mu \xi_2^\nu \xi_3^\rho \equiv v(\xi_1, \xi_2, \xi_3)$$

$$v(\xi_1, \xi_2, \xi_3) = (\xi_1 k_2)(\xi_2 \xi_3) - (\xi_2 k_1)(\xi_1 \xi_3)$$
$$+(\xi_2 k_3)(\xi_3 \xi_1) - (\xi_3 k_2)(\xi_2 \xi_1) + (\xi_3 k_1)(\xi_1 \xi_2) - (\xi_1 k_3)(\xi_3 \xi_2)$$

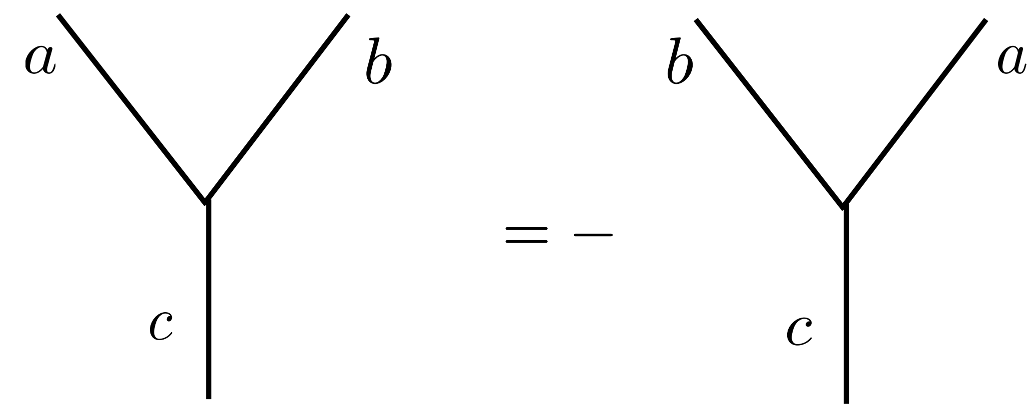
Symmetry

Vertex factor completely symmetric

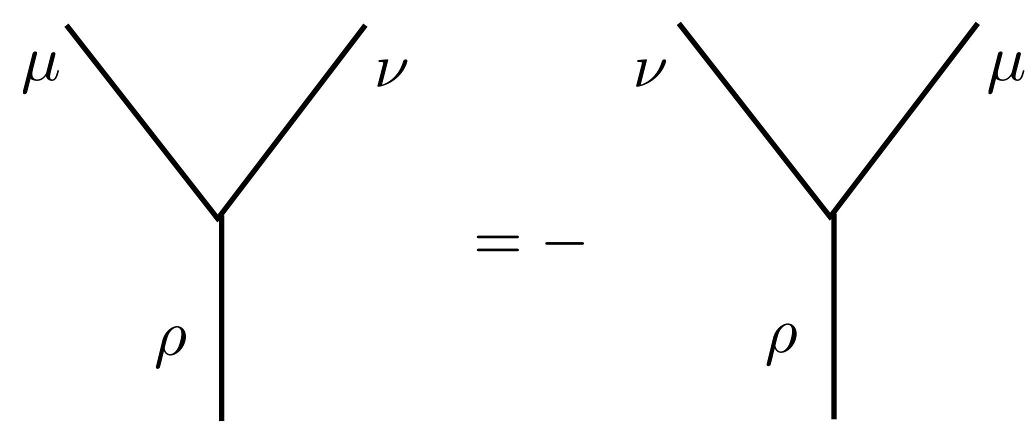


Structure constant anti-symmetric

$\Rightarrow$



Kinematic factor anti-symmetric



Interpreting kinematic factor as a bracket

Define a bracket on vector fields

$$[\cdot,\cdot]_{YM} : TM \times TM \to TM$$

Does not satisfy Jacobi!

via

$$v(\xi_1,\xi_2,\xi_3) = ([\xi_1,\xi_2]_{YM}\xi_3)$$

Explicitly, using momentum conservation $k_3 = -k_1 - k_2$

$$[\xi_1,\xi_2] = 2(\xi_1 k_2)\xi_2 - 2(\xi_2 k_1)\xi_1 + (\xi_1\xi_2)(k_1 - k_2) + (\xi_1 k_1)\xi_2 - (\xi_2 k_2)\xi_1$$

Can introduce such a bracket for any theory where the cubic vertex factor factorises into the product of anti-symmetric structure constant and the kinematic factor - e.g. NLSM

Lie bracket

There exists a different bracket that does satisfy Jacobi

$$[\cdot,\cdot] : TM \times TM \to TM$$

$$[\xi_1, \xi_2] = (\xi_1 k_2)\xi_2 - (\xi_2 k_1)\xi_1$$

Relation between two brackets

$$v(\xi_1, \xi_2, \xi_3) \equiv ([\xi_1, \xi_2]_{YM} \xi_3) =$$
$$([\xi_1, \xi_2]\xi_3) + ([\xi_2, \xi_3]\xi_1) + ([\xi_3, \xi_1]\xi_2)$$

YM bracket is constructed from the Lie bracket and the metric

Diffeos are seen relevant already in YM: May be this is why double copy works?

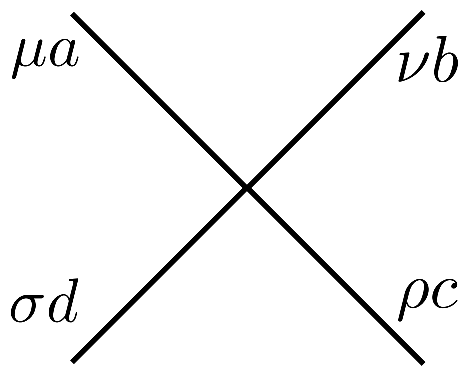
Some terminology:

$$v(\xi_1, \xi_2, \xi_3) = v(\xi_2, \xi_3, \xi_1) = -v(\xi_2, \xi_1, \xi_3)$$

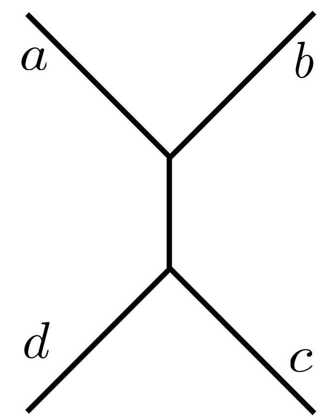
completely anti-symmetric and thus 3-cochain, i.e. object in $\Lambda^3 \mathfrak{g}^*$

Interpreting the 4-vertex

$$\langle A^a_\mu(k_1) A^b_\nu(k_2) A^c_\rho(k_3) A^d_\sigma(k_4)\rangle = f^{abe} f^{ecd} v^s_{\mu\nu\rho\sigma} + f^{bce} f^{ead} v^t_{\mu\nu\rho\sigma} + f^{cae} f^{ebd} v^u_{\mu\nu\rho\sigma}$$

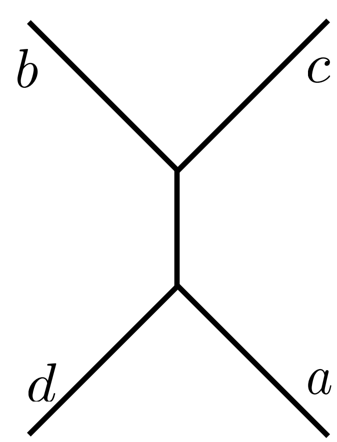


$=$



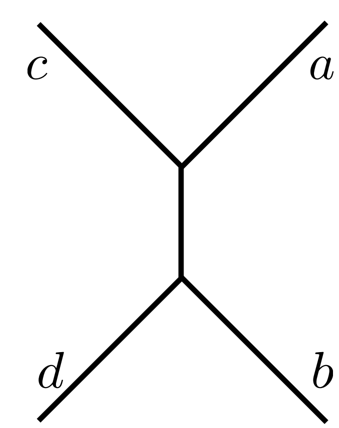
$v^s_{\mu\nu\rho\sigma}$

$+$



$v^t_{\mu\nu\rho\sigma}$

$+$



$v^u_{\mu\nu\rho\sigma}$

The 4-cochain

This is formed by multiplying/dividing by the missing propagators

$$v(\xi_1,\xi_2,\xi_3,\xi_4) := (k_1+k_2)^2 v^s(\xi_1,\xi_2,\xi_3,\xi_4)$$
$$+(k_2+k_3)^2 v^t(\xi_1,\xi_2,\xi_3,\xi_4) + (k_3+k_1)^2 v^u(\xi_1,\xi_2,\xi_3,\xi_4)$$

with the obvious notation

$$v^s(\xi_1,\xi_2,\xi_3,\xi_4) := v^s_{\mu\nu\rho\sigma}\xi_1^\mu \xi_2^\nu \xi_3^\rho \xi_4^\sigma$$

Explicitly

$$v^s(\xi_1,\xi_2,\xi_3,\xi_4) = (\xi_1\xi_3)(\xi_2\xi_4) - (\xi_2\xi_3)(\xi_1\xi_4)$$

The resulting $v_4(\xi_1,\xi_2,\xi_3,\xi_4)$ is 4-cochain, i.e. object in $\Lambda^4\mathfrak{g}^*$

An identity

$$(k_1+k_2)^2-(k_2+k_3)^2=k_1^2+2(k_1k_2)-k_3^2-2(k_2k_3)$$

By momentum conservation also equals

$$(k_3+k_4)^2-(k_1+k_4)^2=k_3^2+2(k_3k_4)-k_1^2-2(k_1k_4)$$

Adding get

$$(k_1+k_2)^2-(k_2+k_3)^2=((k_1-k_3)(k_2-k_4))$$

Hence

$$-v_4(\xi_1,\xi_2,\xi_3,\xi_4)=((k_1-k_2)(k_3-k_4))(\xi_1\xi_2)(\xi_3\xi_4)$$
$$+((k_2-k_3)(k_1-k_4))(\xi_2\xi_3)(\xi_1\xi_4)+((k_3-k_1)(k_2-k_4))(\xi_3\xi_1)(\xi_2\xi_4)$$

Explanation of the C/K duality in the self-dual sector

Lemma 1: On vector fields - polarisations of same helicity, YM bracket coincides with the Lie bracket

Lemma 2: Vector fields - polarisations of same helicity - form a sub algebra wrt Lie bracket

These are just the area preserving diffeos of Riccardo and Donal

Since 4-valent vertex is irrelevant in self-dual sector computations (e.g. Berends-Giele current), above lemmas imply C/K duality in the self-dual sector

Consistent with finding diffeos as relevant in YM self-dual sector by Monteiro O'Connell

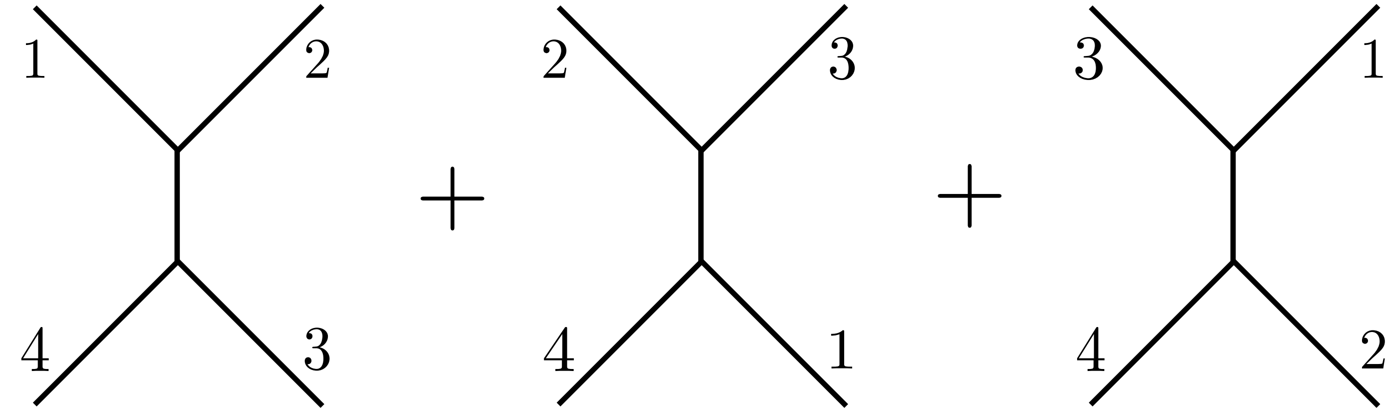
Jacobi for the YM bracket

Can now compute the Jacobiator of the YM bracket $J : \Lambda^3 \mathfrak{g} \to \mathfrak{g}$

$$[\xi_1, \xi_2, \xi_3] := [[\xi_1, \xi_2]_{YM}, \xi_3]_{YM}$$
$$+[[\xi_2, \xi_3]_{YM}, \xi_1]_{YM} + [[\xi_3, \xi_1]_{YM}, \xi_2]_{YM}$$

$$J(\xi_1, \xi_2, \xi_3, \xi_4) := ([\xi_1, \xi_2, \xi_3]\xi_4)$$

$$J(\xi_1, \xi_2, \xi_3, \xi_4) =$$



Result

$$J(\xi_1,\xi_2,\xi_3,\xi_4) = -v_4(\xi_1,\xi_2,\xi_3,\xi_4)$$
$$+(\xi_1 k_1)v(\xi_2,\xi_3,\xi_4) - (\xi_2 k_2)v(\xi_1,\xi_3,\xi_4)$$
$$+(\xi_3 k_3)v(\xi_1,\xi_2,\xi_4) - (\xi_4 k_4)v(\xi_1,\xi_2,\xi_3)$$

Modulo on-shell vanishing terms, the Jacobiator is cancelled by the contribution of the 4-valent vertex

This is why C/K duality works at 4 points

Is there some algebraic structure that "explains" the above Jacobi-like identity?

Part II: Drinfeld Doubles

Vast subject, can't possibly even begin to review here

This is the classical counterpart of the quantum group

In one of the several possible axiomatic versions, a Lie-algebra structure on some Lie algebra and its dual

$$D = \mathfrak{g} \oplus \mathfrak{g}^* \qquad \text{as vector space}$$

There is a canonical metric on the Drinfeld Double

Let us choose a basis in both spaces $\qquad e_i \in \mathfrak{g}, \quad e^i \in \mathfrak{g}^*$

Then the metric is given by

$$G := e_i \otimes e^i + e^i \otimes e_i \in D^* \otimes D^*$$

Bracket on the double

The bracket on D is such that the metric G is invariant

$$\langle X, Y \rangle := G(X, Y) \qquad \langle [Z, X], Y \rangle + \langle X, [Z, Y] \rangle = 0$$

The bracket on D reduces to the bracket on $\mathfrak{g}$

The simplest possible double is when $[\mathfrak{g}^*, \mathfrak{g}^*] = 0$

Then get $[\mathfrak{g}, \mathfrak{g}^*]$ from the requirement that the metric G is invariant

Explicitly if $[e_i, e_j] = C_{ij}^k e_k$

Then $\langle [e_i, e^j], e_k \rangle = -\langle e^j, [e_i, e_k] \rangle = -\langle e^j, C_{ik}^l e_l \rangle = -C_{ik}^j$

$$\Rightarrow \quad [e_i, e^j] = -C_{ik}^j e^k$$

Theorem: the bracket so constructed satisfies Jacobi identity

Drinfeld Double of the Lie algebra of vector fields

$$D = TM \oplus \Lambda^1 M$$

$$[\xi_1, \xi_2] = \mathcal{L}_{\xi_1}\xi_2 - \mathcal{L}_{\xi_2}\xi_1$$

$$[\xi, \eta] = \mathcal{L}_\xi \eta + \eta(\partial \xi)$$

$$[\eta_1, \eta_2] = 0$$

This term depends on a volume form on M

Flat metric volume form

Have an invariant metric on D

$$G(\xi, \eta) = \int_M dv\, (i_\xi \eta)$$

In momentum space

$$[\xi, \eta] = (\xi(k_1 + k_2))\eta + (\xi\eta)k_1$$

Interpreting the YM bracket

We have metric $g : TM \to \Lambda^1 M$ $\quad \xi^\mu \to \xi^*_\mu = g_{\mu\nu}\xi^\nu$

Consider elements of the form $\xi + \xi^* \in D$

The pairing of two such elements with DD metric

$$\langle \xi_1 + \xi_1^*, \xi_2 + \xi_2^* \rangle = 2(\xi_1 \xi_2)$$

is multiple of their metric pairing

Let us compute their DD bracket

$$\langle [\xi_1 + \xi_1^*, \xi_2 + \xi_2^*], \xi_3 + \xi_3^* \rangle = \langle [\xi_1, \xi_2], \xi_3^* \rangle + \langle [\xi_1, \xi_2^*], \xi_3 \rangle + \langle [\xi_1^*, \xi_2], \xi_3 \rangle$$

using the invariance of the metric

$$= \langle [\xi_1, \xi_2], \xi_3^* \rangle + \langle [\xi_3, \xi_1], \xi_2^* \rangle + \langle [\xi_2, \xi_3], \xi_1^* \rangle$$

Precisely the YM bracket

Orthogonal complement

To elements of the form $\xi + \xi^* \in D$

are elements of the form $\xi - \xi^* \in D$

$$D = \mathfrak{u} \oplus \mathfrak{u}^\perp \qquad \begin{array}{l} \xi + \xi^* \in \mathfrak{u} \\ \xi - \xi^* \in \mathfrak{u}^\perp \end{array}$$

The key point is that $\mathfrak{u}$ is not sub algebra

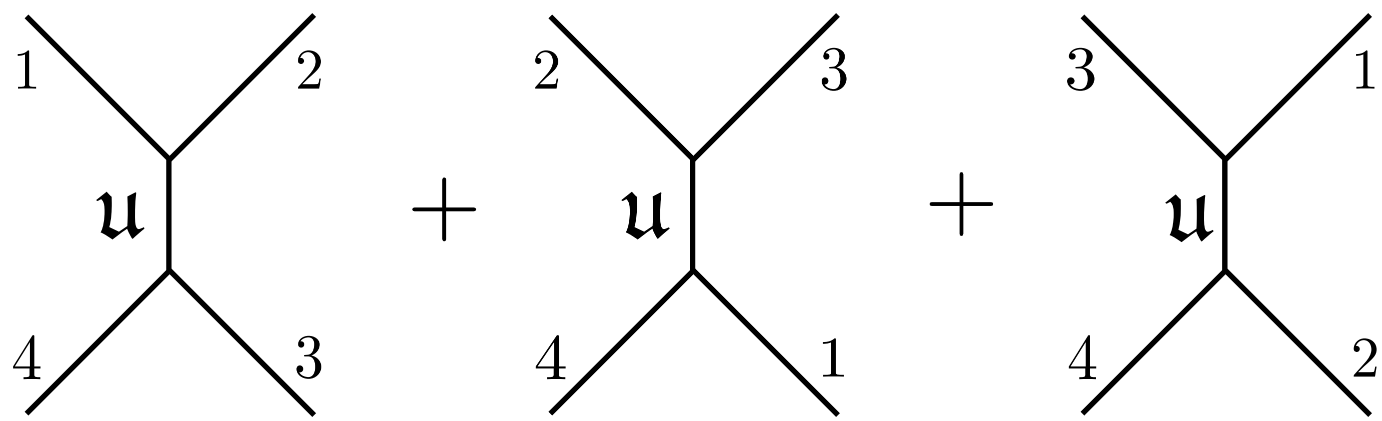
$$\langle [\xi_1 + \xi_1^*, \xi_2 + \xi_2^*], \xi_3 - \xi_3^* \rangle = -\langle [\xi_1, \xi_2], \xi_3^* \rangle + \langle [\xi_3, \xi_1], \xi_2^* \rangle + \langle [\xi_2, \xi_3], \xi_1^* \rangle$$

Explicitly

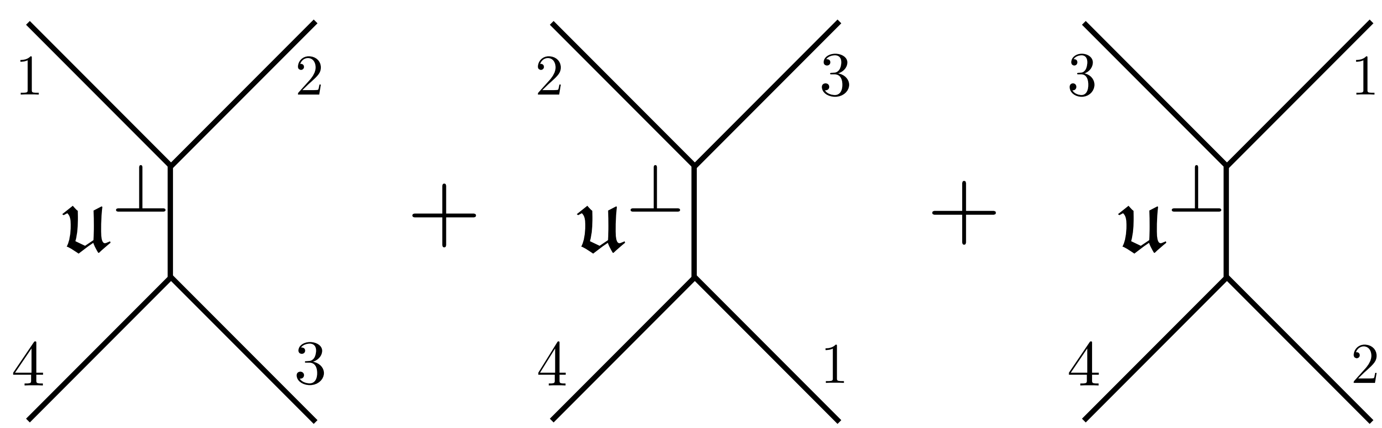
$$[\xi_1 + \xi_1^*, \xi_2 + \xi_2^*]\Big|_{\mathfrak{u}^\perp} = (\xi_1 \xi_2)(k_1 - k_2) + (\xi_1 k_1)\xi_2 - (\xi_2 k_2)\xi_1$$

Jacobi identity

$$0 = J(\xi_1 + \xi_1^*, \xi_2 + \xi_2^*, \xi_3 + \xi_3^*, \xi_4 + \xi_4^*) =$$



$\longleftarrow \quad J(\xi_1, \xi_2, \xi_3, \xi_4)$



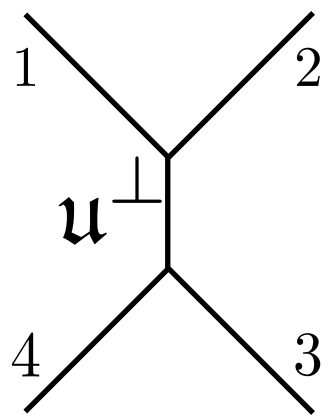
$\longleftarrow \quad v_4(\xi_1, \xi_2, \xi_3, \xi_4)$

plus on-shell vanishing terms

Exactly the Jacobi-like identity seen from YM Feynman rules

Part III: 5 points

The problem



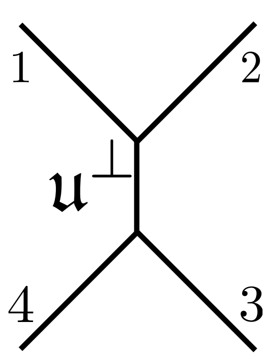
$$= -((k_1 - k_2)(k_3 - k_4))(\xi_1\xi_2)(\xi_3\xi_4)$$

+ terms vanishing on-shell

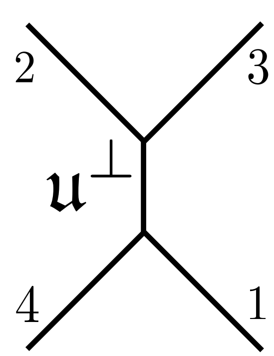
Different!

$$(k_1 + k_2)^2 v^s(\xi_1, \xi_2, \xi_3, \xi_4)$$
$$= (k_1 + k_2)^2((\xi_1\xi_3)(\xi_2\xi_4) - (\xi_1\xi_4)(\xi_2\xi_3))$$

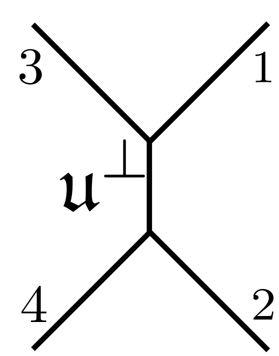
Only the sums are the same



$+$



$+$



$$= (k_1 + k_2)^2 v^s(\xi_1, \xi_2, \xi_3, \xi_4)$$
$$+ \; (k_2 + k_3)^2 v^t(\xi_1, \xi_2, \xi_3, \xi_4)$$
$$+ \; (k_3 + k_1)^2 v^u(\xi_1, \xi_2, \xi_3, \xi_4)$$

+ terms vanishing on-shell

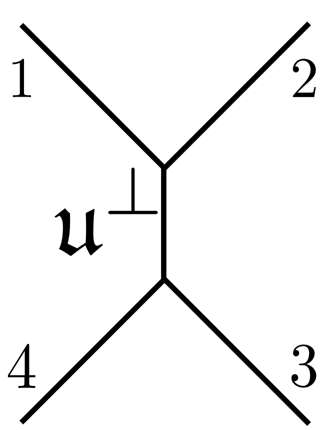
C/K at 5 points

C/K duality does not hold for Feynman rule produced numerators

Taking the sum of 3 numerators, some terms coming from 4-valent vertices can be combined into objects with Lie-algebra interpretation, but some can't

One can only conclude that $n^3_{12} + n^1_{23} + n^2_{31} \sim (k_4 + k_5)^2$

To proceed further I will assume that there exists a Lie-algebraic structure where



$$= (k_1 + k_2)^2 v^s(\xi_1, \xi_2, \xi_3, \xi_4)$$

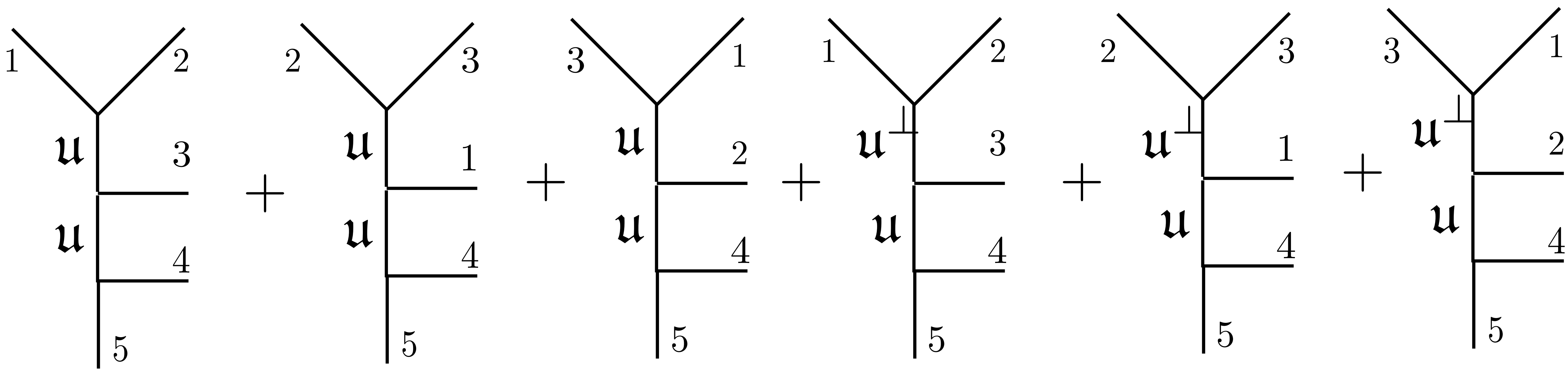
+ terms vanishing on-shell

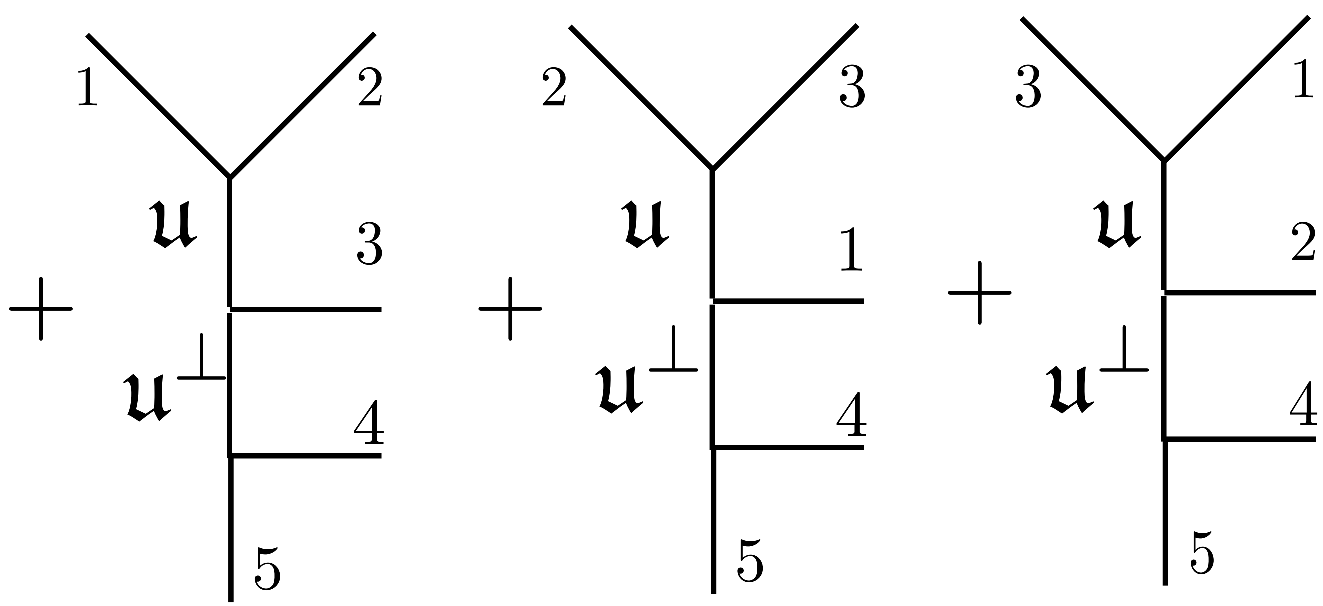
See comments later on how to achieve this

With this assumption all terms receive Lie-algebraic interpretation

$$n^3_{12} + n^1_{23} + n^2_{31} =$$

these terms arise without any assumptions made

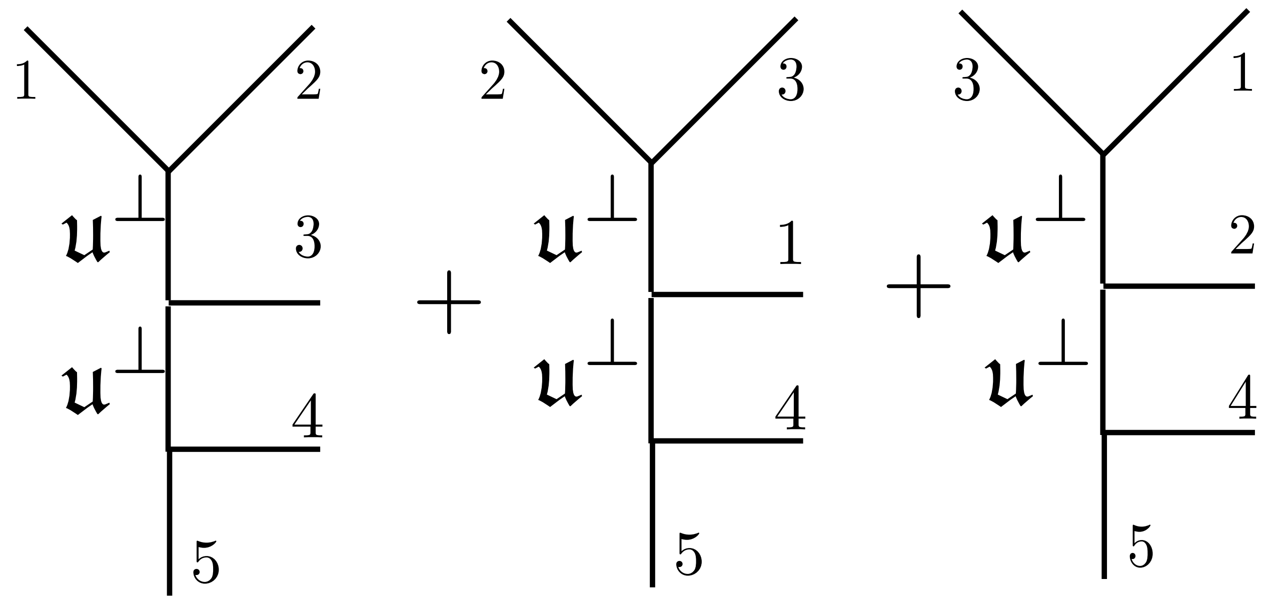




What is missing here?

for these terms the assumption was needed

The sum of 3 numerators cannot be zero because the double projection on $\mathfrak{u}^{\perp}$ terms are missing



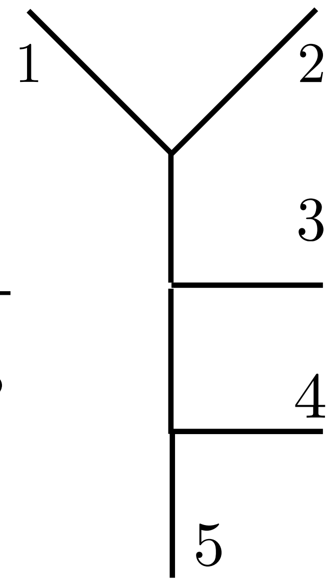
These terms should be added as the new 5-valent vertex changing the Feynman rules but not changing the amplitudes

as in Bern et al '10

Thus confirms that need to change Feynman rules for the C/K to work

If this idea can be made to work, the numerators would be given by successive commutators

$$\frac{1}{s_{12}s_{123}}$$



$$\langle[[[\xi_1+\xi_1^*,\xi_2+\xi_2^*],\xi_3+\xi_3^*],\xi_4+\xi_4^*],\xi_5+\xi_5^*\rangle$$

One way to achieve the presentation desired

$$[\xi_1,\xi_2]^*_\mu := (\eta_{\mu\rho}\eta_{\nu\sigma}-\eta_{\mu\sigma}\eta_{\nu\rho}+\epsilon_{\mu\nu\rho\sigma})(k_1+k_2)^\nu\xi_1^\rho\xi_2^\sigma$$

Then

$$-([\xi_1,\xi_2]^*[\xi_3,\xi_3]^*)=\underline{(k_1+k_2)^2((\xi_1\xi_3)(\xi_2\xi_4)-(\xi_1\xi_4)(\xi_2\xi_3)}+(\epsilon\xi_1\xi_2\xi_3\xi_4))$$

this is just $v^s(\xi_1,\xi_2,\xi_3,\xi_4)$

Summary

- Non-trivial Lie-algebraic structure behind the YM Feynman rules:

YM cubic vertex is just the Drinfeld double bracket on

$$\xi + \xi^* \in D$$

- C/K works at 4 points because of the Drinfeld double Jacobi

- May be can represent the individual parts of the 4-valent vertex in Lie-algebraic terms

- If this is possible, then numerators would be given by successive commutators

(and correspond to Feynman rules modified by new vertices)