Renormalized Volume of Hyperbolic 3-Manifolds

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Fefferman-Graham expansion

From M. Anderson hep-th/0403087

Let g be a complete, conformally compact Einstein metric on M^{n+1}

i.e. $\exists \rho : \rho^2 g$ extends to ∂M

$$\rho\Big|_{\partial M} = 0, \quad d\rho\Big|_{\partial M} \neq 0$$

Then:

M asymptotically hyperbolic i.e. $|R_g + 1| = O(\rho^2)$

can define boundary metric $\gamma := \rho^2 g\Big|_{\partial M}$

with conformal class $[\gamma]$ canonically defined

Foliate M (near ∂M) by $\rho = const$ surfaces

$$ds^2 = \frac{d\rho^2}{\rho^2} + \frac{1}{\rho^2}\Big(\gamma + \rho^2 g_{(2)} + \rho^4 g_{(4)} + \dots$$

$$+ \begin{cases} n_{odd} & \rho^{n-1} g_{(n-1)} + \rho^n g_{(n)} + \rho^{n+1} g_{(n+1)} + \dots\Big) \\ n_{even} & \rho^n g_{(n)} + \rho^n \log\rho\, h_{(n)} + \dots\Big) \end{cases}$$

n_{odd} $\quad g_{(2k)}$ $\quad k \leq (n-1)/2$ computable in terms of γ

$g_{(n)}$ free apart from $\nabla^\gamma g_{(n)} = 0, \quad \mathrm{Tr}_\gamma g_{(n)} = 0$

n_{even} $\quad g_{(2k)}$ $\quad k \leq (n-2)/2$ and $h_{(n)}$ traceless, transverse: computable in terms of γ

$g_{(n)}$ free apart from $\nabla^\gamma g_{(n)} = \tau, \quad \mathrm{Tr}_\gamma g_{(n)} = \delta$

$\forall n \qquad \{\gamma, g_{(n)}\}$ free data

can integrate Einstein eqs in a finite neighborhood of ∂M

However, if ∂M connected and $\pi_1(M, \partial M) = 0$

Then γ can be freely varied, while $g_{(n)}$ is determined by γ

Hard to characterize explicitely

Interesting to compare with bounded metrics

$$ds^2 = d\rho^2 + \gamma + \rho B + \ldots$$

no linear term - no second fundamental form in AH case!

Volume

$$V(\epsilon) = \int_\epsilon \frac{d\rho}{\rho^{n+1}} \sqrt{\det(\gamma + \rho^2 g_{(2)} + \ldots)} =$$

$$\begin{cases} \epsilon^{-n} v_{(0)} + \epsilon^{-n+2} v_{(2)} + \ldots + \epsilon^{-1} v_{(n-1)} + V + O(\epsilon) \\ \epsilon^{-n} v_{(0)} + \epsilon^{-n+2} v_{(2)} + \ldots + \epsilon^{-2} v_{(n-2)} + \log(\epsilon)\, L + V + O(\epsilon) \end{cases}$$

$$\left. \begin{matrix} n_{odd} & V \\ n_{even} & L \end{matrix} \right\}$$ canonical invariants of M

Define Renormalized volume $RVol(M) := V$

depends on the foliation for n_{even}

Example: $n = 3$ Anderson'01

$$\int_{M^4} |W|^2 = 8\pi^2 \chi(M) - RVol(M)$$

renormalized volume controls the L^2 norm of the Weyl curvature

Very special case: $n = 2$

Weyl curvature vanishes identically, AH=hyperbolic

FG expansion stops Skenderis, Solodukhin hep-th/9910023

$$ds^2 = \frac{d\rho^2}{\rho^2} + \frac{1}{\rho^2}\left(1 + \frac{\rho^2}{2} g_{(2)}\gamma^{-1}\right)\gamma\left(1 + \frac{\rho^2}{2}\gamma^{-1}g_{(2)}\right)$$

$g_{(2)}$ free apart from $\nabla^\gamma g_{(2)} = 0, \quad \mathrm{Tr}_\gamma g_{(2)} = R_\gamma$

There is no $\log$ term!

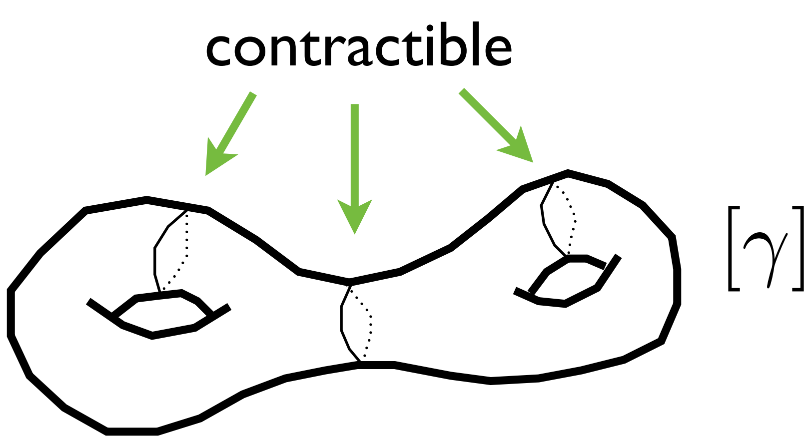
Compare with the known fact that in 3D given the first and second fundamental forms can write down the equidistant foliation metric explicitly

"Geometrization"

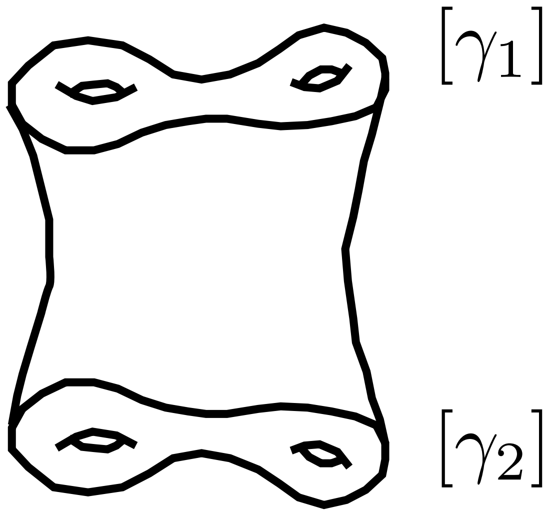
M is completely determined by $[\gamma]$

(plus possibly topological information)

E.g. Schottky



E.g. quasi-Fuchsian



$g_{(2)}$ is determined by $[\gamma]$ of all boundary components

Tracefree part of $g_{(2)}$ - holomorphic quadratic differential - projective structure

$RVol(M,\rho)$ depends on foliation (ρ -coordinate)

Lemma: Given γ on all components of ∂M $\exists!$ foliation of M by equidistant surfaces S_ρ such that

$$\frac{1}{2}\left(\gamma_\rho + B_\rho + \gamma_\rho^{-1} B_\rho \gamma_\rho^{-1}\right) = \frac{1}{\rho^2}\gamma$$

here γ_ρ, B_ρ are fundamental forms of S_ρ

Can be described explicitly - Epstein surfaces in the covering space. Breaks down inside (typically)

$$RVol(M,\rho) = RVol(\gamma)$$

Can be computed explicitly, but even without an explicit result variational formulas can be obtained

de Haro, Skenderis, Solodukhin hep-th/0002230

$$\delta RVol(\gamma) = -\frac{1}{4}\int da_\gamma \langle \tilde{g}_{(2)} - Ric_\gamma, \delta\gamma \rangle$$

where

$$\tilde{g}_{(2)} = g_{(2)} - \frac{1}{2}R_\gamma \gamma \qquad \text{tracefree}$$

subcase of a general n result, computation involving a regularization before the variation is taken

Important formula!

Consider variations not changing the conformal structure: $\delta\gamma = 2u\gamma$ (with fixed total area)

$$\mathcal{F} = RVol(\gamma) + \frac{\lambda}{2}\int da_\gamma$$

variation

$$\delta\mathcal{F} = \frac{1}{4}\int da_\gamma \langle Ric_\gamma + \lambda\gamma, \delta\gamma\rangle$$

$$\Longrightarrow$$

$$Ric_\gamma = -\lambda\gamma$$

critical points (local maxima) - constant curvature metrics

Define:

$$RVol(M) = RVol(\gamma)$$

computed on $\gamma : R_\gamma = -1$

renormalized volume maximized among all volumes for the same conformal structure and area of the boundary

canonically defined for M (with the normalization chosen) otherwise up to a constant

Consider variations of the conformal class

$$\delta RVol(M) = -\frac{1}{4}\int da_\gamma \langle \tilde{g}_{(2)}, \delta\gamma \rangle$$

Well-known that $\nabla^\gamma \tilde{g}_{(2)} = 0, \quad \mathrm{Tr}_\gamma \tilde{g}_{(2)} = 0$

$$\Longrightarrow \qquad g_{(2)} = \mathrm{Re}(Q)$$

where Q holomorphic quadratic differential on ∂M

(with respect to complex structure of γ)

Can characterize Q explicitly -

encodes the projective structure of M

$\exists \phi_i : \partial_i M \to H_2$ holomorphic w.r.t. $[\gamma_i]$

ϕ_i - uniformization map for the $\partial_i M$ component

$Q_i = S(\phi_i)$ Schwarzian derivative

difference of two projective structures on $\partial_i M$

Q_i - holomorphic cotangent vector on $\mathcal{T}_g$

$$\Longrightarrow \quad Q_i = -4\, \partial_i \, RVol(M)$$

Can now be shown that

Schottky - Takhtajan-Zograf' 88

$$\bar{\partial}_i Q_i = \omega_{WP}$$

$\Longrightarrow$ Weil-Petersson symplectic form

Schottky - KK' 00

Renormalized Volume is a Kaehler potential for WP form on the appropriate moduli space - Schottky, Teichmueller

for Teichmueller uses quasi-Fuchsian manifolds; the quantity $\bar{\partial}_1 Q_1$ does not depend on the complex structure of the other component - use Fuchsian

McMullen' 00

can be established by complex-analytic methods in both cases
needs a technical result on reciprocity in quasi-Fuchsian case

Remark:

The variational formula is reminiscent of that of Schlaefli

If V is a volume of a hyperbolic polyhedron

$$dV = \frac{1}{2}\sum_e L_e d\theta_e$$

and so if defines dual volume

$$V^* = V - \frac{1}{2}\sum_e L_e \theta_e$$

then

$$dV^* = -\frac{1}{2}\sum_e \theta_e dL_e$$

more than an analogy - can be derived from a version of the Schlaefli formula

KK & J. M. Schlenker
math/0607081

Historical remarks:

The Renormalized Volume can be computed "explicitly"

Schottky manifolds KK hep-th/0005106

Quasi-Fuchsian, Kleinian manifolds Takhtajan-Teo math/0204318

$$RVol(\gamma) = S_{\rm Liouv}(\gamma)$$

Liouville action as defined by Takhtajan and Zograf' 88
(Takhtajan and Teo '02)

The Kaehler potential property follows from
this relation to Liouville and results of TZ-TT

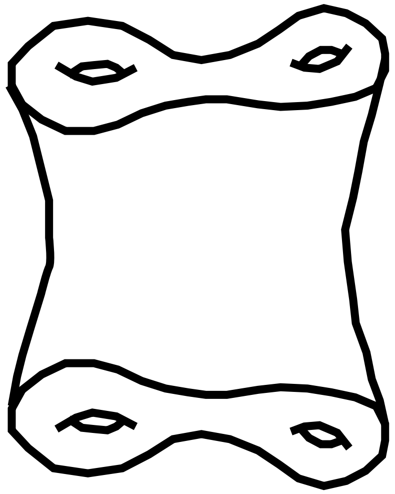
Kaehler potential on $\mathcal{T}_g$ is an essentially 3D quantity!

Application I: McMullen's QF reciprocity

A stronger statement can be formulated as:

Consider $\phi_1 : \mathcal{T}_g \times \mathcal{T}_g \to T^*\mathcal{T}_g \times T^*\mathcal{T}_g$

Simultaneous uniformization,
corresponding projective structure



Theorem: $\phi_1(\mathcal{T}_g \times \mathcal{T}_g)$ - Lagrangian submanifold in $T^*\mathcal{T}_g \times T^*\mathcal{T}_g$

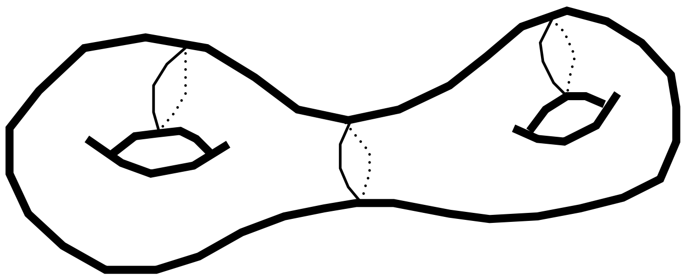
A version of $\mathcal{L} : p^i(x) = \dfrac{\partial V}{\partial x_i}$

$\Longrightarrow$

$$\sum_i dp^i \wedge dx_i \Big|_{\mathcal{L}} = \sum_i \frac{\partial^2 V}{\partial x_i \partial x_j} dx_j \wedge dx_i = 0$$

Application 2: Schottky manifolds

Consider $\phi_2 : \mathcal{S}_g \to T^*\mathcal{S}_g$



Schottky space, corresponding projective structure

Theorem: $\phi_2(\mathcal{S}_g)$ Lagrangian submanifold in $T^*\mathcal{S}_g$

Application 3: Grafting map is symplectic

Consider $\phi_3 : \mathcal{T}_g \times \mathcal{M}L \to T^*\mathcal{T}_g = \mathcal{C}P$

projective structure obtained by grafting a conformal metric as specified by a measured geodesic lamination

Consider the corresponding hyperbolic end; the inner boundary is a surface pleated along $\mathcal{M}L$; a version of renormalized volume proves

Theorem: ϕ_3 is symplectic; alternatively, the image in

$$\mathcal{T}_g \times \mathcal{M}L \times \mathcal{C}P$$

corresponding to hyperbolic ends is a Lagrangian submanifold

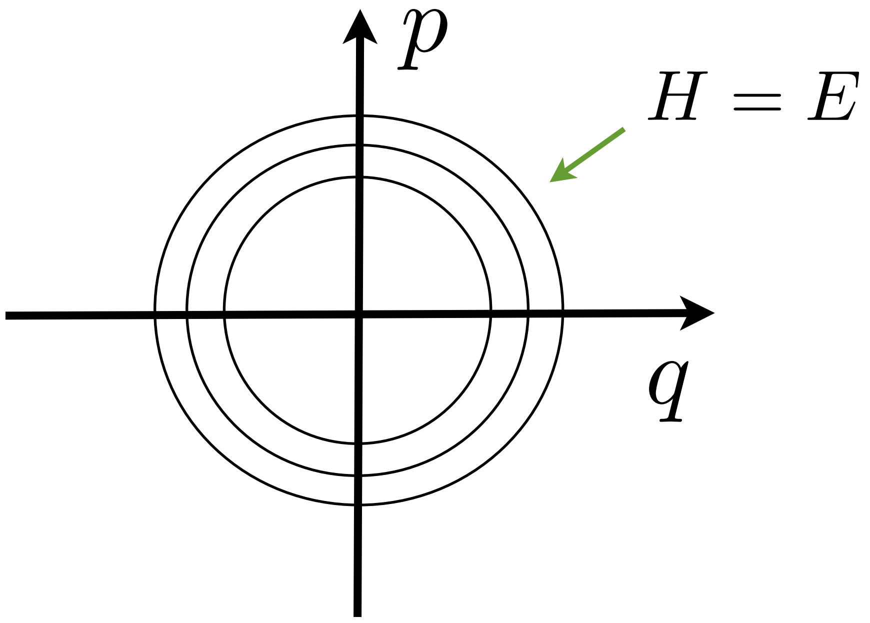
Why is this interesting to a physicist?

Geometric quantization in real polarization:

Foliation of $\mathcal{P}$ by Lagrangian submanifolds; states as integral leaves

E.g. Harmonic Oscillator

$$H = p^2 + q^2$$

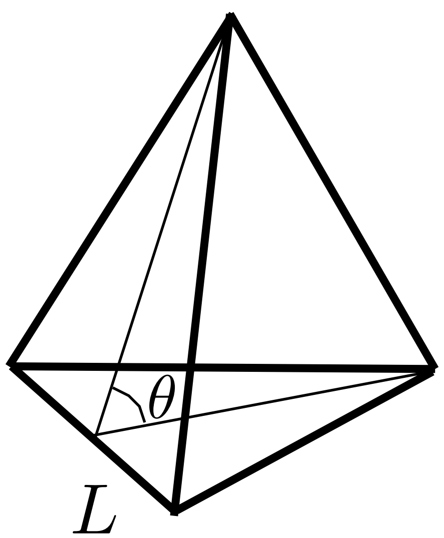


Integral leaves

$$H = E_n \sim n + 1/2$$

This program is realized in the context of flat (or hyperbolic) polyhedra!

$$S_{\mathrm{Regge}} = \sum_e \theta_e(L) L_e$$



$$\delta S_{\mathrm{Regge}} = \sum_e \theta_e(L) \delta L_e$$

Phase space $\mathcal{P}_{tet} = \{L_e, \theta_e\}$

Flat tetrahedron gives a Lagrangian submanifold $\theta_e = \theta_e(L)$

$$\exists\, \Psi(L_e) \sim e^{(i/\hbar) \sum_e \theta_e(L) L_e}$$

Ponzano-Regge

$\mathrm{SU}(2)$ irreps (6j)-symbol of Wigner

Can also be done for hyperbolic tets - (6j)-symbol of the quantum group $\mathrm{SL}_q(2,\mathbb{C})$ Freidel, Roche

Back to hyperbolic manifolds:

$\Lambda < 0$ 3D gravity can be formulated as a Hamiltonian system

phase space - space of complex projective structures

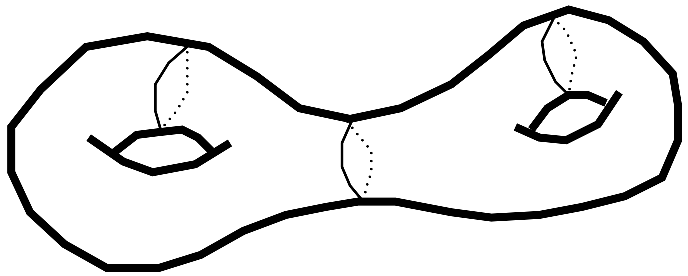
$$\mathcal{P} = \{\gamma, \tilde{g}_{(2)}\} = T^*\mathcal{T}_g = \mathcal{CP}$$

Can show that the gravitational symplectic structure coincides with that in $T^*\mathcal{T}_g$

Quantization of $\mathcal{P} \Rightarrow \mathcal{H} = L^2(\mathcal{T}_g)$

According to axioms of TQFT, 2-surfaces - Hilbert spaces, 3-manifolds - states

E.g. Schottky



What is the corresponding state?

Same question can be asked in the context of Chern-Simons theory on a handlebody, compare Weitsman '91

Do we have a Lagrangian foliation of $T^*\mathcal{T}_g$?

Can one use level surfaces of traces of the 3g-3 holonomies of the flat $\mathrm{PSL}(2,\mathbb{C})$ connection?

reminiscent to the Lagrangian foliation in Hitchin '88 The self-duality equations on a Riemann surface

Possible to construct "Schottky states" in $L^2(\mathcal{T}_g)$?

$\Rightarrow$

Interesting functions on $\mathcal{T}_g$ if expanded in powers of $\hbar$

Semi-classically $\Psi \sim e^{i\, RVol(c)/\hbar}$

"Quantization of hyperbolic 3-manifolds"

Conclusions:

- 3D Renormalized Volume as a Kaehler potential for the 2D Teichmueller space
- (Hyperbolic) 3-Manifolds correspond to Lagrangian submanifolds in the boundary phase space - natural setup for quantization