Estimation and metrology
in quantum input-output systems

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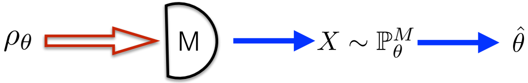
Quantum control and feedback
foundations and applications
IHP Paris 2018

The University of Nottingham

Outline

- Quantum estimation
- The quantum Fisher information of input-output systems
- Estimation with ‘simple’ measurements
- Quantum post-processing with coherent absorbers
- Dynamical phase transitions and Heisenberg scaling

Quantum parameter estimation



- Estimation problem: estimate θ by performing a measurement M on system in state ρ_θ

- What is quantum about this ?
  - fixed measurement: "classical stats" problem with special probabilistic structure
  - "optimal" measurement: need to understand structure of quantum statistical model
  - quantum enhance precision when θ is encoded with "sensitive states"

- Classical and quantum Cramér-Rao bounds[1]: if $\hat{\theta}$ is unbiased

$$\mathbb{E}\left[(\hat{\theta}-\theta)^T \cdot (\hat{\theta}-\theta)\right] \geq I^M(\theta)^{-1} \geq F(\theta)^{-1}$$

Classical Fisher info (under $I^M(\theta)^{-1}$) — Quantum Fisher info (under $F(\theta)^{-1}$)

[1] A. Holevo. *Probabilistic and Statistical Aspects of Quantum Theory* (1982); S. L. Braunstein, C. M. Caves, P.R.L. (1994)

Pure states models

- one parameter pure state rotation model: $|\psi_\theta\rangle := e^{-i\theta G}|\psi\rangle, \qquad \langle\psi|G|\psi\rangle = 0$
- Quantum Fisher Information: $F(\theta) = 4\left\|\frac{d\psi_\theta}{d\theta}\right\|^2 = 4\mathrm{Var}_\psi(G) = 4\left\langle\psi\,\middle|\,G^2\,\middle|\,\psi\right\rangle$

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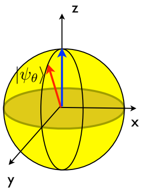
- Example: One-dimensional qubit rotation

$|\psi_\theta\rangle := \exp\left(i\theta\sigma_x\right)|\uparrow\rangle = \cos(\theta)|\uparrow\rangle + \sin(\theta)|\downarrow\rangle$

  - Quantum Fisher information $F = 4\langle\uparrow|\sigma_x^2|\uparrow\rangle = 4$
  - 'most informative' observable σ_y



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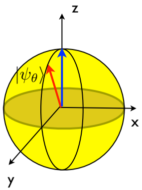
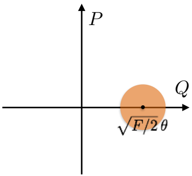


- Example: Gaussian shift model CV system $[\mathbf{Q},\mathbf{P}] = i\mathbf{1}$
  - $\left|\sqrt{F/2}\,\theta\right\rangle$ coherent state with mean $\left(\sqrt{F/2}\,\theta, 0\right)$
  - quantum Fisher information $= 4\mathrm{Var}\left(\sqrt{F/2}\mathbf{P}\right) = F$
  - Cramér-Rao bound achieved by measuring $\mathbf{Q}$



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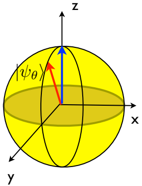

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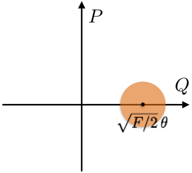

- Tutorial: Convergence of IID ensemble converges to Gaussian shift model

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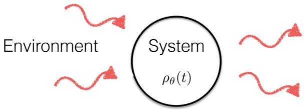
Markovian quantum open systems in continuous time



- Dissipative evolution of open system

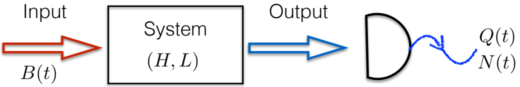
$$\frac{d}{dt}\rho(t) = \mathcal{L}\rho(t) = -i[H, \rho(t)] + \sum_i L^i \rho(t) L^{i*} - \frac{1}{2}\{\rho(t), L^{i*}L^i\}$$

- Ergodicity: system converges to stationary state ρ_{ss} ($\mathcal{L}\rho_{ss} = 0$)

$$\rho(t) = e^{t\mathcal{L}}\rho_{in} \longrightarrow \rho_{ss}$$

- Estimating unknown "dynamical parameters" $\theta \mapsto D_\theta = (H_\theta, L^i_\theta)$ by direct probing
  - system may not be accessible (e.g. in quantum control applications)
  - system would need to be initialised repeatedly
  - Information about dynamical parameters "leaks" continuously into the environment

System identification and estimation with input-output open systems



- Unitary dynamics: singular coupling with incoming input fields (Q Stoch Diff Eq[2])

$$dU(t) = \left(-iHdt + LdA^*(t) - L^*dA(t) - \frac{1}{2}L^*Ldt\right)U(t)$$

- System identification: if $\theta \to (H_\theta, L_\theta)$, estimate θ by measuring the output[3]
  - which parameters can be identified ?
  - how does the output QFI scale with time t ?
  - how does this relate to dynamical properties, e.g. ergodicity, spectral gap...?
  - which measurements are informative ?
  - how to achieve high estimation accuracy ?

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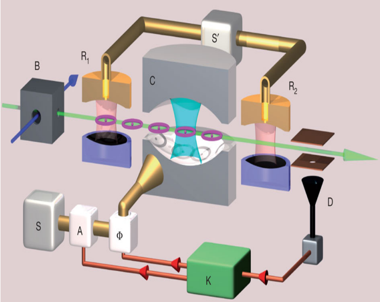
[2] K. R. Parthasarathy, *An introduction to quantum stochastic calculus*, Springer Birkhäuser (1992)

[3] H. Mabuchi *Quant. Semiclass. Optics* (1996); J. Gambetta and H. M. Wiseman *Phys. Rev. A* (2001); S. Gammelmark and K. Molmer *Phys. Rev. A* (2013), S.Bonnabel, M.Mirrahimi, P.Rouchon, *Automatica* (2009)...

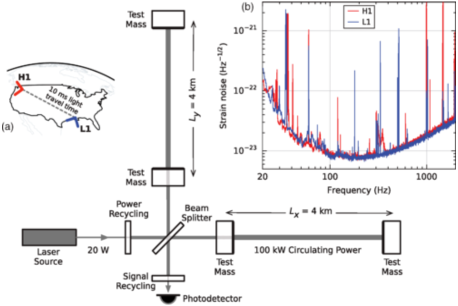
Quantum input-output systems[4]

- Input-output formalism describes controlled open system dynamics
- Quantum filtering, feedback control, quantum networks
- Control and system identification: two sides of the coin



Feedback control of cavity state in the atom maser

C. Sayrin *et al*, *Nature* (2011)

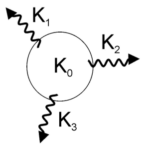


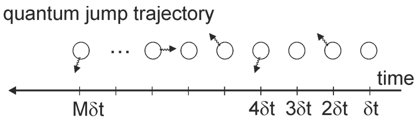
Advanced LIGO

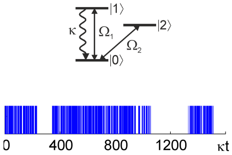
B. P. Abbott *et al. Phys. Rev. Lett.* (2016)

[4] C. W. Gardiner and P. Zoller, *Quantum Noise* (2004)
H. M. Wiseman and G. J. Milburn, *Quantum measurements and control* (2010)

Output state as superposition of quantum trajectories







- Monitoring the environment produces jump trajectories with infinitesimal Kraus operators
  - "no emission": $K_\theta^0 = e^{-i\delta t H_\theta} \sqrt{1 - \delta t \sum_j L_\theta^{j*} L_\theta^j}$
  - "emission" in channel j: $K_\theta^j = e^{-i\delta t H_\theta} \sqrt{\delta t} L_\theta^j$

- System-output state: coherent superposition of quantum trajectories, (continuous) MPS[5]

$$|\psi_\theta^{s+o}(t)\rangle = U_\theta(t)|\psi_{in}^{s+o}\rangle = \sum_{j_1,\dots,j_n} K_\theta^{j_n} \dots K_\theta^{j_1} |\psi\rangle \otimes |j_n \dots j_1\rangle, \qquad n = t/\delta t$$

[5] M. Fannes, B. Nachtergale and R. Werner, *Commun. Math. Phys.*(1992);
D. Perez-Garcia, F. Verstraete, M. Wolf and I. Cirac, *Quantum Inf. Comput.* (2007)

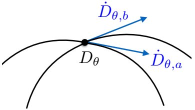
Generator of parameter change in system+output state

- Model dynamics with unknown parameter $\theta \in \mathbb{R}^m$

$$D_\theta = (H_\theta, L_\theta) \quad \longrightarrow \quad \left|\Psi_\theta^{\mathrm{s+o}}(t)\right\rangle = U_\theta(t)|\varphi \otimes \Omega\rangle$$

- Tangent vector at D_θ corresponding to changes in component θ_a

$$\dot{D}_{\theta,a} = (\dot{H}_{\theta,a}, \dot{L}_{\theta,a}) = \left(\frac{\partial H}{\partial \theta_a}, \frac{\partial L}{\partial \theta_a}\right)$$



- Generator of parameter change for component θ_a

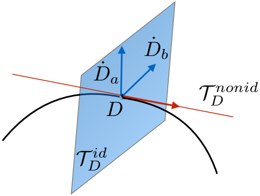
$$\frac{\partial}{\partial \theta_a}\left|\Psi_\theta^{\mathrm{s+o}}(t)\right\rangle = \dot{U}_{\theta,a}(t)|\varphi \otimes \Omega\rangle = U_\theta(t) G_{\theta,a}(t)|\varphi \otimes \Omega\rangle$$

- Generator is a quantum stochastic integral (fluctuation operator)

$$G_{\theta,a}(t) \quad := \quad \sqrt{t}\mathbb{F}_t(\dot{D}_{\theta,a}) = \int_0^t \dot{L}_{\theta,a}(s) dA^*(s) - i\mathcal{E}_D(\dot{D}_{\theta,a})(s)ds$$

$$\mathcal{E}_D(\dot{D}) \quad := \quad \dot{H} + \mathrm{Im}(\dot{L}^* L) - \mathrm{Tr}\left[\rho_{ss}^D(\dot{H} + \mathrm{Im}(\dot{L}^* L))\right]\mathbf{1}$$

Quantum information geometry of stationary output state



Theorem (QFI as Riemanian metric)

The quantum Fisher information matrix $F_{a,b}(t) = 4\mathrm{Re}\left\langle G^*_{\theta,a}(t) \cdot G_{\theta,b}(t)\right\rangle$ *grows linearly in* t *with rate* $F_{a,b}$ *given by the asymptotic Markov covariance of fluctuators*

$$\begin{aligned} F_{a,b} &= 4\mathrm{Re}\left(\dot{D}_{\theta,a}, \dot{D}_{\theta,b}\right)_D \\ &:= 4\mathrm{Re}\,\mathrm{Tr}\left[\rho_{ss}\left(\dot{L}_{\theta,a} - i[L_\theta, \mathcal{L}^{-1} \circ \mathcal{E}_{\mathrm{D}}(\dot{\mathrm{D}}_{\theta,a})]\right)^* \cdot \left(\dot{L}_{\theta,b} - i[L_\theta, \mathcal{L}^{-1} \circ \mathcal{E}_{\mathrm{D}}(\dot{\mathrm{D}}_{\theta,b})]\right)\right]. \end{aligned}$$

The tangent space decomposes into identifiable and unidentifiable subspaces $\mathcal{T}_D = \mathcal{T}_D^{id} \oplus \mathcal{T}_D^{nonid}$

- $\mathcal{T}_D^{nonid} := \{\dot{D} : \dot{D} = i[K, D] + c(\mathbf{1}, 0)\} \quad \longrightarrow \quad (\dot{D}, \dot{D}')_D = 0$
- $\mathcal{T}_D^{id} = \{\dot{D} : \mathcal{E}_D(\dot{D}) = 0\} \quad \longrightarrow \quad (\dot{D}, \dot{D}')_D = \mathrm{Tr}(\rho_{ss}^D \dot{L}^* \dot{L}')$
- $F_{a,b}$ *defines a Riemannian metric on* $\mathcal{P} = \mathcal{D}/G$
- *Tutorial: Convergence to Gaussian model on CCR algebra*

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Counting measurement and total counts statistic[6]

- Let $\Lambda^{(out)}(t)$ be the counting process with stationary emission rate

$$m_\theta = Tr(\rho^\theta_{ss} L^*_\theta L_\theta) = \lim_{t\to\infty} \frac{\Lambda^{(out)}(t)}{t}$$

- Let $\theta = \theta_0 + u/\sqrt{t}$ in terms of local parameter u.

Theorem (asymptotic normality and classical Fisher information for counting)

If dynamics is ergodic, then $\Lambda^{(out)}(t)$ satisfies local asymptotic normality, i.e. the following convergence in distribution holds as $t \to \infty$,

$$\frac{\Lambda^{(out)}(t) - t m_{\theta_0}}{\sqrt{t}} \xrightarrow{\mathcal{D}} N(\mu_c u, V_c)$$

where $\mu_c := \left.\frac{dm_\theta}{d\theta}\right|_{\theta_0}$ and V_c have explicit expressions in terms of $\dot{D} = (\dot{H}, \dot{L})$ and $\mathcal{L}$.

The asymptotic (rescaled) classical Fisher information is given by the SNR

$$I_c = \frac{\mu_c^2}{V_c} \leq F$$

The estimator $\hat{\theta}_t := \theta_0 + \frac{\Lambda^{(out)}(t) - t m_{\theta_0}}{t\mu_c}$ is asymptotically normal and satisfies

$$\lim_{t\to\infty} t\mathbb{E}\left[(\hat{\theta}_t - \theta)^2\right] = I_c^{-1}.$$

[6] C. Catana, L. Bouten, M.G, *J. Phys. A* (2015)

Homodyne measurement and integrated current statistic[7]

- Let $X(t)(t) = A^{(out)}(t) + A^{(out)*}(t)$ be the homodyne process with stationary mean
$$m_\theta = Tr(\rho^\theta_{ss}(L_\theta + L^*_\theta)) = \lim_{t\to\infty} \frac{X(t)}{t}$$
- Let $\theta = \theta_0 + u/\sqrt{t}$ in terms of local parameter u.

Theorem (asymptotic normality and classical Fisher information for homodyne)

If dynamics is ergodic, then $X(t)$ satisfies local asymptotic normality, i.e. the following convergence in distribution holds as $t \to \infty$,

$$\frac{X(t) - tm_{\theta_0}}{\sqrt{t}} \xrightarrow{\mathcal{D}} N(\mu_h u, V_h)$$

where $\mu_h := \left.\frac{dm_\theta}{d\theta}\right|_{\theta_0}$ and V_h have explicit expressions in terms of $\dot{D} = (\dot{H}, \dot{L})$ and $\mathcal{L}$.

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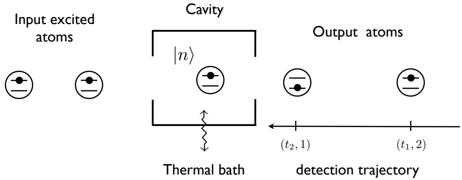
[7] C. Catana, L. Bouten, M.G, *J. Phys. A* (2015)

Example: atom maser



- Atom maser with Jaynes-Cummings interaction

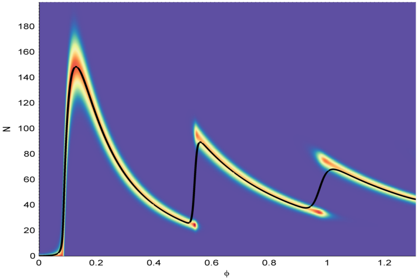
$$U : |1\rangle \otimes |k\rangle \mapsto \cos\left(\phi\sqrt{k+1}\right) |1\rangle \otimes |k\rangle + \sin\left(\phi\sqrt{k+1}\right) |0\rangle \otimes |k+1\rangle$$

- Coarse grained cavity dynamics for Poisson distributed input atoms with rate N_{ex}

$$\frac{d\rho}{dt} = \mathcal{L}(\rho) = \sum_{i=1}^{4} \left(L_i \rho L_i^* - \frac{1}{2}\{L_i^* L_i, \rho\} \right)$$

  - $L_1 : |k\rangle \mapsto \sqrt{N_{ex}} \sin(\phi\sqrt{k+1}) \, |k+1\rangle$ (excitation absorbed from atom)
  - $L_2 : |k\rangle \mapsto \sqrt{N_{ex}} \cos(\phi\sqrt{k+1}) \, |k\rangle$ (atom remains in excited state)
  - $L_3 : |k\rangle \mapsto \sqrt{k(\nu+1)} \, |k-1\rangle$ (photon emitted in the bath)
  - $L_4 : |k\rangle \mapsto \sqrt{(k+1)\nu} \, |k+1\rangle$ (photon absorbed from the bath)

Stationary state and phase transitions



Mean photon number and photon distribution in the stationary state as function of $\alpha = \sqrt{N_{ex}}\phi$

- unique stationary state

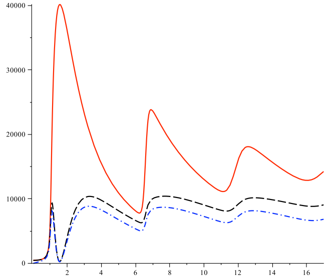
$$\rho_{ss}(n) = \rho_{ss}(0) \prod_{k=1}^{n} \left(\frac{\nu}{\nu+1} + \frac{N_{ex}}{\nu+1} \frac{\sin^2(\phi\sqrt{k})}{k} \right)$$

- jumps in mean photon number around $\alpha = 1, 2\pi, 4\pi$
- bistable stationary distribution around $\alpha = 2\pi, 4\pi$
- can be undestood via large deviations for the counting process[1]

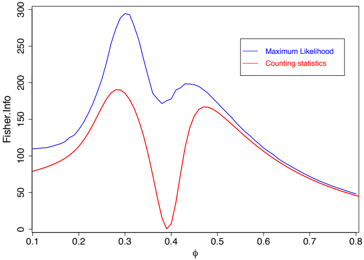
[1] J. P. Garrahan and I. Lesanovsky, Phys. Rev. Lett. 2010

The many Fisher informations of the atom maser[8], [9]

- Quantum Fisher information $F = 4N_{ex}\mathrm{Tr}(\rho_{ss}N)$
- Total counts exhibits zero Fisher information at maximum of QFI
- ML based on full counting trajectory has Fisher close(r) to QFI
- ABC with 'composite statistics' comes close to ML



red: quantum Fisher info
black: observe cavity + bath
blue: observe cavity



red: Fisher info total counts
blue: Fisher info counting process

[8] C. Catana, M van Horssen, M.G., *Phil. Trans. Royal Soc. A* (2012)
[9] C.Catana, T. Kypraios and M.G. *J. Phys. A: Math. Theor.* (2014)

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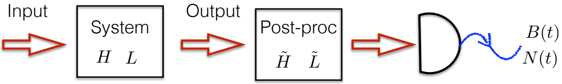
How to design ‘better’ measurements?

- Using full information in measurement trajectory (eg via ML) may be expensive

- Alternative: extend class of ‘simple measurements’ by ‘quantum post-processing’



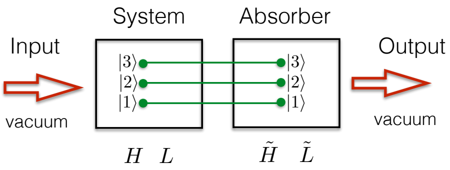
- Dynamical parameters of cascaded system $(L, H) \triangleright (\tilde{L}, \tilde{H})$

$$H_c = H + \tilde{H} - i(L\tilde{L}^* - L^*\tilde{L}) \qquad L_c = L + \tilde{L}$$

- QFI of joint system does not change but allows for more general measurements

- How to design good quantum post-processors, or more general feedback networks ?

Coherent absorbers[10]



- Absorber: stationary state of joint system is pure with Schmidt form

$$|\psi_{ss}\rangle = \sum_n \sqrt{p_n}|n\rangle \otimes |n\rangle, \qquad \rho_{ss} = \sum_n p_n |n\rangle\langle n|$$

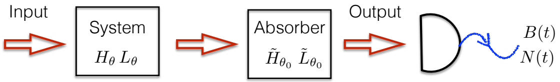
- As consequence, joint output is vacuum
- Absorber's dynamical parameters

$$\tilde{L} = -\sum_{n,m} \sqrt{\frac{p_n}{p_m}} \langle m|L|n\rangle \, |n\rangle\langle m|$$

$$\tilde{H} = -\frac{1}{2}\sum_{n,m} \left(\sqrt{\frac{p_n}{p_m}} H^{(eff)}_{m,n} + \sqrt{\frac{p_m}{p_n}} H^{(eff)*}_{m,n} \right) |n\rangle\langle m|$$

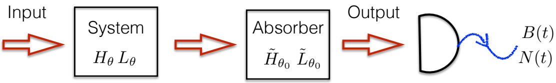
[10] K. Stannigel, P. Rabl, and P. Zoller, *New Journal of Physics* (2012)

Null measurement with compensating absorber



- Let $\theta = \theta_0 + \delta\theta$ with θ_0 a known 'preliminary estimator'
- Use absorber system $(\tilde{H}_{\theta_0}, \tilde{L}_{\theta_0})$ as compensator for unknown system (H_θ, L_θ)
- Is this useful ?
  - null measurement is conceptually simple
  - procedure can be implemented adaptively
  - SNR has the form $\frac{\left(\frac{dm_\theta}{d\theta}\Big|_{\theta_0} \right)^2}{V_{\theta_0}}$. Variance V_{θ_0} is easier to compute wrt vacuum

Homodyne measurement



- Homodyne: stochastic process $\{X(s) = \frac{1}{\sqrt{t}}(A^{(out)}(s) + A^{(out)*}(s)) : s \in [0, t]\}$
  - First moment: integrated homodyne current $X(t)$ has Fisher info rate

$$I_1 = \frac{\mu_{\theta_0}^2}{V_{\theta_0}} = \mu_{\theta_0}^2, \qquad \mu_{\theta_0} = \left. \frac{d\,\mathrm{Tr}(\rho_\theta^{ss}(L_\theta^* + L_\theta))}{d\theta} \right|_{\theta_0}$$

  - Second moment: quadratic functional $Y(t) = \int_0^t \int_0^t k(s,r) dX_s dX_r$

$$SNR = \frac{|\langle k, \dot{c}_{\theta_0} \rangle|^2}{\langle k, k \rangle}$$

  - Maximum Fisher info rate achieved for $k(s,r) = \dot{c}_{\theta_0}(r - s)$

$$I_2 = \left\| \dot{c}_{\theta_0}(\tau) \right\|^2 = \int_0^\infty |\dot{c}_{\theta_0}(\tau)|^2 \, d\tau$$

Simple example

- 'Reference' model

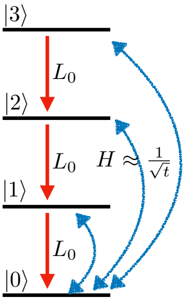
  $L_0 = \epsilon(|0\rangle\langle 1| + |1\rangle\langle 2| + |2\rangle\langle 3|),\ H_0 = 0$

  $\rho_0^{ss} = |0\rangle\langle 0|$

- Perturbed model:

  $L = L_0,\ H = \frac{1}{\sqrt{t}} \sum_i (u_i^x \sigma_x^{(0i)} + u_i^y \sigma_y^{(0i)})$

  $\rho_0^{ss} = |0\rangle\langle 0| + \frac{\mathbf{u}}{\sqrt{t}} \dot{\rho}^{ss} + \dots$

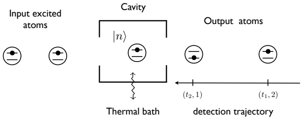


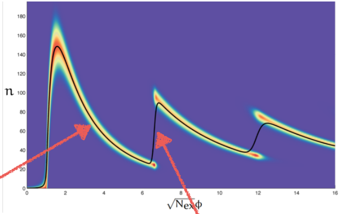
1) Parameters for $\sigma_x^{(ij)}$ and $\sigma_y^{(ij)}$ with $i, j \geq 1$ have zero QFI
2) Perturbations u_1^x, u_1^y have $F = \frac{16}{\epsilon^2}$ and is achieved by first moment homodyne of $P(t), Q(t)$
3) Perturbations u_2^x, u_2^y have $F = \frac{16}{\epsilon^2}$ and is achieved by second moment homodyne functionals
4) Perturbations u_3^x, u_3^y have $F = \frac{16}{\epsilon^2}$ and is achieved by third moment homodyne functionals
5) Wanted: general theory with $H_0|0\rangle = 0$ and $L_0|0\rangle = 0$.

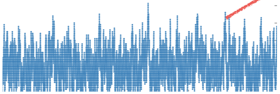
Outline

- Quantum estimation
- The quantum Fisher information of input-output systems
- Estimation with ‘simple’ measurements
- Quantum post-processing with coherent absorbers
- Dynamical phase transitions and Heisenberg scaling

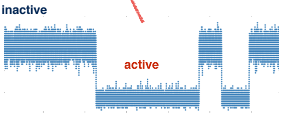
Thermodynamics of quantum trajectories [11] : the atom maser



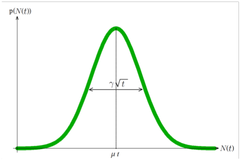




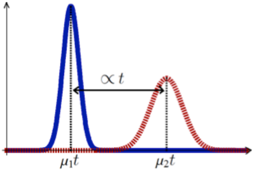
Cavity state (filter) away from DPT



Cavity state (filter) at DPT



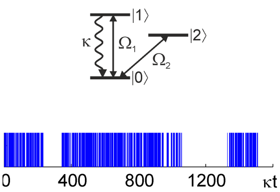
Counting distribution away from DPT

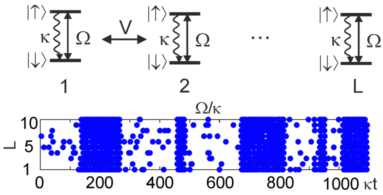


Counting distribution at DPT

[11] J. Garrahan, I. Lesanovsky, *Phys. Rev. Lett.* (2010)

Counting statistics and dynamical phase transitions[12]

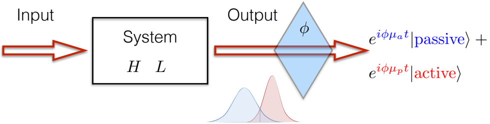




- If $\mathcal{L}$ is ergodic (spectral gap $\Delta\lambda := -\mathrm{Re}\lambda_2 > 0$) then
  - system converges to stationary state $\rho(t) = e^{t\mathcal{L}}(\rho_{\text{in}}) \xrightarrow{t\to\infty} \rho_{ss}$
  - Counting operator $N(t)$ has normal fluctuations ($\Delta N(t) \propto \sqrt{t}$) around mean $t\mu$
- If $\mathcal{L}$ is near phase transition ($\Delta\lambda \approx 0$) then
  - metastability: slow convergence to stationarity, long correlation time $\tau = 1/\Delta\lambda$
  - intermittent trajectories, counting operator $N(t)$ has bimodal distribution up to times τ
- If $\mathcal{L}$ has degenerate stationary states then
  - infinite correlation times
  - counting operator $N(t)$ remains bimodal all times and variance increases as t^2

[12] J. Garrahan, I. Lesanovsky, *Phys. Rev. Lett.* (2010); I. Lesanovsky, M. van Horssen, M. G., J. Garrahan, P. R. L. (2013)

Phase estimation: Heisenberg limit at the DPT[13]



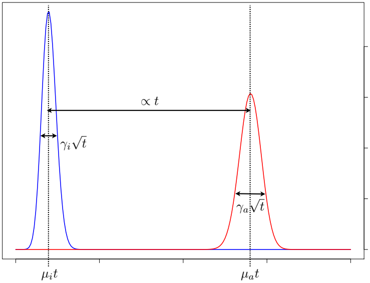
- First order phase transition: system with two "stationary phases" ($\mathcal{H} = \mathcal{H}_i \oplus \mathcal{H}_a$) with different emission rates $\mu_i \neq \mu_a$
- Initial state: superposition $\sqrt{p_i}|\chi_i\rangle + \sqrt{p_a}|\chi_a\rangle$ with $|\chi_{a,i}\rangle \in \mathcal{H}_{i,a}$
- GHZ-type system-output state with generator $N(t)$

  $$|\psi_\phi(t)\rangle = e^{i\phi N(t)}|\psi(t)\rangle \approx \sqrt{p_i} e^{i\phi\mu_i t}|\psi_i(t)\rangle + \sqrt{p_a} e^{i\phi\mu_a t}|\psi_a(t)\rangle$$

- Heisenberg limit wrt time:

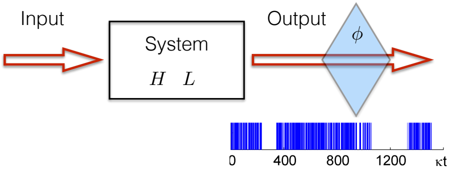
  $$F(t) = 4\mathrm{Var}(N(t)) \approx t^2 p_i p_a (\mu_a - \mu_i)^2$$

- must measure sys+out to achieve QFI

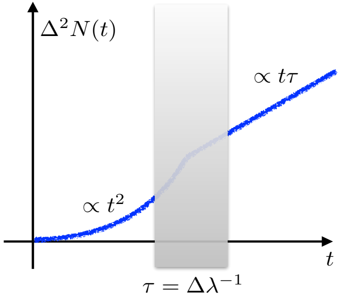


[13] K. Macieszczak, M.G. I. Lesanovsky, J. P. Garrahan *Phys. Rev. A* (2016)

Phase estimation: QFI time behaviour near phase transition



- System near first order DPT: metastability $\Longrightarrow$ counting trajectories exhibit intermittency
- Short time $(t \ll \tau)$ distribution of generator $N(t)$ is bimodal $\Longrightarrow$ quadratic growth of QFI
- Long time $(t \gg \tau)$ ergodicity and normal fluctuations win $\Longrightarrow$ linear growth of QFI

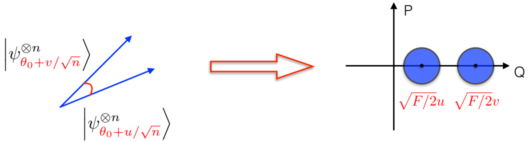


Outlook

- Estimation and identification of input-output systems
  - Identifiable parameters manifold $\mathcal{P} = \mathcal{D}/G$
  - Information Geometry: QFI is real part of covariance of Markov generators
  - Local Asymptotic Normality: CCR Gaussian shift limit model (tutorial)
  - Fisher information depends on measurement AND statistic of trajectory
  - Quantum post-processing with coherent absorber
  - Dynamical phase transitions allow for quadratic scaling of QFI
  - Linear I-O systems: time dependent and stationary theory (tutorial)

- Ongoing / future work
  - General quantum Markov CLT
  - use of coherent feedback in system identification
  - more realistic models with inaccessible channels
  - design of better input states
  - time-dependent paramters

Convergence to Gaussian model for i.i.d. ensembles



- Quantum data: ensemble of n identically prepared systems

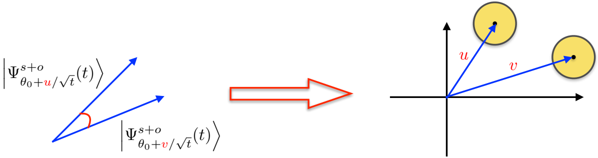
$$|\psi_\theta\rangle^{\otimes n} := \left(e^{i\theta G}|\psi\rangle\right)^{\otimes n}, \qquad \langle\psi|G|\psi\rangle = 0$$

- Local asymptotic normality (Gaussian approximation):

  In an "uncertainty neighbourhood" of size $n^{-1/2}$ around θ_0, the overlaps of joint states are approximately equal to those of a Gaussian model with QFI = F

$$\left\langle \psi^{\otimes n}_{\theta_0+u/\sqrt{n}} \middle| \psi^{\otimes n}_{\theta_0+v/\sqrt{n}} \right\rangle = \underbrace{\left\langle \psi|e^{i(u-v)G/\sqrt{n}}\middle|\psi\right\rangle^n}_{(1-\langle\psi|G^2|\psi\rangle/2n+\dots)^n} \longrightarrow e^{(u-v)^2F/8} = \left\langle \sqrt{F/2}\,u \middle| \sqrt{F/2}\,v \right\rangle$$

Gaussian approximation (LAN) for (system +) output state[6]



- Parameter uncertainty $\approx t^{-1/2} \Rightarrow$ interesting statistical features are local: $\theta = \theta_0 + u/\sqrt{t}$

$$D_{\theta_0+u/\sqrt{t}} = D_{\theta_0} + \frac{1}{\sqrt{t}}\dot{D}_u + O(t^{-1}) = D_{\theta_0} + \frac{1}{\sqrt{t}}\sum_a u_a \dot{D}_{\theta_0,a} + O(t^{-1})$$

Theorem (Local asymptotic normality)

Let $\mathcal{W}_D$ be the CCR algebra over $\mathcal{T}^{id}_D$ (continuous variable system) with Weyl unitaries $W(u)$ and "vacuum" state $|0\rangle$ satisfying

$$W(u)W(v) = e^{-i\mathrm{Im}(\dot{D}_u,\dot{D}_v)_D}\, W(u+v), \qquad \langle 0|W(u)|0\rangle := e^{-\frac{1}{2}\|\dot{D}_u\|^2_D}$$

System+output quantum model $|\Psi^{s+o}_{\theta_0+u/\sqrt{t}}(t)\rangle$ converges locally to coherent states (Gaussian) model $|u\rangle := W(u)|0\rangle$.

$$\lim_{t\to\infty} \left\langle \Psi^{s+o}_{\theta_0+u/\sqrt{t}}(t) \,\middle|\, \Psi^{s+o}_{\theta_0+v/\sqrt{t}}(t) \right\rangle = e^{-\frac{1}{2}\|\dot{D}_u - \dot{D}_v\|^2_D} = \langle u|v\rangle$$

[6] M.G., J. Kiukas, J. Math. Phys. (2017), Similar result for the reduced output state