One-loop β-function for an infinite-parameter family of gauge theories

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This talk is about a calculation

But it is worth putting it into context

Any field theory is renormalisable once all terms compatible with symmetries are added into the Lagrangian

Weinberg

If we could follow the RG flow in the infinite-dimensional space of couplings, the flow could take us to "nice" UV fixed points

The idea of asymptotic safety

In the context of renormalisable theories, realised for asymptotically-free theories

It is generally believed that “nice” UV fixed points of non-renormalisable theories (such as gravity) are of non-perturbative nature

Cannot say anything using perturbative methods

Have to use Wilsonian non-perturbative interpretation of the RG flow

Most of the available results are of this nature, but their interpretation is not without difficulty

In this talk, let me see how far one can go staying perturbative

Perturbative renormalisation:

Divergences appear because propagators are generalised functions, and their products (or products of their derivatives) are ill-defined

Can make sense of such products of generalised functions, but ambiguities arise

These are of the form of a dependence on some arbitrary energy scale, and come in typical logarithmic form

The dependence on the (log of) this energy scale is the same as the dependence on the logarithm of the energy, and this allows to track the energy logarithms

Perturbative RG flow resums the energy logarithms

In practice one computes the "divergencies" to extract the RG flow

In principle, no problem in applying this programme to a non-renormalisable theory

In practice, has to deal with higher-dimension operators, and computations become challenging, to say the least

It seems that to compute even at one-loop, one needs to start with the most general Lagrangian at the tree level

This is clearly impossible in practice

However, and this is the crucial insight:

In perturbative calculations, the higher-dimensional operators (typically) do not affect the RG flow of the lower-dimensional ones

Example:

Imagine one adds an F^3 term to the tree-level Yang-Mills Lagrangian

Effect on the one-loop flow of the g^2 in $\mathcal{L}_{\rm YM} = \frac{1}{4g^2}(F_{\mu\nu})^2$

$$\frac{\partial}{\partial \log(\mu)} \left(\frac{1}{4g^2} \right) = \frac{11 C_2}{6(4\pi)^2}$$

Answer: this operator does not affect the flow, the flow is unaware of a possible presence of this operator in the Lagrangian

But g affects the flow of the coupling constant for F^3

Lower-dimensional operators are relevant for higher-dimensional ones, but not the other way around

Philosophy:

Only applies to perturbative RG flow.
Functional RG behaves completely differently!

Can work with a number of operators, and what has not been added to the tree-level Lagrangian will not affect the flow of what is included

So, can trust the computed (one-loop) RG flow, at least in the regime where the perturbation theory is valid (i.e. when one can neglect the second loop)

One does not need all operators to do meaningful computations

The gravity example:

One loop renormalisable: the arising divergences can be absorbed into the metric redefinition, as well as renormalisation of the EH

GR one-loop counterterm

Christensen and Duff '80

$$\frac{1}{\epsilon}\frac{1}{180(4\pi)^2}\int\left(212(R_{\mu\nu\rho\sigma})^2 - 2088\Lambda^2\right)$$

Or

$$\frac{\partial}{\partial\log(\mu)}\left(-\frac{\Lambda}{8\pi G}\right) = -\frac{58\Lambda^2}{5(4\pi)^2}$$

On-shell EH action (modulo volume)

Interpretation: Λ cannot run

$$\frac{\partial(\Lambda G)}{\partial \log(\mu)} = -\frac{29}{5\pi}(\Lambda G)^2$$

(ΛG) small, becomes even smaller in the UV

Asymptotic freedom, G weakens in the UV

This result must be insensitive to adding $(Weyl)^3$

required at two loops

Can we check this by an explicit calculation?

Leads to fourth order differential operator, challenging but not impossible

Gravity as theory of connections

Formalism that describes geometry using a connection, not metric as the main variable

$$g = \partial A$$
$$F = \partial A$$

Both metric and the curvature are derivatives of the connection

Field equations $\partial^2 A = A$

second order PDE's on the connection

Requires non-zero cosmological constant

On-shell

↓

$$Ricci = \partial^2 g = \partial^3 A = \partial A = g$$
$$Weyl = \partial^2 g = \partial^3 A = \partial A = F$$

Linearised Lagrangian is of the form

$$\mathcal{L}^{(2)} = (\partial A)^2$$

Allows to incorporate $(Weyl)^3$ as $(\partial A)^3$ type term

More generally can consider

$$\mathcal{L} = \sum_{n=2}^{\infty} g_n (\partial A)^n + \text{lower in derivatives terms}$$

An infinite number of interactions interesting from the metric viewpoint

This formalism does not lead to an increase in the order of the operators arising - still second order

“Deformations” of Yang-Mills theory

To gain intuition and develop technology, can play similar games with the usual Yang-Mills theory

Consider

M some energy scale

$$\mathcal{L} = \sum_{n=2}^{\infty} g_n F^n = M^4 f(F/M^2)$$

Just an effective field theory Lagrangian, but without derivatives of the curvature

Euler-Heisenberg

where f is an arbitrary Lorentz and gauge-invariant function of the curvature $F = dA + (1/2)[A, A]$

Field equations $$d_A \left(\frac{\partial f}{\partial F} \right) = 0$$

second-order PDE’s on the connection

Calculations become possible and mimic what happens in "deformations" of General Relativity when takes

$$F = (F^a_{\mu\nu})_{\rm sd} \equiv \frac{1}{2}F^a_{\mu\nu} + \frac{1}{4}\epsilon_{\mu\nu}{}^{\rho\sigma}F^a_{\rho\sigma}$$

self-dual part of the curvature

self-duality makes things simpler

a Lie algebra index

Or, using spinor notations $F = F^a_{AB}$

$A, B = 1, 2$

spinor indices (unprimed)

$$\mathcal{L} = M^4 f(F^a_{AB}/M^2)$$

Linearisation around an arbitrary background

$$(dA)^a_{AB} \equiv d_{(AA'}A^{a\,A'}_{B)}$$

$$[[A,A]]^a_{AB} \equiv f^{abc}A^b_{AA'}A^{c\,A'}_B$$

$$\mathcal{L}^{(2)} = \frac{1}{2}(f'')^{ab\,ABCD}(dA)^a_{AB}(dA)^b_{CD} + M^2(f')^{a\,AB}[[A,A]]^a_{AB}$$

$$(f'')^{ab\,ABCD} \equiv \frac{\partial^2 f}{\partial F^a_{AB}\partial F^b_{CD}} \qquad (f')^{a\,AB} \equiv \frac{\partial f}{\partial F^a_{AB}}$$

Second-order, but non-Laplace type

But the problem can be converted into a Laplace-type one using the first order-formalism

$$\mathcal{L} = B^{a\,AB} F^a_{AB} - M^4 V(B^a_{AB}/M^2).$$

B^{aAB} auxiliary field

$V(B^{aAB}/M^2)$ Legendre transform of $f(F^a_{AB}/M^2)$

Linearised Lagrangian

$$\mathcal{L}^{(2)} = 2b^{a\,AB}(da)^a_{AB} + B^{a\,AB}[[a,a]]^a_{AB} - \frac{1}{2}(V'')^{ab}_{ABCD} b^{a\,AB} b^{b\,CD}$$

After gauge-fixing can write as

$\not{d}$ - Dirac operator

$$\mathcal{L}^{(2)} = \begin{pmatrix} b & a \end{pmatrix} \begin{pmatrix} V'' & \not{d} \\ \not{d} & B \end{pmatrix} \begin{pmatrix} b \\ a \end{pmatrix}$$

The square of the arising operator is of Laplace-type

$$\begin{pmatrix} V'' & \not{d} \\ \not{d} & B \end{pmatrix}^2 = \begin{pmatrix} \not{d}^2 + (V'')^2 & V''\not{d} + \not{d}B \\ \not{d}V'' + B\not{d} & \not{d}^2 + B^2 \end{pmatrix}$$

$$\not{d}^2 = \Delta + F$$

Can now compute the divergent parts of the regularised determinant in the usual way, using the heat kernel

Result: after a (local) field redefinition, this family of "deformed" non-renormalisable Yang-Mills theories is one-loop renormalisable

like GR at one-loop!

The arising renormalisation group flow

1501.00849

$$\frac{\partial f(x)}{\partial \log \mu} = \beta_f(x)$$

$$x^a_{AB} \equiv F^a_{AB}/M^2$$

what flows at one loop is the function f

$$\beta_f(x) = \frac{1}{(4\pi)^2}\frac{1}{6}\Big[x^{ab}_{AB}x^{baAB} - 3((f'')^{-1})^{ab}_{AB}{}^{AB}(f')^{bc}_{MN}x^{caMN} \\ +3((f'')^{-1})^{ab}_{AM}{}^{BN}(f')^{bcC}_{B}((f'')^{-1})^{cd}_{CN}{}^{DM}(f')^{daA}_{D}\Big].$$

$$x^{ab}_{AB} \equiv f^{aeb}x^e_{AB}$$

$$(f'')^{abABCD} := \frac{\partial^2 f}{\partial x^a_{AB}\partial x^b_{CD}}$$

$$(f')^{ab}_{AB} \equiv f^{aeb}(f')^e_{AB}$$

The YM example

$$f_{\mathrm{YM}}(x) = \frac{1}{4g^2}(x^a_{AB})^2,$$

Then

$$(f'_{\mathrm{YM}})^{ab}_{AB} = \frac{1}{2g^2} x^{ab}_{AB}, \qquad ((f''_{\mathrm{YM}})^{-1})^{ab}_{ABCD} = 2g^2 \delta^{ab} \epsilon_{A(C}\epsilon_{|B|D)}$$

and

$$\beta_f^{\mathrm{YM}} = \frac{11 C_2}{6(4\pi)^2}(x^a_{AB})^2$$

More involved example

$$\mathcal{L} = \frac{1}{4g^2}(F^a_{AB})^2 + \frac{\alpha}{3!g^2M^2} f^{abc} F^{a\ B}_{A} F^{b\ C}_{B} F^{c\ A}_{C}$$

in agreement with our philosophy, the flow of g^2 is unchanged

For the flow of α we obtain

$$\frac{\partial(\alpha g)}{\partial \log(\mu)} = \frac{\alpha g^3 C_2}{(4\pi)^2}$$

It is g^2 that tells α how to change

α grows in the UV as could be expected from a non-renormalisable interaction

Discussion

Can do meaningful (one-loop) calculations even in non-renormalisable theories

Results can be trusted (when perturbation theory can be) because what has not been included at the tree level should not affect the flow of what has been included

Interpretation is clean, RG flow for "observable" on-shell couplings

With some tricks, even the problem for higher-derivative operators can be reduced to Laplace-type operators

The calculation for $$\mathcal{L} = GR + \sum_{n=3}^{\infty} g_n (Weyl)^n$$

using such tricks is in progress

What can one hope to learn this way?

Such perturbative caclulations for non-renormalisable theories have been hardly ever done: surely surprises await us

It is not impossible that there are some sufficiently large families of non-renormalisable theories (but smaller than the general EFT) that are renormalisable, in the sense that if one is in, one stays in

Then one can compute

Some yet unknown UV fixed points may be of perturbative nature and thus discoverable by doing one-loop computations for sufficiently large families of theories

The computed RG flow for "deformations" of YM is still to be understood

Take home message

The one-loop renormalisability of GR is just the tip of an iceberg

Perturbative (one-loop) RG flow calculations with non-renormalisable theories can be done, and results can be trusted (in appropriate regimes)

Potentially there is new physics discoverable this way

Thank you!