The q-product of generalised Brownian motions

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Abstract

The notion of generalised Brownian motion is extended to multiple processes indexed by a set $\mathcal{I}$. For $-1 \leq q \leq 1$ the q-product of positive definite functions on pair partitions having the multiplicative property is defined, and shown to be a positive definite function on $\mathcal{I}$-coloured pair partitions. The resulting $\mathcal{I}$-indexed generalised Brownian motion interpolates between graded tensor product ($q = -1$), reduced free product ($q = 0$) and tensor product ($q = 1$) of the given Brownian motions.

1 Introduction

Two different notions of independence stand out in non-commutative probability: tensor and free independence. The quantum stochastic calculus of Hudson and Parhasarathy [23] generalises the classical independence to tensor independence, and the free probability of Voiculescu [28] relies on free independence.

Beyond the well established frameworks of these two theories, an investigation concerning the notions of quantum white noise, Brownian motion and Markov processes is developed by Köstler [17] in the spirit of Kümmerer's approach to quantum probability [18, 19, 20]. Related to this, is the theory of generalised Brownian motion initiated by Bożejko and Speicher which provides concrete examples of quantum white noises but also raises interesting operator algebraic questions.

The generalised Brownian motions [4] are non-commutative processes $\omega(f)$ indexed by elements f of a real Hilbert space $\mathcal{K}$ and endowed with a distinguished "gaussian" positive linear functional $\rho_{\mathbf{t}}$ given by the following pair prescription:

$$\rho_{\mathbf{t}}(\omega(f_1)\ldots\omega(f_n)) = \sum_{\mathcal{V}\in\mathcal{P}_2(n)} \mathbf{t}(\mathcal{V}) \prod_{(k,l)\in\mathcal{V}} \langle f_k, f_l\rangle \tag{1.1}$$

where the sum runs over all pair partitions of the ordered set $\{1, \ldots, n\}$ and $\mathbf{t} : \bigcup_{k\in\mathbb{N}} \mathcal{P}_2(2k) \to \mathbb{C}$ is a function defined on all possible pair partitions called positive definite. We regard $\omega(f)$ as symmetric operators on the Hilbert space arising from the GNS construction.

The most important examples of positive definite functions are $\mathbf{t}(\mathcal{V}) = 1$ for all $\mathcal{V}$ which characterises the classical Brownian motion [24] and, $\mathbf{t}(\mathcal{V}) = 0$ for crossing partitions and $\mathbf{t}(\mathcal{V}) = 1$ for non-crossing partitions which characterises the free Brownian motion [28].

A remarkable interpolation between these cases arises from the algebra of q-deformed commutation relations $a(f)a^*(g) - qa^*(g)a(f) = \langle f, g\rangle \mathbf{1}$ for $-1 \leq q \leq 1$ investigated in a number of papers [5, 6, 7, 8, 9, 10, 13, 14, 21, 25, 29]. The monomials in fields $\omega(f) := a(f) + a^*(f)$ have vacuum expectations as in 1.1 with $\mathbf{t}_q(\mathcal{V}) = q^{\mathrm{cr}(\mathcal{V})}$ and $\mathrm{cr}(\mathcal{V})$ being the number of crossings of the pair partition $\mathcal{V}$. Further functorial and operator algebraic properties of this Brownian motion are studied in [3]. It turns out that we have a functor of white noise [18, 19] that is, a functor from the category of real Hilbert spaces with contractions to the category of non-commutative probability spaces.
Another interpolation between the classical and free Brownian motion is found in [4] : $\mathbf{t}_p(\mathcal{V}) = p^{|\mathcal{V}| - \mathrm{B}(\mathcal{V})}$ where $|\mathcal{V}|$ is the number of pairs of $\mathcal{V}$ and $\mathrm{B}(\mathcal{V})$ is the number of connected components or blocks of $\mathcal{V}$, and $0 \leq p \leq 1$.

Ispired by ideas from combinatorics, such as that of species of structures [1, 15] and analytic functors [16], the author together with Hans Maassen [11, 12] have found a more functorial approch to the study of generalised Brownian motion. The GNS representation space of the algebra of fields $\omega(f)$ over a real Hilber space $\mathcal{K}$, with respect to the gaussian state $\rho_{\mathbf{t}}$ has the Fock-like form

$$\mathcal{F}_{\mathbf{t}}(\mathcal{K}_{\mathbb{C}}) := \bigoplus_{n=0}^{\infty} \frac{1}{n!} V_n \otimes_s \mathcal{K}_{\mathbb{C}}^{\otimes n}, \tag{1.2}$$

where V_n is a Hilbert space carrying a unitary representation U_n of the symmetric group $\mathrm{S}(n)$, and $\otimes_s$ denotes the closed subspace of the tensor product $V_n \otimes \mathcal{K}_{\mathbb{C}}^{\otimes n}$ whose orthogonal projection is

$$P_n = \frac{1}{n!} \sum_{\tau \in \mathrm{S}(n)} U_n(\tau) \otimes \tilde{U}_n(\tau), \tag{1.3}$$

and

$$\tilde{U}_n f_1 \otimes \ldots \otimes f_n = f_{\tau^{-1}(1)} \otimes \ldots \otimes f_{\tau^{-1}(n)} \tag{1.4}$$

for $f_i \in \mathcal{K}_{\mathbb{C}}$. The factor $\frac{1}{n!}$ in 1.2 refers to the inner product on $V_n \otimes_s \mathcal{K}_{\mathbb{C}}^{\otimes n}$. We note that $\mathcal{F}_{\mathbf{t}}$ is an endofunctor of the category of Hilbert spaces with contractions called *analytic functor* [16]. The fields $\omega(f)$ can be represented as the sum of creation and annihilation operators $a(f) + a^*(f)$ which are defined with the help of a sequence of operators $j_n : V_n \to V_{n+1}$ satisfying the intertwining relations with respect to the representation of the symmetric groups U_n and U_{n+1}: $j_n U_n(\tau) = U_{n+1}(\iota(\tau)) j_n$, with ι the natural embedding of U_n into U_{n+1} by keeping the last letter fixed. Concretely,

$$a^*(f) : P_n(v_n \otimes \psi_n) \mapsto P_{n+1}(j_n v_n \otimes r(f)\psi_n). \tag{1.5}$$

where $v_n \in V_n$, $\psi_n \in \mathcal{K}_C^{\otimes n}$ and $r(f)$ is the right creation operator as defined on the full Fock space over $\mathcal{K}_{\mathbb{C}}$. For more details we refer to the first 3 sections of [12]. Using this insight, a new class of positive functions on pair partitons has been found [2], which extend the indecomposable characters of the infinite symmetric group and for which the gaussian state is not tracial.

In the present work we answer a question of Roland Speicher concerning the existence of the q-product of generlised Brownian motions. In his paper [26] Speicher has analysed the existence of universal products on the category of unital algebras with normalised linear functionals. The universal product should satisfy some natural requirements such as associativity and universal calculation rule for mixed moments. The result is that the only possibilities are the tensor product and the reduced free product. In the case of q-deformations, a concrete result in this direction has been proved in [21]. However one can still define a q-product for the algebras of generalised Brownian motions with gaussian states $\rho_{\mathbf{t}_a}$ characterised by the positive definite functions $\mathbf{t}_a$ with a in a index set $\mathcal{I}$, by the calculational rule:

$$(*_{a\in\mathcal{I}}^{(q)}\rho_{\mathbf{t}_a})\left(\prod_{i=1}^{n}\omega_{c(i)}(f_i)\right) = \sum_{\mathcal{V}} q^{\mathrm{cr}(\mathcal{V},c)} \prod_{a\in\mathcal{I}} \mathbf{t}_a(\mathcal{V}_a) \prod_{(i,j)\in\mathcal{V}} \langle f_i, f_j\rangle \,. \tag{1.6}$$

The sum is taken over those pair partitions $\mathcal{V}$ such that if $(i,j) \in \mathcal{V}$ then $c(i) = c(j) \in \mathcal{I}$, which we call the "colour" of the pair (i,j), the pair partition $\mathcal{V}_a$ is the subset of pairs in $\mathcal{V}$ which are coloured in the colour a, and the coefficient $\mathrm{cr}(\mathcal{V}, c)$ counts the number of crossings between pairs of different colours. The main objective of the paper is to prove the positivity of this functional for functions $\mathbf{t}_a$ which have a certain multiplicativity property. This is done in section 3, but as a preparation for that we develop the framework for $\mathcal{I}$-indexed generalised Brownian motion which is a fairly straightforward extension of the usual theory to the case of more than one processes "coexisting", this time the key notion being that of positive definite function on $\mathcal{I}$-coloured pair partions.

The q-product interpolates between graded tensor product [22] (q=-1), reduced free product [28] (q=0) and tensor product [24] (q=1).

Finally, a central limit theorem is proved showing that as $n \to \infty$, the functional obtained by taking the q-product of a function $\mathbf{t}$ a number n of times and restricting it to the algebra of fields $\omega(f) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \omega_i(f)$, converges in law to that of the q-deformed fields given by $\mathbf{t}_q(\mathcal{V}) = q^{\mathrm{cr}(\mathcal{V})}$.

2 Generalised Brownian motions indexed by $\mathcal{I}$ and the *-semigroup $\mathcal{BP}_2^{\mathcal{I}}(\infty)$

Let $\mathcal{I}$ be an arbitrary index set. In this section we will extend the notions of generalised Brownian motion and the associated *-semigroup of broken pair partitions [12] to the case of multidimensional processes indexed by the set $\mathcal{I}$.

Definition 2.1 *Let P be a finite ordered set. By $\mathcal{P}_2^I(P)$ we denote the set of I-coloured pair partitions, that is pairs $(\mathcal{V}, c)$ with $\mathcal{V} \in \mathcal{P}_2(P)$ and $c : \mathcal{V} \to I$.*

Definition 2.2 *Let X be an arbitrary finite ordered set and $(L_a, P_a, R_a)_{a\in\mathcal{I}}$ a disjoint partition of X into triples of subsets indexed by elements of $\mathcal{I}$. For each $a \in \mathcal{I}$ we consider a triple $(\mathcal{V}_a, f_a^{(l)}, f_a^{(r)})$ where $\mathcal{V}_a \in \mathcal{P}_2(P_a)$ and*

$$f_a^{(l)} : L_a \to \{1, \ldots, |L_a|\}, \qquad f_a^{(r)} : R_a \to \{1, \ldots, |R_a|\} \tag{2.1}$$

are bijections. Any order preserving bijection $\alpha : X \to Y$ induces a map

$$\alpha_a : (\mathcal{V}_a, f_a^{(l)}, f_a^{(r)}) \to (\alpha \circ \mathcal{V}_a,\ f_a^{(l)} \circ \alpha^{-1},\ f_a^{(r)} \circ \alpha^{-1}) \tag{2.2}$$

where $\alpha \circ \mathcal{V} := \{(\alpha(i), \alpha(j)) : (i,j) \in \mathcal{V}\}$. This defines an equivalence relation. The corresponding element $\mathbf{d}$ of $\mathcal{BP}_2^{\mathcal{I}}(\infty)$ is such an equivalence class of collections of triples $(\mathcal{V}_a, f_a^{(l)}, f_a^{(r)})$ for $a \in \mathcal{I}$.

In other words the elements of $\mathcal{BP}_2^{\mathcal{I}}(\infty)$ are broken pair partitions as in definition 3.1 in [12], with additional labeling with indices from $\mathcal{I}$ of the pairs and legs. For each $a \in \mathcal{I}$ there is a broken pair partition $\mathbf{d}_a$, however only a finite number of them are not empty. The product and involution are defined as for the *-semigroup $\mathcal{BP}_2(\infty)$ with the additional condition that the left legs and right legs which are joined must be indexed by the same element of $\mathcal{I}$.
Let $\mathbf{d}_1 = (\mathcal{V}_{a,1},\ f_{a,1}^{(l)},\ f_{a,1}^{(r)})_{a\in\mathcal{I}}$ and $\mathbf{d}_2 = (\mathcal{V}_{a,2},\ f_{a,2}^{(l)},\ f_{a,2}^{(r)})_{a\in\mathcal{I}}$ be two elements of $\mathcal{BP}_2^{\mathcal{I}}(\infty)$ with the notations from definition 2.2. Let $\mathrm{M} = \min(|R_{a,1}|, |L_{a,2}|)$ be the number of legs which "join" by taking the product of $\mathbf{d}_1$ and $\mathbf{d}_2$. Then we define

$$\mathbf{d}_1 \cdot \mathbf{d}_2 = (\mathcal{V}_a, f_a^{(l)}, f_a^{(r)})_{a\in\mathcal{I}} \tag{2.3}$$

with

$$\mathcal{V}_a = \mathcal{V}_{a,1} \cup \mathcal{V}_{a,2} \cup \Big\{ \Big((f_{a,1}^{(r)})^{-1}(i),\ (f_{a,2}^{(l)})^{-1}(i) \Big)\ :\ i \leq \mathrm{M} \Big\}, \tag{2.4}$$

and the map $f_a^{(l)}$ defined on the disjoint union of the sets $L_{a,1}$ of left a-coloured legs from the left diagram $\mathbf{d}_1$, and $L_{a,2} \setminus (f_{a,2}^{(l)})^{-1}(\{1, \ldots, \mathrm{M}\})$ consisting of the unpaired left a-coloured legs of the diagram $\mathbf{d}_2$ by

$$\begin{cases} f_a^{(l)}(i) = f_{a,1}^{(l)}(i) & \text{for } i \in L_{a,1} \\ f_a^{(l)}(j) = f_{a,2}^{(l)}(j) - \mathrm{M} + |L_{a,1}| & \text{for } j \in L_{a,2} \setminus (f_{a,2}^{(l)})^{-1}(\{1, \ldots, \mathrm{M}\}) \end{cases}.$$

The function $f_a^{(r)}$ is defined similarly. The product does not depend on the chosen representatives for $\mathbf{d}_i$ in their equivalence class and is associative. The diagrams with no legs are the $\mathcal{I}$-coloured pair partitions, thus $\mathcal{P}_2^{\mathcal{I}}(\infty) \subset \mathcal{BP}_2^{\mathcal{I}}(\infty)$.

The involution is given by mirror reflection. If $\mathbf{d} = (\mathcal{V}_a, f_a^{(l)}, f_a^{(r)})$ then $\mathbf{d}^* = (\mathcal{V}_a^*, f_a^{(r)}, f_a^{(l)})$ with the underlying set X^* obtained by reversing the order on X and

$$\mathcal{V}_a^* := \{(i,j) : (j,i) \in \mathcal{V}_a\} \tag{2.5}$$

is the adjoint of $\mathcal{V}_a$. It can be checked that

$$(\mathbf{d}_1 \cdot \mathbf{d}_2)^* = \mathbf{d}_2^* \cdot \mathbf{d}_1^*. \tag{2.6}$$

We are interested in positive functionals $\hat{\mathbf{t}}$ on the *-semigroup $\mathcal{BP}_2^{\mathcal{I}}(\infty)$ which have the form

$$\hat{\mathbf{t}}(\mathbf{d}) = \begin{cases} \mathbf{t}(\mathbf{d}) & \text{if } \mathbf{d} \in \mathcal{P}_2^{\mathcal{I}}(\infty) \\ 0 & \text{otherwise.} \end{cases} \tag{2.7}$$

The function $\mathbf{t} : \mathcal{P}_2^{\mathcal{I}}(\infty) \to \mathbb{C}$ will be called *positive definite* on $\mathcal{I}$-coloured pair partitions and will be shown to characterise generalised Brownian motions indexed by $\mathcal{I}$.
Let $\mathbf{n} : \mathcal{I} \to \mathbb{N}$ be a function which is equal to zero except a finite number of elements. We denote by $\mathcal{BP}_2^{\mathcal{I}}(\mathbf{n}, \mathbf{0})$ the set of diagrams $\mathbf{d}$ for which $|R_a| = 0$ and $|L_a| = \mathbf{n}(a)$. The GNS-representation $(\chi_{\mathbf{t}}, V, \xi_{\mathbf{t}})$ with respect to $\hat{\mathbf{t}}$ is characterised by

$$\langle \chi_{\mathbf{t}}(\mathbf{d}_1)\xi_{\mathbf{t}}, \chi_{\mathbf{t}}(\mathbf{d}_2)\xi_{\mathbf{t}} \rangle_V = \hat{\mathbf{t}}(\mathbf{d}_1^* \cdot \mathbf{d}_2) \tag{2.8}$$

which implies

$$V = \bigoplus_{\mathbf{n}} V_{\mathbf{n}} \qquad \text{where} \qquad V_{\mathbf{n}} = \overline{\operatorname{lin}\{\chi_{\mathbf{t}}(\mathbf{d})\xi_{\mathbf{t}} : \mathbf{d} \in \mathcal{BP}_2^{\mathcal{I}}(\mathbf{n}, \mathbf{0})\}}. \tag{2.9}$$

On $V_{\mathbf{n}}$ there is a unitary representation of the direct product group

$$\mathrm{S}(\mathbf{n}) := \mathsf{X}_{a \in \mathcal{I}} \mathrm{S}(\mathbf{n}(a)) \tag{2.10}$$

each of the terms $\mathrm{S}(\mathbf{n}(a))$ permuting the a-coloured left legs of the diagrams in $\mathcal{BP}_2^{\mathcal{I}}(\mathbf{n}, \mathbf{0})$. We denote by $\pi(\mathbf{d})$ the diagram obtained by applying the permutation π to the element $\mathbf{d}$ and the representation by $U_{\mathbf{n}}$. We distinguish the operators $j_a := \chi_{\mathbf{t}}(\mathbf{d}_{a,0})$ where $\mathbf{d}_{a,0}$ is the diagram containing one left leg indexed by $a \in \mathcal{I}$. The following intertwining relation will be important in later constructions:

$$\begin{aligned} & j_a : V_{\mathbf{n}} \to V_{\mathbf{n}+\delta_a} \\ & j_a \cdot U_{\mathbf{n}}(\tau) = U_{\mathbf{n}+\delta_a}(\iota_{\mathbf{n}}^{(a)}(\tau)) \cdot j_a \end{aligned} \tag{2.11}$$

with $(\mathbf{n} + \delta_a)(b) := \mathbf{n}(b) + \delta_{a,b}$ and $\iota_{\mathbf{n}}^{(a)}$ the natural embedding of $\mathrm{S}(\mathbf{n})$ into $\mathrm{S}(\mathbf{n} + \delta_a)$.

The function $\mathbf{t}$ on a $\mathcal{I}$-coloured pair partition $(\mathcal{V}, c)$ can be calculated by putting the pair partition in the "standard form", as a sequence (from the right to left) consisting of left legs, followed by permutations acting on the legs of the same index in $\mathcal{I}$ and right legs which connect with the left legs having the same index, etc. The permutations can be fixed by asking that if two pairs of the same colour cross, then the crossing should be performed once and only once by the right-most permutation possible.

For example, let $X = \{1, \ldots, 6\}$ and $\mathcal{V} = \{(1,4), (2,5), (3,6)\}$ with the colouring $c((1,4)) = c((3,6)) = a$ and $c((2,5)) = b$. Then

$$(\mathcal{V}, c) = \mathbf{d}_{a,0}^* \cdot \mathbf{d}_{b,0}^* \cdot \mathbf{d}_{a,0}^* \cdot (\pi_{1,2},\ \mathrm{e})(\mathbf{d}_{a,0} \cdot \mathbf{d}_{b,0} \cdot \mathbf{d}_{a,0}) \tag{2.12}$$

where $(\pi_{1,2}, e)$ is an element of $\mathrm{S}(2) \times \mathrm{S}(1)$. The function $\mathbf{t}$ can be calculated as follows

$$\mathbf{t}((\mathcal{V}, c)) = \langle \xi_{\mathbf{t}},\ j_a^* \ j_b^* \ j_a^* \ U_{2,1}(\pi_{1,2},\ \mathrm{e}) \ j_a \ j_b \ j_a \ \xi_{\mathbf{t}} \rangle \,. \tag{2.13}$$

In general, let $\mathcal{V} \in \mathcal{P}_2(2n)$ be a pair partition and $c : \mathcal{V} \to \mathcal{I}$ a colouring. Then c can be also seen as defined on $\{1, \ldots, 2n\}$ with the condition that it takes the same value in points belonging to the same pair of $\mathcal{V}$. We split the ordered set $\{1, \ldots, 2n\}$ in a number $2m$ of disjoint subsets $B_i^{(r)} := \{k_{i-1}, \ldots, p_i\}$ and $B_j^{(l)} := \{p_i + 1, \ldots, k_i\}$, with $k_0 = 1$ and $k_r = 2n$ such that the blocks $B_j^{(l)}$ contain left legs of pairs in $\mathcal{V}$ and $B_j^{(r)}$ contain right legs. Finally the value of $\mathbf{t}$ can be written as the expectation

$$\mathbf{t}((\mathcal{V}, c)) = \left\langle \xi_{\mathbf{t}},\ \prod_{l=1}^{p_1} j_{c(l)}^* \ U_{\mathbf{n}_1}(\pi_1) \prod_{l=p_1+1}^{k_1} j_{c(l)} \ \ldots U_{\mathbf{n}_r}(\pi_r) \prod_{l=p_m+1}^{2n} j_{c(l)} \ \xi_{\mathbf{t}} \right\rangle . \tag{2.14}$$

On the basis of this structure we pass now to the construction of the generalised Brownian motion indexed by $\mathcal{I}$. Let $\mathbf{t} : \mathcal{P}_2^{\mathcal{I}}(\infty) \to \mathbb{C}$ be a positive definite function on $\mathcal{I}$-coloured pair partitions, and $\mathcal{H}$ a Hilbert space. We define the Fock-like space

$$\mathcal{F}_{\mathbf{t}}(\mathcal{H}) := \bigoplus_{\mathbf{n}} \frac{1}{\mathbf{n}!} V_{\mathbf{n}} \otimes_s \bigotimes_{a \in \mathcal{I}} \mathcal{H}^{\otimes \mathbf{n}(a)} \tag{2.15}$$

with the factor $\frac{1}{\mathbf{n}!} := \prod_a \frac{1}{\mathbf{n}(a)!}$ refering to the inner product. The symbol $\otimes_s$ is a short notation for the subspace consisting of vectors ψ which lie in the range of the projection $P_{\mathbf{n}}$

$$P_{\mathbf{n}} := \frac{1}{\mathbf{n}!} \sum_{\tau \in \mathrm{S}(\mathbf{n})} U(\tau) \otimes \tilde{U}(\tau). \tag{2.16}$$

The operators $\tilde{U}(\tau)$ act on the space $\bigotimes_{a \in \mathcal{I}} \mathcal{H}^{\otimes \mathbf{n}(a)}$ by permuting the vectors in each term $\mathcal{H}^{\otimes \mathbf{n}(a)}$.

On $\mathcal{F}_{\mathbf{t}}(\mathcal{H})$ we define creation and annihilation operators for each index $a \in \mathcal{I}$ and vector $f \in \mathcal{H}$. Let $r_b(f)$ be the creation operator which acts on the tensor product

$$r_b(f) : \bigotimes_{a \in \mathcal{I}} \mathcal{H}^{\otimes \mathbf{n}(a)} \to \bigotimes_{a \in \mathcal{I}} \mathcal{H}^{\otimes \mathbf{n}(a) + \delta_{a,b}} \tag{2.17}$$

as identity on $\mathcal{H}^{\otimes \mathbf{n}(a)}$ for $a \neq b$ and as *right* creation operator on $\mathcal{H}^{\otimes \mathbf{n}(b)}$.
The annihilation operator $a_b(f)$ acts on a vector of the level $\mathbf{n}$ of the Fock space as follows:

$$\begin{aligned} a_b(f) : V_{\mathbf{n}} \otimes_s \bigotimes_{a \in \mathcal{I}} \mathcal{H}^{\otimes \mathbf{n}(a)} \quad &\to \quad V_{\mathbf{n} - \delta_b} \otimes_s \bigotimes_{a \in \mathcal{I}} \mathcal{H}^{\otimes \mathbf{n}(a) - \delta_{a,b}} \\ a_b(f) : \psi_{\mathbf{n}} \quad &\mapsto \quad (j_b^* \otimes r_b^*(f)) \psi_{\mathbf{n}} \end{aligned} \tag{2.18}$$

As a consequence of 2.11 we have

$$P_{\mathbf{n}-\delta_b}(j_b^* \otimes r_b^*(f))P_{\mathbf{n}} = (j_b^* \otimes r_b^*(f))P_{\mathbf{n}} \tag{2.19}$$

which means that $a_b(f)$ is a well defined operator on the dense domain $\mathcal{F}_{\mathbf{t}}^{(\mathrm{fin})}(\mathcal{H})$ of the Fock space $\mathcal{F}_{\mathbf{t}}(\mathcal{H})$ consisting of "finite number of particles" states. The creation operator acts on a simple vector $v_{\mathbf{n}} \otimes_s \mathbf{f} := P_{\mathbf{n}} v_{\mathbf{n}} \otimes \mathbf{f}$ as

$$a_b^*(f)\ v_{\mathbf{n}} \otimes_s \mathbf{f} = (j_b v_{\mathbf{n}}) \otimes_s (r_b(f)\mathbf{f}). \tag{2.20}$$

We give without proof the following result which is a straightforward extension of the similar one in the case of one generalised Brownian motion [12]:

Theorem 2.3 *Let $f_1, \ldots, f_n$ be vectors in a Hilbert space $\mathcal{H}$. Then the expectation values with respect to the vacuum state $\rho_{\mathbf{t}}$ of the monomials in creation and annihilation operators have the expression*

$$\rho_{\mathbf{t}}(\prod_{i=1}^{n} a_{b_i}^{\sharp_i}(f_i)) = \sum_{(\mathcal{V},c)\in\mathcal{P}_2^{\mathcal{I}}(n)} \mathbf{t}((\mathcal{V},c)) \prod_{(i,j)\in\mathcal{V}} \langle f_i, f_j\rangle \cdot \delta_{b_i,b_j} \cdot Q(\sharp_i, \sharp_j) \tag{2.21}$$

where

$$Q = \begin{pmatrix} \rho_{\mathbf{t}}(a_b a_b) & \rho_{\mathbf{t}}(a_b a_b^*) \\ \rho_{\mathbf{t}}(a_b^* a_b) & \rho_{\mathbf{t}}(a_b^* a_b^*) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Furthermore consider $\mathcal{H} = \mathcal{K}_{\mathbb{C}}$ with $\mathcal{K}$ a real Hilbert space and define the fields $\omega_b(f) := a_b(f) + a_b^*(f)$ for vectors f in $\mathcal{K}$. Then the restriction of $\rho_{\mathbf{t}}$ to the *-algebra generated by the fields is the gaussian state characterising the $\mathcal{I}$-indexed generalised Brownian motion.

Corollary 2.4 *Let $f_1, \ldots, f_n$ be vectors in the real Hilbert space $\mathcal{K}$. Then*

$$\rho_{\mathbf{t}}(\prod_{i=1}^{n} \omega_{b_i}(f_i)) = \sum_{(\mathcal{V},c)\in\mathcal{P}_2^{\mathcal{I}}(n)} \mathbf{t}((\mathcal{V},c)) \prod_{(i,j)\in\mathcal{V}} \langle f_i, f_j\rangle \cdot \delta_{b_i,b_j}. \tag{2.22}$$

Remark 2.5 *The restriction of the state $\rho_{\mathbf{t}}$ to the algebra generated by the fields $\omega_b(f)$ for a fixed $b \in \mathcal{I}$ is a generalised Brownian motion in the usual sense. For different indices b one obtains in general different Brownian motions.*

An important class of positive definite functions on pair partitions are those which have the *multiplicativity* property

$$\mathbf{t}(\mathcal{V}) = \mathbf{t}(\mathcal{V}_1) \cdot \mathbf{t}(\mathcal{V}_2) \tag{2.23}$$

for any pair partition $\mathcal{V}$ which is the reunion of two subpartitions $\mathcal{V}_1$ and $\mathcal{V}_2$ not crossing each other. For a multiplicative function $\mathbf{t}$, the operators j_a are isometric (see lemma 5.9 in [12]). Furthermore by proposition 5.10 of [12], the field operators $\omega_b(f)$ are in this case essentially selfadjoint.

3 The q-product of generalised Brownian motions

Definition 3.1 *Let $(\mathcal{V}, c) \in \mathcal{P}_2^{\mathcal{I}}(\infty)$ be a $\mathcal{I}$-coloured pair partition. The number of inter-crossings of $(\mathcal{V}, c)$ is defined by the total number of crossings between the pairs of different colours in $\mathcal{V}$*

$$\mathrm{cr}(\mathcal{V}, c) = \frac{1}{2}\natural\{(p, q) \mid p, q \in \mathcal{V} \text{ crossing}, \ c(p) \neq c(q)\}. \tag{3.1}$$

The main result of this work is the existence of the q-product of generalised Brownian motions which satisfy the multiplicativty property.

Theorem 3.2 *Let $\mathcal{I}$ be an index set and $-1 \leq q \leq 1$. Let $\mathbf{t}_a$ be a given multiplicative positive definite function on $\mathcal{P}_2(\infty)$ for every $a \in \mathcal{I}$. Then the function*

$$(*_{a\in\mathcal{I}}^{(q)} \mathbf{t}_a)((\mathcal{V}, c)) := q^{\mathrm{cr}(\mathcal{V},c)} \prod_{a\in\mathcal{I}} \mathbf{t}_a(c^{-1}(a)). \tag{3.2}$$

is positive definite on $\mathcal{P}_2^{\mathcal{I}}(\infty)$.

Proof. We will firstly prove that for each $\mathbf{n} : \mathcal{I} \to \mathbb{N}$ the kernel $k_{\mathbf{n}}$ defined on the set of diagrams in $\mathcal{BP}_2^{\mathcal{I}}(\mathbf{n}, \mathbf{0})$ is positive definite. Let $\mathbf{d}_i = (\mathcal{V}_{a,i}, f_{a,i}^{(l)})_{a\in\mathcal{I}}$ for $i = 1, 2$, be two such diagrams with legs only to the left. Then we have

$$k_{\mathbf{n}}(\mathbf{d}_1, \mathbf{d}_2) = (*_{a\in\mathcal{I}}^{(q)} \mathbf{t}_a)(\mathbf{d}_1^* \cdot \mathbf{d}_2). \tag{3.3}$$

The kernel $k_{\mathbf{n}}$ can be written as a product of three kernels

$$k_{\mathbf{n}}(\mathbf{d}_1, \mathbf{d}_2) = \prod_{a\in\mathcal{I}} \mathbf{t}_a((\mathbf{d}_1^* \cdot \mathbf{d}_2)_a) \cdot q^{\mathrm{cr}(\mathbf{d}_1)+\mathrm{cr}(\mathbf{d}_2)} \cdot q^{\mathrm{cr}(\mathbf{d}_1,\mathbf{d}_2)}. \tag{3.4}$$

where by $(\mathbf{d}_1^* \cdot \mathbf{d}_2)_a$ we denote the a-colured component of $\mathbf{d}_1^* \cdot \mathbf{d}_2$. The first product is a positive kernel by the positivity of each of the functions $\mathbf{t}_a$. The second product is also a positive definite kernel. The exponent $\mathrm{cr}(\mathbf{d}_i)$ stands for the number of inter-crossings of $\mathbf{d}_i$, that is number of crossings between pairs p, q with different indices - i.e., $p \in \mathcal{V}_{a,i}$ and $q \in \mathcal{V}_{b,i}$ with $a \neq b$ - plus the number of crossings between pairs and left legs with different indices - i.e., $p = (l, r) \in \mathcal{V}_{a,i}$ and $k \in L_{a,i} = \mathrm{Dom}(f_b^{(l)})$ such that $l < k < r$ and $a \neq b$. To obtain the total number of inter-crossings $\mathrm{cr}(\mathbf{d}_1^* \cdot \mathbf{d}_2)$ of the $\mathcal{I}$-indexed pair partition $\mathbf{d}_1^* \cdot \mathbf{d}_2$, we need to add the number of crossings of right legs of $\mathbf{d}_1^*$ and left legs of $\mathbf{d}_2$ which have different colours. This is the exponent $\mathrm{cr}(\mathbf{d}_1, \mathbf{d}_2)$ of the last term of the product 3.4. The factor $q^{\mathrm{cr}(\mathbf{d}_1,\mathbf{d}_2)}$ depends only on the functions $f_{a,i}^{(l)}$ which determine the positions of the left legs, and does not depend on the pair partitions $\mathcal{V}_{a,i}$. The positivity of $\mathrm{cr}(\mathbf{d}_1, \mathbf{d}_2)$ is thus equivalent to that of the vacuum representation of an algebra with commutation relations which is described below.

Consider the algebra generated by the operators $a_{b,i}$ with $i = 1, \ldots, \mathbf{n}(b)$ satisfying the commutation relations

$$a_{b,i}a^*_{c,j} - q_{a,b}\ a^*_{c,j}a_{b,i} = \delta_{a,b}\delta_{i,j} \tag{3.5}$$

with $q_{a,b} = 1$ if $a = b$ and $q_{a,b} = q$ if $a \neq b$. Such algebras have been investigated in [13, 14, 25] and more generally in [7]. There it is proved that for $|q| \leq 1$ the algebra can be represented on a Hilbert space with vacuum vector $\tilde{\Omega}$ satisfying $a_{b,i}\tilde{\Omega} = 0$. In particular this implies that the third kernel in 3.4 is positive definite and thus $k_{\mathbf{n}}$ as well.

We denote the Hilbert spaces generated by the kernels $k_{\mathbf{n}}$ by $V_{\mathbf{n}}$. Let

$$\lambda_{\mathbf{n}} : \mathcal{BP}_2^{\mathcal{I}}(\mathbf{n}, \mathbf{0}) \to V_{\mathbf{n}} \tag{3.6}$$

be the Gelfand map, i.e. $\langle \lambda_{\mathbf{n}}(\mathbf{d}_1), \lambda_{\mathbf{n}}(\mathbf{d}_2)\rangle = k_{\mathbf{n}}(\mathbf{d}_1,\ \mathbf{d}_2)$. On $\mathcal{BP}_2^{\mathcal{I}}(\mathbf{n}, \mathbf{0})$ there is an action of the group $\mathrm{S}(\mathbf{n})$ and $k_{\mathbf{n}}$ is invariant under this action, thus it gives rise to the unitary representation $U_{\mathbf{n}}$ on $V_{\mathbf{n}}$. On the Hilbert space $V := \bigoplus_{\mathbf{n}} V_{\mathbf{n}}$ we define the operators j_a by

$$j_a\lambda_{\mathbf{n}}(\mathbf{d}_1) = \lambda_{\mathbf{n}+\delta_a}(\mathbf{d}_{a,0} \cdot \mathbf{d}_1). \tag{3.7}$$

The multiplicative property of the function $\mathbf{t}_a$ implies that

$$k_{\mathbf{n}}(\mathbf{d}_{a,0} \cdot \mathbf{d}_1,\ \mathbf{d}_{a,0} \cdot \mathbf{d}_2) = k_{\mathbf{n}}(\mathbf{d}_1,\ \mathbf{d}_2) \tag{3.8}$$

which means that j_a is a well defined isometry. Moreover j_a satisfies the intertwining property 2.11. We have thus constructed a representation of the *-semigroup $\mathcal{BP}_2^{\mathcal{I}}(\infty)$ on the Hilbert space V, with respect to the positive functional $\hat{\mathbf{t}}$ where $\mathbf{t} = *^{(q)}_{a\in\mathcal{I}}\mathbf{t}_a$.

□

As described in the previous section we define now the creation and annihilation operators $a^\sharp_b(f)$ on the Fock space $\mathcal{F}_{*^{(q)}_{a\in\mathcal{I}}\mathbf{t}_a}(\mathcal{H})$ for $f \in \mathcal{H}$ in an arbitrary Hilbert space $\mathcal{H}$ and $b \in \mathcal{I}$. Similarly, the fields are $\omega_b(f) := a^*_b(f) + a_b(f)$.

Definition 3.3 *Let $\mathbf{t}$ be a multiplicative positive definite function on ($\mathcal{I}$-indexed) pair partitions. We denote by $\Gamma_{\mathbf{t}}(\mathcal{K})$ the von Neumann algebra on $\mathcal{F}_{\mathbf{t}}(\mathcal{K}_{\mathbb{C}})$ generated by the selfadjoint operators $\omega_b(f)$ with f in a real Hilbert space $\mathcal{K}$ and $b \in \mathcal{I}$. If the state $\rho_{\mathbf{t}}$ on $\Gamma_{\mathbf{t}}(\mathcal{K})$ is tracial then we call the function $\mathbf{t}$ tracial. For a complex Hilbert space $\mathcal{H}$ we denote the von Neumann algebra on $\mathcal{F}_{\mathbf{t}}(\mathcal{H})$ generated by all the fileds $\omega_b(f)$ with $f \in \mathcal{H}$ by $\Gamma_{\mathbf{t}}(\mathcal{H})$.*

Graded tensor product for $q = -1$. We take a closer look at the case $q = -1$. Let F be the unitary operator $Ff = -f$ on $\mathcal{H}$. Then

$$\Gamma_{\mathbf{t}}(F) : X \mapsto \mathcal{F}_{\mathbf{t}}(F)X\mathcal{F}_{\mathbf{t}}(F)^* \tag{3.9}$$

is an order two *-automorphism of $\Gamma_{\mathbf{t}}(\mathcal{H})$ which we call $\mathbb{Z}_2$-grading. The vacuum state $\rho_{\mathbf{t}}$ is invariant under $\Gamma_{\mathbf{t}}(F)$. This makes $(\Gamma_{\mathbf{t}}(\mathcal{H}), \rho_{\mathbf{t}})$ a $\mathbb{Z}_2$-graded noncommutative probability space [22].

Definition 3.4 *[22] Let $(\mathcal{A}, \phi)$ be a $\mathbb{Z}_2$-graded probability space with grading γ. Two von Neumann subalgebras $\mathcal{A}_1$ and $\mathcal{A}_2$ of $\mathcal{A}$ which are invariant under γ, are called graded independent if the gradedly commute, i.e. $a_1 a_2 = (-1)^{\partial a_1 \partial a_2}$ for all $a_i \in \mathcal{A}_i$ which satisfy $\gamma a_i = (-1)^{\partial a_i} a_i$, where $\partial a_i \in \{0, 1\}$ is the grading of a_i, and moreover $\phi(a_1 a_2) = \phi(a_1)\phi(a_2)$. If $\mathcal{A} = \mathcal{A}_1 \bigvee \mathcal{A}_2$ then we call $\mathcal{A}$ the graded tensor product of $\mathcal{A}_1$ and $\mathcal{A}_2$.*

From the definition of the -1-product function $*^{(-1)}_{a\in\mathcal{I}}\mathbf{t}_a$ we can conclude that the creation and annihilation operators of different index anticommute, i.e.

$$a_b^{\sharp_1}(f_1)a_c^{\sharp_2}(f_2) = -a_c^{\sharp_2}(f_2)a_b^{\sharp_1}(f_1) \tag{3.10}$$

for $b \neq c$. This implies that the algebra $\Gamma_{*^{(q)}_{a\in\mathcal{I}}\mathbf{t}_a}(\mathcal{H})$ is the graded tensor product of the non-commutative probability spaces $\Gamma_{\mathbf{t}_a}(\mathcal{H})$ for $a \in \mathcal{I}$.

Corollary 3.5 *The q-product $*^{(q)}_{a\in\mathcal{I}}\mathbf{t}_a$ of multiplicative positive definite functions $\mathbf{t}_a$ is a positive definite multiplicative function on $\mathcal{P}_2^{\mathcal{I}}(\infty)$ and interpolates between the graded tensor product (q=-1), reduced free product (q=0) and the tensor product (q=1). If all $\mathbf{t}_a$ are tracial then $*^{(q)}_{a\in\mathcal{I}}\mathbf{t}_a$ is tracial.*

The q-product provides a method for obtaining new positive definite functions on pair partitions by taking the product of known ones and restricting to a subalgebra generated by the sums of creation operators $a_b(f)$ over the same vector. Let $\mathcal{I}$ be a finite index set and $\mathbf{t}_a$ be positive multiplicative functions for each $a \in \mathcal{I}$. On $\mathcal{F}_{*^{(q)}_{b\in\mathcal{I}}\mathbf{t}_b}(\mathcal{K})$ we define the new creation operators $a^*(f) := \frac{1}{\sqrt{|\mathcal{I}|}}\sum_{b\in\mathcal{I}} a_b^*(f)$. The restriction of the state $*^{(q)}_{b\in\mathcal{I}}\mathbf{t}_b$ to the algebra generated by $a^\sharp(f)$ is a Fock state and the associated positive definite function on pair partitions $(*^{(q)}_{b\in\mathcal{I}}\mathbf{t}_b)^{(\mathrm{r})}$ has the following expression in terms of $\mathbf{t}_a$ and q:

$$(*^{(q)}_{b\in\mathcal{I}}\mathbf{t}_b)^{(\mathrm{r})}(\mathcal{V}) = \left(\frac{1}{|\mathcal{I}|}\right)^{|\mathcal{V}|} \sum_{c:\mathcal{I}\to\mathcal{V}} q^{\mathrm{cr}(c,\mathcal{V})} \prod_{a\in\mathcal{I}} \mathbf{t}_a(c^{-1}(a)) \tag{3.11}$$

The restriction of a q-product of n functions which are equal to $\mathbf{t}$ will be denoted by $\mathbf{t}_q^{*n}$. We denote by $\mathbf{t}_q$ the positive definite function arising from the algebra of q-commutation relations:

$$\mathbf{t}_q(\mathcal{V}) = q^{\mathrm{cr}(\mathcal{V})}. \tag{3.12}$$

The following central limit theorem states that for any positive definite multiplicative $\mathbf{t}$, the q-product $\mathbf{t}_q^{*n}$ converges to $\mathbf{t}_q$ as n goes to infinity.

Theorem 3.6 (Central Limit) *Let $\mathbf{t}$ be a positive definite multiplicative function on pair partitions. Then $\mathbf{t}_q^{*n}$ converges pointwise to $\mathbf{t}_q$ when $n \to \infty$.*

Proof. Let $\mathcal{V}$ be a pair partition. Then

$$\mathbf{t}_q^{*n}(\mathcal{V}) = \left(\frac{1}{n}\right)^{|\mathcal{V}|} \sum_{c:\underline{n}\to\mathcal{V}} q^{\mathrm{cr}(c,\mathcal{V})} \prod_{a\in\mathcal{I}} \mathbf{t}(c^{-1}(a)). \tag{3.13}$$

for $\underline{n} := \{1, \ldots, n\}$. We consider n big enough such that $n > |\mathcal{V}|$. From the sum we isolate the terms which give to each pair in $\mathcal{V}$ a different color, $c(p_1) \neq c(p_2)$ for $p_1 \neq p_2$. Any such term brings a contribution equal to $\left(\frac{1}{n}\right)^{|\mathcal{V}|} q^{\mathrm{cr}(\mathcal{V})}$ which in total gives $q^{\mathrm{cr}(\mathcal{V})}$.
The rest of the terms can be grouped according to the partition of $\mathcal{V}$ in subpartitions determined by the colour of the pairs $\mathcal{V}_a = \{c^{-1}(a)\}$ for $a = 1 \ldots, n$. A fixed such partitioning of $\mathcal{V}$ has at most $|\mathcal{V}| - 1$ sets and thus the number of possibilities of atributing one of the n colours to each set is smaller that $n^{|\mathcal{V}|-1}$. In the limit $n \to \infty$ the contribution of this group of terms in the sum 3.13 tends to zero.

$\square$

Example. Consider the free Brownian motion, i.e. $\mathbf{t}(\mathcal{V}) = 0$ if $\mathcal{V}$ is crossing and $\mathbf{t}(\mathcal{V}) = 1$ if $\mathcal{V}$ is non-crossing. Then the product of n such functions gives

$$\mathbf{t}_q^{*n}(\mathcal{V}) = \left(\frac{1}{n}\right)^{|\mathcal{V}|} \cdot q^{\mathrm{cr}(\mathcal{V})} \cdot \sharp\{c : \mathcal{V} \to \underline{n} \mid \mathrm{cr}(c^{-1}(i)) = 0, \ \forall i \in \underline{n}\}. \tag{3.14}$$

Remark. At the moment is it not clear to us in what extent the q-product defined here for generalised Brownian motions can be extended to other algebras. For example let $\mathcal{A}, \mathcal{B}$ be two algebras as considered in [7] with generators satisfying commutations relations of the type

$$a_i a_j^* - \sum_{k,l} t_{j,l}^{i,k} a_k^* a_l = \delta_{i,j} \tag{3.15}$$

for $\mathcal{A}$ and similarly for $\mathcal{B}$. Then the q-product could be defined by adding the commutation relations $a_i b_m^* = q b_m^* a_i$ for all generators a_i of $\mathcal{A}$ and b_m of $\mathcal{B}$. The new algebra will satisfy the conditions formulated in corollary 3.2 of [7] if the given algebras $\mathcal{A}, \mathcal{B}$ do satisfy them.

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