Gravity
vs
Yang-Mills theory

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This is a meeting about Planck scale

The problem of quantum gravity

Many models for physics at Planck scale

This talk: attempt at re-evaluation of the problem in light of developments of the last 10 years

There is a deep relationship between gravity and Yang-Mills theory, as has been emerging over the last 10 years

We have discarded as terribly complicated and not making sense a QFT that is related to and in many ways analogous to the best QFT we have - YM

Gravity is the most symmetric theory we have - diffeomorphisms. Should be the most beautiful QFT.

Do we just use the wrong language?

Plan

- Gravity vs. Yang-Mills: The old story
- Gravity = (YM)^2
- Gravity as a diffeomorphism invariant gauge theory
- The two loop divergence

General Relativity

$$S[g] = \frac{1}{16\pi G} \int \sqrt{g}\,(R - 2\Lambda)$$

GR is the unique diffeomorphism invariant theory of the metric with second order field equations

- GR Lagrangian is linear in second derivatives $L \sim \ddot{x}$

  Rather than non-linear in first as for all other theories $L \sim (\dot{x})^2$

- Unlike in YM, gravitational instanton condition is second-order in derivatives $W^+_{\mu\nu\rho\sigma} = 0$

  self-dual part of Weyl

Einstein-Hilbert action: Linearisation

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} \qquad \kappa^2 = 32\pi G$$

$$\mathcal{L}_{\mathrm{EH}}^{(2)} = -\frac{1}{2}(\partial_\mu h_{\rho\sigma})^2 + \frac{1}{2}(\partial_\mu h)^2 + (\partial^\mu h_{\mu\nu})^2 + h\partial^\mu\partial^\nu h_{\mu\nu}$$

$$h := h^\mu{}_\mu$$

Invariant under

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_{(\mu}\xi_{\nu)}$$

Unique linearised Lagrangian with this invariance (modulo surface term)

The linearised operator appearing in the EH case is not a square of any first order operator (unlike in YM)

Interactions

Second derivative interactions

Schematically, the cubic vertex $\mathcal{L}^{(3)} \sim \frac{1}{M_p} h(\partial h)^2$

Negative mass dimension coupling constant

Hence power-counting non-renormalisable

Because of so many derivatives in the vertex, 2-to-2 graviton scattering amplitude $\mathcal{M} \sim \frac{E^2}{M_p^2}$

E energy of the process

$\mathcal{M} \sim 1$ apparent breakdown of perturbative unitarity

$E \sim M_p$

Theory seems to break down at some energy scale:

Problem of quantum gravity

Perturbation theory is extremely complicated:

Expansion around an arbitrary background $g_{\mu\nu}$

quadratic order (together with the gauge-fixing term)

$$L_{g.f.} = -\sqrt{-g}\left(h^{\mu\nu}{}_{;\nu} - \frac{1}{2}h_{\nu}{}^{\nu;\mu}\right)\left(h^{\rho}{}_{\mu;\rho} - \frac{1}{2}h^{\rho}{}_{\rho;\mu}\right)$$

$$\begin{aligned} L_2 = \sqrt{-g}\Big\{ & -\frac{1}{2}h^{\alpha\beta}{}_{;\gamma}h_{\alpha\beta}{}^{;\gamma} + \frac{1}{4}h^{\alpha}{}_{\alpha;\gamma}h_{\beta}{}^{\beta;\gamma} + h_{\alpha\beta}h_{\gamma\delta}R^{\alpha\gamma\beta\delta} - h_{\alpha\beta}h^{\beta}{}_{\gamma}R^{\delta\alpha\gamma}{}_{\delta} \\ & + h^{\alpha}{}_{\alpha}h_{\beta\gamma}R^{\beta\gamma} - \frac{1}{2}h_{\alpha\beta}h^{\alpha\beta}R + \frac{1}{4}h^{\alpha}{}_{\alpha}h^{\beta}{}_{\beta}R\Big\}. \end{aligned}$$

from Goroff-Sagnotti "2-loop" paper

cubic order

$$\begin{aligned} L_3 = \sqrt{-g}\Big\{ & -\frac{1}{2}h^{\alpha\beta}h^{\gamma\delta}{}_{;\alpha}h_{\gamma\delta;\beta} + 2h^{\alpha\beta}h^{\gamma\delta}{}_{;\alpha}h_{\beta\gamma;\delta} - h^{\alpha\beta}h^{\gamma}{}_{\gamma;\alpha}h^{\delta}{}_{\beta;\delta} - \frac{1}{2}h^{\alpha}{}_{\alpha}h^{\beta\gamma;\delta}h_{\beta\delta;\gamma} \\ & + \frac{1}{4}h^{\alpha}{}_{\alpha}h^{\beta\gamma;\delta}h_{\beta\gamma;\delta} - h^{\alpha\beta}h^{\gamma}{}_{\gamma;\delta}h^{\delta}{}_{\alpha;\beta} + \frac{1}{2}h^{\alpha\beta}h^{\gamma}{}_{\gamma;\alpha}h^{\delta}{}_{\delta;\beta} - h^{\alpha\beta}h_{\alpha\beta;\gamma}h^{\gamma\delta}{}_{;\delta} \\ & + \frac{1}{2}h^{\alpha}{}_{\alpha}h^{\beta}{}_{\beta;\gamma}h^{\gamma\delta}{}_{;\delta} + h^{\alpha\beta}h_{\alpha\beta;\gamma}h_{\delta}{}^{\delta;\gamma} + \frac{1}{4}h^{\alpha}{}_{\alpha}h^{\beta}{}_{\beta;\gamma}h_{\delta}{}^{\delta;\gamma} - h^{\alpha\beta}h^{\gamma}{}_{\alpha;\delta}h_{\beta\gamma}{}^{;\delta} \\ & + h^{\alpha\beta}h^{\gamma}{}_{\alpha;\delta}h^{\delta}{}_{\beta;\gamma} + R_{\alpha\beta}(2h^{\alpha\gamma}h_{\gamma\delta}h^{\beta\delta} - h^{\gamma}{}_{\gamma}h^{\alpha\delta}h^{\beta}{}_{\delta} - \frac{1}{2}h^{\alpha\beta}h^{\gamma\delta}h_{\gamma\delta} \\ & + \frac{1}{4}h^{\alpha\beta}h^{\gamma}{}_{\gamma}h^{\delta}{}_{\delta}) + R(-\frac{1}{3}h^{\alpha\beta}h_{\beta\gamma}h^{\gamma}{}_{\alpha} + \frac{1}{4}h^{\alpha}{}_{\alpha}h^{\beta\gamma}h_{\beta\gamma} - \frac{1}{24}h^{\alpha}{}_{\alpha}h^{\beta}{}_{\beta}h^{\gamma}{}_{\gamma})\Big\} \end{aligned}$$

quartic order

$$
\begin{aligned}
L_4 = \sqrt{-g}\,\Big\{ & (h^{\alpha}{}_{\alpha} h^{\beta}{}_{\beta} - 2h^{\alpha\beta} h_{\alpha\beta}) \Big(\frac{1}{16} h^{\gamma\delta;\sigma} h_{\gamma\delta;\sigma} - \frac{1}{8} h^{\gamma\delta;\sigma} h_{\gamma\sigma;\delta} + \frac{1}{8} h^{\gamma}{}_{\gamma;\delta} h^{\delta\sigma}{}_{;\sigma} \\
& - \frac{1}{16} h^{\gamma}{}_{\gamma;\delta} h_{\sigma}{}^{\sigma;\delta}\Big) + h^{\alpha}{}_{\alpha} h^{\beta\gamma}\Big(-\frac{1}{2} h_{\beta\gamma;\delta} h^{\delta\sigma}{}_{;\sigma} + \frac{1}{2} h_{\beta\gamma;\delta} h_{\sigma}{}^{\sigma;\delta} - \frac{1}{2} h^{\delta}{}_{\delta;\beta} h^{\sigma}{}_{\sigma;\gamma} \\
& + \frac{1}{4} h^{\delta}{}_{\delta;\beta} h^{\sigma}{}_{\sigma;\gamma} + h^{\delta}{}_{\beta;\sigma} h^{\sigma}{}_{\delta;\gamma} - \frac{1}{4} h^{\delta\sigma}{}_{;\beta} h_{\delta\sigma;\gamma} - \frac{1}{2} h^{\delta}{}_{\beta;\sigma} h_{\delta\gamma}{}^{;\sigma} - \frac{1}{2} h^{\delta}{}_{\delta;\sigma} h^{\sigma}{}_{\beta;\gamma} \\
& + \frac{1}{2} h_{\beta\delta;\sigma} h_{\gamma}{}^{\sigma;\delta}\Big) + h^{\alpha}{}_{\beta} h^{\beta\gamma}\Big(h^{\delta}{}_{\delta;\alpha} h^{\sigma}{}_{\alpha;\gamma} - h_{\alpha\gamma;\delta} h_{\sigma}{}^{\sigma;\delta} + \frac{1}{2} h^{\delta\sigma}{}_{;\alpha} h_{\delta\sigma;\gamma} \\
& - h^{\delta}{}_{\alpha;\sigma} h^{\sigma}{}_{\gamma;\delta} - 2h^{\delta}{}_{\alpha;\sigma} h^{\sigma}{}_{\delta;\gamma} + h_{\alpha\gamma;\delta} h^{\delta\sigma}{}_{;\sigma} + h^{\delta}{}_{\delta;\alpha} h^{\sigma}{}_{\gamma;\sigma} - \frac{1}{2} h^{\delta}{}_{\delta;\alpha} h^{\sigma}{}_{\sigma;\gamma} \\
& + h^{\delta}{}_{\alpha;\sigma} h_{\gamma\delta}{}^{;\sigma}\Big) + h^{\alpha\gamma} h^{\beta\delta}\Big(h_{\alpha\gamma;\beta} h^{\sigma}{}_{\delta;\sigma} - h_{\alpha\gamma;\delta} h^{\sigma}{}_{\sigma;\beta} + \frac{1}{2} h_{\alpha\beta;\sigma} h_{\gamma\delta}{}^{;\sigma} \\
& - \frac{1}{2} h_{\alpha\gamma;\sigma} h_{\beta\delta}{}^{;\sigma} + h^{\sigma}{}_{\alpha;\beta} h_{\gamma\sigma;\delta} - h^{\sigma}{}_{\alpha;\beta} h_{\delta\sigma;\gamma} + h_{\alpha\beta;\delta} h^{\sigma}{}_{\sigma;\gamma} - 2h^{\sigma}{}_{\alpha;\beta} h_{\delta\gamma;\sigma} \\
& + h_{\alpha\gamma;\sigma} h^{\sigma}{}_{\beta;\delta}\Big) + R_{\alpha\beta}\Big(-2h^{\alpha\gamma} h_{\gamma\delta} h^{\delta\sigma} h_{\sigma}{}^{\beta} + h^{\gamma}{}_{\gamma} h^{\alpha\delta} h_{\delta\sigma} h^{\sigma\beta} + \frac{1}{2} h^{\alpha\gamma} h_{\gamma}{}^{\beta} h^{\delta\sigma} h_{\delta\sigma} \\
& - \frac{1}{4} h^{\alpha\gamma} h_{\gamma}{}^{\beta} h^{\delta}{}_{\delta} h^{\sigma}{}_{\sigma} + \frac{1}{3} h^{\alpha\beta} h^{\gamma\delta} h_{\delta\sigma} h^{\sigma}{}_{\gamma} - \frac{1}{4} h^{\alpha\beta} h^{\gamma}{}_{\gamma} h^{\delta\sigma} h_{\delta\sigma} + \frac{1}{24} h^{\alpha\beta} h^{\gamma}{}_{\gamma} h^{\delta}{}_{\delta} h^{\sigma}{}_{\sigma}\Big) \\
& + R\,\Big(-\frac{1}{192} h^{\alpha}{}_{\alpha} h^{\beta}{}_{\beta} h^{\gamma}{}_{\gamma} h^{\delta}{}_{\delta} + \frac{1}{16} h^{\alpha}{}_{\alpha} h^{\beta}{}_{\beta} h^{\gamma\delta} h_{\gamma\delta} + \frac{1}{4} h^{\alpha\beta} h_{\beta\gamma} h^{\gamma\delta} h_{\delta\alpha} \\
& - \frac{1}{16} h^{\alpha\beta} h_{\alpha\beta} h^{\gamma\delta} h_{\gamma\delta} - \frac{1}{6} h^{\alpha}{}_{\alpha} h^{\beta\gamma} h_{\gamma\delta} h^{\delta}{}_{\beta}\Big)\Big\}
\end{aligned}
$$

very far from the simplicity of YM theory

Imagine having to do calculations with these interaction vertices!

Some surprise:

In spite of being very badly divergent already at one loop, the theory is actually one loop finite

‘t Hooft and Veltman

(“pure” GR, for zero cosmological constant, in 4 dimensions)

When $\Lambda \neq 0$ there is a divergence that can be absorbed into the tree-level action

Originally, raised hopes that may be miracles continue to higher loops as well

Explicit heroic two loop calculation by Goroff and Sagnotti, and then by van de Ven gave a non-zero result

Metric GR is perturbatively non-renormalizable starting at two loops - This is why we are here

Linearization of Yang-Mills

$$\mathcal{L} = \frac{1}{4g^2}(F^a_{\mu\nu})^2$$

Maxwell: (part of) de Rham complex

$$\Lambda^0 \xrightarrow{d} \Lambda^1 \xrightarrow{d} \Lambda^2$$

$$F^a = dA^a + \frac{1}{2} f^{abc} A^b \wedge A^c$$

field strength (two-form with values in the Lie algebra)

Lagrangian $\mathcal{L} \sim (dA)^2, \qquad A \in \Lambda^1$

In 4 spacetime dimensions

Yang-Mills is the same, just everything tensored with the Lie algebra

$$\begin{array}{ccccccccc} & & & & \Lambda^+ & & & & \\ & & & \overset{d_+}{\nearrow} & & \overset{d_+^*}{\searrow} & & & \\ \Lambda^0 & \xrightarrow{d} & \Lambda^1 & & & & \Lambda^3 & \xrightarrow{d} & \Lambda^4 \\ & & & \underset{d_-}{\searrow} & & \underset{d_-^*}{\nearrow} & & & \\ & & & & \Lambda^- & & & & \end{array}$$

Maxwell equations (in vacuum) $d_+^* d_+ A = 0$

Negative helicity photons $d_+ A = 0$

Yang-Mills interactions

$$\mathcal{L}^{(3)} \sim gAA(\partial A)$$

just one derivative in the vertex - much less divergent than GR

dimensionless coupling - may expect renormalisability

The theory diverges already at one loop, the divergence may be shown to be related to the β-function

$$\frac{\partial}{\partial \log(\mu)} \left(\frac{1}{4g^2} \right) = \frac{11 C_2}{6(4\pi)^2}$$

C_2 - quadratic Casimir

the factor in front of the Lagrangian grows in the UV (asymptotic freedom)

Summary so far:

Yang-Mills is nice, a perfect theory

Gravity is a mess

Seem very far from each other

But glimpses of GR being not just a random non-renormalizable theory:

- GR uniqueness
- One loop finiteness in 4 spacetime dimensions

The new story part I: Gravity=(YM)^2

Over the last 20, and increasingly so 10 years it has been realised that a lot of progress can be achieved if one looks not at the Lagrangian and the resulting Feynman rules, but directly at the on-shell scattering amplitudes

Often these can be determined completely from surprisingly minimal input

New simple proofs of GR and YM uniqueness

GR is the only parity invariant interacting theory of massless spin 2 particles with second order field equations

Colour/Kinematis duality

Bern, Carrasco, Johansson '08

Surprisingly, at the level of the on-shell scattering amplitudes (definitely not at the level of the Lagrangian), tree-level graviton amplitudes are appropriate squares of the YM ones

Many ways to state Gravity= (YM)^2. One of the conceptually simplest is the colour/kinematics duality

$$\mathcal{A}^{a_1,\dots,a_n}(1,\dots,n) = \sum_{3-\text{valent trees}} \prod_v f_v n_v \prod_e \frac{1}{s_e}$$

n-gluon scattering amplitude depending on gluon momenta and helicities

Lie algebra structure constants

kinematic "numerators"

propagators

Tri-valent graphs only!

Kinematic numerators satisfy the same Jacobi identities as the structure constants do

Such a prescription is proven to be correct (at tree level) for some choice of numerators, which are not unique

If a set of "numerators" is known, then the gravity amplitude is

$$\mathcal{M}(1,\ldots,n) = \sum_{3-\text{valent trees}} \prod_v n_v n_v \prod_e \frac{1}{s_e}$$

strip off colour, replace with kinematics

Nobody knows why it works, but it works!

Open/closed string duality if embedded into string theory

Remarks:

The “worst” possible QFT (gravity) appears to be the square of the best possible QFT (YM)

If one uses the modern on-shell methods, gravity is not as bad as it seems from the expansion of EH Lagrangian

Gravity has a very powerful group of gauge symmetries - diffeomorphisms

Gravity is in a certain sense best behaved theory in the UV:

$1/z^2$ behaviour of tree level amplitudes under the BCFW shift as compared to $1/z$ in YM theory

Unexpected loop level cancellations (in SUGRA) as emphasized by Bern and collaborators

New developments. Part II:

Gravity as a diffeomorphism invariant gauge theory

Exotic reformulation of General Relativity (in 4 spacetime dimensions) as a theory of connections rather than metrics

Makes GR quite analogous to YM at off-shell level

Puts problems of GR as a QFT in a different perspective

The idea of construction

4D GR is the unique diffeomorphism invariant theory of metrics with second order field equations

Can write diffeomorphism invariant theories of connections with second order field equations (see below)

Any such theory will contain gravity - there are propagating massless spin 2 particles in the spectrum

Particular diffeomorphism invariant gauge theory is GR

Diffeomorphism invariant gauge theories

Let f be a function on $\mathfrak{g} \otimes_S \mathfrak{g}$

satisfying

$\mathfrak{g}$ - Lie algebra of G

$$f: X \to \mathbb{R}(\mathbb{C})$$

$$X \in \mathfrak{g} \otimes_S \mathfrak{g}$$

defining function

1) $$f(\alpha X) = \alpha f(X)$$ homogeneous degree 1

2) $$f(gXg^T) = f(X), \quad \forall g \in G$$ gauge-invariant

Then $f(F \wedge F)$ is a well-defined 4-form (gauge-invariant)

$$F = dA + (1/2)[A, A]$$

Can define a gauge and diffeomorphism invariant action

$$S[A] = \mathrm{i} \int_M f(F \wedge F)$$

Diffeomorphism invariant gauge theories

second order feqs

no dimensionful coupling constants!

can show that linearisation around appropriate backgrounds always contains gravitons

Always non-renormalizable!
(because non-polynomial)

Metric from connections

The metric owes its existence to the "twistor" isomorphism

$$\mathrm{SO}(6,\mathbb{C}) \sim \mathrm{SL}(4,\mathbb{C})$$

The isomorphism implies

$$\mathrm{SL}(4)/\mathrm{SO}(4) \qquad \Leftrightarrow \qquad \mathrm{SO}(3,3)/\mathrm{SO}(3)\times\mathrm{SO}(3)$$

conformal metrics on M $\Leftrightarrow$ Grassmanian of 3-planes in Λ^2

Conformal metrics can be encoded into the knowledge of which 2-forms are self-dual

Definition of the metric:

Let A be an SU(2) connection

(SL(2,C) connection for Lorentzian signature)

$$F \wedge (F)^* = 0$$

reality conditions

declare F to be self-dual 2-forms $\Rightarrow$ conformal metric

To complete the definition of the metric need to specify the volume form

$$\Lambda^2 (\mathrm{vol}) = i\, f(F \wedge F)$$

$$S[A] = \Lambda^2 \int_M (\mathrm{vol})$$

Any diffeomorphism invariant SU(2) gauge theory is a theory of interacting gravitons

GR as a diffeomorphism invariant gauge theory

$$S_{\mathrm{GR}}[A] = \frac{\mathrm{i}}{16\pi G\Lambda}\int_M \left(\mathrm{Tr}\sqrt{F\wedge F}\right)^2$$

$\Lambda \neq 0$

Lagrangian is a function of the first derivatives of the basic field

(only) on-shell equivalent description:

connection satisfying the resulting Euler-Lagrange equations $\Rightarrow$ Einstein metric (of non-zero scalar curvature)

Final result: Gravity as theory of connections

Formalism that describes geometry using an SO(3) connection, not metric as the main variable

$$g = \partial A$$
$$F = \partial A$$

Both metric and the curvature are derivatives of the connection

Field equations $\partial^2 A = A$

second order PDE's on the connection

Requires non-zero cosmological constant

On-shell

↓

$$Ricci = \partial^2 g = \partial^3 A = \partial A = g$$
$$Weyl = \partial^2 g = \partial^3 A = \partial A = F$$

Gravitational instantons

These are particularly simple in the language of connections

$$F^i \wedge F^j \sim \delta^{ij}$$

first-order condition!

Claim: for connections satisfying this first-order PDE, the metric obtained by declaring F's self-dual is anti-self-dual Einstein with non-zero scalar curvature

New description of gravitons

Linearization gives rise to the following complex

$$S_+^2 \xrightarrow{d} S_+^3 \otimes S_- \xrightarrow{\delta} S_+^4$$

describes two spin 2 propagating DOF

$$\mathcal{L} \sim (\delta a)^2, \quad a \in S_+^3 \otimes S_-$$

<u>non-negative Lagrangian</u>

description of GR without the conformal mode problem

field equations

$$\delta^* \delta a = 0$$

gravitons of negative helicity

$$\delta a = 0$$

Works on any instanton space!

One loop behaviour

GR continues to be one loop renormalisable in the language of connections

$$\frac{\partial}{\partial \log(\mu)} \left(\frac{1}{16\pi G \Lambda} \right) = \frac{121}{5(4\pi)^2}$$

The coefficient in front of the action grows in the UV

Compare with the YM one loop result!

Summary

In the language of connections, GR becomes in many ways analogous to YM theory

- Lagrangian function of first derivatives $L \sim (\dot{x})^2$
- Linearised field equations operator is a square of appropriate first order one (and complex of operators arises)

$$\mathcal{L} \sim (\delta a)^2, \quad a \in S_+^3 \otimes S_-$$

- Instantons is a first order condition
- One loop divergence makes the coefficient in front of the action flow logarithmically with energy, and increase in the UV

Principal difference: diffeomorphism invariance in gravity

Results in gravity interactions being controlled by a negative mass dimension coupling

Remarks:

One should not be scared of negative mass dimension couplings

They are good for field redefinitions!

$$\phi \to \phi + (1/M)\phi^2 + (1/M)\partial\phi + \ldots$$

Only divergences modulo field redefinitions matter

The two loop divergence in GR

Goroff, Sagnotti '85

Van de Ven '91

Scary calculation, ever done just by 3 people

Algebraic manipulation (computers) is essential

$$\Gamma^{(2)}_{\infty} = \frac{1}{120(4\pi)^4} \frac{209}{24} \frac{1}{\epsilon} \int (\text{Riemann})^3$$

The new calculation uses on-shell methods

Bern et al '15

Possible to add non-propagating 3-form fields

$$\frac{209}{24} \to \frac{209}{24} - \frac{15}{2} n_3$$

On-shell irrelevant modes change the UV divergence

Divergence is sensitive to the off-shell details of the theory

Renormalisation scale dependence is insensitive to this

Concluding remarks

- Gravity is much closer to gauge theory than could have been anticipated. Either (YM)^2 on-shell, or particular diffeomorhism invariant gauge theory off-shell

- Gravity also behaves like YM in many ways. The principal difference is the dimensionality of the coupling

- The two loop behaviour of gravity is poorly understood.

  Can the divergence be an artefact of a particular off-shell version?

  Can it be an artefact of a particular regularization that is used?

The 2-loop integrand vanishes if evaluated in 4 dimensions!

Perturbative gravity used to be a mess. Everybody was happy that it diverges at 2 loops - don't have to deal with it!

But may be it is time to change the frame of mind and accept that gravity is in some sense most symmetric and beautiful QFT there is - we just don't understand it yet

Is there a diffeomorphism invariant QFT in 4D that makes sense?

Thank you!