Convergence of Quantum Statistical Experiments

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Statistical experiments and statistical decision problems

Statistical experiment

- Measure space: $(\Omega, \Sigma, \mathbb{P})$
- Statistical experiment over Θ (dominated): $\mathcal{E} = \{\mathbb{P}_\theta \ll \mathbb{P} : \theta \in \Theta\}$

Statistical decision problem

- Decision space: $(D, \mathcal{D}, \nu)$
- Loss function: $W : D \times \Theta \to \mathbb{R}_+$
- Decision procedure: $T : L^1(\Omega, \Sigma, \mathbb{P}) \to L^1(D, \mathcal{D}, \nu)$
  (randomization, e.g. Markov kernel, randomized statistic)
- Risk of the decision procedure T: $R(T, \theta) = T\mathbb{P}_\theta W_\theta := \int W(y, \theta) T\left(\frac{d\mathbb{P}_\theta}{d\mathbb{P}}\right) \nu(dy)$

Standard examples:

- Estimation: $D = \Theta,\ W(\hat{\theta}, \theta) = \text{“}(\hat{\theta} - \theta)^2\text{”}$
- Hypothesis testing: $D = \{0, 1\},\ W(h, \theta) = 1\,(0)$ if $\theta \in\ (\notin)\, \Theta_h$ where $\Theta = \Theta_0 \cup \Theta_1$

Statistical experiments and statistical decision problems

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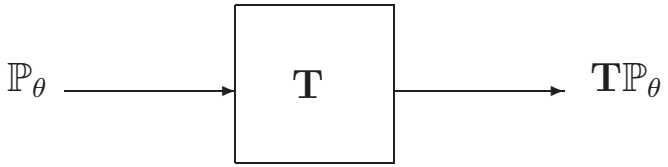
Statistical decision problem



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Comparison of experiments

Experiments which are "close" to each other should have similar optimal risks

- Weak convergence of experiments: $\mathcal{E}_n = \{\mathbb{P}_\theta^n : \theta \in \Theta\} \rightsquigarrow \mathcal{E} = \{\mathbb{P}_\theta : \theta \in \Theta\}$ if

$$\left\{ \frac{d\mathbb{P}_\theta^n}{d\mathbb{P}_{\theta_0}^n} : \theta \in \Theta \right\} \rightsquigarrow \left\{ \frac{d\mathbb{P}_\theta}{d\mathbb{P}_{\theta_0}} : \theta \in \Theta \right\}$$

Convergence in distribution of finite marginals

- Strong convergence of experiments: $\mathcal{E}_n \to \mathcal{E}$ iff $\max(\delta(\mathcal{E}_n, \mathcal{E}), \delta(\mathcal{E}, \mathcal{E}_n)) \to 0$,

$$\delta(\mathcal{E}_n, \mathcal{E}) = \inf_T \sup_\theta \|T\mathbb{P}_\theta^n - \mathbb{P}_\theta\|_{\text{tv}}$$

Le Cam deficiency

Theorem: Weak and strong convergence are equivalent for **finite** sets Θ.

Local asymptotic normality: let $Y_1, \ldots, Y_n$ i.i.d. with $\mathbb{P}_{\theta_0 + u/\sqrt{n}}$ be a "smooth" family with $u \in \mathbb{R}$. Then

$$\left\{ \left(\mathbb{P}_{\theta_0 + u/\sqrt{n}}\right)^n : u \in \mathbb{R} \right\} \rightsquigarrow \left\{ N\left(u, I(\theta_0)^{-1}\right) \right\}$$

Quantum systems

Quantum system: light in a microcavity, atom, nanomechanical systems...

- Observables = quantities which can be measured
  form a non-commutative algebra $\mathcal{A}$ (for ex. $\mathrm{M}(\mathbb{C}^d)$)

- State = preparation: determines the prob. distribution of results of any measurements

$$\mathcal{S}_{\mathcal{A}} = \mathcal{A}_*^{+,1} = \{\rho \in \mathrm{M}(\mathbb{C}^d) \,:\, \rho \geq 0,\, \mathrm{Tr}(\rho) = 1\}$$

- Duality between $\mathcal{A}_*$ and $\mathcal{A}$:

$$\begin{aligned} \langle \cdot, \cdot \rangle \quad &: \mathcal{A}_* \times \mathcal{A} \to \mathbb{C} \\ \langle \rho, A \rangle \quad &:= \mathbb{E}_\rho(A) = \mathrm{Tr}(\rho\, A) \end{aligned}$$

"Classical" case: $\mathcal{A} = L^\infty(\Omega, \Sigma, \mathbb{P})$, $\mathcal{A}_* = L^1(\Omega, \Sigma, \mathbb{P})$

$$\begin{aligned} \langle \cdot, \cdot \rangle_{\mathbb{P}} \quad &: L^1(\Omega, \Sigma, \mathbb{P}) \times L^\infty(\Omega, \Sigma, \mathbb{P}) \to \mathbb{C} \\ \langle p, X \rangle_{\mathbb{P}} \quad &:= \int p(\omega)\, X(\omega)\, \mathbb{P}(d\omega) \end{aligned}$$

Cl. Statistical Experiments

Measure space: $(\Omega, \Sigma, \mathbb{P})$

Space of random variables: $L^\infty(\Omega, \Sigma, \mathbb{P})$

Space of probability densities: $L^1(\Omega, \Sigma, \mathbb{P})$

$$\mathbb{Q} \ll \mathbb{P} \Longrightarrow q := \frac{d\mathbb{Q}}{d\mathbb{P}} \in L^1(\Omega, \Sigma, \mathbb{P})$$

Duality between the L^1 and L^∞ spaces:

$$\begin{aligned} \langle \cdot, \cdot \rangle_{\mathbb{P}} \quad &: L^1(\Omega, \Sigma, \mathbb{P}) \times L^\infty(\Omega, \Sigma, \mathbb{P}) \to \mathbb{C} \\ \langle p, X \rangle_{\mathbb{P}} \quad &:= \int p(\omega)\, X(\omega)\, \mathbb{P}(d\omega) \end{aligned}$$

Statistical experiment over Θ (dominated):

$$\mathcal{E} = \{\mathbb{P}_\theta \ll \mathbb{P} : \theta \in \Theta\}$$

Q. Statistical Experiments

"No little $\omega's$"

Algebra of Observables: $\mathcal{A} = \mathrm{M}(\mathbb{C}^d)$

Space of States (density matrices):

$$\mathcal{S} = \{\rho \in \mathrm{M}(\mathbb{C}^d) \ : \ \rho \geq 0, \mathrm{Tr}(\rho) = 1\}$$

Duality between $\mathcal{A}_*$ and $\mathcal{A}$:

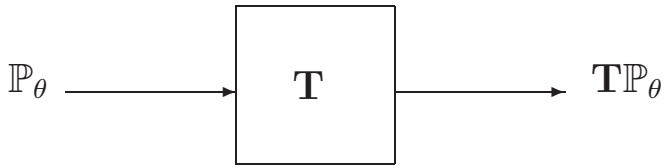
$$\begin{aligned} \langle \cdot, \cdot \rangle \quad &: \mathcal{A}_* \times \mathcal{A} \to \mathbb{C} \\ \langle \rho, A \rangle \quad &= \mathbb{E}_\rho(A) := \mathrm{Tr}(\rho\, A) \end{aligned}$$

Quantum statistical experiment over Θ:

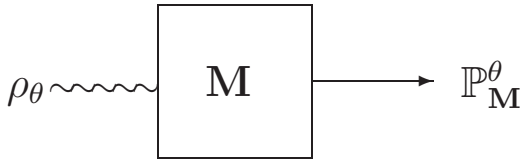
$$\begin{aligned} &\mathcal{Q} = \{\phi_\theta : \theta \in \Theta\} \\ &\phi_\theta : \mathcal{A} \to \mathbb{C} \quad \phi_\theta(A) = \mathrm{Tr}(\rho_\theta\, A) \end{aligned}$$

Quantum and Classical Randomizations

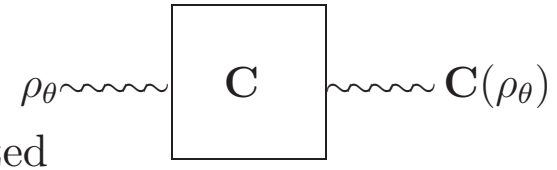
Classical-Classical: $T : L^1(\Omega, \Sigma, \mathbb{P}) \to L^1(\Omega', \Sigma', \mathbb{P}')$
linear, positive, normalized



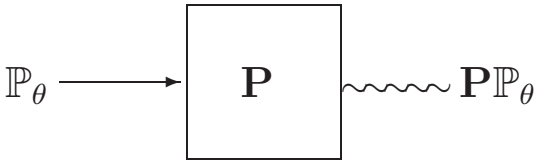
Quantum-Classical (measurement): $M : \mathcal{A}_* \to L^1(\Omega, \Sigma, \mathbb{P})$
linear, positive, normalized



Quantum-Quantum (channel): $M : \mathcal{A}_* \to \mathcal{A}'_*$
linear, c. positive, normalized

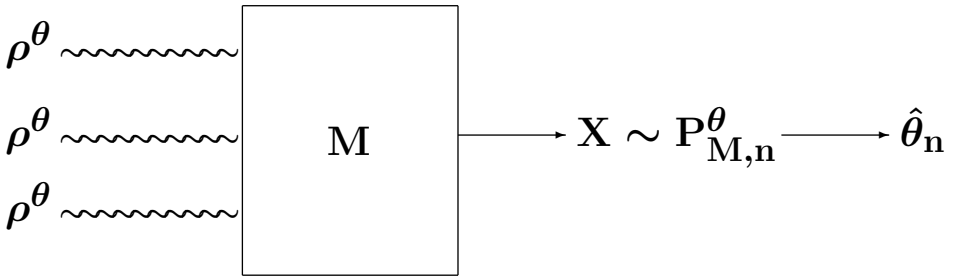


Classical-Quantum (preparation): $P : L^1(\Omega, \Sigma, \mathbb{P}) \to \mathcal{A}_*$
linear, positive, normalized

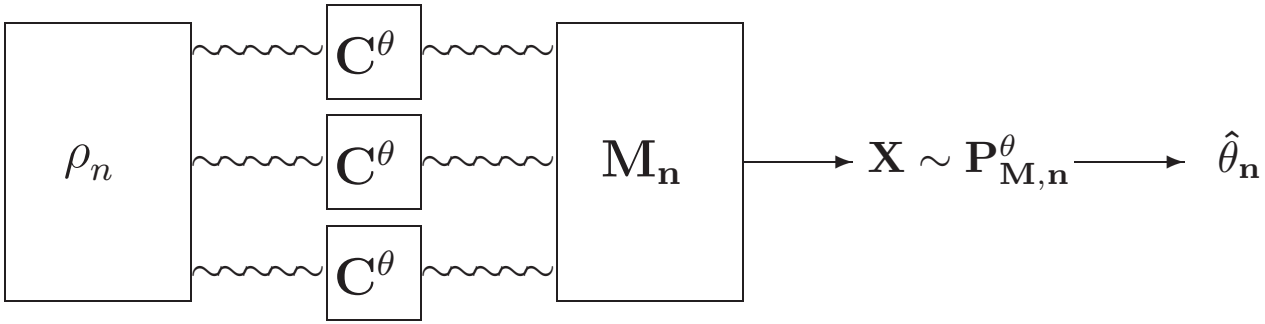


Some quantum statistical problems

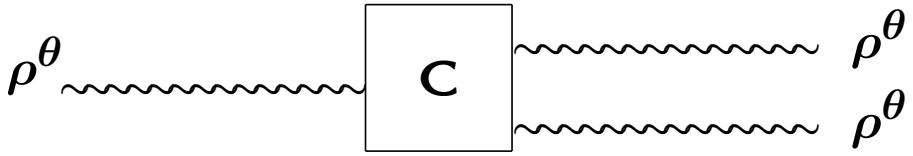
Estimation of an unknown state (preparation) of a quantum system through measurements



Estimation of a quantum channel (transformation)



Cloning (copying) an unknown quantum state



Connes cocycles as likelihood ratio process

- Likelihood ratio process: Experiment $\mathcal{E} = (\Omega, \Sigma, \{\mathbb{P}_\theta \,:\, \theta \in \Theta\})$

$$\Lambda_{\theta_0} : \theta \mapsto \frac{d\mathbb{P}_\theta}{d\mathbb{P}_{\theta_0}}(\omega)$$

is sufficient statistic for $\mathcal{E}$: if $\Sigma_0 := \sigma(\Lambda_{\theta_0})$ then $\mathcal{E}_0 := \mathcal{E}|_{\Sigma_0} \simeq \mathcal{E}$.

Quantum likelihood ratio is not uniquely defined: Experiment $\mathcal{Q} := (\mathcal{A}, \{\rho_\theta : \theta \in \Theta\})$

$$\rho_\theta \left(\rho_{\theta_0}\right)^{-1} \neq \left(\rho_{\theta_0}\right)^{-1} \rho_\theta$$

- "Quantum likelihood ratio process": Connes cocycles

$$[D\rho_\theta,\, D\rho_{\theta_0}]_t := \left(\rho_\theta\right)^{it} \left(\rho_{\theta_0}\right)^{-it} \qquad \left(\text{analogue of } \left(\frac{d\mathbb{P}_\theta}{d\mathbb{P}_{\theta_0}}\right)^{it}\right)$$

Theorem: [Petz] Let $\mathcal{A}_0 \subset \mathcal{A}$ be the algebra generated by $\{[D\rho_\theta,\, D\rho_{\theta_0}]_t : \theta \in \Theta, t \in \mathbb{R}\}$. Then

$$\mathcal{Q}_0 := \mathcal{Q}|_{\mathcal{A}_0} \simeq \mathcal{Q}.$$

Convergence of quantum statistical experiments

- Weak convergence of quantum experiments:

$$\mathcal{Q}_n := (\mathcal{A}_n, \{\rho_\theta^n : \theta \in \Theta\}) \rightsquigarrow \mathcal{Q} := (\mathcal{A}, \{\rho_\theta : \theta \in \Theta\})$$

if for all $k \in \mathbb{N}, \theta_i \in \Theta, t_i \in \mathbb{R}$

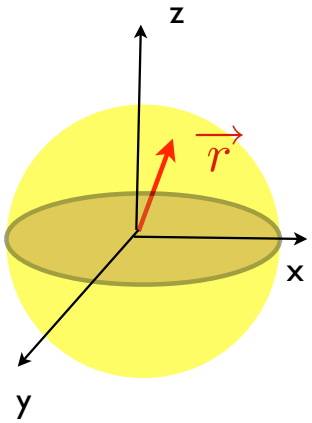
$$\lim_{n\to\infty} \mathrm{Tr}\left(\rho_{\theta_0}^n \prod_{i=1}^k [D\rho_{\theta_i}^n \,,\, D\rho_{\theta_0}^n]_{t_i}\right) \;=\; \mathrm{Tr}\left(\rho_{\theta_0} \prod_{i=1}^k [D\rho_{\theta_i} \,,\, D\rho_{\theta_0}]_{t_i}\right),$$

- Strong convergence of quantum experiments: $\mathcal{Q}_n \to \mathcal{Q}$ if $\max(\delta(\mathcal{Q}_n, \mathcal{Q}) \,,\, \delta(\mathcal{Q}, \mathcal{Q}_n)) \to 0$

$$\delta(\mathcal{Q}_n, \mathcal{Q}) = \inf_C \sup_\theta \|C(\rho_\theta^n) - \rho_\theta\|_1$$

Theorem: If $\mathcal{Q}_n$ are based on discrete sums of matrix algebras, then weak and strong convergence are equivalent for **finite** sets Θ.

Local asymptotic normality for quantum states

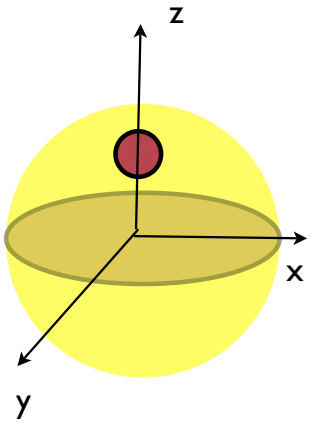


- Qubit state

$$\rho = \frac{\mathbf{1} + \vec{r}\,\vec{\sigma}}{2} := \begin{pmatrix} 1+r_z & r_x - ir_y \\ r_x + ir_y & 1-r_z \end{pmatrix} \geq 0, \quad \text{for } |\vec{r}| \leq 1.$$

- Consider a local neighborhod (small ball) of the state

$$\rho_0 := \begin{pmatrix} \mu & 0 \\ 0 & 1-\mu \end{pmatrix},$$



- Parametrize the local neighborhood

$$\rho = \rho_{\mathbf{u}/\sqrt{n}} := U\left(\mathbf{u}/\sqrt{n}\right) \begin{pmatrix} \mu + u_z/\sqrt{n} & 0 \\ 0 & 1-\mu-u_z \end{pmatrix} U\left(\mathbf{u}/\sqrt{n}\right)^*,$$

where $\mathbf{u} = (u_x, u_y, u_z) \in \mathbb{R}^3$ is the local parameter, $U(\mathbf{u}) := \exp(i(u_x\sigma_x + u_y\sigma_y))$.

- The i.i.d. experiment : n identically prepared quatum systems

$$\mathcal{Q}_n = \left\{ \left(\rho_{\mathbf{u}/\sqrt{n}}\right)^{\otimes n} : \mathbf{u} \in \mathbb{R}^3 \right\}$$

The two convergence Theorems

Locally, the joint state of n qubits converges to a quantum Gaussian shift model

$$\text{“} \quad \rho_{\mathbf{u}/\sqrt{n}}^{\otimes n} \rightsquigarrow N^{\mathbf{u}} \otimes \phi^{\mathbf{u}} \quad \text{”}$$

Theorem: Let $\rho_{\mathbf{u}}^{(n)} := \left(\rho_{\mathbf{u}/\sqrt{n}}\right)^{\otimes n}$. Then as $n \to \infty$

$$\left\{\rho_{\mathbf{u}}^{(n)} \,:\, \mathbf{u} \in \mathbb{R}^3\right\} \rightsquigarrow \left\{N^{\mathbf{u}} \otimes \phi^{\mathbf{u}} \,:\, \mathbf{u} \in \mathbb{R}^3\right\}.$$

Theorem: Let $\rho_{\mathbf{u}}^{(n)} := \left(\rho_{\mathbf{u}/\sqrt{n}}\right)^{\otimes n}$. Then there exist quantum randomizations (q. channels) T_n, S_n such that for any $\eta < 1/4$ and $\epsilon > 0$:

$$\lim_{n\to\infty} \sup_{\|\mathbf{u}\|<n^{\eta}} \left\| T_n(\rho_{\mathbf{u}}^{(n)}) - \phi_{\mathbf{u}} \right\|_1 = O(n^{-1/4+\eta+\epsilon}),$$

and

$$\lim_{n\to\infty} \sup_{\|\mathbf{u}\|<n^{\eta}} \left\| \rho_{\mathbf{u}}^{(n)} - S_n\left(\phi_{\mathbf{u}}\right) \right\|_1 = O(n^{-1/4+\eta+\epsilon}).$$

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- M. Guță:
  Quantum statistical experiments and quantum decision problems
  in preparation