# Moduli space of shapes of a tetrahedron and SU(2) intertwiners

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Main theme:

Relationship between representation theory of classical groups and geometry

Old subject: e.g. Gelfand et al 50's-60's

decomposition of the regular representation into irreducibles $\Leftrightarrow$ integral transforms associated with homogeneous group spaces

Our setup as a toy model for the problem of quantization of the moduli space of flat connections on Riemann surfaces

A complete description is possible by elementary methods (no quantum groups arise)

# Our main object of interest:

$$\mathcal{H}_{\vec{j}} := \left(V^{j_1} \otimes \ldots \otimes V^{j_n}\right)^{\mathrm{SU(2)}}$$

the space of SU(2) invariant tensors - intertwiners

here $V^j$ is the irreducible representation of dimension

$$\dim(V^j) = 2j + 1$$

Finite-dimensional vector space

multiplicity of the trivial representation in the tensor product

$$\dim(\mathcal{H}_{\vec{j}}) = \frac{2}{\pi} \int_0^\pi d\theta \sin^2(\theta/2) \chi^{j_1}(\theta) \ldots \chi^{j_n}(\theta)$$

where

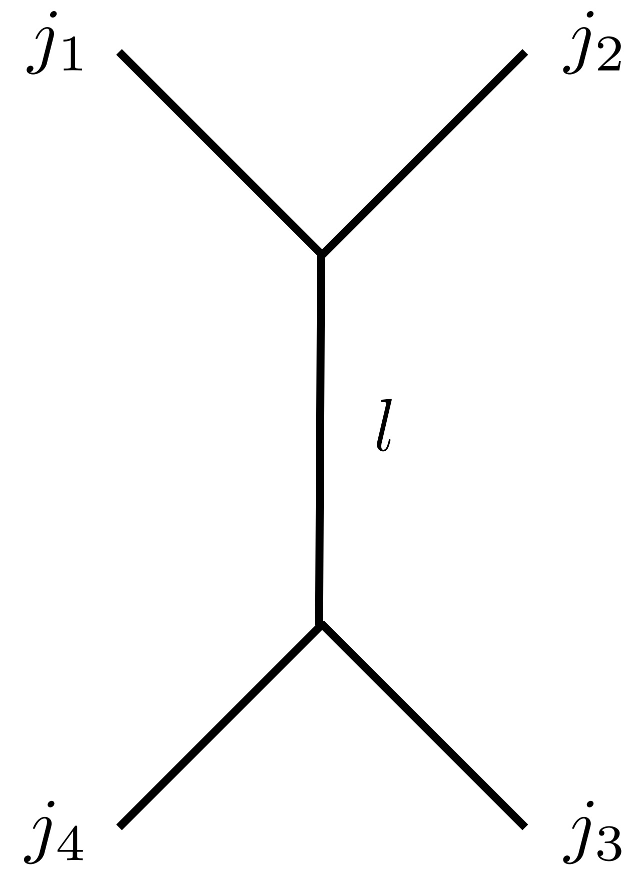
$$\chi^j(\theta) = \frac{\sin(j + 1/2)\theta}{\sin \theta/2}$$

A convenient basis:

use uniqueness (up to normalization) of the 3-valent intertwiner

Example: n=4

$$|j_1 - j_2| \leq l \leq j_1 + j_2$$



$\mathcal{H}_{\vec{j}}$ is closely related to other important spaces:

$k \to \infty$ limit of the space of SU(2) WZW conformal blocks (states of SU(2) CS theory) on an n-punctured sphere

Dimension is the $k \to \infty$ limit of the Verlinde formula

$\mathcal{H}_{\vec{j}}$ also arises as the quantization of a certain moduli space

## $V^j$ can be obtained as the quantization of $S^2$ of radius $j$

canonical example of geometric quantization
or Kirillov's orbit method

$S^2$ is a symplectic manifold

$$\omega_j = j \sin(\theta) d\theta \wedge d\phi$$

$$\int_{S^2} \omega_j = 4\pi j \quad \Longrightarrow$$

expect a finite-dimensional Hilbert space

Action of SU(2) on $S^2$ is Hamiltonian

moment map $\quad \mu(\theta, \phi) = j \, \vec{n}(\theta, \phi) \in \mathbb{R}^3$

$\left(V^{j_1} \otimes \ldots \otimes V^{j_n}\right)^{\mathrm{SU}(2)}$ can be obtained as the quantization of

$$\mathcal{S}_{\vec{j}} := S^2 \otimes \ldots \otimes S^2 /\!/ \mathrm{SU}(2)$$

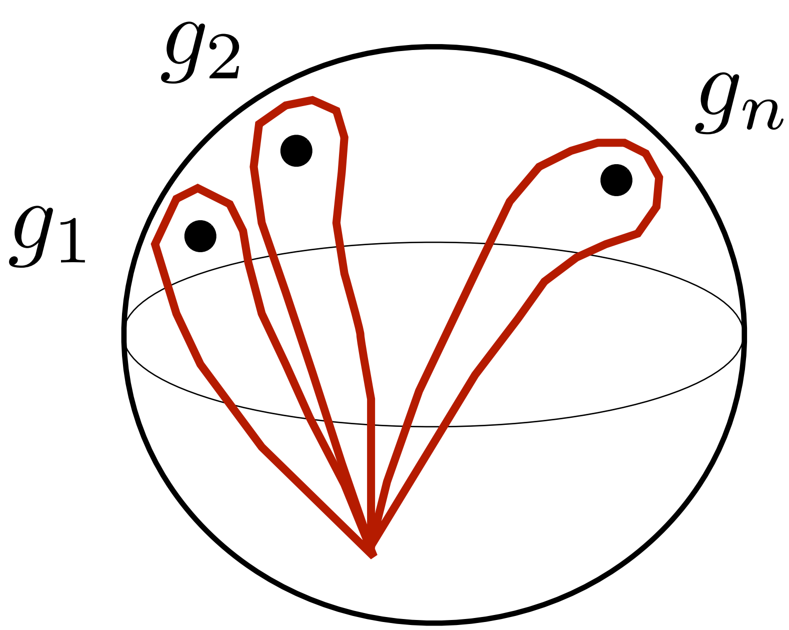
i.e. symplectic reduction $\mu^{-1}(0)/\mathrm{SU}(2)$

where $\mu = \sum_{i=1}^{n} j_i \vec{n}_i$

Note $\dim(\mathcal{S}_{\vec{j}}) = 2n - 6$

similar to, but much simpler than the moduli spaces of flat connections on an n-punctured sphere

# Moduli space of flat SU(2) connections on $S^2$




fixed conjugacy classes around punctures $\mathrm{Tr}(g_i) = \lambda_i$

$$\mathcal{M} = \{g_i : \prod_{i=1}^{n} g_i = 1\} / \, \mathrm{SU}(2)$$

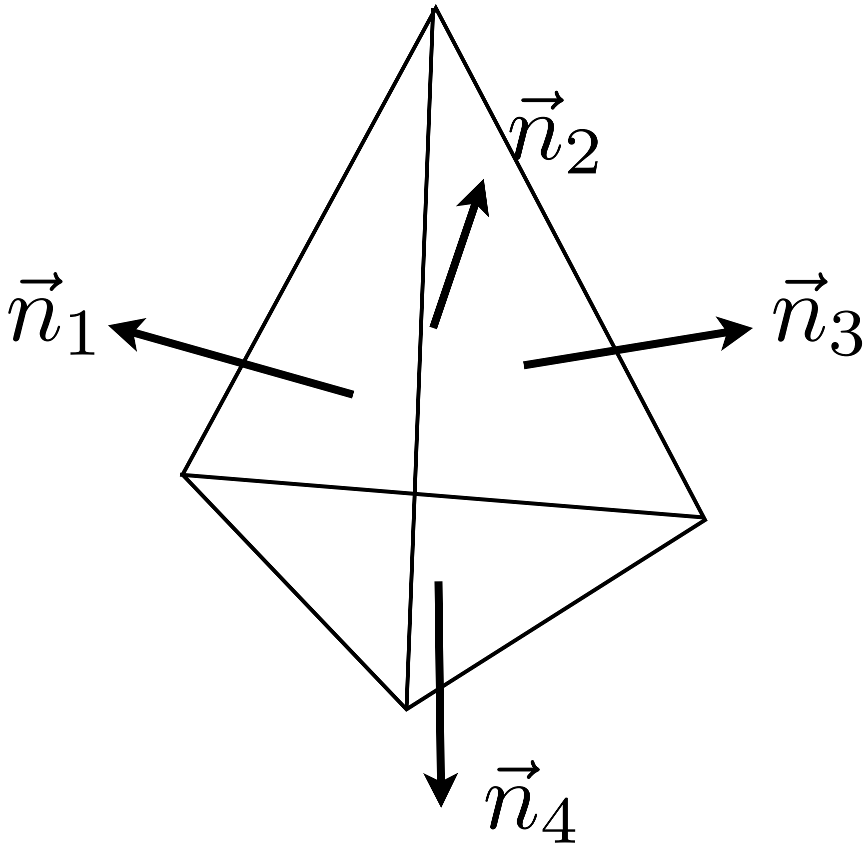
different $g$'s do not commute $\Longrightarrow$ quantum group

The difference between $\mathcal{S}_{\vec{j}}$ and $\mathcal{M}$ is that between the Lie algebra and the group

different $\vec{n}$'s commute!

## Geometric interpretation of the moduli space $\mathcal{S}_{\vec{j}}$

particularly relevant for n=4




if identify $A_i = j_i$

then $$\sum_{i=1}^{4} j_i \vec{n}_i = 0$$

modulo rotations and translations, a tetrahedron with fixed face areas can be described by

$$\{\vec{n}_1, \vec{n}_2, \vec{n}_3, \vec{n}_4 : \sum_i j_i \vec{n}_i = 0\} / \, \mathrm{SU}(2)$$

explicit description is possible for all areas equal

How to quantize $\mathcal{S}_{\vec{j}}$?

Apart from simple cases, no explicit description of the moduli space is available

Strategy: first quantize then reduce

$\Longrightarrow$ invariant tensors

Would like a more explicit description of the arising Hilbert space

Same as if first reduce and then quantize?

## $S^2$ is a Kaehler manifold

$$z = \tan(\theta/2)e^{-i\phi}$$

holomorphic coordinate
(stereographic projection)

symplectic form

$$\omega_j = (2j/\mathrm{i})\frac{dz \wedge d\bar{z}}{(1+|z|^2)^2} = (1/\mathrm{i})\partial\bar{\partial}\Phi\, dz \wedge d\bar{z}$$

where

$$\Phi(z,\bar{z}) = 2j\ \log(1+|z|^2)$$

holomorphic quantization is possible

$V^j$ space of sections of a Hermitian line bundle over $S^2$

of curvature $\mathrm{i}\omega_j$

$$\langle \Psi_1, \Psi_2 \rangle = \frac{d_j}{2\pi} \int \frac{d^2 z}{(1+|z|^2)^{2(j+1)}} \overline{\Psi_1(z)} \Psi_2(z)$$

here $d^2 z := |dz \wedge d\bar{z}|$

the group action

$$\left(T(k^t)\Psi\right)(z) = (-\bar{\beta} z + \bar{\alpha})^{2j} \Psi(z^k)$$

where

$$k = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \qquad z^k = \frac{\alpha z + \beta}{-\bar{\beta} z + \bar{\alpha}}$$

leaves the inner product invariant

The holomorphic description of $V^j$ is very useful because related to coherent states

$$\Psi(z) = \langle \Psi | j, z \rangle$$

where

$$|j, z\rangle = |z\rangle^{\otimes 2j} \qquad |z\rangle = e^{zJ_+}|1/2, -1/2\rangle$$

$$|j, z\rangle = \sum_{m=-j}^{j} \sqrt{\frac{(2j)!}{(j-m)!(j+m)!}}\, z^{j-m} |j, m\rangle$$

and $|j, m\rangle$ is the usual basis in $V^j$

$|j,z\rangle$ are coherent states

$$\frac{\langle j,z|\vec{J}\,|j,z\rangle}{\langle j,z|j,z\rangle} = j\,\vec{n}(z,\bar{z})$$

$\vec{n}(z,\bar{z})$ -the corresponding point on $S^2$

Decomposition of identity formula

$$1_{\vec{j}} = \frac{d_j}{2\pi}\int \frac{d^2 z}{(1+|z|^2)^{2(j+1)}}|j,z\rangle\langle j,z|$$

$|j,z\rangle$ an overcomplete basis of states in $V^j$

$$\langle j,z|j,z\rangle = (1+|z|^2)^{2j}$$

A similar (explicit) holomorphic quantization is possible for $\mathcal{S}_{\vec{j}}$

States - holomorphic functions $\Psi(Z_4, \ldots, Z_n)$ with appropriate behavior under permutations

$Z_i$ cross-ratios

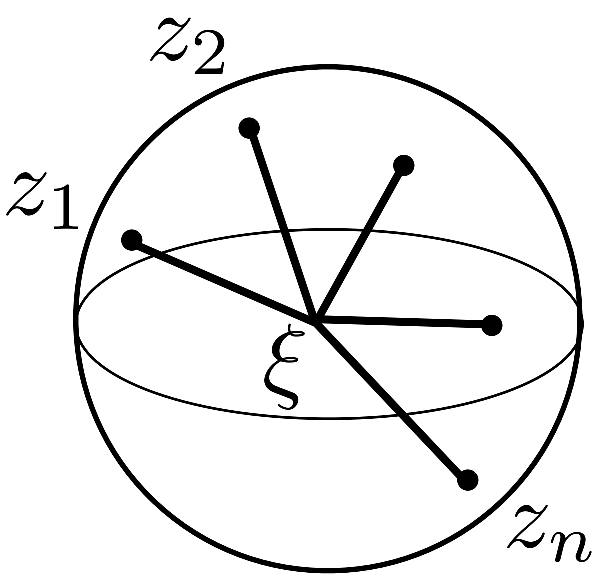
Inner product

$$\langle \Psi_1 | \Psi_2 \rangle = 8\pi^2 \prod_{i=1}^{n} \frac{d_{j_i}}{2\pi} \int d^2 Z \, \hat{K}_{\vec{j}}(Z, \bar{Z}) \overline{\Psi_1(Z)} \Psi_2(Z)$$

where

$$\hat{K}_{\vec{j}}(Z, \bar{Z}) = \lim_{X \to \infty} |X|^{2\Delta_3} K_{\vec{j}}(0, 1, X, Z_4, \ldots, Z_n)$$

AdS/CFT n-point function




$$K_{\vec{j}}(z_1, \ldots, z_n) = \int_{H^3} d\xi \prod_{i=1}^{n} K_{\Delta_i}(\xi, z_i)$$

where $\Delta_i = 2(j_i + 1)$

$$K_\Delta(\xi, z) = \frac{\rho^\Delta}{(\rho^2 + |z - y|^2)^\Delta}$$

$\xi = (\rho, y)$ upper half-space model coordinates of a point in $H^3$

# A coherent state interpretation of the inner product formula

can introduce coherent intertwiners $|\vec{j}, Z\rangle \in \mathcal{H}_{\vec{j}}$

$$1_{\mathcal{H}_{\vec{j}}} = 8\pi^2 \prod_{i=1}^{n} \frac{d_{j_i}}{2\pi} \int d^2Z \, \hat{K}_{\vec{j}}(Z, \bar{Z}) |\vec{j}, Z\rangle\langle\vec{j}, Z|$$

In general in holomorphic quantization

$$\langle \Psi_1 | \Psi_2 \rangle = \int \omega^k e^{-\Phi(Z,\bar{Z})} \overline{\Psi_1(Z)} \Psi_2(Z)$$

Thus, the inner product formula implies (in the limit of large spins)

$$\hat{K}_{\vec{j}}(Z,\bar{Z}) \sim e^{-\Phi_{\vec{j}}(Z,\bar{Z})}$$

where $\Phi_{\vec{j}}(Z,\bar{Z})$ is the Kaehler potential on the moduli space $\mathcal{S}_{\vec{j}}$

Can be checked explicitly, or can appeal to "quantization commutes with reduction" of Guillemin and Sternberg

As far as we know this interpretation of the boundary n-point functions is new

# A sketch of the proof:

Consider SU(2) invariant states

$$\Psi(z_1, \ldots, z_n) \in \left(V^{j_1} \otimes \ldots \otimes V^{j_n}\right)^{\mathrm{SU}(2)}$$

the action of SU(2) can be continued to that of SL(2,C)

$$(T(g^t)\Psi)(z_1, \ldots, z_n) = \prod_{i=1}^{n} (cz + d)^{2j_i} \Psi(z_1^g, \ldots, z_n^g)$$

where

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad z^g = \frac{az + b}{cz + d}$$

being holomorphic such states are automatically SL(2,C) invariant

$$(T(g^t)\Psi)(z_1, \ldots, z_n) = \Psi(z_1, \ldots, z_n)$$

Immediately get transformation properties

$$\Psi(z_1^g, \ldots, z_n^g) = \prod_{i=1}^{n} (cz+d)^{-2j_i}\, \Psi(z_1, \ldots, z_n)$$

In the inner product formula in $V^{j_1} \otimes \ldots \otimes V^{j_n}$

$$\langle \psi_1 | \psi_2 \rangle = \prod_{i=1}^{n} \frac{d_i}{2\pi} \int \frac{d^2 z_i}{(1+|z_i|^2)^{2(j_i+1)}} \overline{\psi_1(z_1, \ldots, z_n)} \psi_2(z_1, \ldots, z_n)$$

restricted to invariant states can change coordinates

$$(z_1, \ldots, z_n) \to (g, Z_4, \ldots, Z_n)$$

where

$$g : (0^g, 1^g, \infty^g) = (z_1, z_2, z_3) \qquad Z_i = \frac{(z_i - z_1)(z_2 - z_3)}{(z_i - z_3)(z_2 - z_1)}$$

Can now introduce new states

$$\psi(z_1,\ldots,z_n) = \psi(0^g, 1^g, \infty^g, Z_4^g, \ldots, Z_n^g)$$
$$= d^{-2j_1}(c+d)^{-2j_2}c^{-2j_3}\prod_{i=4}^{n}(cZ_i+d)^{-2j_i}\Psi(Z_4,\ldots,Z_n)$$

where

$$\Psi(Z_4,\ldots,Z_n) := \lim_{X\to\infty} X^{-2j_1}\psi(0,1,X,Z_4,\ldots,Z_n)$$

Remains only to integrate over g

$$\int_{\mathbb{C}^n}\prod_{i=1}^{n}\mathrm{d}^2 z_i\; F(z_i,\overline{z_i}) = 8\pi^2\int_{\mathbb{C}^{n-3}}\prod_{i=4}^{n}\mathrm{d}^2 Z_i\int_{\mathrm{SL}(2,\mathbb{C})}\mathrm{d}^{\mathrm{norm}}g\;\;\frac{F(z_i(g,Z_j),\overline{z_i(g,Z_j)})}{|d|^4|c+d|^4|c|^4\prod_{i=4}^{n}|cZ_i+d|^4}.$$

An analogous argument gives inner product of CS states as an integral over $G_{\mathbb{C}}/G$

For n=4 understand coherent intertwiners very explicitly

Define overlaps between coherent and usual intertwiners

$$C_{\vec{j}}^{k}(Z) \equiv \langle \vec{j}, k | \vec{j}, Z \rangle$$

Can show that

$$C_{\vec{j}}^{k}(Z) = N_{\vec{j}}^{k} \, \hat{P}_{k-j_{34}}^{(j_{34}-j_{12},\, j_{34}+j_{12})}(Z)$$

where $\hat{P}_{n}^{(a,b)}(Z) \equiv P_{n}^{(a,b)}(1-2Z)$ is the shifted Jacobi polynomial

and

$$N_{\vec{j}}^{k} = (-1)^{s-2k} \sqrt{\frac{(2j_1)!(2j_2)!(2j_3)!(2j_4)!(k+j_{34})!(k-j_{34})!}{(j_1+j_2+k+1)!(j_3+j_4+k+1)!(j_1+j_2-k)!(j_3+j_4-k)!(k+j_{12})!(k-j_{12})!}}.$$

## Case of all equal areas

One cross-ratio $Z$

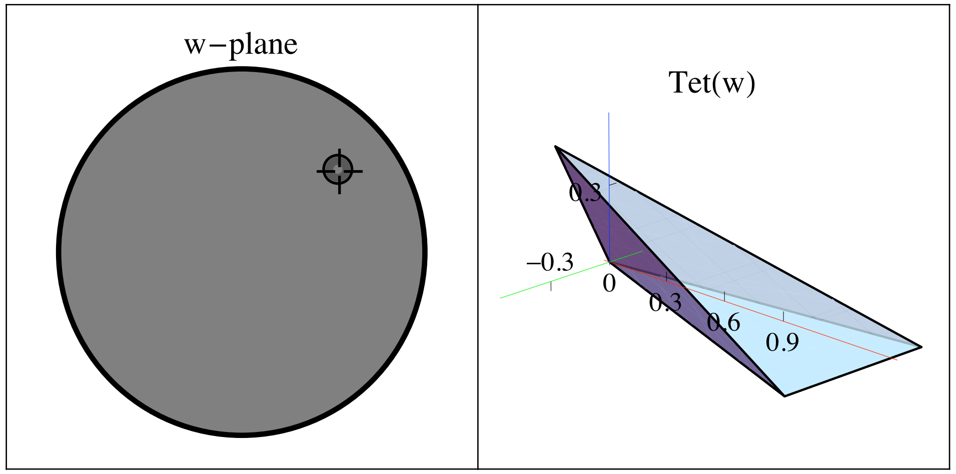
The regular tetrahedron corresponds to $Z = \exp(i\pi/3)$

Convenient to define $z = (2Z - 1)/\sqrt{3}$

so that the regular tetrahedron corresponds to $z = \mathrm{i}$

and then map upper half z-plane to the unit disc $w = \dfrac{\mathrm{i} - z}{\mathrm{i} + z}$

Unit disc as the moduli space of shapes



Can characterize this moduli space explicitly

Let $\theta$ be the dihedral angle between faces 1 and 2

Let $\phi$ be the angle between edges (12) and (34)

Define $k := 2j\cos(\theta/2)$

the value of spin k at which $C^{k}_{\vec{j}}(Z)$ is peaked

Symplectic form $\omega = dk \wedge d\phi$

To show this introduce $\Theta(Z)$ via $Z = \cosh^2\Theta(Z)$

Then $$e^{2\Theta(Z)} = \frac{2j+k}{2j-k}e^{i\phi}$$

and the Kaehler potential $\Phi_j(Z,\bar{Z}) = 8j\ln\left[\cosh \mathrm{Re}(\Theta)\right]$

# Summary

- A toy model for the quantization of the moduli space of flat SU(2) connections was described. Different “holonomies” commute, quantization reduces to the classical representation theory. Everything can be described explicitly.
- A new interpretation for the boundary n-point functions of AdS_3/CFT_2 as the exponential of the Kaehler potential on the moduli space.
- “Quantum geometry” of the tetrahedron with fixed areas.