New directions in Quantum Statistics

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Minimal informationally complete observables and sequential measurements

Teiko Heinosaari
University of Turku
Finland

Operational Quantum Physics

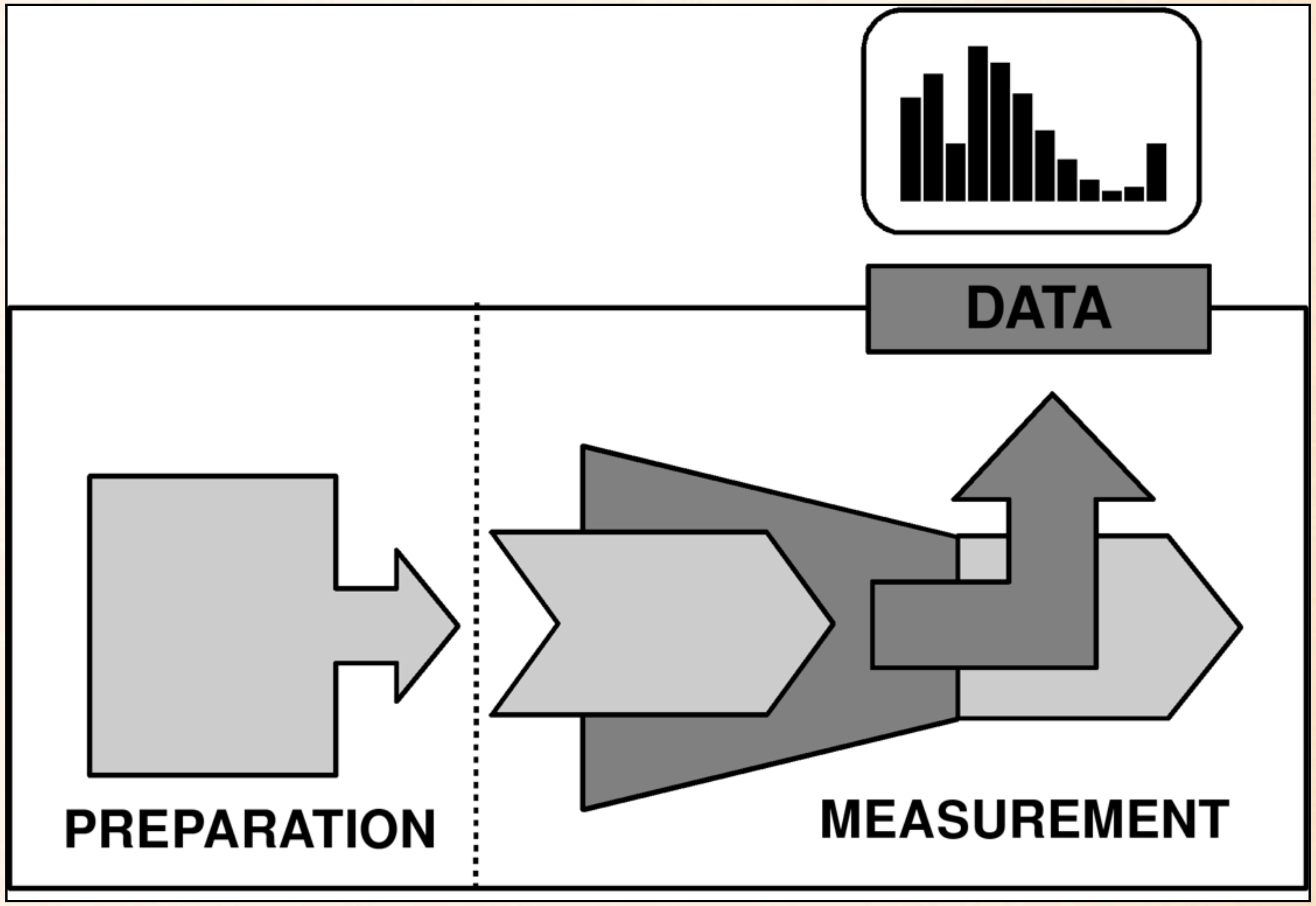
TCQP

Turku Centre for Quantum Physics

Turun yliopisto
University of Turku

STARTING POINT: INFORMATIONALLY COMPLETE MEASUREMENT

Measurement is **info-complete** if two different states (preparations) lead to different measurement outcome distributions.



If system is d-dimensional, then we need d^2 outcomes.

PRELIMINARY QUESTIONS

Abstract

- Labels of measurement outcomes do not matter -- how to get rid of them?
- How to calculate the minimal number of measurement outcomes needed to identify some subset of states?

Concrete

- Observable (POVM) can be a strange mathematical object -- how to implement it?
- What are simple and useful “building blocks”?

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- Labels of measurement outcomes do not matter -- how to get rid of them?
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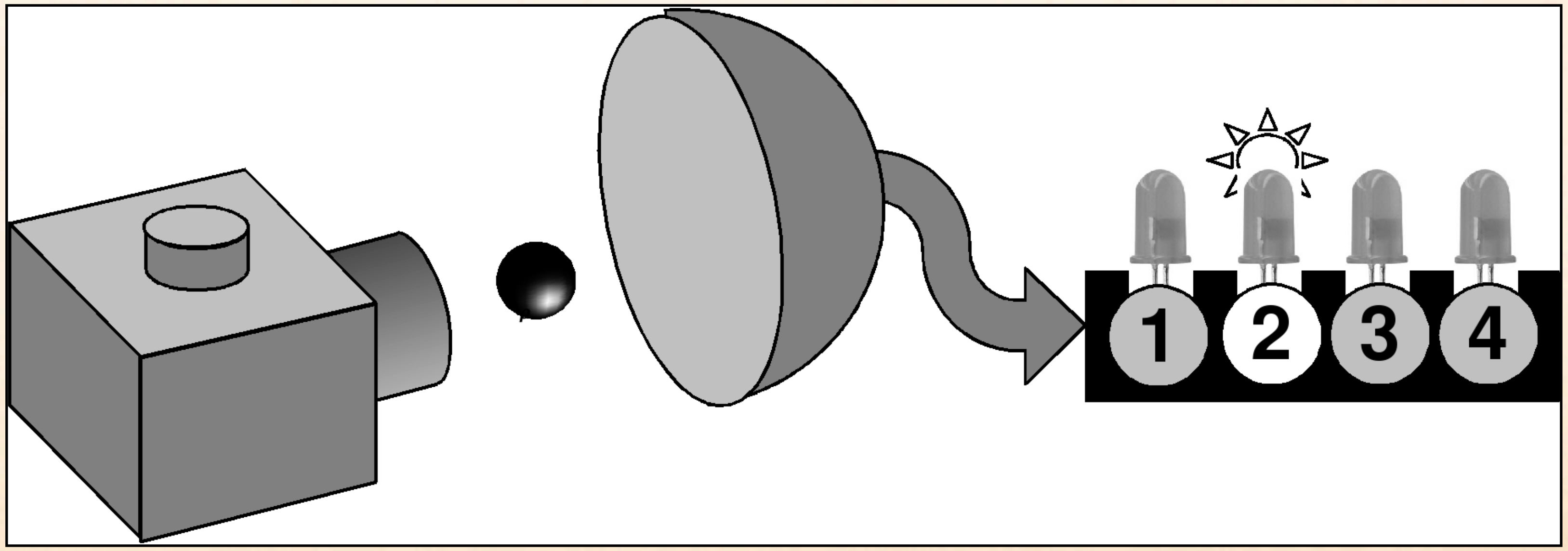
- What are simple and useful “building blocks”?

General aim: to understand the structure and properties of quantum observables

USUAL DEFINITIONS - OBSERVABLE

Observable (POVM) with finite number of outcomes

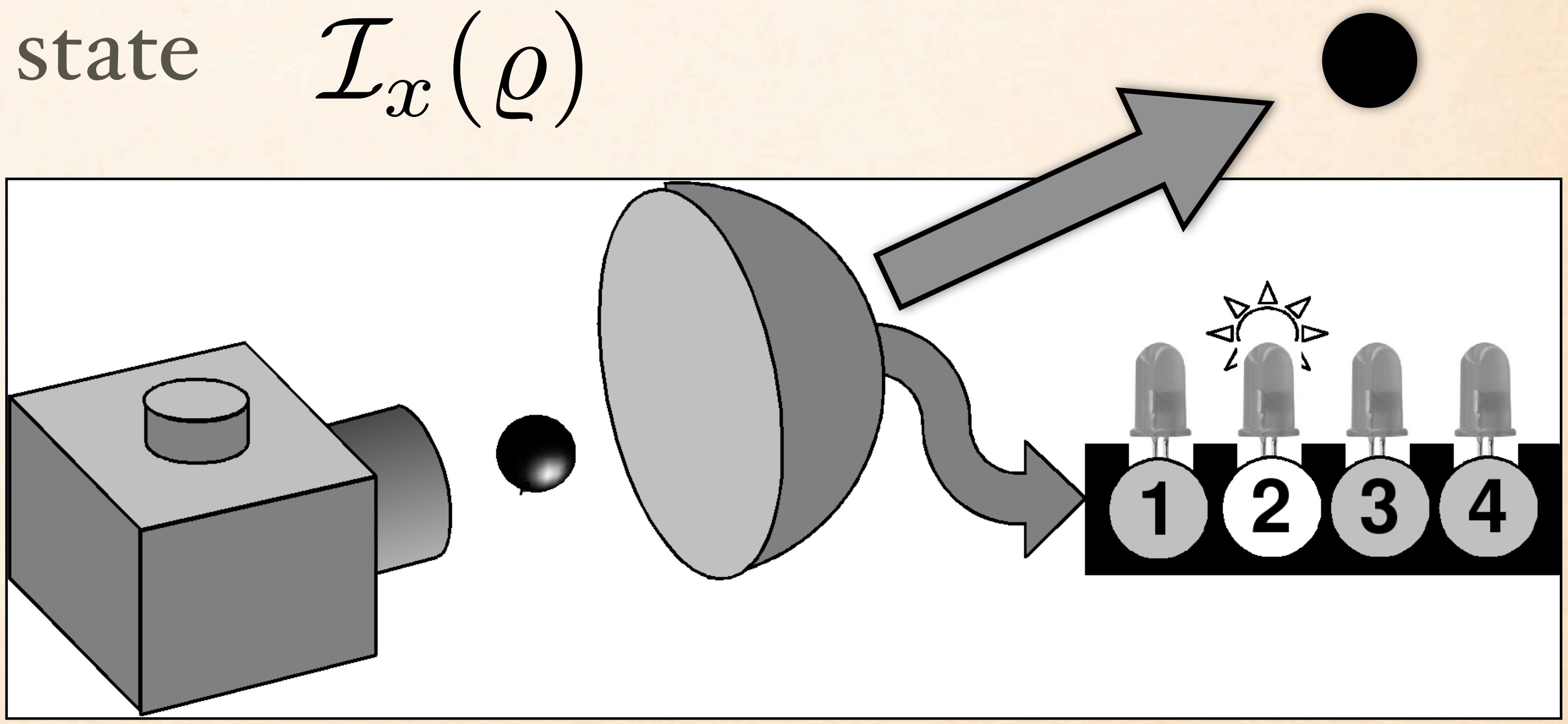
- consists of positive operators $M(x) \geq 0$
- normalization $\Sigma_x M(x) = 1$
- rule for calculating probabilities $tr[\varrho M(x)]$



USUAL DEFINITIONS - INSTRUMENT

Instrument with finite number of outcomes

- consists of CP maps $\mathcal{I}_x$
- normalization $\Sigma_x tr[\mathcal{I}_x(\varrho)] = 1$
- conditional output state $\mathcal{I}_x(\varrho)$

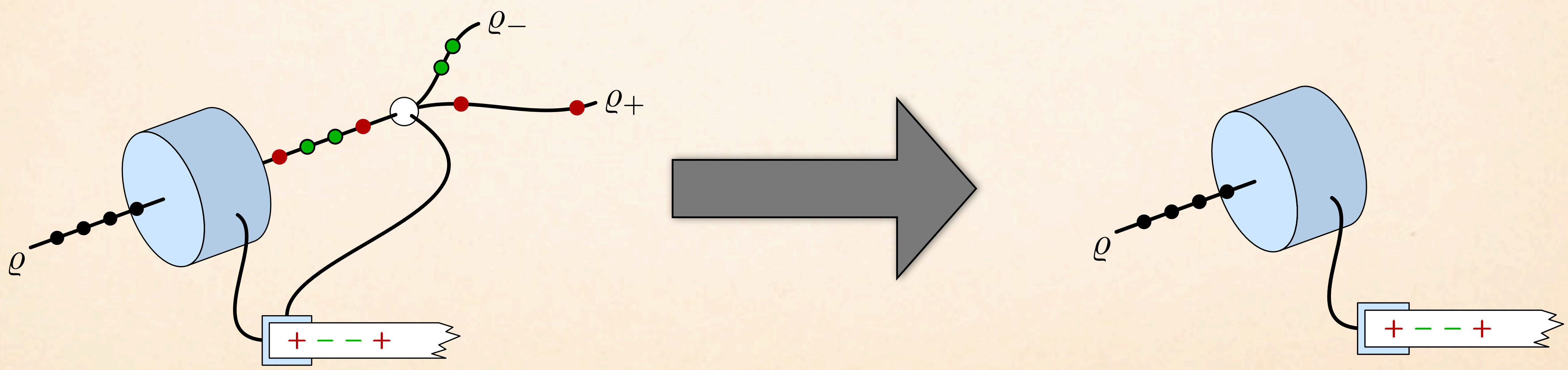


INSTRUMENT → OBSERVABLE

Instrument determines a unique observable.

Schrödinger: $$tr[\varrho M(x)] = tr[\mathcal{I}_x(\varrho)] \quad \forall \varrho$$

Heisenberg: $$M(x) = \mathcal{I}_x^*(1)$$



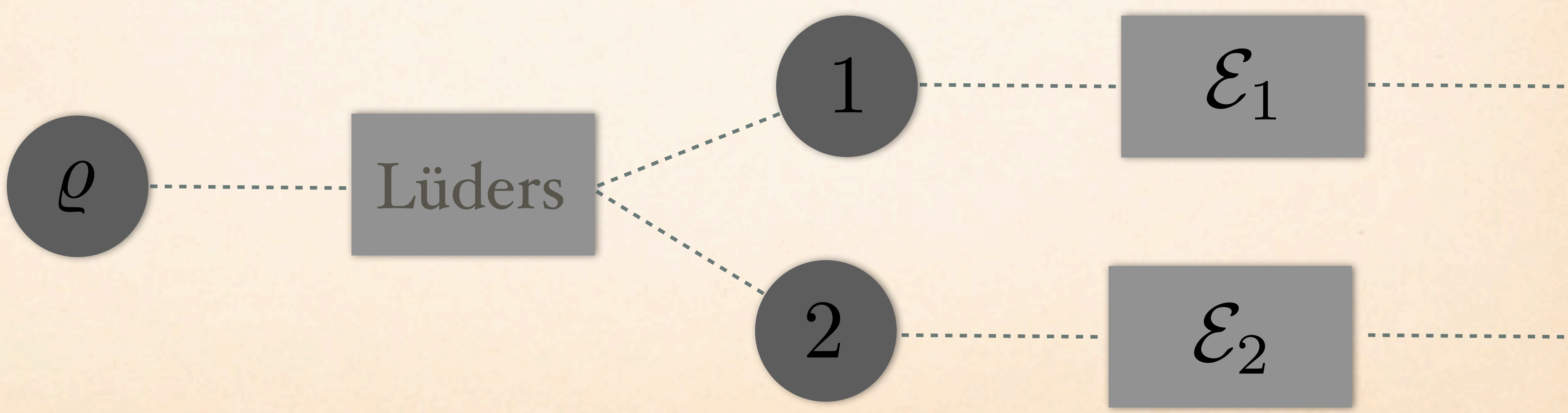
OBSERVABLE ➙ INSTRUMENT

Observable determines many instruments.

One choice: $$\mathcal{I}_x(\varrho) = \sqrt{M(x)}\varrho\sqrt{M(x)}$$

(This is called **Lüders instrument**.)

All instruments: $$\mathcal{I}_x(\varrho) = \mathcal{E}_x(\sqrt{M(x)}\varrho\sqrt{M(x)})$$



PRELIMINARY QUESTIONS

Abstract

Concrete

SEQUENTIAL MEASUREMENT

If two observables could be measured without disturbance, then their sequential measurement would be a realization of both of them.

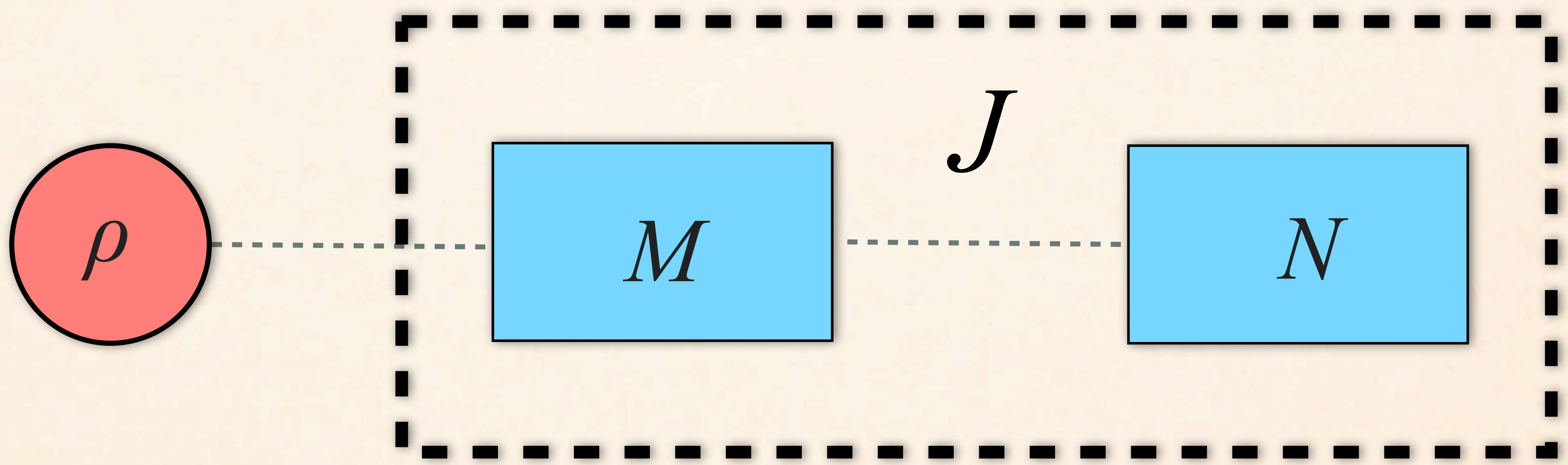
But there is unavoidable disturbance, hence this does not generally work...

+ We obtain correlations between M and N.
- Since M disturbs N, it is not so clear what we get.

SEQUENTIAL MEASUREMENT

Instrument of the first measurement: $\mathcal{I}_x^*(1) = M(x)$

Joint observable describing the overall setup: $J(x,y) = \mathcal{I}_x^*(N(y))$



Second observable is disturbed: $\widetilde{N}(y) = \Sigma_x \, \mathcal{I}_x^*(N(y))$

EXAMPLE: SEQUENTIAL QUBIT MEASUREMENT

$$M(\pm 1) = \frac{1}{2}(1 \pm m\sigma_x) \qquad N(\pm 1) = \frac{1}{2}(1 \pm \sigma_y)$$

unsharpness

$$\mathcal{I}_{\pm 1}(\varrho) = U_{\pm 1}\sqrt{M(\pm 1)}\varrho\sqrt{M(\pm 1)}U_{\pm 1}$$

$$U_{\pm 1} = \cos\frac{\theta}{2}1 \mp i\sin\frac{\theta}{2}\sigma_x$$

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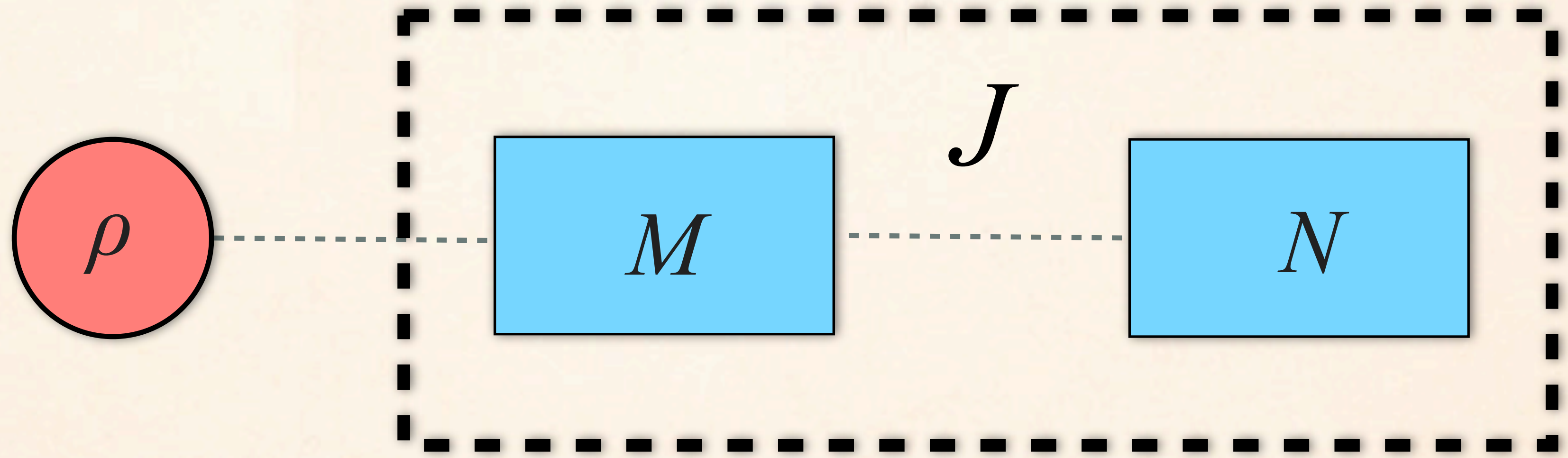


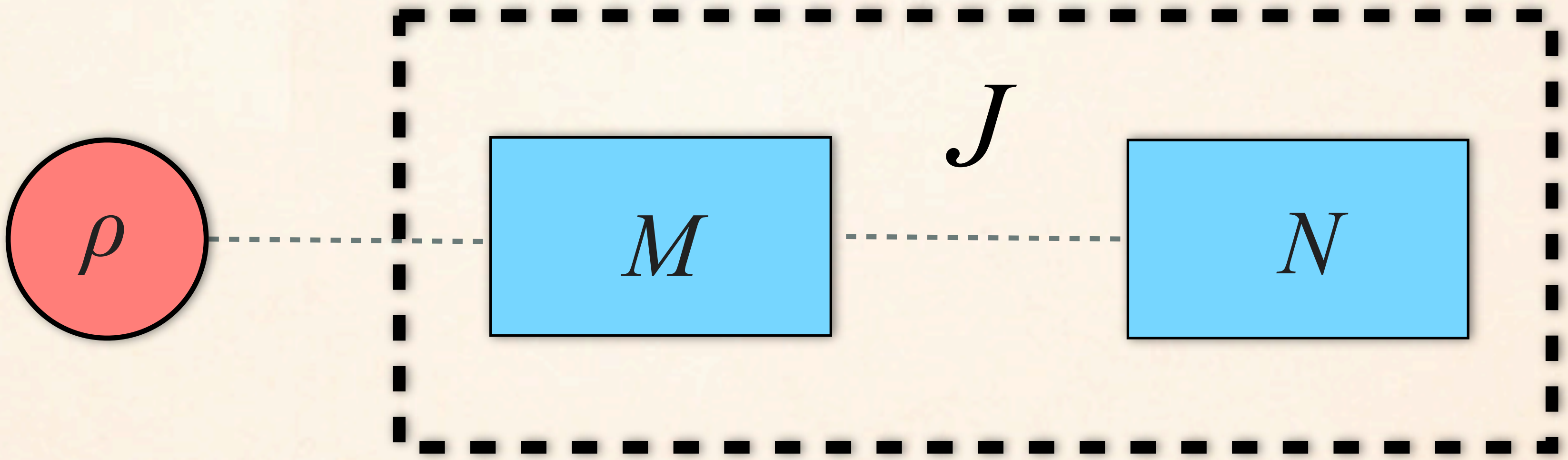
$$J(j,k) = \frac{1}{4}\left(1 + j\ m\sigma_x + k\ \cos\theta\sqrt{1-m^2}\sigma_y + jk\ \sin\theta\sqrt{1-m^2}\sigma_z\right)$$

J is informationally complete if

0<m<1 and *0<θ<π/2*

EXAMPLE: SEQUENTIAL QUBIT MEASUREMENT

<u>Remark</u>: The second marginal of *J* is not *N* but a disturbed version of *N*.



$$N(\pm 1) = \frac{1}{2}(1 \pm \sigma_y) \qquad \widetilde{N}(\pm 1) = \frac{1}{2}(1 \pm n\sigma_y)$$

disturbance depends on the instrument

$$n = \cos\theta \sqrt{1 - m^2}$$

We have seen that a sequential measurement of two **commutative** observables can give a joint observable which is:

- info-complete
- **non-commutative**
- rank-1
- extremal

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Another example: sequential measurement of position and momentum.

But how to do this kind of sequential scheme in dimension $2<d<\infty$?

SEQUENTIAL QUDIT MEASUREMENT

- Fix orthonormal basis $\{\varphi_j\}_{j=0}^{d-1}$
- Fix dth root of unity $\omega = e^{2\pi i/d}$
- Use Fourier transform $\psi_k = \frac{1}{\sqrt{d}} \sum_{h=0}^{d-1} \omega^{hk} \varphi_h$
- Define observable $M(j) = |\varphi_j\rangle\langle\varphi_j|$
- Define observable $N(k) = |\psi_k\rangle\langle\psi_k|$

SEQUENTIAL QUDIT MEASUREMENT

M and *N* are complementary observables:

$$tr[\varrho M(j)] = 1 \Rightarrow tr[\varrho N(k)] = 1/d$$

In other words, the orthonormal bases are mutually unbiased :

$$|\langle \varphi_j \,|\, \psi_k \rangle| = 1/\sqrt{d}$$

SEQUENTIAL QUDIT MEASUREMENT

First measurement must be unsharp, that's why we take

$$M_\lambda(j) = \lambda M(j) + (1-\lambda)/d\ I$$

There is an instrument for M_λ such that the sequential measurement of M_λ and N gives a joint observable J with the properties that
- J is info-complete
- the elements $J(j,k)$ are rank-1 and have the same nonzero eigenvalue $1/d$

Useful thing in the proof: symmetry between M and N

SEQUENTIAL QUDIT MEASUREMENT

First measurement must be unsharp, that's why we take

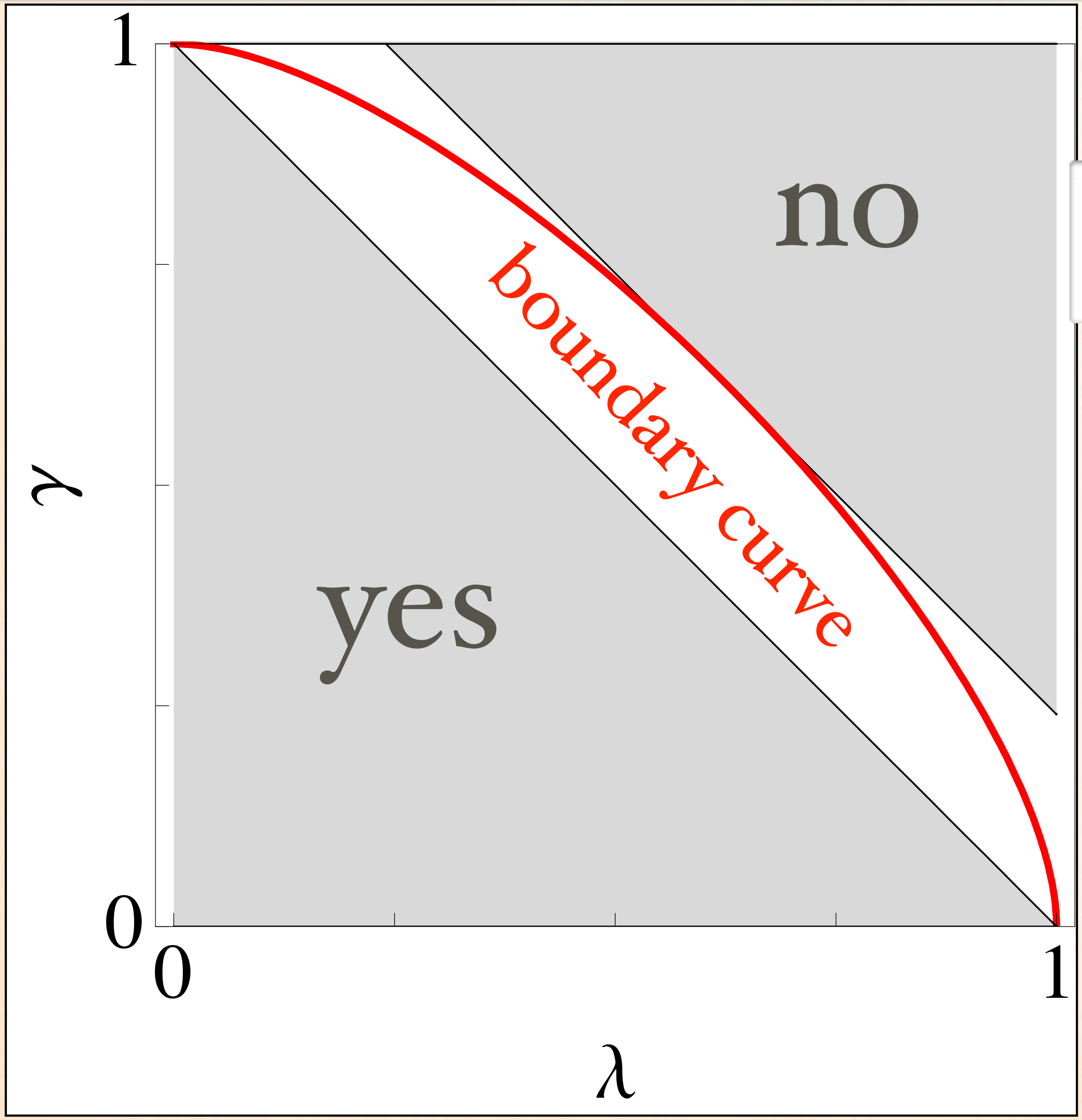
$$M_\lambda(j) = \lambda M(j) + (1-\lambda)/d\ I$$

Second measurement is disturbed, let's look for the simplest form

$$N_\gamma(k) = \gamma N(k) + (1-\gamma)/d\ I$$

If M_λ and N_γ are given, could they have arisen from the same sequential measurement?

SEQUENTIAL QUDIT MEASUREMENT



ellipse

$$\gamma^2 + \lambda^2 + \frac{2(d-2)}{d}(1-\gamma)(1-\lambda) = 1$$

yes-line

$$\gamma + \lambda = 1$$

no-line

$$\gamma + \lambda = 1 + \frac{\sqrt{d}-1}{d-1}$$

SEQUENTIAL QUDIT MEASUREMENT

If M_λ and N_γ are given, when they have a sequential realization that is informationally complete?

- If (λ,γ) **is not a boundary point**, then we can choose the first measurement in a way that the joint observable is informationally complete.
- If $\lambda=1$ or $\gamma=1$, then there is no info-complete joint observable.

SEQUENTIAL QUDIT MEASUREMENT

If M_λ and N_γ are given, when they have a sequential realization that is informationally complete?

- If (λ,γ) **is a boundary point**, then the answer depends on the dimension!

boundary condition:

$$\gamma = \frac{1}{d}\left[(d-2)(1-\lambda) + 2\sqrt{(1-d)\lambda^2 + (d-2)\lambda + 1}\right]$$

SEQUENTIAL QUDIT MEASUREMENT

If (λ,γ) is a boundary point (but neither $\lambda=1$ nor $\gamma=1$), then:

- Joint observable is unique and it corresponds to the Lüders measurement.

- Joint observable is info-complete iff d is odd.

SEQUENTIAL QUDIT MEASUREMENT

If (λ,γ) is a boundary point (but neither $\lambda=1$ nor $\gamma=1$), then:

- ❖ Joint observable is unique and it corresponds to the Lüders measurement.
- ❖ Joint observable is info-complete iff d is odd.

Why the parity of the dimension matters?

PRELIMINARY QUESTIONS

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Concrete

OPERATOR SYSTEM

For each observable M, we denote

$$S_M = \{\Sigma_j \; c_j M(j) : c_j \in \mathbb{C}\}$$

This is a linear subspace and

i) $A \in S_M \Rightarrow A^\dagger \in S_M$

ii) $1 \in S_M$

In other words, S_M is an **operator system**.

OPERATOR SYSTEM

If S is an operator system, there exists an observable M such that $S=S_M$ and M has $dim(S)$ outcomes.

Example: $S=span\{1,T\}$ and T selfadjoint.

$$M(\pm 1) = \frac{1}{2}(1 \pm T/\|T\|)$$

OPERATOR SYSTEM

If S is an operator system, there exists an observable M such that $S=S_M$ and M has $dim(S)$ outcomes.

Any observable N satisfying $S=S_N$ has at least $dim(S)$ outcomes.

OPERATOR SYSTEM

For each operator system S, we denote by $S^{\perp}$ its orthogonal complement in the Hilbert-Schmidt inner product:

$$S^{\perp} = \{X : tr[X^{\dagger}Y] = 0 \ \forall Y \in S\}$$

- $dim(S)+dim(S^{\perp})=d^2$
- The measurement outcome distributions for two states ρ_1 and ρ_2 are different in M-measurement iff $\rho_1-\rho_2 \in S_M{}^{\perp}$.
- M is info-complete iff $S_M{}^{\perp}=\{0\}$

INFO-COMPLETE MEASUREMENT ON PURE STATES

An observable M can distinguish all pure states iff every nonzero selfadjoint operator in $S_M{}^{\perp}$ has rank greater than 2.

dim 3 Two options to satisfy this condition:

- $S_M{}^{\perp} = \{0\} \Rightarrow M$ is informationally complete
- $S_M{}^{\perp} = \{cT: c \in \mathbb{C}\}$ for some invertible operator T

Conclusion: observable must have 8 outcomes in order to distinguish all pure states in dimension 3.

INFO-COMPLETE MEASUREMENT ON PURE STATES

The minimal number of measurement outcomes necessary to identify all pure states in a d-dimensional Hilbert space is $4d - 4 - \delta(d)$, where $0 \leq \delta(d) \leq 2\log_2(d)$.

INFO-COMPLETE MEASUREMENT ON PURE STATES

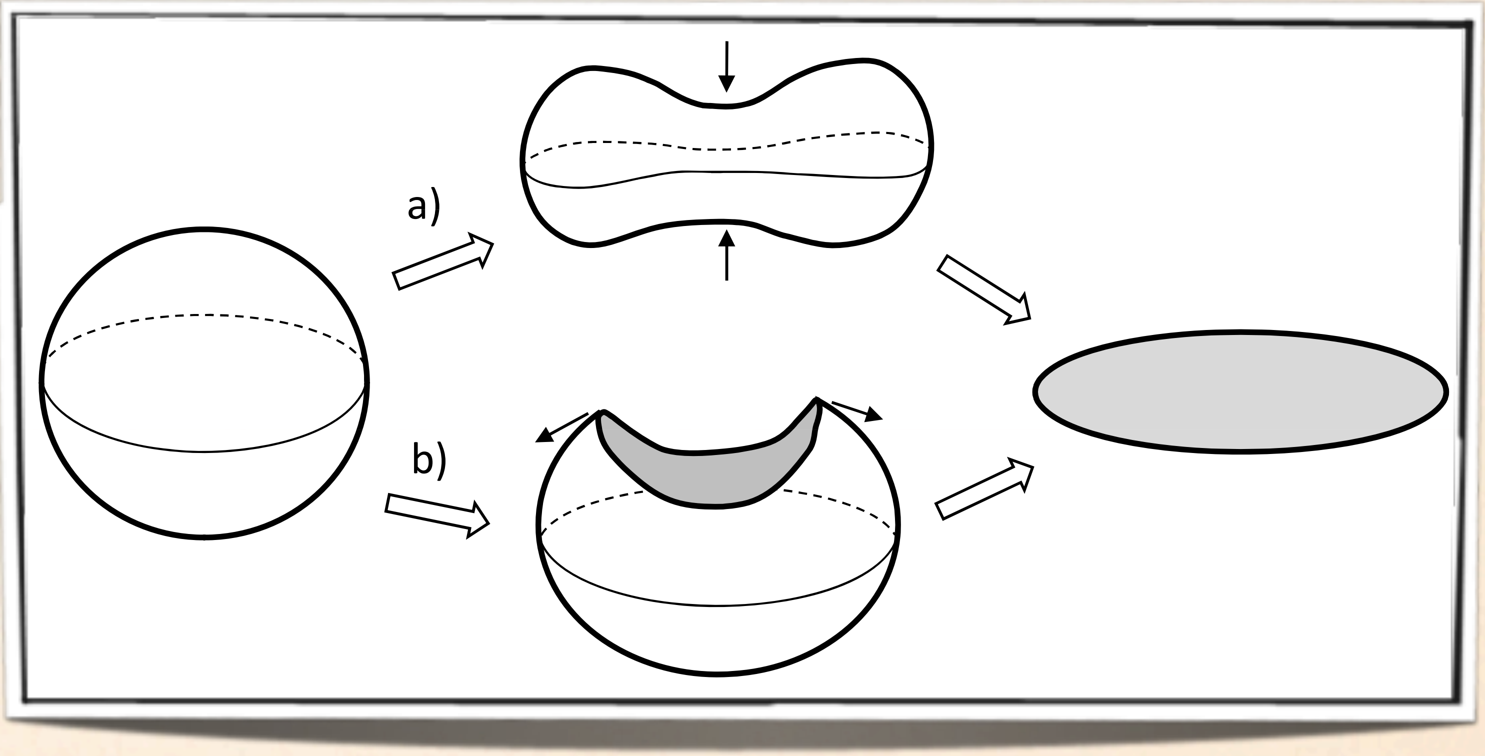
The minimal number of measurement outcomes necessary to identify all pure states in a d-dimensional Hilbert space is $4d - 4 - \delta(d)$, where $0 \leq \delta(d) \leq 2\log_2(d)$.

easy upper bound

INFO-COMPLETE MEASUREMENT ON PURE STATES

The minimal number of measurement outcomes necessary to identify all pure states in a d-dimensional Hilbert space is $4d - 4 - \delta(d)$, where $0 \leq \delta(d) \leq 2\log_2(d)$.

Lower bound comes from differential geometry.



Borsuk-Ulam Theorem

PRELIMINARY QUESTIONS

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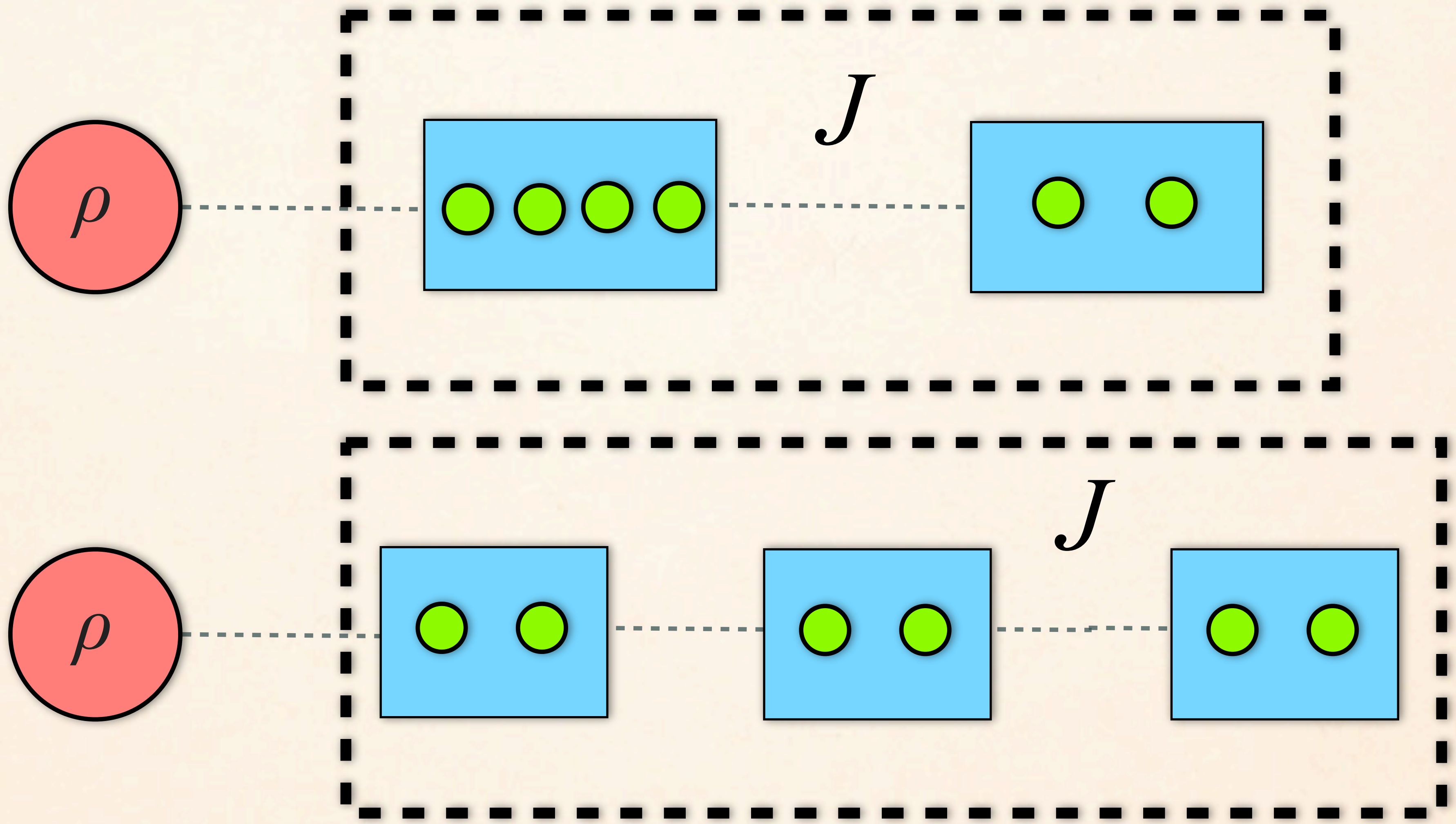
INFO-COMPLETE MEASUREMENT ON PURE STATES



What is a natural/canonical way to implement a minimal pure state info-complete observable?

INFO-COMPLETE MEASUREMENT ON PURE STATES

dimension	min number
2	4
3	8
4	10
5	16
6	18
7	23

first measurement has to be non-commutative

4∗2	2∗2∗2
5∗2	-
8∗2	2∗2∗2∗2
9∗2	-

INFO-COMPLETE MEASUREMENT ON PURE STATES

dimension	min number
2	4
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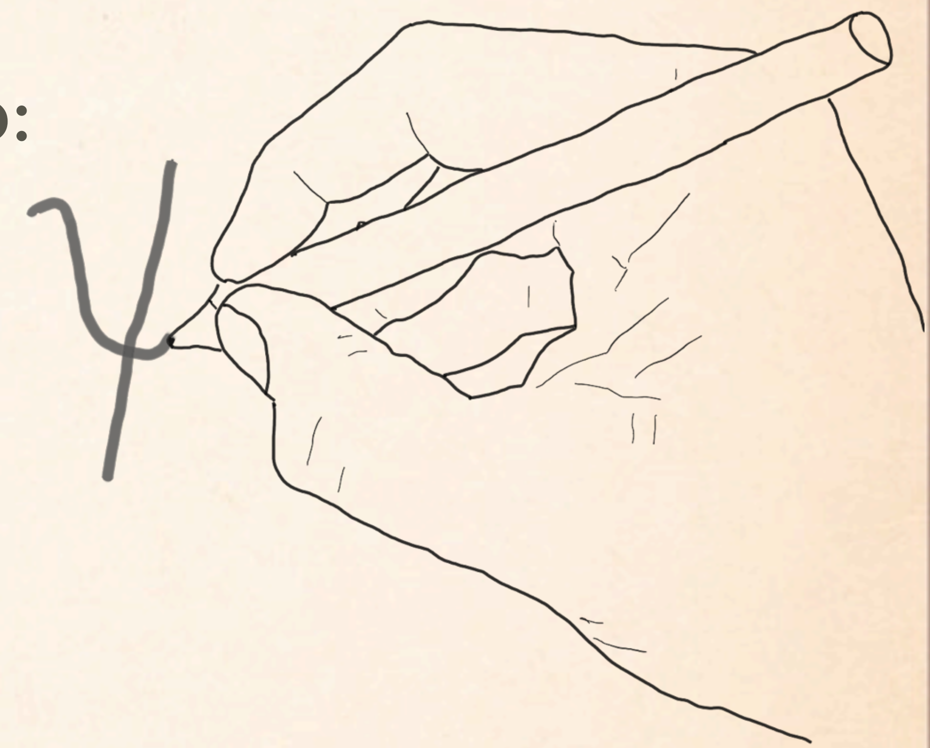
prime!

?

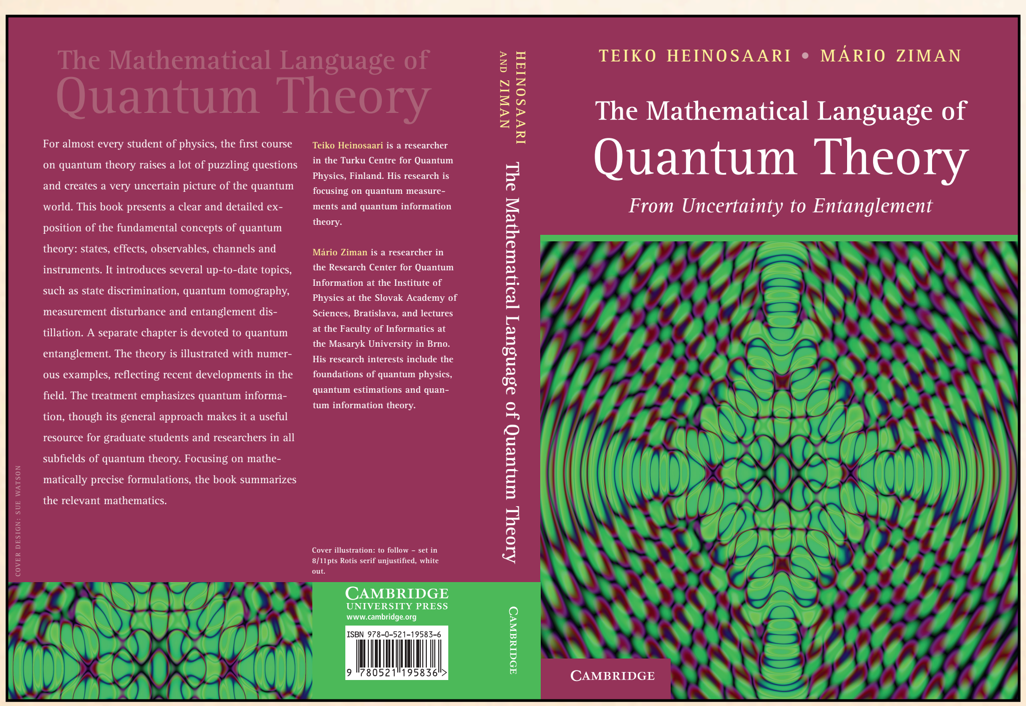
- Why the parity of the dimension matters?
- What is a natural way to implement a minimal pure state info-complete observable?
- What one can and cannot do with sequential measurements?
- Minimal observable identifying all pure states in a certified way?

THIS TALK IS BASED ON:

- Claudio Carmeli, TH & Alessandro Toigo: 1105.4976 , 1111.3509



- TH, Luca Mazzarella & Michael Wolf: 1109.5478



- TH & Mario Ziman: The Mathematical Language of Quantum Theory