# Gaussian processes in non-commutative probability theory

# Gaussian processes in non-commutative probability theory

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CHAPTER I

# Introduction and results

The notions of Brownian motion and Gaussian processes play an important role in probability theory. This thesis is a contribution to the understanding of non-commutative Gaussian processes in the framework of quantum probability theory.

Quantum mechanics describes the laws governing the microscopic world, and represents one of the main inspiration sources for non-commutative probability theory. It has been developed around 1925 independently by Heisenberg, who created what was then called 'matrix mechanics', and by Schrödinger who named his version 'wave mechanics'. They realized that in order to explain the discrete configuration of energy levels in atoms, the physical quantities must be governed by non-commutative algebras. In 1932 von Neumann published his book *Mathematische Grundlagen der Quantenmechanik* [66] proving that the two versions are equivalent. Motivated by quantum mechanics he defined the notion of ring of operators nowadays called *von Neumann algebra*, and developed their theory in a series of papers in collaboration with Murray. In the same time Gel'fand and Neumark initiated the study of $C^*$-algebras, their work being continued by Segal who gave the 'GNS-construction' relating states with representations. Since then the theory of operator algebras has developed immensely, the philosophy of encoding mathematical structures into $C^*$ or von Neumann algebras leading to the program on *non-commutative geometry* pioneered by Connes [17].

In his book *Local Quantum Physics* [30], Haag offers a view of quantum relativistic physics in which all these mathematical structures find their place. The theory of operator algebras is equally important in quantum statistical physics [14, 15] as well as in the study of open quantum systems [19]. Here one describes the dissipative evolution of a small system as reduced hamiltonian flow of the system coupled with a 'bath'. Mathematically this translates into the problem of constructing *dilations of completely positive semigroups* on von Neumann algebras. Based on the intuition offered by physics, Kümmerer has shown [41, 39, 40, 43] that the Markov dilations arise as couplings with *quantum white noises*. His analysis leads to a general definition of quantum Lévy processes and white noises with respect to which, Markov processes can be constructed as solutions of stochastic differential equations [38]. Particular cases of such stochastic calculi have been developed by Hudson and Parthasarathy [32] for the bosonic noise, and by Applebaum and Hudson [3] for fermionic noise, generalizing the classical theory of Itô integration and stochastic differential equations.

In the early 80's Voiculescu [65] discovered the theory of *free probability* in which the concept of freeness plays a similar role to that of independence in classical probability, but this time for non-commuting random variables. The free noise can be represented through creation and annihilation operators on the full Fock space over $\mathrm{L}^2(\mathbb{R})$ and the stochastic integration [44, 7] parallels the one of Hudson and Partahsarathy.

The vectors of the bosonic Fock space are completely symmetric under the permutation of the 'one-particle states', while those of the free Fock spaces behave as totally ordered sets. Can we construct Fock spaces with the same symmetry as that of 'trees' or 'cycles' ? This question was the entering point of my investigation together with Hans Maassen in the study of other noises in quantum probability. Our approach was inspired by Joyal's theory of *combinatorial species of structures* [35, 5]. It turned out that we had a different perspective at the type of algebras of creation and annihilation operators called generalised Brownian motion by Bożejko and Speicher [9, 13, 10, 11]. The results of our analysis are described in section 6 of this chapter. In the next sections we will review some facts on non-commutative probability, Gaussian processes over Hilbert spaces, Fock spaces and generalised Brownian motion.

# 1 Some general definitions

Non-commutative probability theory is an inhomogeneous field lying at the crossings between quantum physics, probability theory and operator algebras. The aims range from describing photon counting measurements in quantum optics [58] to random matrices and $\mathrm{II}_1$ factors of free groups [65]. In doing this, one tries to integrate concepts and techniques from probability into the theory of operator algebras. We will give a few necessary definitions and refer to [6, 19, 22, 31, 46, 50] for further reading on quantum probability and its application in quantum physics.

In a purely algebraic setting, a *non-commutative probability space* consists of a pair $(A, \rho)$ where $A$ is a unital algebra over $\mathbb{C}$, and $\rho$ is a linear functional $\rho : A \to \mathbb{C}$ such that $\rho(\mathbf{1}) = 1$. If $A$ is a $^*$-algebra we require that $\rho(a^*) = \overline{\rho(a)}$ and that $\rho$ is positive, i.e. $\rho(a^*a) \geq 0$ for all $a \in A$. An element of $a \in A$ is called *random variable*, and the distribution of $a$ is the functional $\mu_a$ on the algebra of complex polynomials in one variable $\mathbb{C}[X]$, defined by $\mu_a(P) = \rho(P(a))$. In this work we deal mainly with $W^*$-probability spaces, in which case $A$ is a von Neumann algebra, i.e. an algebra of bounded operators on a Hilbert space which is closed in the $\sigma$-weak topology (or equivalently in the weak or strong operator topology), and $\rho$ a normal state, i.e. a positive functional continuous with respect to this topology [37]. The distribution of a selfadjoint element $a \in A$ is then given by a measure $d\mu_a$ with support contained in the spectrum of $a$ such that

$$\int P(t)d\mu_a(t) = \rho(P(a)) \qquad \forall P \in \mathbb{C}[X].$$

and similarly for measurable functions on the spectrum of $a$. In particular if $(X, \Sigma, \mu)$ is a 'classical' probability space then the corresponding object in our algebraic formulation is the $W^*$-probability space $(L^\infty(X), E)$ where $E(f) = \int f d\mu$. The events are identified with the orthogonal projections in $L^\infty(X)$. Occasionally we will work with $C^*$-probability spaces $(A, \rho)$ in which case $A$ is a $C^*$-algebra and $\rho$ a state on $A$.

When dealing with a family $\{a_i\}_{i\in\mathcal{I}}$ of random variables, we define their *joint distribution* as the linear functional $\mu : \mathbb{C}\langle X_i | i \in \mathcal{I}\rangle \to \mathbb{C}$ on non-commutative polynomials in $|\mathcal{I}|$ variables defined by $\mu(P) = \rho(h(P))$ where $h : \mathbb{C}\langle X_i | i \in \mathcal{I}\rangle \to A$ is the unique homomorphism such that $h(x_i) = a_i$. The joint $^*$-distribution can be defined along the same lines for $^*$-probability spaces.

The notion of independence is essential in probability theory and has an obvious extension to the non-commutative context. In a non-commutative probability space $(A, \rho)$, a family of subalgebras $A_i \subset A$ is *independent* if the algebras commute with each other (i.e. $[A_i, A_j] = 0$ if $i \neq j$) and $\rho(a_1 \dots a_n) = \rho(a_1)\dots\rho(a_n)$ whenever $a_k \in A_{i_k}$ and $k \neq l$ implies $i_k \neq i_l$. This means that we can calculate the joint distribution of $\{a_i\}_{i=1}^n$ in terms of the individual distributions of the variables $a_i$. In particular for two real random variables $X, Y$ in a $C^*$-probability space, the measure $\mu_{X+Y}$ which determines the distribution of the sum $X + Y$ is the convolution $\mu_X * \mu_Y$ of the individual measures.

# 2 Gaussian processes in classical probability

A random variable $q$ is called *Gaussian of variance $a$* if its distribution is given by $d\mu_q(x) := (2\pi a)^{-\frac{1}{2}} \exp(-\frac{1}{2a}x^2)dx$. The importance of this distribution stems from the *central limit theorem*, which in its simplest form says the following:

**Theorem 2.1** *Let $\{q_i\}_{i=1}^\infty$ be a family of independent, identically distributed random variables with $\mathbb{E}(q_i) = 0$, $\mathbb{E}(q_i^2) = a$. Let*

$$Q_n = n^{-1/2} \sum_{i=1}^n q_i,$$

*then as $n \to \infty$, $Q_n$ approaches a Gaussian random variable of variance $a$ in the sense that for any bounded continuous function on $\mathbb{R}$,*

$$\mathbb{E}(F(Q_n)) \to (2\pi a)^{-\frac{1}{2}} \int e^{-y^2/2a} F(y)dy.$$

When dealing with a family of random variables or a process, one is interested in the joint distribution of n-tuples $q_1, \ldots, q_n$ of random variables in that process. The *Gaussian processes* are characterized by the Fourier transform of the joint probability distributions

$$C_{q_1,\ldots,q_n}(t_1, \ldots, t_n) := \int e^{i \sum x_j t_j} d\mu_{q_1,\ldots,q_n}(x_1, \ldots, x_n) = e^{-\frac{1}{2} \sum c(i,j) t_i t_j},$$

where $c(i, j) = \mathbb{E}(q_i q_j)$ is the covariance of $q_i$ and $q_j$. The *Brownian motion* is the Gaussian process $\{B(t)\}_{t \geq 0}$ with covariance $\mathbb{E}(B(t)B(s)) = \min(s, t)$. Intuitively, $B(t)$ is the limit $\lim_{n\to\infty} n^{-1/2} Q_{[nt]}$ where $[s]$ is the integer part of $s$. The increments $B(t) - B(s)$ corresponding to the time intervals $[s, t]$ have stationary gaussian distribution with variance $t-s$ and are independent for disjoint intervals. An elegant way to deal with Gaussian processes is by making use of the following theorem.

**Theorem 2.2** [54] *Let $\mathcal{K}$ be a real Hilbert space. Then there exists a probability measure space $(X_{\mathcal{K}}, \mathcal{F}_{\mathcal{K}}, \mu_{\mathcal{K}})$ and for each $f \in \mathcal{K}$ a random variable $B(f)$ such that $f \to B(f)$ is linear and for any $f_1, \ldots, f_n \in \mathcal{K}$, $(B(f_1), \ldots, B(f_n))$ are jointly Gaussian with covariance $\langle f_i, f_j \rangle$, where $\langle \cdot, \cdot \rangle$ is the inner product on $\mathcal{K}$. In particular if $f \perp g$ then $B(f)$ and $B(g)$ are independent.*

Then the existence of a process with a given covariance $c(\cdot, \cdot)$ is equivalent to $c(\cdot, \cdot)$ being positive definite as a kernel. The Brownian motion is obtained by taking $B(t) := \omega(\chi_{[0,t]})$ for $\chi_{[0,t]}$ the characteristic function of the interval $[0, t]$, an element of the real Hilbert space $\mathcal{K} := \mathrm{L}^2_{\mathbb{R}}(\mathbb{R}_+)$.

The Gaussian process $\{B(f)\}_{f \in \mathcal{K}}$ is 'equivalent' to a well known construction in quantum physics, the symmetric Fock space and the vacuum representation of the algebra of canonical commutation relations (C.C.R.) which we briefly describe here. Let $\mathcal{H}$ be a complex Hilbert space. The *symmetric Fock space* over $\mathcal{H}$ is

$$\mathcal{F}_s(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes_s n},$$

where $\mathcal{H}^{\otimes_s n}$ denotes the $n$-fold symmetric tensor product of $\mathcal{H}$, that is the subspace of $\mathcal{H}^{\otimes n}$ consisting of vectors which are invariant under the unitary action of the symmetric group $\mathrm{S}(n)$ permuting the vectors in the tensor product. Let $P_n$ denote the projection onto $\mathcal{H}^{\otimes_s n}$, then for any vector $f \in \mathcal{H}$ we define the *creation operator* acting on a vector $\psi \in \mathcal{H}^{\otimes_s n}$

$$a^*(f)\psi = \sqrt{n+1}\; P_{n+1}(f \otimes \psi).$$

The adjoint $a(f)$ of $a^*(f)$ is called *annihilation operator* and the $^*$-algebra formed by all $a^*(f), a(g)$ satisfies the commutation relations

$$\begin{aligned} a(f)a^*(g) - a^*(g)a(f) &= \langle f, g \rangle \mathbf{1} \\ a(f)a(g) - a(g)a(f) &= 0. \end{aligned}$$

The distinguished unit vector $\Omega$ such that $\mathcal{H}^{\otimes 0} = \mathbb{C}\Omega$ satisfies $a(f)\Omega = 0$ and is called vacuum while the associated state $\rho$, vacuum state. Another important operator is the *field* $\omega(f) := a(f) + a^*(f)$ which is essentially selfadjoint on the domain of 'finite number of particles' and can thus be extended to an unbounded selfadjoint operator denoted by the same symbol. The rigorous study of the algebra of creation and annihilation operators and its representations has to be carried out in the framework of C*-algebras. One constructs the unitary Weyl operators $W(f) = \exp(i\omega(f))$ satisfying the relations

$$W(f)W(g) = e^{-i\mathrm{Im}\langle f,g\rangle} W(f+g),$$

and the C*-algebra generated by $\{W(f)\}_{f\in\mathcal{H}}$ which we denote by CCR($\mathcal{H}$). The vacuum expectation of a Weyl operator is $\rho(W(f)) = e^{-||f||^2/2}$. For a detailed analysis of the C.C.R. algebra and its representations we refer to the monographs [14, 15].

The map CCR has nice properties, i.e. it is a functor from the category of Hilbert spaces to the category of $C^*$-probability spaces having the state preserving completely positive maps as morphisms. For any contraction $T : \mathcal{H}_1 \to \mathcal{H}_2$ the vacuum preserving map $\mathrm{CCR}(T) : \mathrm{CCR}(h_1) \to \mathrm{CCR}(h_2)$ given by

$$\mathrm{CCR}(T) : W(f) \mapsto e^{(||Tf||^2 - ||f||^2)/2} W(Tf)$$

is completely positive. Moreover $\mathrm{CCR}(T_1)\mathrm{CCR}(T_2) = \mathrm{CCR}(T_1T_2)$. Such functors are called *white noise functors* and the reason for giving such a name is explained below.

Let us leave the $C^*$-algebraic framework and consider the $^*$-algebra of (unbounded) creation and annihilation operators and the vacuum expectation of the monomials in these operators. In particular, if we choose $\mathcal{H} = \mathcal{K}_{\mathbb{C}}$, the complexification of the real Hilbert space $\mathcal{K}$, then the fields $\{\omega(f)\}_{f\in\mathcal{K}}$ form a commutative algebra and their joint distribution with respect to the vacuum are of the form

$$\rho\left(\prod_{i=1}^{n} e^{it_i\omega(f_i)}\right) = \rho\left(e^{i\sum t_i\omega(f_i)}\right) = e^{-||\sum t_i f_i||^2/4} = e^{-\frac{1}{2}\sum t_i t_j \langle f_i, f_j\rangle},$$

which is precisely that prescribed by theorem 2.2. In other words $\{\omega(f)\}_{f\in\mathcal{K}}$ form a Gaussian process over the Hilbert space $\mathcal{K}$ with covariance $\langle\cdot,\cdot\rangle$. If we denote by $\Gamma_s(\mathcal{K})$ the von Neumann algebra generated by the selfadjoint operators $\{\omega(f)\}_{f\in\mathcal{K}}$ then we obtain another functor $\Gamma_s$ this time from the category of real Hilbert spaces to the category of $W^*$-probability spaces which is called the Gaussian functor, or functor of white noise. For this reason we will name functors of white noise all functors from (real) Hilbert spaces to non-commutative probability spaces which map the zero dimensional Hilbert space into the algebra $\mathbb{C}$.

**Remark.** In fact for any real Hilbert sub-space $\mathcal{K}'$ of the Hilbert space $\mathcal{H}$, we obtain an associated Gaussian process $\{\omega(f)\}_{f\in\mathcal{K}'}$, but in general the different processes do not commute with each other. A stochastic theory for such non-commuting Brownian motions, and other 'quantum noises' has been developed by Hudson and Parthasarathy [50] and is called *quantum stochastic calculus*.

An explicit expression of the expectations of monomials of field operators is given by the following formula which is the starting point of our investigation of generalised Brownian motion:

$$\rho(\omega(f_1)\dots\omega(f_n)) = \sum_{\mathcal{V}\in\mathcal{P}_2(n)} \prod_{(l,r)\in\mathcal{V}} \langle f_l, f_r\rangle \, .$$

The sum is taken over all *pair partitions* i.e., the partitions of the ordered set $(1,\dots,n)$ into subsets with two elements, and by convention it is set to zero for odd $n$.

# 3 Free probability

The notion of freeness has been introduced by Voiculescu around 1985, and is the key concept of an essentially non-commutative probability theory called free probability. The results presented in this section are standard and can be found in [65].

**Definition 3.1** *Let $(A,\rho)$ be a non-commutative probability space. A family $(A_i)_{i\in\mathcal{I}}$ of unital subalgebras of $A$ is called free if $\rho(a_1\dots a_n) = 0$ whenever $a_j \in A_{i_j}$, $i_1 \neq i_2 \neq \cdots \neq i_n$, and $\rho(a_{i_j}) = 0 \forall j$. A family $(b_i)_{i\in\mathcal{I}}$ of random variables is free ($^*$-free) if the algebras ($^*$-algebras) which they generate form a free family.*

**Remark.** This definition is at the first sight, of a different nature than that of independence, but here also one can compute mixed moments $\rho(a_1\dots a_n)$ for arbitrary $a_j \in A_{i_j}$ by writing $a_i = \rho(a_i)\mathbf{1} + \mathring{a}_i$ with $\rho(\mathring{a}_i) = 0$ using induction by $n$ and the rule given above. Speicher has shown that the combinatorics involved in such calculations can be beautifully expressed in terms of *non-crossing partitions* and *free cumulants* [56].

Freeness is a notion of independence equally interesting to its classical counterpart. In fact all the objects defined in the previous section have their 'free' counterpart.

If $X$ and $Y$ are two free random variables in a non-commutative probability space, then the distribution $\mu_{X+Y}$ depends only on $\mu_X$ and $\mu_Y$ and it is called the *free convolution* of $\mu_X$ and $\mu_Y$: $\mu_{X+Y} = \mu_X \boxplus \mu_Y$ [65]. The so called R-transform plays a similar role to that of the Fourier transform for the classical convolution, in the 'linerization' of the free convolution.

The *semicircle law* centered at $a \in \mathbb{R}$ and of radius $r$ is the distribution $\gamma_{a,r} : \mathbb{C}[X] \to \mathbb{C}$ defined by

$$\gamma_{a,r}(P) = \frac{2}{\pi r^2} \int_{a-r}^{a+r} P(t)\sqrt{r^2 - (t-a)^2}dt.$$

Such distributions appear in the free central limit theorem.

**Theorem 3.2** *Let $(A, \rho)$ be a non-commutative probability space, and let $\{q_j\}_{j=1}^{\infty}$ be a free family of identically distributed random variables in $A$ such that $\mathbb{E}(q_i) = 0$, $\mathbb{E}(q_i^2) = r^2/4$. Let*

$$Q_n = n^{-1/2} \sum_{i=1}^{n} q_i,$$

*then as $n \to \infty$, $Q_n$ converges in distribution to the semicircle law $\Gamma_{0,r}$.*

Again, let $\mathcal{H}$ be a Hilbert space. We consider now the free analog of the $CCR(\mathcal{H})$ algebra, that is the algebra of creation and annihilation operators on the *full Fock space* over $\mathcal{H}$,

$$\mathcal{F}(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes n}.$$

The left creation and annihilation operators are defined by $l(f)\Omega = f,\ l^*(f)\Omega = 0$ and

$$\begin{aligned} l(f) \quad &: f_1 \otimes \ldots \otimes f_n \mapsto f \otimes f_1 \otimes \ldots \otimes f_n, \\ l^*(f) \quad &: f_1 \otimes \ldots \otimes f_n \mapsto \langle f, f_1 \rangle\, f_2 \otimes \ldots \otimes f_n. \end{aligned}$$

The 'commutation relations' are now $l^*(h_1)l(h_2) = \langle h_1, h_2 \rangle\, \mathbf{1}$ implying that $l(e)$ is an isometry for any unit vector $e$ and the $C^*$-algebra $C^*(l(\mathcal{H}))$ is isomorphic to the Cuntz algebra $\mathcal{O}_\infty$ [18] for infinite dimensional $\mathcal{H}$, and an extension of $\mathcal{O}_{|\mathcal{H}|}$ by the compact operators for $|\mathcal{H}| < \infty$. The fields are denoted by $s(f) := l(f)+l^*(f)$ and are bounded selfadjoint operators whose distribution with respect to the vacuum state $\rho_{\mathcal{H}}$, is the semicircle law $\gamma_{0,2||f||}$. Moreover, for any family of pairwise orthogonal vectors $\{f_i\}_{i\in\mathcal{I}}$, the family $\{s(f_i)\}_{i\in\mathcal{I}}$ is free.

Let us consider $\mathcal{H} = \mathcal{K}_{\mathbb{C}}$ for a real Hilbert space $\mathcal{K}$ and denote by $\Gamma(\mathcal{K})$ the von Neumann algebra generated by the fields $s(f)$ with $f \in \mathcal{K}$. Then the linear map $s : \mathcal{K} \to \Gamma(\mathcal{K})$ is a *free Gaussian process* over $\mathcal{K}$. Let $\mathbf{T}$ be a separable topological space and $c : \mathbf{T} \times \mathbf{T} \to \mathbb{R}$ be a jointly continous function which is a positive definite kernel on $\mathbf{T}$, i.e. for all $n \geq 0$, $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ and all $t_1, \ldots, t_n \in \mathbf{T}$ the following inequality holds

$$\sum_{i,j=1}^{n} \alpha_i \alpha_j \; c(t_i, t_j) \geq 0.$$

In a standard way (corollary 2.4 in [54]) one constructs the separable real Hilbert space $\mathcal{K}_c$ associated to the kernel $c$ and the vectors $f_t \in \mathcal{K}_c$ such that $\langle f_t, f_s \rangle = c(s,t)$. Then $\{s(f_t)\}_{t \in \mathbf{T}}$ is a free Gaussian process with covariance $c$. Such processes are in general non-commutative unless one is dealing with a single random variable, and the von Neumann algebras $\Gamma(\mathcal{K}_c)$ which they generate are type $\mathrm{II}_1$ non-hyperfinite factors for which the vacuum state $\rho_{\mathcal{K}_c}$ is the unique tracial state, i.e. $\rho_{\mathcal{K}_c}(ab) = \rho_{\mathcal{K}_c}(ba)$. These factors are isomorphic to the von Neumann algebras of the free group with $|\mathcal{K}_c|$ generators, $\mathbf{F}_{|\mathcal{K}_c|}$.

The map $\Gamma : \mathcal{K} \to (\Gamma(\mathcal{K}), \rho_{\mathcal{K}})$ is a functor from the category of real Hilbert spaces with contractions to the category of $W^*$-probability spaces with state preserving completely positive maps. Any contraction $T : \mathcal{K}_1 \to \mathcal{K}_2$ between two real Hilbert spaces induces a natural map $T_{\mathbb{C}} : (\mathcal{K}_1)_{\mathbb{C}} \to (\mathcal{K}_2)_{\mathbb{C}}$ and its second quantisation at the Hilbert space level

$$\begin{aligned} &\mathcal{F}(T_{\mathbb{C}}) : \mathcal{F}((\mathcal{K}_1)_{\mathbb{C}}) \to \mathcal{F}((\mathcal{K}_2)_{\mathbb{C}}) \\ &\mathcal{F}(T_{\mathbb{C}}) : f_1 \otimes \ldots \otimes f_n \mapsto T_{\mathbb{C}} f_1 \otimes \ldots \otimes T_{\mathbb{C}} f_n. \end{aligned}$$

There exists a unique unital, trace preserving completely positive map $\Gamma(T) : \Gamma(\mathcal{K}_1) \to \Gamma(\mathcal{K}_2)$ which satisfies

$$\Gamma(T) X \Omega = \mathcal{F}(T_{\mathbb{C}}) X \Omega.$$

If $T$ is a isometry then $\Gamma(T)$ is a faithful $^*$-homomorphism, and if $T$ is the orthogonal projection onto a subspace, then $\Gamma(T)$ is a conditional expectation.

The joint distribution of the field operators $\{s(f)\}_{f \in \mathcal{K}}$ is given by a similar formula to that of the Gaussian case

$$\rho(s(f_1) \ldots s(f_n)) = \sum_{\mathcal{V} \in \mathcal{P}_2^{(\mathrm{NC})}(n)} \; \prod_{(l,r) \in \mathcal{V}} \langle f_l, f_r \rangle \,,$$

but with the sum running only over the *non-crossing* pair partitions, i.e. those which do not contain pairs $p_i = (l_i, r_i)$ for $i = 1, 2$ such that $l_1 < l_2 < r_1 < r_2$.

# 4 Fermions and graded independence

Another fundamental example of Fock space in physics is the fermionic or *anti-symmetric Fock space*. This space appears in the description of multi-particle states of identical fermions, which are particles with semi-integer spin. For a Hilbert space $\mathcal{H}$, the associated Fock space is

$$\mathcal{F}_a(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes_a n}$$

where $\mathcal{H}^{\otimes_a n}$ is the $n$-fold anti-symmetric tensor product of $\mathcal{H}$. The creation and annihilation operators can be defined similarly to the bosonic case and satisfy the canonical anti-commutation relations, (C.A.R.):

$$a(f)a^*(g) + a^*(g)a(f) = \langle f, g\rangle \mathbf{1}$$
$$a(f)(g) + a(g)a(f) = 0.$$

The fermionic $C^*$-algebra $\mathrm{CAR}(\mathcal{H})$ generated by the bounded operators $\{a^\sharp(f)\}_{f\in\mathcal{H}}$ is isomorphic to $\bigotimes^{|\mathcal{H}|} M(\mathbb{C}^2)$. By completion in the strong operator topology in certain representations one obtains the type $\mathrm{III}_\lambda$ factors $R_\lambda$ of Powers [51] for $0 < \lambda < 1$ which are the unique hyperfinite factors of this type.

As in the bosonic and free cases, we take $\mathcal{H} = \mathcal{K} \oplus i\mathcal{K}$ where $\mathcal{K}$ is a real Hilbert space and construct the von Neumann algebra $\Gamma_a(\mathcal{K})$ generated by the fields $\omega(f) = a(f) + a^*(f)$ with $f \in \mathcal{K}$. If $|\mathcal{K}| = \infty$ then $\Gamma_a(\mathcal{K})$ is isomorphic to the hyperfinite factor of type $\mathrm{II}_1$ and the vacuum state $\rho$ is the unique trace of $\Gamma_a(\mathcal{K})$. In a similar fashion to the functors $\Gamma_s$ and $\Gamma$, the map $\Gamma_a$ is a functor of white noise. The joint distributions are given by

$$\rho(\omega(f_1)\ldots\omega(f_n)) = \sum_{\mathcal{V}\in\mathcal{P}_2(n)} (-1)^{\mathrm{cr}(\mathcal{V})} \prod_{(l,r)\in\mathcal{V}} \langle f_l, f_r\rangle\,,$$

where $\mathrm{cr}(\mathcal{V})$ is the *number of crossings* of $\mathcal{V}$, that is the number of pairs $(p_1, p_2)$ of elements $p_i = (l_i, r_i)$ of $\mathcal{V}$ for which $l_1 < l_2 < r_1 < r_2$.

Two random variables $\omega(f)$ and $\omega(g)$ corresponding to orthogonal vectors in $\mathcal{K}$ should be 'independent' in some sense, like it was the case for bosonic and free Gaussian processes. In order to define the 'fermionic independence' we need to introduce one more structure, namely a $\mathbb{Z}_2$-grading.

**Definition 4.1** [47] *Let $(\mathcal{A}, \rho)$ be a probability space. An automorphism $\gamma$ of $\mathcal{A}$ which preserves the state $\rho$, is called $\mathbb{Z}_2$-grading if $\gamma^2 = \mathrm{id}_{\mathcal{A}}$. Two subalgebras $\mathcal{A}_1$ and $\mathcal{A}_2$ of $\mathcal{A}$ which are invariant under $\gamma$, are called graded independent if they gradely commute, i.e. $a_1 a_2 = (-1)^{\partial a_1 \partial a_2} a_2 a_1$ for all $a_i \in \mathcal{A}_i$ which satisfy $\gamma a_i = (-1)^{\partial a_i} a_i$, where $\partial a_i \in \{0, 1\}$ is called the grading of $a_i$, and moreover $\phi(a_1 a_2) = \phi(a_1)\phi(a_2)$. In the case of $W^*$-probability spaces, if $\mathcal{A} = \mathcal{A}_1 \bigvee \mathcal{A}_2$ then we call $\mathcal{A}$ the graded tensor product of $\mathcal{A}_1$ and $\mathcal{A}_2$.*

On the probability space $\Gamma_a(\mathcal{K})$, we have the $\mathbb{Z}_2$-grading given by $\gamma = \Gamma_a(-\mathbf{1})$, and for any subspaces $\mathcal{K}_1, \mathcal{K}_2$ such that $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$ the corresponding algebras $\Gamma_a(\mathcal{K}_1), \Gamma_a(\mathcal{K}_2)$ are gradely independent in $\Gamma_a(\mathcal{K})$. Similar results hold for the functor $\mathcal{H} \to \mathrm{CAR}(\mathcal{H})$.

# 5 Generalised Brownian motion

After having presented the three notions of independence and their associated Gaussian processes, we formulate the following natural questions: what is the general form of a 'Gaussian process', and is there always a notion of independence attached to it ?

A reasonable answer to the first question is: a Gaussian process over a real Hilbert space $\mathcal{K}$ is uniquely determined by a positive functional $\rho$ on the free unital $^*$-algebra $\mathcal{A}(\mathcal{K})$ over the real Hilbert space $\mathcal{K}$ (see definition 2.1 in chapter III), such that the joint distribution is invariant under all the orthogonal transformations $O$ on $\mathcal{K}$:

$$\rho(\omega(Of_1)\ldots\omega(Of_n)) = \rho(\omega(f_1)\ldots\omega(f_n)).$$

Let $e_1, e_2$ be two orthogonal normalized vectors in $\mathcal{K}$. Then $\rho(\omega(e_1)+\omega(e_2)/\sqrt{2}) = \rho(\omega(e_1)) = \sqrt{2}\rho(\omega(e_1))$ which implies $\rho(\omega(f)) = 0$ for any vector $f$. By using the linearity of $\omega$ and the invariance of $\rho$ one can show that the general form of the joint distribution is

$$\rho_{\mathbf{t}}(\omega(f_1)\ldots\omega(f_n)) = \sum_{\mathcal{V}\in\mathcal{P}_2(n)} \mathbf{t}(\mathcal{V}) \prod_{(l,r)\in\mathcal{V}} \langle f_l, f_r\rangle\,, \tag{5.1}$$

where $\mathbf{t} : \mathcal{P}_2(\infty) \to \mathbb{C}$ is a complex valued function defined on the set of all pair partitions $\mathcal{P}_2(\infty) := \bigcup_{n=0}^{\infty} \mathcal{P}_2(2n)$, which characterizes completely the Gaussian process. In particular the expectation of monomials containing an odd number of random variables is zero. Following the original term coined by Bożejko and Speicher [10, 11], we will call such a Gaussian process *generalised Brownian motion* and the functional $\rho_{\mathbf{t}}$, *Gaussian state*. The characterization of the positivity of $\rho_{\mathbf{t}}$ directly in terms of the function $\mathbf{t}$ will be one of the main tasks in chapter III.

The first example of generalised Brownian motion has been treated in [10, 11], and arises from the Fock representation of the $q$-deformed commutation relations

$$a(f)a^*(g) - qa^*(g)a(f) = \langle f, g\rangle\,\mathbf{1}.$$

These relations interpolate between the bosonic ($q = 1$), free ($q = 0$), and fermionic ($q = -1$) ones. Various aspects have been investigated in the literature [21, 23, 26, 33, 34, 45, 55, 67] to which we refer for further details. The cyclic representation with respect to the vacuum vector $\Omega$ satisfying $a(f)\Omega = 0$ for all $f \in \mathcal{H}$, can be described in terms of a deformation of the representation of the algebra of creation and annihilation operators on the full Fock space over $\mathcal{H}$. On the space $\mathcal{F}^{(\mathrm{fin})}(\mathcal{H})$ of linear combinations of tensor products of vectors in $\mathcal{H}$, one defines the inner product:

$$\langle f_1 \otimes \ldots \otimes f_n, g_1 \otimes \ldots \otimes g_m \rangle_q := \delta_{m,n} \sum_{\pi \in \mathrm{S}(n)} q^{i(\pi)} \left\langle f_1, g_{\pi(1)} \right\rangle \ldots \left\langle f_n, g_{\pi(n)} \right\rangle,$$

where $\mathrm{S}(n)$ denotes the symmetric group of permutations of $n$ elements and $i(\pi)$ is the number of inversions of the permutation $\pi$

$$i(\pi) := |\{(i,j) \ : \ 1 \le i < j \le n, \ \pi(i) > \pi(j)\}|.$$

The positivity of the inner product follows from the fact that the function $q^{i(\cdot)}$ : $\mathrm{S}(n) \to \mathbb{C}$ is positive definite for $-1 \le q \le 1$. The $q$-Fock space $\mathcal{F}_q(\mathcal{H})$ is obtained by (dividing out the kernel for $q = \pm 1$ and) completion of $\mathcal{F}^{(\mathrm{fin})}(\mathcal{H})$ with respect to $\langle \cdot, \cdot \rangle_q$. The creation operator is defined as for the full Fock space by tensoring on the left side. Due to the $q$-dependent inner product the action of the adjoint will depend on $q$:

$$a(f) f_1 \otimes \ldots f_n = \sum_{i=1}^{n} q^i \langle f, f_i \rangle f_1 \otimes \ldots \otimes \check{f}_i \otimes \ldots \otimes f_n,$$

and $a(f)\Omega = 0$. The field operators $\omega(f)$ are bounded for $-1 < q < 1$, and have joint distributions as in equation (5.1), with

$$\mathbf{t}_q(\mathcal{V}) = q^{\mathrm{cr}(\mathcal{V})}.$$

In reference [9] the authors analyze the operator-algebraic and functorial properties of the von Neumann algebra $\Gamma_q(\mathcal{K})$ generated by the fields $\{\omega(f)\}_{f \in \mathcal{K}}$ acting on $\mathcal{F}_q(\mathcal{K}_C)$. They prove that $\Gamma_q(\mathcal{K})$ is a non-hyperfinite type $\mathrm{II}_1$ factor for infinite dimensional $\mathcal{K}$. The vacuum state is faithful for the von Neumann algebra and thus one can define the injective map

$$\begin{aligned} \Gamma_q(\mathcal{K}) \quad &\to \mathcal{F}_q(\mathcal{K}_{\mathbb{C}}), \\ X \quad &\mapsto X\Omega. \end{aligned}$$

The inverse image of a vector $f_1 \otimes \ldots \otimes f_n$ is called Wick product and denoted $\Psi(f_1 \otimes \ldots \otimes f_n)$. The Wick products behave nicely under the second quantisation maps. Let $T : \mathcal{K} \to \mathcal{K}'$ be a contraction between real Hilbert spaces, then

$$\Gamma_q(T) : \Psi(f_1 \otimes \ldots \otimes f_n) \mapsto \Psi(Tf_1 \otimes \ldots \otimes Tf_n)$$

extends to a completely positive map from $\Gamma_q(\mathcal{K})$ to $\Gamma_q(\mathcal{K}')$ which preserves the vacuum state. Moreover $\Gamma_q(T_1)\Gamma_q(T_2) = \Gamma_q(T_1 T_2)$ and thus, $\Gamma_q$ is a functor of white noise.

An interesting question is that regarding the stability of the $C^*$-algebra of creation and annihilation operators $C^*(a^\sharp(h) : h \in \mathcal{H})$ when the deformation parameter $q$ moves away from zero. It has been shown in [20, 33] that in a certain interval around zero, the algebra $C^*(a^\sharp(h) : h \in \mathcal{H})$ is isomorphic to the extension of the Cuntz algebra $\mathcal{O}_{|\mathcal{H}|}$ by the compact operators for finite dimensional $\mathcal{H}$. A similar question for the case of the von Neumann algebra $\Gamma_q(\mathcal{K})$ has not been answered yet.

Another example of interpolation between the free and classical Gaussian processes with parameter $0 \leq r \leq 1$ is given in [13]:

$$\mathbf{t}_r(\mathcal{V}) = r^{|\mathcal{V}|-\mathrm{B}(\mathcal{V})},$$

where $\mathrm{B}(\mathcal{V})$ denotes the number of blocks, or connected components of $\mathcal{V}$. The state $\rho_{\mathbf{t}_r}$ is tracial for the algebra of Gaussian fields over a real Hilbert space $\mathcal{K}$. A particular feature of this case is that the cyclic representation space is not a 'deformation' of the full Fock space over $\mathcal{K}_\mathbb{C}$ like it was the case for the $q$-deformations, but a bigger space containing more copies of the tensor products $\mathcal{K}_\mathbb{C}^{\otimes n}$ on the level $n$ of the representation space.

We notice that both examples are based on representations of a larger unital free $^*$-algebra $\mathcal{C}(\mathcal{K}_\mathbb{C})$ over the Hilbert space $\mathcal{K}_\mathbb{C}$ which we call algebra of 'creation and annihilation operators'. We regard $\mathcal{A}(\mathcal{K})$ as sub-algebra of $\mathcal{C}(\mathcal{K}_\mathbb{C})$ through the identification $\omega(f) = a(f) + a^*(f)$. Then the Gaussian state $\rho_\mathbf{t}$ on $\mathcal{A}(\mathcal{K})$ is the restriction of the *Fock state* on $\mathcal{C}(\mathcal{K}_\mathbb{C})$ which for simplicity we denote by the same symbol $\rho_\mathbf{t}$. The precise definitions can be found on page 64. The functions $\mathbf{t}$ on pair partitions which give rise to positive Fock functionals $\rho_\mathbf{t}$ will be called *positive definite*. In chapter III it will be show that $\mathbf{t}$ is positive definite already if $\rho_\mathbf{t}$ is positive on $\mathcal{A}(\mathcal{K})$, which explains why in all examples we encounter creation and annihilation operators.

The second question from the beginning of this section inquires about the existence of a notion of independence attached to a given generalised Brownian motion. This question was answered negatively [45] in the case of $q$-deformed commutation relations, in the sense that there exists no $q$-convolution of measures: the distribution $\mu_{X+Y}$ of the sum of of two $q$-independent real valued random variables $X$ and $Y$ cannot be calculated in terms of the individual distributions $\mu_X$ and $\mu_Y$.
A more general result proved by Speicher [57] shows that under certain universality conditions (associativity, universal calculation rule for mixed moments) there are no notions of products of probability spaces (unital algebras with normalized linear functionals), other than the tensor and the free product. We have seen however that the Fermionic independence can be defined if the probability spaces have more structure, escaping the no-go theorem of Speicher. Also, there exist notions of $q$-convolutions of measures such as those treated in [2, 48]. Alterna-

tively, one can relax the independence to the notion of statistical independence or that of pyramidal independence [13].

The generalised Brownian motion can be obtained in the limit $n \to \infty$, by taking partial sums $Q_{[nt]}$ of 'identically distributed' random variables in the following non-commutative central limit theorem [13].

**Theorem 5.1** *Let $(A, \rho)$ be a ${}^*$-probability space. Consider selfadjoint elements $q_i = q_i^* \in \mathcal{A}$ which fulfill the following assumptions:*
*i) we have $\rho(q_{i(1)} \dots q_{i(n)}) = 0$ for all $n \in \mathbb{N}$ and all $i(1), \dots, i(n)$ with the property that one of the $i(k)$ is different from all others.*
*ii) for each permutation $\pi$ of the natural numbers we have*

$$\rho(q_{i(1)} \dots q_{i(n)}) = \rho(q_{\pi(i(1))} \dots q_{\pi(i(n))}),$$

*for all $n \in \mathbb{N}$ and $i(1), \dots, i(n) \in \mathbb{N}$.*
*Let $Q_N = \sum_{i=1}^{N} q_i/\sqrt{N}$. Then in the limit $N \to \infty$ the process $Q_{[Nt]}$ converges in law to the generalised Brownian motion with associated positive definite function on pair partitions given by*

$$\mathbf{t}(\mathcal{V}) := \rho(q_{i(1)} \dots q_{i(2n)}),$$

*where $\mathcal{V} \in \mathcal{P}_2(2n)$ and the indices are chosen such that for any $l \neq r \in \{1, \dots, 2n\}$ then $i(l) = i(r)$ if and only if $(l, r) \in \mathcal{V}$.*

Finally, we would like to mention the work of Köstler [38] who has developed a general theory of non-commutative Markov processes, white noise and Lévy processes. His contribution is in the spirit of Kümmerer's approach to quantum probability [42, 41, 39]. The white noise is described by a *finite* probability space of $\mathcal{A}_0$-valued random variables i.e., a von Neumann algebra $\mathcal{A}$, endowed with a tracial normal state $\rho$ together with a subalgebra $\mathcal{A}_0$ and the state preserving conditional expectation $\mathrm{P}_0$ from $\mathcal{A}$ to $\mathcal{A}_0$ [61]. There exists a filtration of subalgebras $\mathcal{A}_\mathrm{I}$ of $\mathcal{A}$, for all closed intervals I of the time axis $\mathbb{R}$. A group $(S_t)_{t\in\mathbb{R}}$ of automorphisms of $(\mathcal{A}, \rho)$ acts as a shift on the local algebras $S_t(\mathcal{A}_\mathrm{I}) = \mathcal{A}_{\mathrm{I}+t}$ and lets $\mathcal{A}_0$ pointwise invariant. For disjoint intervals $\mathrm{I}, \mathrm{J}$ the local algebras $\mathcal{A}_\mathrm{I}$ and $\mathcal{A}_\mathrm{J}$ are *statistically independent* over $\mathcal{A}_0$ i.e., $\mathrm{P}_\mathrm{I} \circ \mathrm{P}_\mathrm{J} = \mathrm{P}_0$ a notion in which we recognize a commuting square of von Neumann algebras [24]. The quantum Brownian motion is an additive cocycle $(B_t)_{t\in\mathbb{R}}$ with respect to the white noise $(\mathcal{A}, \rho, S_t, \mathcal{A}_\mathrm{I})$ over $\mathcal{A}_0$ that is, a process which is adapted to the filtration $\mathcal{A}_{[0,t]}$, satisfying $B_{s+t} = B_t + S_t(B_s)$ and certain continuity requirements in the $\mathrm{L}^p$-norms (see definitions in chapter 3 of [38]).

The Gaussian process $\{\omega_\mathbf{t}(f)\}_{f\in L^2_\mathbb{R}(\mathbb{R}_+)}$ associated to a positive definite function $\mathbf{t}$ for which the vacuum state $\rho_\mathbf{t}$ is tracial for the von Neumann algebra $\mathcal{A} := \Gamma_\mathbf{t}(L^2(\mathbb{R}_+))$ fits into this framework. The local algebras are defined by

$$\mathcal{A}_\mathrm{I} := \mathrm{vN}\{\omega_\mathbf{t}(f) : f = \chi_I f \in L^2_\mathbb{R}(\mathbb{R}_+)\} \subset \Gamma_\mathbf{t}(L^2(\mathbb{R}_+)),$$

and $A_0 := \mathbb{C}$. In this case $\Gamma_{\mathbf{t}}$ is a functor of white noise (for this result we refer to chapter III) and the conditional expectations $\mathrm{P_I}$ are given by $\Gamma_{\mathbf{t}}(P(I))$ where $P(I)$ is the orthogonal projection onto the subspace of functions with support in the interval $I$.
But not all Gaussian states $\rho_{\mathbf{t}}$ are tracial. A class of non-tracial generalised Brownian motion is analyzed in chapter IV.

# 6 Outline of results

**Chapter II** employs the combinatorial concept of species of structures to construct generalisations of the symmetric and free Fock spaces, and define the creation and annihilation operators on these spaces which are derived from certain 'actions' on the structures.

A *species of structures* $F$ is a functor from the category $\mathbb{B}$ which has as objects the finite sets and as morphisms bijections between finite sets, to the category $\mathbb{E}$ of finite sets with functions as morphisms. Let $U$ be a finite set. Then $F[U]$ is the set of $F$-structures over $U$; for any bijection $\sigma : U \to V$ there exists a map $F[\sigma] : F[U] \to F[V]$ called transport along the bijection $\sigma$, such that $F[\sigma \cdot \tau] = F[\sigma] \circ F[\tau]$ and $F[\mathrm{id}_U] = \mathrm{id}_{F[U]}$. Some examples of species are given on page 28.

The calculus with species of structures has been developed by Joyal in [35] where he defines the following operations between species: sum/difference, product, cartesian product, substitution and derivation. This allows one to write equations with species just in the same way one does for formal power series. We have to do here with an instance of categorification [4]. For example the equation

$$\mathcal{A} = X \cdot E(\mathcal{A})$$

is an implicit definition of the species $\mathcal{A}$ of rooted trees, and can be read off as follows: a rooted tree is composed of a root ($X$) and a set (the species of sets is denoted by $E$) of rooted trees (growing out of the root). Many more applications can be found in the monograph [5].

For a given species $F$ we define an endofunctor $\mathcal{F}_F$ of the category of Hilbert spaces which maps a Hilbert space $\mathcal{H}$ to the *symmetric Hilbert space* $\mathcal{F}_F(\mathcal{H})$ given by

$$\mathcal{F}_F(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \frac{1}{n!}\ \ell^2_{\mathrm{sym}}(F[n] \to \mathcal{H}^{\otimes n}),$$

where $\ell^2_{\mathrm{sym}}(F[n] \to \mathcal{H}^{\otimes n})$ denotes the functions which intertwine with the natural actions of $\mathrm{S}(n)$ on the two sides, and $\frac{1}{n!}$ is a factor referring to the inner product. In particular the species $E$ of sets, the species $L$ of linear orders and the species $E^{\pm}$ of ordered sets, give rise to the symmetric Fock space, the full Fock space, and respectively the anti-symmetric Fock space over $\mathcal{K}$. The construction of the functor $\mathcal{F}_F$ is inspired by the *analytic functor* $F(\cdot)$ of Joyal [36] which maps a set of colors $J$ to the set of $J$-colored unlabeled $F$-structures

$$F(J) := \sum_{n=0}^{\infty} F[n] \times J^n \ / \ \mathrm{S}(n).$$

which can also be interpreted as orthogonal basis for $\mathcal{F}_F(\mathcal{H})$ if $(e_j)_{j\in J}$ is an orthogonal basis of $\mathcal{H}$.

On $\mathcal{F}_F(\mathcal{H})$ we can define creation and annihilation operators by choosing a *weight* $\mathbf{j}$ on the species $F \times F'$ where $F'$ is the derivative of $F$. By definition $F'[U] = F[U \cup \{*\}]$ where '$*$' is a distinguished point and the transport is done along bijections which keep the point '$*$' fixed. A weight $w$ on a species $G$ attaches in a functorial way a coefficient $w(s)$ to any structure $s$ of $G$. Alternatively $\mathbf{j}$ defines a map

$$\mathbf{j} : \bigoplus_{n=0}^{\infty} \ell^2(F[n]) \to \bigoplus_{n=0}^{\infty} \ell^2(F[n])$$

which sends a basis vector $\delta_s \in \ell^2(F[n])$ to a vector in $\ell^2(F[n+1])$ and whose matrix coefficients are $\langle \mathbf{j}\ \delta_s, \delta_t \rangle := \mathbf{j}(s,t)$. Let $h \in \mathcal{H}$, and $\varphi \in \mathcal{F}^{(n+1)}(\mathcal{H})$ a vector belonging to the $n+1$-th level of the symmetric Hilbert space $\mathcal{F}^{(n+1)}(\mathcal{H})$, then

$$\begin{aligned} a(h) \qquad &: \ell^2_{\mathrm{sym}}(F[n+1] \to \mathcal{H}^{\otimes n+1}) \to \ell^2_{\mathrm{sym}}(F[n] \to \mathcal{H}^{\otimes n}), \\ (a(h)\varphi)(s) \qquad &:= \sum_{t \in F[n+1]} \mathbf{j}(s,t)\ \mathrm{inp}_n(h, \varphi(g)), \end{aligned}$$

with $\mathrm{inp}_n(h, h_0 \otimes \ldots \otimes h_n) = \langle h, h_n \rangle\, h_0 \otimes \ldots \otimes h_{n-1}$. Various properties of the creation and annihilation operators are discussed on pages 37-41 and subsequently applied to concrete examples:

1) the species $\mathcal{A}$ of rooted trees (page 43) with weight $\mathbf{j}_{\mathcal{A}}(t_1, t_2) = 1$ if the tree $t_1$ can be obtained from $t_2$ by 'removing the leaf labeled by $*$', and otherwise zero, gives rise to creation and annihilation operators satisfying the commutation relations

$$a(h_1)a^*(h_2) - a^*(h_2)a(h_1) = \langle h_1, h_2 \rangle\, N,$$

where $N$ denotes the number operator which assigns the value 1 to the vacuum.

2) the species $\mathcal{D}_s$ of simple directed graphs (page 45) with a weight $\mathbf{j}^{\mathcal{D}_s,q}$ depending on the real parameter $0 \geq q \geq 1$ gives rise to the $q$-commutation relations

$$a(h_1)a^*(h_2) - qa^*(h_2)a(h_1) = \langle h_1, h_2 \rangle\, \mathbf{1}.$$

The last section makes the connection with the generalised Brownian motion. It turns out that the algebras of creation and annihilation operators constructed from pairs $(F, \mathbf{j})$ as above are always of the type defined by Bożejko and Speicher in[13], and in particular the fields $\omega_F(f) = a_F(f) + a_F^*(f)$ form a Gaussian process. Some operations with species such as the sum, the product, the cartesian product and the composition have counterparts in the symmetric Hilbert spaces.

In the end we use the species Bal of 'ballots' to obtain the generalised Brownian motion with positive definite function $\mathbf{t}(\mathcal{V}) = q^{|\mathcal{V}|-\mathrm{B}(\mathcal{V})}$ for $0 \leq q \leq 1$, which has been found in [13].

**Chapter III** builds on the combinatorial results from the previous one. The first aim is to have a general understanding of the GNS representation of the algebra of creation and annihilation $\{a^\sharp(f)\}_{f\in\mathcal{H}}$ over a Hilbert space $\mathcal{H}$, with respect to a Fock state $\rho_\mathbf{t}$ where $\mathbf{t}$ is a positive definite function on pair partitions. Subsequently we concentrate on the Gaussian process $\{\omega(f)\}_{f\in\mathcal{K}}$ over a real Hilbert space $\mathcal{K}$ and its functorial properties.

Let $\rho_\mathbf{t}$ be a positive definite function and $\mathcal{H}$ a Hilbert space. Then the representation of $\mathcal{C}(\mathcal{H})$ with respect to $\rho_\mathbf{t}$ is given by:

1) The Hilbert space

$$\mathcal{F}_V(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \frac{1}{n!}\; V_n \otimes_s \mathcal{H}^{\otimes n}$$

where $(V_n)_{n=0}^\infty$ are Hilbert spaces on which there exists a unitary representation $U_n$ of the symmetric group $\mathrm{S}(n)$. The symbol $\otimes_s$ denotes the subspace of the tensor product consisting of vectors which are invariant under the action of $U_n(\tau)\otimes\tilde{U}_n(\tau)$ for all $\tau$ in $\mathrm{S}(n)$. Note that the spaces $\mathcal{F}_F(\mathcal{H})$ used in chapter II are naturally isomorphic with the space $\mathcal{F}_V(\mathcal{H})$ for $V_n = \ell^2(F[n])$. Let $P_n$ be the orthogonal projection onto this subspace and $v_n \otimes_s \psi_n := n! P_n v \otimes \psi_n$

2) The creation and annihilation operators are defined with the help of the linear maps $j_n : V_n \to V_{n+1}$ with the same intertwining properties as their counterparts $\mathbf{j}$ in chapter II

$$a^*(h):\; v_n \otimes_s \psi_n \mapsto (j_n v_n)\otimes_s(\psi_n \otimes h).$$

The correspondence between $\mathbf{t}$ and $(V_n, j_n)_{n\geq 0}$ is bijective as proved in theorems 2.6 and 2.7. In fact the positivity of the functions $\mathbf{t}$ can be directly characterized through an algebraic object, the $^*$-*semigroup of broken pair partitions* denoted $\mathcal{BP}_2(\infty)$. The elements of this semigroup can be described graphically as segments of pair partitions, that is the part of the graphic of a pair partition which is contained between two vertical lines. A precise definition is given on page 71. The product of two such segments is obtained by joining the legs which point to each other according to their order on the vertical. The $^*$-operation is the reflexion with respect to a vertical line. In particular $\mathcal{P}_2(\infty)$ is a subset of $\mathcal{BP}_2(\infty)$.

The function $\mathbf{t}$ is positive definite if and only if its extension $\hat{\mathbf{t}}$ to $\mathcal{BP}_2(\infty)$ by defining $\hat{\mathbf{t}}(d) = 0$ on the complement of $\mathcal{P}_2(\infty)$, is a positive functional on the $^*$-semigroup (theorem 3.2 on page 73). The GNS-like representation of $\mathcal{BP}_2(\infty)$ with respect to $\hat{\mathbf{t}}$ produces the Hilbert space $V := \bigoplus_{n=0}^{\infty} V_n$, a distinguished unit vector $\xi \in V_0$, and the representation $\chi_{\mathbf{t}}$ such that $\hat{\mathbf{t}}(d) = \langle \xi, \chi_{\mathbf{t}}(d)\xi\rangle_V$. The space $V_n$ contains the dense domain of linear combinations of vectors of the form $\chi_{\mathbf{t}}(d)\xi$, with $d \in \mathcal{BP}_2^{(n,0)}(\infty)$ the set of diagrams having $n$ left legs and no right legs. The permutation of the left legs produces the unitary representations $U_n$ on $V_n$. The diagram $d_0 \in \mathcal{BP}_2^{(1,0)}(\infty)$ containing just one left leg gives rise to the operator $j := \chi(d_0)$ whose restriction to $V_n$ is $j_n$.

If $\mathbf{t}$ is positive definite, i.e. the functional $\rho_{\mathbf{t}}$ is positive on $\mathcal{C}(\mathcal{K}_C)$ for any real Hilbert space $\mathcal{K}$, then by restriction, $\rho_{\mathbf{t}}$ is also positive on the $^*$-algebra $\mathcal{A}(\mathcal{K})$ of the fields $\{\omega(f)\}_{f\in\mathcal{K}}$. Using the notion of *generalised Wick products* we prove in section 4 that the converse is also true, namely for any Gaussian process with joint distribution as in eq. (5.1), the Fock functional $\rho_{\mathbf{t}}$ is positive on $\mathcal{C}(\mathcal{K}_C)$ as well.

Let $(\tilde{\mathcal{F}}_{\mathbf{t}}(\mathcal{K}), \tilde{\pi}_{\mathbf{t}}, \tilde{\Omega}_{\mathbf{t}})$ be the GNS representation of $(\mathcal{A}(\mathcal{K}), \rho_{\mathbf{t}} \restriction_{\mathcal{A}(\mathcal{K})})$. The generalised Wick products (see definition 4.1) are operators acting on $\tilde{\mathcal{F}}_{\mathbf{t}}(\mathcal{K})$ and denoted by $\Psi(\mathcal{V}, \mathbf{f})$ with $\mathcal{V}$ a pair partition on a subset $P$ of an ordered set $X$, and $\mathbf{f} : X \setminus P \to \mathcal{K}$. The following decomposition holds for monomials in field operators:

$$\omega_{\mathbf{t}}(\mathbf{f}(1)) \dots \omega_{\mathbf{t}}(\mathbf{f}(n)) = \sum_{P\subset \underline{n}} \sum_{\mathcal{V}\in\mathcal{P}_2(P)} \eta_{\mathbf{f}}(\mathcal{V})\ \Psi(\mathcal{V}, \mathbf{f} \restriction_{\underline{n}\setminus P})$$

with $\underline{n} := \{1, \dots, n\}$, and the coefficients $\eta_{\mathbf{f}}(\mathcal{V}) := \prod_{(l,r)\in\mathcal{V}} \langle \mathbf{f}(l), \mathbf{f}(r)\rangle$. The Wick products are important because they reveal the underlying Fock-structure of the representation space $\tilde{\mathcal{F}}_{\mathbf{t}}(\mathcal{K})$. Creation and annihilation operators can be defined such that for any $\mathbf{f} : \underline{n} \to \mathcal{K}$

$$\prod_{i=1}^{n} a^{\sharp_i}(\mathbf{f}(i))\tilde{\Omega}_{\mathbf{t}} = \sum_{\mathcal{V}} \eta_{\mathbf{f}}(\mathcal{V})\ \Psi(\mathcal{V}, \mathbf{f} \restriction_{\underline{n}\setminus \mathrm{supp}(\mathcal{V})})\tilde{\Omega}_{\mathbf{t}},$$

where the sum is taken over those sets of pairs $(l, r)$ in $\underline{n}$ for which $a^{\sharp}_r(\mathbf{f}(r)) = a^*(\mathbf{f}(r))$ and $a^{\sharp}_l(\mathbf{f}(l)) = a(\mathbf{f}(l))$. The analysis of the Wick operator algebra shows that the the function $\mathbf{t}$ is positive definite and $\omega(f) = a(f) + a^*(f)$ ( theorem 4.6). For this reason we can drop the 'tilde' superscript and identify $\tilde{\mathcal{F}}_{\mathbf{t}}(\mathcal{K})$ with the symmetric Hilbert space $\mathcal{F}_{\mathbf{t}}(\mathcal{K}_C)$.

Section 5 makes an analysis of the functorial properties of the gaussian processes $\{\omega_{\mathbf{t}}(f)\}_{f\in\mathcal{K}}$. If is clear from its definition that the map $\mathcal{F}_{\mathbf{t}} : \mathcal{H} \to \mathcal{F}_{\mathbf{t}}(\mathcal{H})$ is an endofunctor of the category of Hilbert spaces with contractions. We are interested primarily in functors at the algebraic level, that is functors from (real)

Hilbert spaces to non-commutative probability spaces in their $W^*$-version. The morphisms between two probability spaces $(\mathcal{A}, \rho_{\mathcal{A}})$ and $(\mathcal{B}, \rho_{\mathcal{B}})$ are the state and unit preserving completely positive maps from $\mathcal{A}$ to $\mathcal{B}$. An isometry is mapped under such functors into an injective $^*$-homomorphism, while an orthogonal projection, into a state preserving *conditional expectation* or equivalently, a norm-one projection [63]. This is the generalisation of the the classical notion from probability theory. Note that in contrast to the usual conditional expectation, a state preserving norm-one projection between two von Neumann algebras does not always exist [60].

We investigate three possible functors: $\Gamma_{\mathbf{t}}, \Gamma_{\mathbf{t}}^{\infty}$ and $\Delta_{\mathbf{t}}$. In all cases we need to assume that the fields are selfadjoint operators. With this in mind, we prove in proposition 5.10 that if the function $\mathbf{t}$ is *multiplicative*, i.e. its value on a pair partition is equal to the product of the values on its connected components, then the fields are essentially selfadjoint operators. Let us define after Kümmerer [42], a *functor of white noise* as functor from (real) Hilbert spaces to probability spaces which map the zero dimensional Hilbert space $\{0\}$ into the algebra $\mathbb{C}$. We want to construct such functors using the framework of generalised Brownian motion, loosely speaking, the algebras must be 'build out of fields'. To distinguish from the previous concept we call this *functor of second quantisation.* We then show in lemma 5.8 that multiplicativity of the function $\mathbf{t}$ is a necessary condition in order to construct functors of second quantisation. However we identify three possible choices defined on page 81, which must be analyzed separately. The most obvious one is the algebra $\Gamma_{\mathbf{t}}(\mathcal{K})$ generated by the Gaussian process $\{\omega_t(f)\}_{f\in\mathcal{K}}$ acting on $\mathcal{F}_{\mathbf{t}}(\mathcal{K}_{\mathbb{C}})$. This is the case of the $q$-second quantisation functor from [9] and in particular the bosonic, fermionic and free functors. However in general this is not the best definition, but one should consider an algebra acting on $\mathcal{F}_{\mathbf{t}}(\mathcal{K}_{\mathbb{C}} \oplus \ell^2(\mathbb{Z}))$ where the $\ell^2(\mathbb{Z})$ 'modes' are only passive. The two possibilities are:

1) $\Gamma_{\mathbf{t}}^{\infty}(\mathcal{K})$ is the von Neumann sub-algebra of $\Gamma_{\mathbf{t}}(\mathcal{K} \oplus \ell^2_{\mathbb{R}}(\mathbb{Z}))$ of operators commuting with $\mathcal{F}_{\mathbf{t}}(\mathbf{1} \oplus O)$ for all $O \in \mathcal{O}(\ell^2_{\mathbb{R}}(\mathbb{Z}))$;

2) $\Delta_{\mathbf{t}}(\mathcal{K})$ is the von Neumann generated by the Wick products $\Psi(\mathcal{V}, \mathbf{f})$ with Im $\mathbf{f} \subset \mathcal{K}$ acting on $\mathcal{F}_{\mathbf{t}}(\mathcal{K} \oplus \ell^2(\mathbb{Z}))$. We consider here only the case of bounded Wick operators.

The first result is that $\Delta_{\mathbf{t}}$ is a functor of second quantisation. The Wick products have the desired behavior under second quantisation morphisms

$$\Delta_{\mathbf{t}}(T) : \Psi(\mathcal{V}, \mathbf{f}) \mapsto \Psi(\mathcal{V}, T \circ \mathbf{f}),$$

for any contraction $T$ between real Hilbert spaces. For $\Gamma_{\mathbf{t}}^{\infty}$ we prove that it is also functor from Hilbert spaces to probability spaces but $\Gamma_{\mathbf{t}}^{\infty}(\{0\}) = \mathbb{C}$ if and only if the vacuum state $\rho_{\mathbf{t}}$ is *faithful* for $\Gamma_{\mathbf{t}}(\ell^2_{\mathbb{R}}(\mathbb{Z}))$ (see theorem 5.16). In this case the two coincide if the fields are bounded (see corollary 5.17).

The last section of chapter III deals with a concrete example of tracial positive definite function on pair partitions which has been found in [13] and rediscovered in chapter II:

$$\mathbf{t}_q(\mathcal{V}) = (-1)^{\mathrm{cr}(\mathcal{V})}|q|^{|\mathcal{V}|-\mathrm{B}(\mathcal{V})},$$

with $-1 < q \leq 0$. The field operators are in this case bounded. The aim of the investigation is to show that the von Neumann algebra $\Gamma_{\mathbf{t}_q}(\mathcal{K})$ is a type $\mathrm{II}_1$ factor for infinite dimensional spaces $\mathcal{K}$. The developed technique is inspired from [9] and can be applied to any function $\mathbf{t}$ for which the state $\rho_{\mathbf{t}}$ is faithful. We define the map

$$\Phi : X \mapsto \operatorname*{w-lim}_{n\to\infty} \omega(e_n) X \omega(e_n)$$

which is a completely positive trace preserving contraction on $\Gamma_{\mathbf{t}}(\mathcal{K})$. At the Hilbert space level we have the contraction $\Theta$ which has a transparent action on Wick product vectors. Let $\{P, F\}$ be a partition of the set $\{1, \ldots, n\}$ and $\mathcal{V} \in \mathcal{P}_2(P)$, $\mathbf{f} : F \to \mathcal{K}$. Then

$$\Theta : \Psi(\mathcal{V}, \mathbf{f})\Omega_{\mathbf{t}} \mapsto \Psi(\underline{\mathcal{V}}, \mathbf{f})\Omega_{\mathbf{t}},$$

where $\underline{\mathcal{V}} = \mathcal{V} \cup \{(0, n+1)\}$ adds to $\mathcal{V}$ one pair which 'embraces' the whole ordered set $\{1, \ldots, n\}$. This contraction has a counterpart $\theta$ on the space $V = \oplus_{n=0}^{\infty} V_n$. By lemma 6.2, the operator $\theta$ is selfadjoint if the state $\rho_{\mathbf{t}}$ is tracial. If $\theta$ has no eigenvector with eigenvalue 1 other than $\xi \in V_0 = \mathbb{C}$, then $\theta^n \to P_\xi$ as $n \to \infty$ and in consequence $\Phi^n(X) \to \rho_{\mathbf{t}}(X)\mathbf{1}$ which implies that $\rho_{\mathbf{t}}$ is the only trace on $\Gamma_{\mathbf{t}}(\mathcal{K})$.
In the case of the function $\mathbf{t}_q$ the map $\theta$ has eigenvalues in the set $\{0, q, -q\}$ on $V \ominus V_0$. This implies that $\Gamma_{\mathbf{t}_q}$ is a factor of type $\mathrm{II}_1$.

**Chapter IV** deals with a class of positive definite functions on pair partitions $\mathbf{t}_{\alpha,\beta}$ which extend in a natural way the characters $\phi_{\alpha,\beta}$ of the infinite symmetric group $\mathrm{S}(\infty)$.

The symmetric group $\mathrm{S}(n)$ embeds in the set of pair partitions over the ordered set $\{1, \ldots, 2n\}$ by

$$\mathrm{S}(n) \ni \tau \mapsto \mathcal{V}_\tau := \{(i, 2n+1-\tau(i)) \ : \ i = 1, \ldots, n\} \in \mathcal{P}_2(2n).$$

The restriction to the symmetric group of a positive definite function $\mathbf{t}$ is positive definite, as function on the group [13]. For multiplicative functions $\mathbf{t}$, the restriction to the symmetric group is itself multiplicative for permutations whose supports lie in disjoint segments on the ordered set $\{1, 2, \ldots\}$. The converse is still an open question: can a multiplicative positive definite function on the symmetric group be extended in a natural way to the pair partitions $\mathcal{P}_2(\infty)$ ?
The characters of the infinite symmetric group are indecomposable positive definite central functions on the group. The theorem of Thoma [62] gives the concrete

expression of all the characters:

$$\phi_{\alpha,\beta}(\sigma) = \prod_{m\geq 2}\left(\sum_{i=1}^{\infty}\alpha_i^m \ + \ (-1)^{m+1}\sum_{i=1}^{\infty}\beta_i^m\right)^{\rho_m(\sigma)},$$

where $\alpha_1 \geq \alpha_2 \geq \cdots \geq 0$, $\beta_1 \geq \beta_2 \geq \cdots \geq 0$, $\sum \alpha_i + \sum \beta_i \leq 1$, and $\rho_m(\sigma)$ is the number of cycles of length $m$ in the permutation $\sigma$.

The GNS-representation with respect to $\phi_{\alpha,\beta}$ has been described by Veršik and Kerov in [64] by giving a probabilistic interpretation to the coefficients $\alpha_i, \beta_i$. We use their construction to define the spaces $V_n$ and the maps $j_n$. The details can be found on page 101. In the simpler case when $\sum_i \alpha_i = 1$ we consider $V_n^{(\alpha)} = \ell^2(\tilde{\mathcal{X}}_n, \tilde{m}_n)$ where $\tilde{\mathcal{X}}_n$ is the space of pairs $(x, y)$ such that $x, y \in \mathcal{N}^n$ and $x = \sigma y$ for some permutation $\sigma \in \mathrm{S}(n)$ action on the $\mathcal{N}^n$. The measure $\tilde{m}_n$ is $\alpha^n$ for the $x$ coordinate and counting measure over the $y$ coordinate for a given $x$. We see $\alpha$ as measure over $\mathcal{N} := \{1, 2, \dots\}$. On $\tilde{\mathcal{X}}_n$ the permutation group acts on the first coordinate, and produces the unitary representation

$$(U_n^{(\alpha)}(\sigma)h)(x, y) = h(\sigma^{-1}x, y).$$

We fix the unit vector $\mathbf{1}_n$, the indicator function of the diagonal $\{(x, x) \mid x \in \mathcal{X}_n\} \subset \tilde{\mathcal{X}}_n$. This gives the positive definite function $\phi_\alpha(\sigma)$ and the isometry $j_n : V_n \to V_{n+1}$ given by $(j_n h)(x, y) = \delta_{x_{n+1}, y_{n+1}} h(x^{(n)}, y^{(n)}))$ where $x = (x_1, \dots, x_n, x_{n+1}) = (x^{(n)}, x_{n+1})$. We obtain thus a representation of the $^*$-semigroup $\mathcal{BP}_2(\infty)$ on $\bigoplus_{n=0}^{\infty} V_n^{(\alpha)}$. A similar construction can be made in the general case. We denote by $\mathbf{t}_{\alpha,\beta}$ the corresponding positive definite function on pair partitions. Its expression is similar to that of $\phi_{\alpha,\beta}$:

$$\mathbf{t}_{\alpha,\beta}(\mathcal{V}) = \prod_{m\geq 2}\left(\sum_{i=1}^{\infty}\alpha_i^m + (-1)^{m+1}\sum_{i=1}^{\infty}\beta_i^m\right)^{\rho_m(\mathcal{V})}.$$

**Definition.**Let $\mathcal{V} \in \mathcal{P}_2(2n)$. There exists a unique *non-crossing* pair partitions $\hat{\mathcal{V}} \in \mathcal{P}_2(2n)$ such that the set of left points of the pairs in $\mathcal{V}$ and $\hat{\mathcal{V}}$ coincide. A *cycle* in $\mathcal{V}$ is a sequence $((l_1, r_1), \dots, (l_m, r_m))$ of pairs of $\mathcal{V}$ such that the pairs $(l_1, r_2), (l_2, r_3), \dots, (l_m, r_1)$ belong to $\hat{\mathcal{V}}$. The length of this cycle is $m$. We denote by $\rho_m(\mathcal{V})$ the number of cycles of length $m$ in the pair partition $\mathcal{V}$.

In particular $\mathbf{t}_{\alpha,\beta}$ is multiplicative and thus the fields $\omega_{\alpha,\beta}$ are selfadjoint, so we can define the von Neumann algebra $\Gamma_{\alpha,\beta}(\mathcal{K})$ generated by the Gaussian process $\{\omega_{\alpha,\beta}(f)\}_{f\in\mathcal{K}}$ on $\mathcal{F}_{\alpha,\beta}(\mathcal{K}_{\mathbb{C}})$. We prove that with the exception of the cases $\alpha_1 = 1$ (bosonic), $\beta_1 = 1$ (fermionic), and $\alpha_i = \beta_i = 0$ (free), there is no tracial normal state on $\Gamma_{\alpha,\beta}(\ell^2_{\mathbb{R}}(\mathbb{Z}))$ and the vacuum state $\rho_{\alpha,\beta}$ is not faithful (lemma 4.3 and lemma 4.4).

The last section of chapter IV concentrates on the special case in which the function on pair partitions is $\mathbf{t}_N(\mathcal{V}) = \left(\frac{1}{N}\right)^{|\mathcal{V}|-\mathrm{c}(\mathcal{V})}$, with $N \in \mathbb{Z} \setminus \{0\}$ and $\mathrm{c}(\mathcal{V})$ the number of cycles of $\mathcal{V}$. This arises from the choice $\alpha_1 = \ldots, \alpha_N = \frac{1}{N}$ for positive $N$ and $\beta_1 = \ldots, \beta_{|N|} = \frac{1}{|N|}$ for negative $N$. The following 'commutation relations' are found

$$a_N(f)a_N^*(g) = \langle f,\ g\rangle\,\mathbf{1} + \frac{1}{N}\mathrm{d}\Gamma(T_{f,g}),$$

where $T_{f,g}$ is the finite rank operator defined by $T_{f,g}h := \langle f, h\rangle\, g$, and $\mathrm{d}\Gamma(A)$ is a differential second quantisation operator for $A$ bounded operator on the one particle space. In the case $N < 0$ this leads to the *exclusion principle* which says in physical language that at most $|N|$ particles can be in the same one-particle state. However we do not know if this algebra has any relevance in physics.

The fact that the vacuum state is not faithful, can have drastic consequences for the algebra of fields: for infinite dimensional real Hilbert spaces $\mathcal{K}$ and negative $N$, the von Neumann algebra $\Gamma_N(\mathcal{K})$ is the whole algebra of bounded operators on $\mathcal{F}_N(\mathcal{K}_\mathbb{C})$. This is proved in proposition 5.3, by showing that the number operators $\mathbf{N}_i$ which count the particles in state $e_i \in \mathcal{K}$, belong to the algebra $\Gamma_N(\mathcal{K})$ and thus also the projection on the cyclic vacuum vector.

In the last part of the chapter we analyze the functors of white noise $\Delta_N$ (for negative $N$) generated by the generalised Wick products $\Psi(\mathcal{V}, \mathbf{f})$ acting on $\mathcal{F}_N(\mathcal{K}_\mathbb{C} \oplus \ell^2(\mathbb{Z}))$ for which $\mathrm{Im}(\mathbf{f}) \subset \mathcal{K}$. For finite dimensional $\mathcal{K}$, the von Neumann algebra $\Delta_N(\mathcal{K})$ is finite dimensional. In particular in the one dimensional case $\Delta_N(\mathbb{R}) = \bigoplus_{p=2}^{N+1} \mathrm{M}_p(\mathbb{C})$ which is non-commutative, in contrast with the known tracial cases of functors of white noise. This result is proved in theorem 5.8. For infinite dimensional $\mathcal{K}$, $\Delta_N(\mathcal{K})$ is a discrete sum of type $\mathrm{I}_\infty$ factors, according to theorem 5.7.

**Chapter V** answers affirmatively the following question: does there exist a $q$-product of two generalised Brownian motions ?

As it was mentioned already, the no-go theorem of Speicher [57] shows that only the free and tensor products of probability spaces are 'universal'. The generalised Brownian motions are defined in terms of pairings for which there is a clear notion of 'crossings'. This allows the definition of *$q$-product of Brownian motions* by inserting a factor of the type '$q^{\mathrm{cr}}$' in the joint distributions. More precisely, let $\{\mathbf{t}_a\}_{a\in\mathcal{I}}$ be a family of multiplicative positive definite functions. On the free product of algebras of fields $\mathcal{A}_a(\mathcal{K})$ indexed by elements of $a \in \mathcal{I}$ we define the functional

$$\Big(\mathop{*}_{a\in\mathcal{I}}^{(q)} \rho_{\mathbf{t}_a}\Big)\left(\prod_{i=1}^{n} \omega_{c(i)}(f_i)\right) = \sum_{\mathcal{V}} q^{\mathrm{cr}(\mathcal{V},c)} \prod_{a\in\mathcal{I}} \mathbf{t}_a(\mathcal{V}_a) \prod_{(i,j)\in\mathcal{V}} \langle f_i, f_j\rangle\,,$$

where the sum is taken over those pair partitions $\mathcal{V}$ such that if $(i,j) \in \mathcal{V}$ then $c(i) = c(j) \in \mathcal{I}$, which we call the 'color' of the pair $(i,j)$, the pair partition $\mathcal{V}_a$ is the subset of pairs in $\mathcal{V}$ which are colored in the color $a$, and the coefficient $\mathrm{cr}(\mathcal{V}, c)$ counts the number of crossings between pairs of *different* colors. We call this functional the $q$-product of $\{\rho_{\mathbf{t}_a}\}_{a\in\mathcal{I}}$. Its positivity is again determined by a $^*$-semigroup $\mathcal{BP}_2^{\mathcal{I}}(\infty)$ of $\mathcal{I}$-indexed broken pair partitions which is similar to the 'mono-color' one except that the pairs and legs get an index of their own and the legs join according to their index (see definition 2.2). The $q$-product of the functions $\{\mathbf{t}_a\}_{a\in\mathcal{I}}$ is defined by:

$$(\mathop{*}_{a\in\mathcal{I}}^{(q)} \mathbf{t}_a)((\mathcal{V}, c)) := q^{\mathrm{cr}(\mathcal{V},c)} \prod_{a\in\mathcal{I}} \mathbf{t}_a(c^{-1}(a)).$$

where $(\mathcal{V}, c)$ is an element of $\mathcal{P}_2^{\mathcal{I}}(\infty)$ with $c : \mathcal{V} \to \mathcal{I}$. This extends to the whole semigroup $\mathcal{BP}_2^{\mathcal{I}}(\infty)$ by taking the value zero on diagrams with legs. The representation space of the $^*$-semigroup is the direct sum of the spaces $V_{\mathbf{n}}$ constructed from the kernels

$$k_{\mathbf{n}}(\mathbf{d}_1, \mathbf{d}_2) = (\mathop{*}_{a\in\mathcal{I}}^{(q)} \mathbf{t}_a)(\mathbf{d}_1^* \cdot \mathbf{d}_2),$$

over all $\mathbf{n} : \mathcal{I} \to \mathbb{N}$ such that $\sum_a \mathbf{n}(a) < \infty$. The diagrams $\mathbf{d}_i$ have $\mathbf{n}(a)$ left legs indexed by $a \in \mathcal{I}$ and no right legs. Each of these kernels is a product of 3 positive kernels:

1) $\prod_{a\in\mathcal{I}} \mathbf{t}_a((\mathbf{d}_1^* \cdot \mathbf{d}_2)_a)$ is positive because all $\mathbf{t}_a$ are positive;

2) $q^{\mathrm{cr}(\mathbf{d}_1)+\mathrm{cr}(\mathbf{d}_2)}$ counts the number of crossings inside each of the diagrams, and it is obviously positive;

3) $q^{\mathrm{cr}(\mathbf{d}_1,\mathbf{d}_2)}$ is the number of crossings which appear 'when taking the product' of $\mathbf{d}_1^*$ and $\mathbf{d}_2$ by joining the legs of the same index which point to each other in the two diagrams. The positivity of this kernel is essentially equivalent to the existence of the vacuum representation for an algebras generated by the operators $a_{b,i}$ with $i = 1, \dots, \mathbf{n}(b)$, and satisfying the commutation relations

$$a_{b,i}a_{c,j}^* - q_{a,b}\ a_{c,j}^* a_{b,i} = \delta_{a,b}\delta_{i,j}\mathbf{1},$$

with $q_{a,b} = 1$ if $a = b$ and $q_{a,b} = q$ if $a \neq b$. Such representations have been studied in [12] and also in [33, 34, 55].

We define the operators $j_a : V_{\mathbf{n}} \to V_{\mathbf{n}+\delta_a}$ which have the right intertwining relations with respect to the unitary representations $U_{\mathbf{n}}$ of $\mathrm{S}(\mathbf{n}) := \mathsf{X}_{a\in\mathcal{I}}\mathrm{S}(\mathbf{n}(a))$. This gives the representation of $\mathcal{BP}_2^{\mathcal{I}}(\infty)$ on $\bigoplus_{\mathbf{n}} V_{\mathbf{n}}$. The creation and annihilation operators are defined on the Fock-like space

$$\mathcal{F}_{\mathbf{t}}(\mathcal{H}) := \bigoplus_{\mathbf{n}} \frac{1}{\mathbf{n}!} V_{\mathbf{n}} \otimes_s \bigotimes_{a\in\mathcal{I}} \mathcal{H}^{\otimes \mathbf{n}(a)}$$

with similar notations and definitions as in the usual case.

Some easy consequences of the construction are enumerated below:

1) the vacuum state $*^{(q)}_{a\in\mathcal{I}}\rho_{\mathbf{t}_a}$ on the field algebra generated by the selfadjoint operators $\{\omega_a(f)\}_{a\in\mathcal{I},f\in\mathcal{K}}$ is tracial if and only if all $\rho_{\mathbf{t}_a}$ are tracial.
2) the $q$-product of two functions $\mathbf{t}_1, \mathbf{t}_2$ produces the positive definite function $\mathbf{t}_1 *^{(q)} \mathbf{t}_2$ whose restriction on the algebra generated by sums $\omega_1(f) + \omega_2(f)$ has the form

$$(\mathbf{t}_1 *^{(q)} \mathbf{t}_2)^{(r)}(\mathcal{V}) = \sum_{\mathcal{V}_1\cup\mathcal{V}_2=\mathcal{V}} q^{\mathrm{cr}(\mathcal{V}_1,\mathcal{V}_2)}\mathbf{t}_1(\mathcal{V}_1)\mathbf{t}_2(\mathcal{V}_2).$$

3) the central limit theorem 3.6 states that the properly normalized sums of '$q$-independent' fields having the same distribution $\rho_{\mathbf{t}}$ converges in law to the field $\omega = a + a^*$ in the algebra of $q$-commutation relations $a(f)a^*(g) - qa^*(g)a(f) = \langle f, g\rangle \mathbf{1}$.
4) the $q$-product interpolated between graded tensor product ($q = -1$), reduced free product ($q = 0$) and tensor product ($q = 1$) of the given Brownian motions.

# 7 Open problems

With regard to chapter II one would like to know if it is possible to make a stronger connection between combinatorial tools such as the generating series of the species of structures, and the algebra of creation and annihilation operators.

Chapter III leaves the following questions unanswered:

1) how can one characterize the faithfulness of the state $\rho_{\mathbf{t}}$ directly in terms of the *-semigroup $\mathcal{BP}_2(\infty)$ ?

2) do there exist faithful states $\rho_{\mathbf{t}}$ which are non-tracial ?

3) can the functor $\Delta_{\mathbf{t}}$ be defined for unbounded field operators ? (we suspect that the answer is positive)

4) can a positive definite multiplicative function on the infinite symmetric group be extended to the *-semigroup $\mathcal{BP}_2(\infty)$ in a natural way ?

The analysis of the algebras $\Delta_N$ made in chapter IV shows that if the vacuum state is not faithful then the von Neumann algebra of the fields can contain all bounded operators, and we have type $I$ von Neumann algebras. It would be interesting to see if this is the case in general.

The $q$-product for generalised Brownian motions defined in chapter V points in the direction of a $q$-product of algebras with additional 'pairing' structure. Can this be formulated precisely ?

Finally, the quasi-free representations, the stochastic integration and Lévy processes have not been investigated. In this direction we mention the work of

Anshelevich on Lévy processes on $q$-deformed Fock spaces [2].

CHAPTER II

# Symmetric Hilbert spaces arising from species of structures [1]

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**ABSTRACT**

Symmetric Hilbert spaces such as the bosonic and the fermionic Fock spaces over some 'one particle space' $\mathcal{K}$ are formed by certain symmetrization procedures performed on the full Fock space. We investigate alternative ways of symmetrization by building on Joyal's notion of a combinatorial species. Any such species $F$ gives rise to an endofunctor $\mathcal{F}_F$ of the category of Hilbert spaces with contractions mapping a Hilbert space $\mathcal{K}$ to a symmetric Hilbert space $\mathcal{F}_F(\mathcal{K})$ with the same symmetry as the species $F$. A general framework for annihilation and creation operators on these spaces is developed, and compared to the generalised Brownian motions of R. Speicher and M. Bożejko. As a corollary we find that the commutation relation $a_i a_j^* - a_j^* a_i = f(N)\delta_{ij}$ with $Na_i^* - a_i^* N = a_i^*$ admits a realization on a symmetric Hilbert space whenever $f$ has a power series with infinite radius of convergence and positive coefficients.

[1] This chapter is based on reference [28].

# 1 Introduction

Symmetric Hilbert spaces play a role in physics as the state spaces of many particle systems. The type of particle dictates the type of symmetrization: bosons require complete symmetrization and fermions complete antisymmetrization.
More general ways of symmetrization, although apparently not realized in nature, have been studied for their own sake: parastatistics [49] and interpolations by a parameter $q \in [-1, 1]$ between the above two cases [26, 21, 67, 10, 45].
All these constructions lead to quantum fields or generalized Brownian motions, each with their own generalized Gauss distributions [9, 13, 10, 11, 45]. One particularly important case is $q = 0$: the free Brownian motion, exhibiting the Wigner distribution. This case is related to free independence in the same way as the case $q = 1$ of complete symmetrization is related to ordinary commutative independence.
Although there are results [57] indicating that these two are the only notions of independence, more relaxed conditions such as the weak factorization property [42], or pyramidal independence [13] are satisfied in a variety of examples.
In this paper we study combinatorial ways of symmetrization. Our starting point is the following observation. The category $E$ of finite sets has as its isomorphy classes the natural numbers $\mathbb{N}$, and for each object $U$ in class $n \in \mathbb{N}$ there are $n!$ symmetries. This leads to the Fock space

$$l^2\left(\mathbb{N}, \frac{1}{n!}\right) =: \mathcal{F}_E(\mathbb{C}).$$

Taking for an annihilation operator $a$ the left shift on this space, we find that the field operator $X := a + a^*$ has distribution given by ([53])

$$\langle \delta_\emptyset, X^n \delta_\emptyset \rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^n e^{-\frac{1}{2}x^2}\, dx.$$

On the other hand, if we consider the category $L$ of finite sequences (or linear orderings on a set), we obtain

$$\mathcal{F}_L(\mathbb{C}) = l^2(\mathbb{N})$$

and, since $a^*$ is now the right shift [65],

$$\langle \delta_\emptyset, X^n \delta_\emptyset \rangle = \frac{1}{\pi} \int_{-2}^{2} x^n \sqrt{4 - x^2}\, dx\ .$$

We conclude that the Gauss and Wigner distributions are produced by the concepts of ‘set’ and ‘sequence’. Our program in this paper is to generalize the Fock space construction to such combinatorial concepts as ‘tree’, ‘graph’ and ‘cycle’.
The proper framework for this undertaking turns out to be Joyal’s notion of a *combinatorial species of structures* [35]. These are defined as functors from the

category of finite sets with bijections to the category of finite sets with maps. Combinatorial species of structures can be viewed as coefficients of the Taylor expansion of analytic functors [36] and lead to Joyal's notion of a tensorial species, very close to the our $\mathcal{F}_F(\mathcal{K})$. This circle of ideas is introduced in Sections 2 and 3.

A natural way to introduce an annihilation operator into this context is via the operation of removal of a point from a structure, which is called *differentiation* $F \mapsto F'$ by Joyal. We thus arrive at operators

$$a(k) : \mathcal{F}_F(\mathcal{K}) \to \mathcal{F}_{F'}(\mathcal{K}), \quad a^*(k) : \mathcal{F}_{F'}(\mathcal{K}) \to \mathcal{F}_F(\mathcal{K}), \quad (k \in \mathcal{K})$$

However, these operators can only be added, in order to yield field operators, if the species $F$ and $F'$ are the same. This holds in two cases: the species $E$ of sets and the species $E_\pm$ of oriented sets, related to the Bose and Fermi symmetries. Natural as this may be, we cannot move any further if we do not modify the operation of removal of a point in some way.

Now in fact, already in the case of sequences it is required that *the last* point is removed. In the same way, we may require in the case of trees that only leaves may be picked off (points that leave the tree connected when removed). In the case of cycles we may require that the chain, coming from a broken, cycle must be connected up again. All of this leads to the study of suitable transformations between $F$ and $F'$, which are the subject of Section 4.

Our approach to symmetric Hilbert spaces and field operators provides a tool for creating new examples, and is particularly transparent due to the use of combinatorial objects which are easy to visualize. The two examples of 'q-deformations' appearing in [13] and [10] are cast in the form of combinatorial Fock spaces for the species of ballots and the species of simple directed graphs respectively. In section 5 we point out the connection between this combinatorial approach and the one based on positive definite functions on pair partitions [13]. We describe how the operations between species can be extended to the weights, illustrating this by examples.

## 2 Species of Structures

This section is a brief introduction to the combinatorial theory of species of structures [5, 35], insofar as is needed here.

We are concerned with the different kinds — or 'species' — of structures that can be imposed on a set $U$. The basic idea is that such a species is characterized by the way it transforms under permutations of the set $U$.

It will be convenient to consequently adhere to von Neumann's construction of the natural numbers according to which $0 = \emptyset$ and $n + 1 = n \cup \{n\}$, so that the number $n$ coincides with the set $\{0, 1, 2, \ldots, n - 1\}$.

**Definition.**[5] A *species of structures* is a rule $F$ which
(i) produces for each finite set $U$ a finite set $F[U]$,
(ii) produces for each bijection $\sigma\ : U \to V$ a function $F[\sigma] : F[U] \to F[V]$. The function $F[\sigma]$ should have the following functorial properties:
(a) for all bijections $\sigma\ : U \to V$, $\tau\ : V \to W$, we have $F[\tau \circ \sigma] = F[\tau] \circ F[\sigma]$,
(b) for the identity map $\mathrm{Id}_U : U \to U$, $F[\mathrm{Id}_U] = \mathrm{Id}_{F[U]}$.
The elements of $F[U]$ are called *$F$-structures* on $U$ and the function $F[\sigma]$ describes the transport of $F$-structures along $\sigma$. Note that $F[\sigma]$ is a bijection by the functorial property of $F$.
We denote by $H_s$ the stabilizer $\{\sigma \in \mathrm{S}(U) : F[\sigma](s) = s\}$ of the structure $s \in F[U]$.

**Examples.** 1. The species $E$ of sets is given by

$$\begin{aligned} E[U] &= \{U\}. \\ E[\sigma] &: U \mapsto V \quad \text{if} \quad \sigma : U \to V. \end{aligned}$$

Thus the only $E$-structure over $U$ is the set $U$ itself. The stabilizer of this structure coincides with the whole permutation group $H_s = \mathrm{S}(U)$.

2. The species $L$ of linear orderings:

$$L[U] = \{f : |U| \to U\,;\ f \text{ bijective}\}$$

where $|U| = \{0, 1, 2, \ldots, |U| - 1\}$ is the cardinality of $U$. The transport along the bijection $\sigma : U \to V$ is given by

$$L[\sigma](f) = \sigma \circ f.$$

The stabilizer of each linear ordering is trivial. The cardinality of the set of structures $L[U]$ is equal to that of the permutation group $\mathrm{S}(U)$.

3. The species $\mathcal{C}$ of cyclic permutations:

$$\begin{aligned} \mathcal{C}[U] &= \{\pi \in \mathrm{S}_U \mid \pi^k(u) \neq u \text{ for all } u \in U, k < |U|\}; \\ \mathcal{C}[\sigma] : \pi &\mapsto \sigma \circ \pi \circ \sigma^{-1}\ . \end{aligned}$$

Each structure $\pi \in \mathcal{C}[U]$ has a nontrivial stabilizer $H_\pi = \{\pi^k \mid k < |U|\}$, the number of structures is

$$|\mathcal{C}[U]| = \frac{|U|!}{|H_\sigma|} = (|U| - 1)!$$

**Definition.** A species of structures $F$ is called *molecular* if the permutation group acts transitively on its structures. A molecular species can be characterized by the conjugacy class of the stabilizer of any of its structures. Indeed for $s, t \in F[U]$

and $s = F[\sigma](t)$ we have $H_s = \sigma \circ H_t \circ \sigma^{-1}$. By a well-known combinatorial lemma we have for each structure $s$:

$$|F[U]| \cdot |H_s| = |U|! \quad \text{for} \quad s \in F[U]$$

In general a species of structure may not be molecular, in which case it is a sum of species:

**Definition.**Let $F,\ G$ be species of structures. Then their *sum* $F + G$ is the species defined by the disjoint union

$$(F + G)[U] = F[U] \cup G[U],$$

and the transport along the bijection $\sigma : U \to V$ is given by:

$$(F+G)[\sigma](s) = \begin{cases} F[\sigma](s) & \text{if } s \in F[U] \\ G[\sigma](s) & \text{if } s \in G[U] \end{cases}$$

The *canonical decomposition* of a species $F$ is its decomposition as a sum $F = F_0 + F_1 + F_2 + \cdots$ where $F_n$ denotes the $n - th$ *level* of $F$:

$$F_n[U] = \begin{cases} F[U] & \text{if } |U| = n \\ \emptyset & \text{if } |U| \neq n \end{cases}$$

The simplest species having structures at only one level is the species of singletons $X$:

$$X[U] = \begin{cases} \{U\} & \text{if } |U| = 1 \\ \emptyset & \text{otherwise} \end{cases}$$

Besides addition, there is a number of other operations between species by which to construct new species out of simpler ones. Following a standard notation [5], we use the sum symbol to denote disjoint reunion.

**Definition.**Let $F,\ G$ be two species of structures. We define the *product* species $F \cdot G$ as:

$$(F \cdot G)[U] = \sum_{(U_1, U_2)} F[U_1] \times G[U_2]$$

where the sum runs over all partitions of the set $U$ into disjoint parts $U_1$ and $U_2$. The transport along the bijection $\sigma : U \to V$ of the structure $s = (f, g) \in (F \cdot G)[U]$ is:

$$(F \cdot G)[\sigma](s) = (F[\sigma_1](f), G[\sigma_2](g))$$

where $f \in F[U_1]$ , $g \in G[U_2]$ and $\sigma_1,\ \sigma_2$ are the restrictions of $\sigma$ to the sets $U_1$ and $U_2$ respectively.

The stabilizer of $s = (f, g)$ is $H_{(f,g)} = H_f \cdot H_g \subset \mathrm{S}(U_1) \cdot \mathrm{S}(U_2) \subset \mathrm{S}(U_1 + U_2)$.

As an example let us consider the n-th power of the species $X$ of singletons:

$$X^n[U] = \begin{cases} \{(u_1, .., u_n) | u_i \in U,\ u_i \neq u_j \text{ for } i \neq j\} & \text{if } |U| = n \\ \emptyset & \text{otherwise} \end{cases}$$

It is clear that the species $X^n$ and $L_n$ are essentially the same. Indeed there exists a natural bijection between $X^n[U]$ and $L_n[U]$:

$$(u_1, .., u_n) \mapsto \big(u : n \to U : i \mapsto u_i\big) \ .$$

**Remark.** In the language of category theory, a species of structures $F$ is a *functor* from the category $\mathbb{B}$ of finite sets with bijections to the category $\mathbb{E}$ of finite sets with functions.

**Definition.**A *morphism* from the species of structures $F$ to the species $G$ is a natural transformation of functors, that is a family of functions $m_U : F[U] \to G[U]$ such that:

$$G[\sigma] \circ m_U = m_V \circ F[\sigma] \quad \text{for all} \quad \sigma : U \to V.$$

An *isomorphism* is an invertible morphism.

**Definition.**The *cartesian product* $F \times G$ of two species of structures $F$ and $G$ is given by:

$$\begin{aligned} (F \times G)[U] &= F[U] \times G[U] \\ (F \times G)[\sigma](f, g) &= (F[\sigma](f), G[\sigma](g)) \end{aligned} \tag{2.1}$$

The canonical decomposition of the cartesian product is:

$$F \times G = \sum_{n=0}^{\infty} F_n \times G_n. \tag{2.2}$$

A structure $(f, g) \in (F \times G)[U]$ has the stabilizer $H_{(f,g)} = H_f \cap H_g \subset \mathrm{S}(U)$.

An operation which will play an important role later is the derivation.

**Definition.**The *derivative* $F'$ of a species $F$ is a species whose set of structures over a finite set $U$ is given by:

$$F'[U] = F[U \cup \{U\}]$$

and $F'[\sigma](s) = F[\sigma^+](s)$ where $\sigma^+ : U \cup \{U\} \to V \cup \{V\}$ is the extension of $\sigma : U \to V$:

$$\sigma^+(x) = \begin{cases} \sigma(x) & \text{if } x \in U, \\ V & \text{if } x = U. \end{cases}$$

**Remark.** The term $\{U\}$ in $U \cup \{U\}$ is just any additional point, not belonging to $U$. In particular for $U = n$ we have $U \cup \{U\} = n + 1$. If no confusion arises, we may write $U \cup \{U\}$ as $U + \{*\}$. The transport along bijections is the one inherited from the species $F$ but it is restricted to those transformations that keep the point $*$ fixed. The stabilizer of a structure $s$ when considered as a $F'$−structure is different from its stabilizer as a $F$−structure:

$$s \in F'[U] = F[U + \{*\}] \ \Rightarrow \ H_s^{F'} = H_s^F \cap \mathrm{S}(U).$$

As explained in the introduction, we wish to compare successive levels of a species, i.e. to compare $F$ with $F'$. In this direction there is a small

**Lemma 2.1** *There are only two species (up to multiplicity) which satisfy $F = F'$.*

*Proof.* Clearly, the species $F$ must have the same number of structures at all levels. For $s \in F[n]$, the stabilizer $H_s$ satisfies $|H_s| \geq \frac{n!}{|F[n]|}$ which for $n$ big enough, reduces the possibilities to either the whole symmetric group $\mathrm{S}(n)$ or the subgroup $A(n)$ of even permutations. In the first case we obtain the species $E$ of sets which has only one structure at each level, in the second we have the species $E^{\pm}$ of *oriented sets* with exactly two structures at each level

$$E^{\pm}[U] = \{U_+,\ U_-\}$$

the stabilizer of each structure being $H_{U_\pm} = A(U)$.

□

Besides these two ideal cases, we are interested in species $F$ whose structure at successive levels 'resemble' each other. That means that $F_n[U]$ and $F_{n+1}[U + \{*\}]$ should contain structures that behave similarly under permutations of $U$. Suppose that we are given a morphism $m$ from a subspecies $F_1$ of $F'$ to $F$ ($F' = F_1 + F_2$). Then the $F$-structures which belong to the image of this morphism are similar to their preimages in the sense that their stabilizers contain those of their preimages. The action of the morphism $m$ can be encoded in a weight on the species $F \times F'$.

**Definition.** A *weighted species* $(F, \mathbf{j})$ consists of

1. a species of structures $F$
2. a family of functions $\mathbf{j}_U : F[U] \to \mathbb{C}$ called *weights*,

such that for a bijection $\sigma : U \to V$ one has $\mathbf{j}_V \circ F[\sigma] = \mathbf{j}_U$.
The weight $\mathbf{j}_m$ associated to the morphism $m : F_1 \to F$ is the indicator function of its graph:

$$\mathbf{j}_{m,U}(f,g) = \begin{cases} \delta_{f,m(g)} & \text{if } g \in F_1[U]\ ,\ f \in F[U], \\ 0 & \text{if } g \notin F_1[U]. \end{cases}$$

One of the most interesting operations between species is the composition.

**Definition.** Let $F$ and $G$ two species of structures such that $G[\emptyset] = \emptyset$. The *composition* $F \circ G$ is a species whose structures on a set $U$ are made in the following way:

1. make a partition $\pi$ of the set $U$;
2. choose an $F$-structure over the set $\pi$: $f \in F[\pi]$;
3. for each $p \in \pi$ choose a structure $g_p \in G[p]$. Then the triple $(\pi, f, (g_p)_{p\in\pi})$ is a structure in $F \circ G[U]$. The transport along $\sigma : U \to V$ is the natural one.

In brief, an $F \circ G$ structure is an $F$-assembly of $G$-structures. As an example consider the following combinatorial equation.

$$\mathcal{A} = X \cdot E(\mathcal{A}) \tag{2.3}$$

This equation implicitly defines the species $\mathcal{A}$ of rooted trees. Here is an explicit definition:

$$\begin{array}{rcl} \mathcal{A}[U] & = & \{f : U \to U \mid \forall_{u\in U} : f^{\circ k}(u) \text{ is eventually constant}\}; \\ \mathcal{A}[\sigma] & : & f \mapsto \sigma \circ f \circ \sigma^{-1}. \end{array}$$

The constant is the root of the tree. The preimage of the root consists of roots of subtrees. One can thus consider the tree $f$ as the pair $(\mathrm{root}(f)\ ,\ \{f_a \mid a \in f^{-1}(\mathrm{root}(f))\})$ with $f_a \in \mathcal{A}[U_a]$ the subtree of $f$ with root $a$:

$$U_a = \{u \in U \mid \exists k \in \mathbb{N} \text{ such that } f^{\circ k} = a\} \quad , \quad f_a = f \restriction_{U_a}$$

We finally note

$$U = \{\mathrm{root}(f)\} \cup \bigcup_{a\in f^{-1}(\mathrm{root}(f))} U_a$$

thus completing the bijection between $\mathcal{A}[U]$ and $X \cdot E(\mathcal{A})[U]$.

# 3 Fock Spaces and Analytic Functors

In this section we will describe how one can associate to a species of structures an endofunctor of the category of Hilbert spaces with contractions. We call the images of this functor symmetric spaces associated to the species $F$ and as we shall see in the following sections, they are suitable for constructing algebras of creation and annihilation operators, by exploiting the symmetry properties of the species $F$.

Following Joyal [36] we define a special class of endofunctors of the category of sets with maps.

**Definition.**Let $F[\cdot]$ be a species of structures. The *analytic functor* $F(\cdot)$ is an endofunctor of the category **Set** of sets with maps, defined by:

$$F(J) = \tilde{\sum_U} F[U] \times J^U \tag{3.1}$$

where $J^U = \{c | c : U \to J\}$ and the symbol $\tilde{\sum}_U$ means the set of equivalence classes under bijective transformations:

$$F[U] \times J^U \ni (s, c) \mapsto (F[\sigma](s), c \circ \sigma^{-1}) \in F[V] \times J^V$$

for $\sigma : U \to V$. We call the elements of $J$ 'colors'. Thus, an element in $F(J)$ is an orbit of J-colored F-structures denoted by $[s, c]$. Alternatively

$$F(J) = \tilde{\sum_U} F[U] \times J^U = \sum_{n=0}^{\infty} F[n] \times J^n \ / \ \mathrm{S}(n).$$

**Remark.** This relation can be viewed as a Taylor expansion of the set $F(J)$, which explains the name 'analytic functor' for $F(\cdot)$ [36].

Parallel to the functor $F(\cdot)$ we define another endofunctor, this time on the category **Hilb** of Hilbert spaces with contractions. For any Hilbert space $\mathcal{H}$ and a finite set $U$ we denote by $\mathcal{H}^{\otimes U}$ the Hilbert space arising from the positive definite kernel on $\mathcal{H}^U$ given by

$$k\left(\bigotimes_{u \in U} \psi_u, \bigotimes_{u' \in U} \varphi_{u'}\right) = \prod_{u \in U} \langle \psi_u, \varphi_u \rangle .$$

For every bijection $\sigma : U \to V$ there is a unitary transformation $U(\sigma) : \mathcal{H}^{\otimes U} \to \mathcal{H}^{\otimes V}$ obtained by linear extension of:

$$U(\sigma) : \bigotimes_{u \in U} \psi_u \to \bigotimes_{v \in V} \psi_{\sigma^{-1}(v)}.$$

**Definition.**Let $F$ be a species of structures. For each Hilbert space $\mathcal{K}$ we construct the *symmetric Hilbert space*

$$\mathcal{F}_F(\mathcal{K}) := \bigoplus_{n=0}^{\infty} \frac{1}{n!} \ \ell^2_{\mathrm{sym}}(F[n] \to \mathcal{K}^{\otimes n}) \tag{3.2}$$

where the subscript 'sym' denotes the invariance under the natural action of the symmetric group $\mathrm{S}(n)$:

$$\Psi \mapsto U(\sigma)\Psi \circ F[\sigma^{-1}].$$

The factor $\frac{1}{n!}$ refers to the inner product on $\ell^2_{\mathrm{sym}}$.

**Remark.** There is an equivalent way of writing $\mathcal{F}_F(\mathcal{K})$:

$$\mathcal{F}_F(\mathcal{K}) = \bigoplus_{n=0}^{\infty} \frac{1}{n!} \ \ell^2(F[n]) \otimes_{\mathrm{S}(n)} \mathcal{K}^{\otimes n} \tag{3.3}$$

where $\otimes_{S(n)}$ means that we consider only the subspace of the tensor product whose vectors are invariant under the action of $S(n)$. If $T$ is a contraction on $\mathcal{K}$ then $\mathcal{F}_F(T)$ is defined by:

$$(\mathcal{F}_F(T)\Psi)(s) = (\underbrace{T \otimes T \otimes \cdots \otimes T}_{\text{n times}})(\Psi(s)). \tag{3.4}$$

for $s \in F[n]$. Note that $\mathcal{F}_F(T)$ is a well defined contraction on $\mathcal{F}_F(\mathcal{K})$ and $\mathcal{F}_F(T_1) \cdot \mathcal{F}_F(T_2) = \mathcal{F}_F(T_1 T_2)$, thus $\mathcal{F}_F$ is a functor from the category of Hilbert spaces with contractions to itself.

Let us choose an orthonormal basis $(e_j)_{j \in J}$ for the Hilbert space $\mathcal{K}$. Let $(e_c)_{c \in J^n}$ be the basis of $\mathcal{K}^{\otimes n}$ given by $e_c := \otimes_{j \in n} e_{c(j)}$, and

$$\begin{array}{rcl} \gamma_{F,J} : F(J) & \to & [0, \infty) \\ \gamma_{F,J}([s, c]) & = & |H_{(s,c)}| \end{array}$$

where $[s, c]$ denotes the orbit of the colored structure $(s, c)$.

**Lemma 3.1** *There is a unitary equivalence between $\ell^2(F(J), \gamma_{F,J})$ and $\mathcal{F}_F(\mathcal{K})$, given by*

$$U\delta_{[s,c]} = \delta_s \otimes_{S(n)} e_c := \sum_{\sigma \in S(n)} \delta_{F[\sigma](s)} \otimes e_{c \circ \sigma^{-1}}$$

*Proof.* Considering $\mathcal{K}^{\otimes n}$ as $\ell^2(J^n)$ we may write

$$\begin{aligned} U\delta_{[s,c]} &= \sum_{\sigma \in S(n)} \delta_{F[\sigma](s)} \otimes e_{c \circ \sigma^{-1}} \\ &= \sum_{\sigma \in S(n)} \delta_{F[\sigma](s), c \circ \sigma^{-1}} = |H_{(s,c)}| \cdot \mathbf{1}_{[s,c]} \end{aligned}$$

It follows that

$$\begin{array}{rcl} \|U\delta_{[s,c]}\|^2 & = & \frac{1}{n!} |H_{(s,c)}|^2 \cdot |[s, c]| = |H_{(s,c)}| \\ & = & \|\delta_{[s,c]}\|^2. \end{array}$$

Since the functions $\mathbf{1}_{[s,c]}$ span the space $\ell^2_{\text{symm}}(F[n] \times J^n)$, the operator $U$ is surjective and hence unitary.

□

**Remark.** For a constant coloring $c$ we have $\|\delta_{[s,c]}\|^2 = |H_s|$, whereas for all colors different, $\|\delta_{[s,c]}\|^2 = 1$.

Certain operations with species of structures extend to analytic species [36] and to the symmetric spaces: addition, multiplication and substitution.

**1) Addition.** As $(F+G)(J)$ is the disjoint union of $F(J)$ and $G(J)$, we have

$$\mathcal{F}_{F+G}(\mathcal{K}) = \mathcal{F}_F(\mathcal{K}) \oplus \mathcal{F}_G(\mathcal{K}).$$

**2) Multiplication.** Similarly, we have

$$\begin{aligned}(F\cdot G)(J) &= \tilde{\sum_U}(\sum_{U_1+U_2=U} F[U_1]\times G[U_2])\times J^U \\ &= \tilde{\sum_U}\sum_{U_1+U_2=U} F[U_1]\times G[U_2]\times J^{U_1+U_2} \\ &= \tilde{\sum_{U_1,U_2}} F[U_1]\times G[U_2]\times J^{U_1+U_2} = F(J)\times G(J).\end{aligned}$$

which suggests the following unitary transformation from $\mathcal{F}_{F\cdot G}(\mathcal{K})$ to $\mathcal{F}_F(\mathcal{K})\otimes\mathcal{F}_G(\mathcal{K})$:

$$T : \delta_{[(f,g),c]} \to \delta_{[f,c_1]}\otimes\delta_{[g,c_2]}$$

for $f\in F[n]$, $g\in G[m]$, $c\in J^{m+n}$ and $c_1, c_2$ the restrictions of $c$ to $n$ respectively $m$. Indeed the map preserves orthogonality and is isometric:

$$\begin{aligned}\|\delta_{[(f,g),c]}\|^2 &= |H_{((f,g),c)}| = |H_{(f,c_1)}\cdot H_{(g,c_2)}| = |H_{(f,c_1)}|\cdot|H_{(g,c_2)}| \\ &= \|\delta_{[f,c_1]}\|^2\cdot\|\delta_{[g,c_2]}\|^2\end{aligned}$$

From now on we will consider $\mathcal{F}_{F\cdot G}(\mathcal{K})$ and $\mathcal{F}_F(\mathcal{K})\otimes\mathcal{F}_G(\mathcal{K})$ as identical, without mentioning the unitary $T$.

**3) Substitution.** we start with the analytic functors:

$$\begin{aligned}(F\circ G)(J) &= \tilde{\sum_U}(F\circ G)[U]\times J^U \\ =\tilde{\sum_U}(\tilde{\sum_\pi}F[\pi]\times G^\pi[U])\times J^U &= \tilde{\sum_\pi}(F[\pi]\times\tilde{\sum_U}G^\pi[U])\times J^U \\ =\tilde{\sum_\pi}F[\pi]\times G^\pi(J) &= F(G(J))\end{aligned}$$

where we have used $G^\pi(J) = G(J)^\pi$, which follows from the multiplication property. At the level of symmetric spaces we have the unitary transformation from $\mathcal{F}_F(\mathcal{F}_G(\mathcal{K}))$ to $\mathcal{F}_{F\circ G}(\mathcal{K})$:

$$T : \delta_{[f,C]} \to \delta_{[f,(g_a)_{a\in\pi},c]}$$

with the following relations for the structures appearing above: $f\in F[\pi]$, $C : \pi\to G(J)$ such that $C(a) = [g_a, c_a]$, and $c\restriction_a = c_a$. Let us check the isometric property:

$$\begin{aligned}\|\delta_{[f,C]}\|^2 &= \prod_{a\in\pi}\|\delta_{C(a)}\|^2\cdot|H_{(f,C)}| = \prod_{a\in\pi}\|\delta_{[g_a,c_a]}\|^2\cdot|H_{f,C}| \\ &= \prod_{a\in\pi}|H_{(g_a,c_a)}|\cdot|H_{f,C}| = \|\delta_{[f,(g_a)_{a\in\pi},c]}\|^2\end{aligned}$$

**Symmetric Fock space.** The symmetric Hilbert space associated to the species of sets is the well known symmetric Fock space:

$$\mathcal{F}_E(\mathcal{K}) = \bigoplus_{n=0}^{\infty}\frac{1}{n!}\ell^2_{\mathrm{sym}}(E[n]\to\mathcal{K}^{\otimes n}) = \bigoplus_{n=0}^{\infty}\frac{1}{n!}\mathcal{K}^{\otimes_s n}.$$

**Full Fock space.** For the linear orders we obtain the full Fock space:

$$\mathcal{F}_L(\mathcal{K}) = \bigoplus_{n=0}^{\infty}\frac{1}{n!}\ell^2_{\mathrm{sym}}(L[n]\to\mathcal{K}^{\otimes n}) = \bigoplus_{n=0}^{\infty}\mathcal{K}^{\otimes n}.$$

**Antisymmetric Fock space.** We recall from lemma 2.1 that the species $E^{\pm}$ of oriented sets has two structures at all levels

$$E^{\pm}[U] = \{U_+,\ U_-\}$$

which are mapped into each other by odd permutations and have as stabilizer the group $A(U)$ of even permutation. The representation of $\mathrm{S}(n)$ on $\ell^2(E^{\pm}[n])$ contains two one-dimensional irreducible sub-representations, the symmetric and the antisymmetric representation. Accordingly the symmetric Hilbert space associated to $E^{\pm}$ is the direct sum of the symmetric and antisymmetric Fock spaces:

$$\mathcal{F}_{E^{\pm}}(\mathcal{K}) = \mathcal{F}_s(\mathcal{K})\oplus\mathcal{F}_a(\mathcal{K}).$$

**Remark.** The set of species as defined in the previous section can be enlarged by defining [5] the *virtual species* as equivalence classes of pair of species of structures under the equivalence relation:

$$(F_1,G_1)\sim(F_2,G_2)\Leftrightarrow F_1+G_2=F_2+G_1$$

One denotes the equivalence class of $(F,G)$ by $F-G$. Thus we can say that the antisymmetric Fock space is associated to the virtual species $E^{\pm}-E$.

# 4 Creation and Annihilation Operators

In this section we will describe a general framework for constructing *-algebras of operators on symmetric Hilbert spaces by giving the action of the generators

of these algebras, the creation and annihilation operators. In particular in the case of the species of sets $E$ and linear orderings $L$, we obtain the well known canonical commutation relations algebra (C.C.R.), respectively the algebra of creation/annihilation operators on the full Fock space.
The starting point is the observation that the operation of derivation of species of structures can be interpreted as removal of point $*$ from a structure. This makes it possible to define operators between the symmetric Hilbert spaces of a species of structure $F$ and its derivative $F'$.

We will consider now 'colored' $F$-structures. Let $J$ be the set of 'colors' and $i \in J$. We have the map

$$a^*(i) : F'[U] \times J^U \to F[U + \{*\}] \times J^{U+\{*\}}$$

such that

$$a^*(i) : (s, c) \to (s, c_i^+)$$

where $c_i^+ : U + \{*\} \to J$ is given by:

$$c_i^+(u) = \begin{cases} c(u) & \text{if } u \in U \\ i & \text{if } u = * \end{cases}$$

As we did in section 3, we pass to the set of orbits of $J$-colored $F$-structures. The map $a^*(i)$ projects to a well defined map from $F'(J)$ to $F(J)$:

$$\begin{array}{rcl} a^*(i) : F'[U] \times J^U/\mathrm{S}(U) & \to & F[U + \{*\}] \times J^{U+\{*\}}/\mathrm{S}(U + \{*\}) \\ a^*(i) : [s, c] & \to & [s, c_i^+] \end{array}$$

But as $F(J)$ determines an orthogonal basis of the space $\mathcal{F}_F(\mathcal{K})$ for $(e_j)_{j\in J}$ orthogonal basis in $\mathcal{K}$, we can extend $a^*(i)$ by linearity to an operator

$$a^*(i) : \mathcal{F}_{F'}(\mathcal{K}) \to \mathcal{F}_F(\mathcal{K}).$$

The adjoint of $a^*(i)$ acts in the opposite direction:

$$a(i) : \mathcal{F}_F(\mathcal{K}) \to \mathcal{F}_{F'}(\mathcal{K}).$$

The problem with this definition is that in general the species $F$ and $F'$ are distinct which means that one cannot take the 'field operators' $a^*(i) + a(i)$ and only certain products of creation and annihilation operators are well defined. In section 2 we pointed out that the 'similarity' of the structures of the species $F$ and $F'$ can be encoded in a weight on the cartesian product $F \times F'$. Let $\mathbf{j}$ be such a weight. Then $\mathbf{j}_U : F[U] \times F'[U] \to \mathbb{C}$ such that for all $s \in F[U],\ t \in F'[U]$ and $\sigma : U \to W$ we have:

$$\mathbf{j}_U(s, t) = \mathbf{j}_W(F[\sigma](s), F'[\sigma](t))$$

We will use this to define creation and annihilation operators which act on the same space $\mathcal{F}_F(\mathcal{K})$. In the sequel we will refer to the pair $(\mathcal{F}_F(\mathcal{K})\ ,\ \mathbf{j})$ as *combinatorial Fock space.* In order to simplify the notation we will ignore the subscript $U$ in $\mathbf{j}_U$ when no confusion can arise.

**Definition.**a) The *annihilation operator* (before symmetrization) associated to the species $F$ and weight $\mathbf{j}$ is defined by:

$$\tilde{a}(h) : \bigoplus_{n=0}^{\infty} \frac{1}{n!}\ell^2(F[n] \to \mathcal{K}^{\otimes n}) \to \bigoplus_{n=0}^{\infty} \frac{1}{n!}\ell^2(F[n] \to \mathcal{K}^{\otimes n})$$

$$(\tilde{a}(h)\varphi)(f) = \sum_{g \in F[n+1]} \mathbf{j}(f,g) \cdot \mathrm{inp}_n(h, \varphi(g))$$

where $f \in F[n]\ ,\ h \in \mathcal{K}$ and $\mathrm{inp}_k(h,\cdot)$ is the operator:

$$\mathrm{inp}_k(h, \psi_0 \otimes .. \otimes \psi_n) = \langle h, \psi_k \rangle\, \psi_0 \otimes .. \otimes \psi_{k-1} \otimes \psi_{k+1} \otimes .. \otimes \psi_n$$

for $k \in \{0, 1, .., n\}$.
b) The *creation operator* (before symmetrization) is:

$$\tilde{a}^*(h) : \bigoplus_{n=0}^{\infty} \frac{1}{n!}\ell^2(F[n] \to \mathcal{K}^{\otimes n}) \to \bigoplus_{n=0}^{\infty} \frac{1}{n!}\ell^2(F[n] \to \mathcal{K}^{\otimes n})$$

$$(\tilde{a}^*(h)\varphi)(f) = (n+1) \cdot \sum_{g \in F[n]} \overline{\mathbf{j}(g,f)} \cdot \mathrm{tens}_n(h, \varphi(g))$$

where $f \in F[n+1]\ ,\ h \in \mathcal{K}$ and $\mathrm{tens}_k(h,\cdot)$ is the operator:

$$\mathrm{tens}_k(h, \psi_0 \otimes .. \otimes \psi_{n-1}) = \psi_0 \otimes .. \otimes \psi_{k-1} \otimes h \otimes \psi_k \otimes .. \otimes \psi_{n-1}$$

for $k \in \{0, 1, .., n\}$.
**Remark.** In order to avoid domain problems for $\tilde{a}, \tilde{a}^*$, we will restrict to weights which are bounded, $|\omega(t,s)| \leq C$ for all $t, s$. Then

$$\|a(h)\psi_n\| \leq n^{\frac{1}{2}} C \|h\| \|\psi_n\| \qquad \|a^*(h)\psi_n\| \leq (n+1)^{\frac{1}{2}} C \|h\| \|\psi_n\|$$

for $\psi_n \in \ell^2(F[n] \to \mathcal{K}^{\otimes n})$ thus $a(h), a^*(h)$ have well defined extensions to the domain $D(N^{\frac{1}{2}})$ $(N\psi_n = n\psi_n)$. As this will not play a major role here, we will omit specifying the domain, usually the the vectors considered should belong to $D(N^{\frac{1}{2}})$.

We consider now the symmetrized creation and annihilation operators which act on the symmetric Hilbert space and which are the main object of our investigation.

**Lemma 4.1** *The unsymmetrized annihilation operator $\tilde{a}(h)$ restricts to a well defined operator $a(h)$ on the symmetric Hilbert space $\mathcal{F}_F(\mathcal{K})$:*

*Proof.* Let $\varphi \in \mathcal{F}_F(\mathcal{K})$. Then $\varphi(F[\sigma](f)) = U(\sigma)\varphi(f)$ for all $\sigma \in \mathrm{S}(n)$, $f \in F[n]$ and

$$\begin{aligned}
(a(h)\varphi)(F[\sigma](f)) &= \sum_g \mathbf{j}(F[\sigma](f), g) \cdot \mathrm{inp}_n(h, \varphi(g)) \\
= \sum_{g'} \mathbf{j}(f, g') \cdot \mathrm{inp}_n(h, \varphi(F[\tilde{\sigma}]g')) &= \sum_{g'} \mathbf{j}(f, g') \cdot U(\sigma)\mathrm{inp}_n(h, \varphi(g')) \\
&= U(\sigma)(a(h)\varphi)(f)
\end{aligned}$$

where $\tilde{\sigma} : n+1 \to n+1$ is given by

$$\tilde{\sigma}(i) = \begin{cases} \sigma(i) & \text{if } i \in n \\ n & \text{if } i = n \end{cases}$$

□

**Lemma 4.2** *The operator $\tilde{a}^*(h)$ is the adjoint of $\tilde{a}(h)$ on the unsymmetrized space $\bigoplus_{n=0}^{\infty} \frac{1}{n!}\ell^2(F[n] \to \mathcal{K}^{\otimes n})$.*

*Proof.* Let $\varphi, \psi$ be two vectors in $\bigoplus_{n=0}^{\infty} \frac{1}{n!}\ell^2(F[n] \to \mathcal{K}^{\otimes n})$ and $\varphi_n, \psi_n \in \frac{1}{n!}\ell^2(F[n] \to \mathcal{K}^{\otimes n})$ their components on level $n$.
Then we have:

$$\begin{aligned}
\langle \psi, \tilde{a}^*(h)\varphi\rangle &= \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \sum_g (n+1) \left\langle \psi_{n+1}(g), (\tilde{a}^*(h)\varphi_n)(g)\right\rangle \\
= & \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{f,g} \left\langle \psi_{n+1}(g), \overline{\mathbf{j}(f,g)} \cdot \varphi_n(f) \otimes h \right\rangle \\
= & \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{f,g} \left\langle \mathbf{j}(f,g) \cdot \mathrm{inp}_n(h, \psi_{n+1}(g)), \varphi_n(f)\right\rangle \\
= & \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{f} \left\langle (\tilde{a}(h)\psi_{n+1})(f), \varphi_n(f)\right\rangle = \langle \tilde{a}(h)\psi, \varphi\rangle
\end{aligned}$$

□

We will restrict our attention to the action of the creation and annihilation operators on the symmetric Hilbert space $\mathcal{F}_F(\mathcal{K})$. From Lemma 4.1 the annihilation operator $a(h)$ is well defined on $\mathcal{F}_F(\mathcal{K})$. We call its adjoint on the symmetric Hilbert space, the symmetrized creation operator. If $P$ is the projection to $\mathcal{F}_F(\mathcal{K})$ then the symmetrized creation operator is:

$$a^*(h)\varphi = P\tilde{a}^*(h)\varphi \text{ for } \varphi \in \mathcal{F}_F(\mathcal{K})$$

**Lemma 4.3** *Let $f \in F[n+1]$ and $\tau_{n,k} \in \mathrm{S}(n+1)$ the transposition of $n$ and $k$. Then for any $\varphi \in \mathcal{F}_F(\mathcal{K})$ the action of the symmetrized creation operator has the expression:*

$$(a^*(h)\varphi)(f) = \sum_{k=0}^{n} \sum_{g} \overline{\mathbf{j}(g, F[\tau_{n,k}](f))} \cdot U(\tau_{n,k})(\varphi(g) \otimes h).$$

*Proof.* We have:

$$(P\tilde{a}^*(h)\varphi)(f) = \frac{1}{(n+1)!} \sum_{\sigma \in \mathrm{S}(n+1)} U(\sigma)(\tilde{a}^*(h)\varphi)(F[\sigma^{-1}]f)$$

$$= \frac{1}{n!} \sum_{\sigma \in \mathrm{S}(n+1)} \sum_{g} \overline{\mathbf{j}(g, F[\sigma](f))} \cdot U(\sigma^{-1})(\varphi(g) \otimes h) \tag{4.1}$$

If $\sigma \in \mathrm{S}(n+1)$ and $\sigma^{-1}(n) = k$ then $\rho = \tau_{n,k} \circ \sigma^{-1} \in \mathrm{S}(n)$. Thus the sum over all permutations can be split into a sum over $k \in n+1$ and one over $\mathrm{S}(n)$. Moreover, from the definitions of $\mathcal{F}_F(\mathcal{K})$ and that of a weight we know that

$$\begin{aligned} U(\rho)\varphi(g) &= \varphi(F[\rho](g)) \\ \mathbf{j}(g, F[\rho^{-1} \circ \tau_{n,k}](f)) &= \mathbf{j}(F[\rho](g), F[\tau_{n,k}](f)) \end{aligned}$$

which substituted into the sum (4.1) gives:

$$\frac{1}{n!} \sum_{k=0}^{n} \sum_{\rho \in \mathrm{S}(n)} \sum_{g} \overline{\mathbf{j}(F[\rho](g), F[\tau_{n,k}](f))} \cdot U(\tau_{n,k})(\varphi(F[\rho](g)) \otimes h)$$

$$= \sum_{k=0}^{n} \sum_{g'} \overline{\mathbf{j}(g', F[\tau_{n,k}](f))} \cdot U(\tau_{n,k})(\varphi(g') \otimes h).$$

□

Sometimes algebras are defined by giving relations among generators as for example commutation relations. We will give next explicit formulas for the product of a creation and an annihilation operator.

**Lemma 4.4** *Let $f \in F[n]$ and $\varphi \in \mathcal{F}_F(\mathcal{K})$. Then*

$$(a^*(h_1)a(h_2)\varphi)(f) = \sum_{k=0}^{n-1} \sum_{f'} (\bar{\mathbf{j}} \cdot \mathbf{j})_k(f, f') \cdot tens_k(h_1, inp_k(h_2, \varphi(f'))) \tag{4.2}$$

*where we have made the notation*

$$(\bar{\mathbf{j}} \cdot \mathbf{j})_k(f, f') = \sum_{g} \overline{\mathbf{j}(g, F[\tau_{n-1,k}](f))} \mathbf{j}(g, F[\tau_{n-1,k}](f')) \tag{4.3}$$

*Proof.* By applying successively the definitions of $a^*(h_1)$ and $a(h_2)$ we have:

$$
\begin{aligned}
&(a^*(h_1)a(h_2)\varphi)(f) = \\
&= \sum_{k=0}^{n-1}\sum_{g} \overline{\mathbf{j}(g, F[\tau_{n-1,k}](f))} \cdot U(\tau_{n-1,k})(a(h_2)\varphi)(g) \otimes h_1 \\
&= \sum_{k=0}^{n-1}\sum_{g,f'} \overline{\mathbf{j}(g, F[\tau_{n-1,k}](f))}\mathbf{j}(g, f') \cdot U(\tau_{n-1,k})(\mathrm{inp}_{n-1}(h_2, \varphi(f')) \otimes h_1) \\
&= \sum_{k=0}^{n-1}\sum_{g,f'} \overline{\mathbf{j}(g, F[\tau_{n-1,k}](f))}\mathbf{j}(g, F[\tau_{n-1,k}]f') \cdot \mathrm{tens}_k(h_1, \mathrm{inp}_k(h_2, \varphi(f'))) \\
&= \sum_{k=0}^{n-1}\sum_{f'} (\bar{\mathbf{j}} \cdot \mathbf{j})_k(f, f') \cdot \mathrm{tens}_k(h_1, \mathrm{inp}_k(h_2, \varphi(f'))).
\end{aligned}
$$

□

**Lemma 4.5** *Let $f \in F[n]$ and $\varphi \in \mathcal{F}_F(\mathcal{K})$. Then:*

$$
\begin{aligned}
(a(h_1)a^*(h_2)\varphi)(f) &= \sum_{k=0}^{n-1}\sum_{f'} (\mathbf{j} \cdot \bar{\mathbf{j}})_k(f, f') \cdot \mathrm{tens}_k(h_2, \mathrm{inp}_k(h_1, \varphi(f'))) \\
&+ \langle h_1, h_2 \rangle \cdot \sum_{f'} (\mathbf{j} \cdot \bar{\mathbf{j}})_n(f, f')\varphi(f')
\end{aligned}
$$

*where we have made the notation*

$$
(\mathbf{j} \cdot \bar{\mathbf{j}})_k(f, f') = \sum_{g} \mathbf{j}(f, g)\overline{\mathbf{j}(f', F[\tau_{n,k}](g))} \tag{4.4}
$$

*Proof.* We use the definitions of $a(h_1)$ and $a^*(h_2)$:

$$
\begin{aligned}
& (a(h_1)a^*(h_2)\varphi)(f) = \sum_{g} \mathbf{j}(f, g) \cdot \mathrm{inp}_n(h_1, (a^*(h_2)\varphi)(g)) \\
= \;& \sum_{g,f'}\sum_{k=0}^{n} \mathbf{j}(f, g)\overline{\mathbf{j}(f', F[\tau_{n,k}](g))} \cdot \mathrm{inp}_n(h_1, U(\tau_{n,k})(\varphi(f') \otimes h_2)) \\
= \;& \sum_{k=0}^{n-1}\sum_{f'} (\mathbf{j} \cdot \bar{\mathbf{j}})_k(f, f') \cdot \mathrm{tens}_k(h_2, \mathrm{inp}_k(h_1, \varphi(f'))) \\
+ \;& \langle h_1, h_2 \rangle \cdot \sum_{f'} (\mathbf{j} \cdot \bar{\mathbf{j}})_n(f, f')\varphi(f')
\end{aligned}
$$

□

## 4.1 Examples

We will describe a few known operator algebras in the language developed so far and a new algebra based on the species $\mathcal{A}$ of rooted trees.

**1) Sets:** The combinatorial Fock space is $(E, \mathbf{j}_E)$ with $E[U] = \{U\}$ and $\mathbf{j}(\{U\}, \{U + \{*\}\}) = 1$. We use lemmas 4.4 and 4.5 to calculate the commutator of the creation and annihilation operator:

$$\begin{aligned} & (a(h_1)a^*(h_2) - a^*(h_2)a(h_1))\varphi(f) = \langle h_1, h_2\rangle \cdot (\mathbf{j} \cdot \bar{\mathbf{j}})_n(f, f')\varphi(f') \\ + \quad & \sum_{k=0}^{n-1}((\mathbf{j} \cdot \bar{\mathbf{j}})_k(f, f') - (\bar{\mathbf{j}} \cdot \mathbf{j})_k(f, f')) \cdot \mathrm{tens}_k(h_2, \mathrm{inp}_k(h_1, \varphi(f'))) \end{aligned}$$

But $(\mathbf{j} \cdot \bar{\mathbf{j}})_k(f, f') = (\bar{\mathbf{j}} \cdot \mathbf{j})_k(f, f') = (\mathbf{j} \cdot \bar{\mathbf{j}})_n(f, f') = \delta_{f,f'}$ for all $k \in n$ which implies the C.C.R.:

$$a(h_1)a^*(h_2) - a^*(h_2)a(h_1) = \langle h_1, h_2\rangle\, \mathbf{1}$$

In particular it is clear that $\mathcal{F}_E(\mathcal{K})$ is the symmetric Fock space over the Hilbert space $\mathcal{K}$.

**2) Linear Orders:** Let $(L, \mathbf{j}_L)$ be the combinatorial Fock space with

$$L[U] = \{f : U \to \{0, 1, .., |U| - 1\}\}$$

and

$$\mathbf{j}_L(f, g) = \delta_{f,g\restriction_U} \text{ for } f \in L[U],\ g \in L[U + \{*\}]$$

where

$$\delta_{f,g\restriction_U} = \begin{cases} 1 & \text{if } f(u) = g(u) \text{ for } u \in U \\ 0 & \text{otherwise} \end{cases}$$

From (4.4) we have:

$$(\mathbf{j} \cdot \bar{\mathbf{j}})_k(f, f') = \sum_g \delta_{f,g\restriction_U} \cdot \delta_{f',L[\tau_{n,k}](g)\restriction_U} = \delta_{k,n} \cdot \delta_{f,f'}$$

Then by applying Lemma 4.5 we obtain

$$a(h_1)a^*(h_2) = \langle h_1, h_2\rangle\, \mathbf{1}$$

which characterizes the algebra of creation and annihilation operators on the full Fock space [65].

**3) Oriented Sets:** We refer to the previous sections for the definition of the species $E^{\pm}$ of oriented sets. The weight $\mathbf{j}_{E^{\pm}}$ is given by

$$\begin{aligned} \mathbf{j}_{E^{\pm}}(U_+, U_+^*) &= \mathbf{j}_{E^{\pm}}(U_-, U_-^*) = 1, \\ \mathbf{j}_{E^{\pm}}(U_+, U_-^*) &= \mathbf{j}_{E^{\pm}}(U_-, U_+^*) = 0 \end{aligned} \tag{4.5}$$

where $U^* = U + \{*\}$. With the help of the 'switching' sign operator

$$\mathbf{g}\varphi(\pm) = \varphi(\mp)$$

we obtain the **g**-commutation relations:

$$a(h_1)a^*(h_2) - \mathbf{g}a^*(h_2)a(h_1) = \langle h_1, h_2 \rangle \mathbf{1} \tag{4.6}$$

As we saw in the previous section, the space $\mathcal{F}_{E^\pm}(\mathcal{K})$ is isomorphic to the direct sum of the symmetric and antisymmetric Fock space over $\mathcal{K}$:

$$\mathcal{F}_{E^\pm}(\mathcal{K}) = \mathcal{F}_s(\mathcal{K}) \oplus \mathcal{F}_a(\mathcal{K})$$

through the transformation:

$$\varphi_s = \varphi(+) + \varphi(-) \ \text{ and } \ \varphi_a = \varphi(+) - \varphi(-)$$

then the **g**-commutation relations can be written equivalently as:

$$(a(h_1)a^*(h_2) - a^*(h_2)a(h_1))\varphi_s = \langle h_1, h_2 \rangle \varphi_s$$

and

$$(a(h_1)a^*(h_2) + a^*(h_2)a(h_1))\varphi_a = \langle h_1, h_2 \rangle \varphi_a$$

**4) Rooted Trees:** We recall the definition of the species $\mathcal{A}$:

$$\mathcal{A}[U] = \{f : U \to U \mid f^{\circ k}(u) = \mathrm{root}(f) \in U \text{ for } k \geq |U|, u \in U\}$$

with the transport along $\sigma$: $\mathcal{A}[\sigma](f) = \sigma \circ f \circ \sigma^{-1}$. We note that $\mathcal{A}[\emptyset] = \emptyset$. We consider a natural weight which can be described as follows: it takes value 1 on those pairs of trees for which the second is obtained by adding a leaf to the first one, and takes value 0 for the rest. Thus for $t_1 \in \mathcal{A}[U]$ and $t_2 \in \mathcal{A}[U + \{*\}]$ the weight is:

$$\mathbf{j}_{\mathcal{A}}(t_1, t_2) = \left\{ \begin{array}{ll} 1 & \text{if } t_1(u) = t_2(u) \text{ for } u \in U \\ 0 & \text{otherwise} \end{array} \right\} := \delta_{t_1, t_2\restriction_U}$$

We will compute the commutator of $a(h_2)$ with $a^*(h_1)$. For this we need to obtain the expressions of $(\bar{\mathbf{j}} \cdot \mathbf{j})_k(\cdot, \cdot)$ and $(\mathbf{j} \cdot \bar{\mathbf{j}})_k(\cdot, \cdot)$. We start with

$$\begin{aligned} (\mathbf{j} \cdot \bar{\mathbf{j}})_n(f, f') &= \sum_g \mathbf{j}_{\mathcal{A}}(f, g) \cdot \overline{\mathbf{j}_{\mathcal{A}}(f', g)} = \sum_g \delta_{f, g\restriction_n} \cdot \delta_{f', g\restriction_n} \\ &= \delta_{f,f'} \sum_g \delta_{f, g\restriction_n} = n\delta_{f,f'} \end{aligned} \tag{4.7}$$

The factor $n$ appears because there are $n$ possible way of attaching a leaf to the tree $f$ each one giving a tree $g$ such that $g\restriction_n = f$. For $k < n$ we have

$$(\mathbf{j}\cdot\bar{\mathbf{j}})_k(t,t') = \sum_g \mathbf{j}_{\mathcal{A}}(t,g)\cdot\overline{\mathbf{j}_{\mathcal{A}}(t',\mathcal{A}[\tau_{n,k}](g))} = \sum_g \delta_{t,g\restriction_n}\cdot\delta_{t',\mathcal{A}[\tau_{n,k}](g)\restriction_n}.$$

At most one term in this sum is different from zero, for the tree $g$ satisfying:

$$\begin{cases} g(i) = t(i) & \text{if } i\in n \\ g(j) = t'(j) & \text{if } j\in n\setminus\{k\} \\ g(n) = t'(k) & \end{cases} \tag{4.8}$$

On the other hand

$$\begin{aligned}(\bar{\mathbf{j}}\cdot\mathbf{j})_k(t,t') &= \sum_{g'}\overline{\mathbf{j}_{\mathcal{A}}(g',\mathcal{A}[\tau_{n-1,k}](t))}\cdot\mathbf{j}_{\mathcal{A}}(g',\mathcal{A}[\tau_{n-1,k}](t')) \\ &= \sum_g \delta_{g',\mathcal{A}[\tau_{n-1,k}](t)\restriction_{n-1}}\cdot\delta_{g',\mathcal{A}[\tau_{n-1,k}](t')\restriction_{n-1}} \\ &= \delta_{\mathcal{A}[\tau_{n-1,k}](t)\restriction_{n-1},\ \mathcal{A}[\tau_{n-1,k}](t')\restriction_{n-1}} \\ &= \begin{cases} 1 & \text{if } t(i) = t'(i) \text{ for all } i\in n, i\neq k \\ 0 & \text{otherwise}\end{cases} \end{aligned} \tag{4.9}$$

Finally from (4.8), (4.9) we conclude that $(\bar{\mathbf{j}}\cdot\mathbf{j})_k(t,t') = (\mathbf{j}\cdot\bar{\mathbf{j}})_k(t,t')$ for $k\in\{0,1,..n-1\}$.
Let us define the 'vertex number' operator $N$ by

$$(N\varphi)(t) = n\varphi(t)$$

for $t\in\mathcal{A}[n]$. The usual commutation relations between $N$ and the creation operator hold:

$$[N, a^*(h)] = a^*(h)$$

By using Lemmas 4.4 and 4.5 we obtain the following:

**Theorem 4.6** *The following commutation relations hold on the combinatorial Fock space* $(\mathcal{A},\mathbf{j}_{\mathcal{A}})$*:*

$$a(h_1)a^*(h_2) - a^*(h_2)a(h_1) = N\langle h_1,h_2\rangle \tag{4.10}$$

**Remark.** Notice that the vacuum of $\mathcal{F}_{\mathcal{A}}(\mathcal{K})$ is an eigenvector of $N$ with eigenvalue 1. For a one dimensional Hilbert space $\mathcal{K}$, the cyclic space generated by applying $a, a^*$ to the vacuum is a one mode interaction Fock space [1]. The vacuum distribution of the selfadjoint operator $a + a^*$ is symmetric and is uniquely determined by the orthogonal polynomials satisfying the relations

$$P_{n+1}(x) = xP_n - \frac{1}{2}n(n+1)P_{n-1} \tag{4.11}$$

for all $n \in \mathbb{N}$, with $P_{-1} = 0, P_0 = 1, P_1 = x$. These are Meixner polynomials of the second kind [16].

**5) Simple Directed Graphs:** Let us define a species whose structures are directed graphs for which any pair of vertices is connected by at most one edge:

$$\mathcal{D}_s[U] = \{g \in U \times U \mid (u,v) \in g \Rightarrow (v,u) \notin g\} \tag{4.12}$$

where the transport along $\sigma$ is given by $\sigma \times \sigma$.

Let $g_1 \in \mathcal{D}_s[U]$ and $g_2 \in \mathcal{D}_s[U + \{*\}]$. Then $\mathbf{j}(g_1, g_2) \neq 0$ if and only if $g_2$ contains $g_1$ as a subset and all edges of $g_2$ connecting the vertex $*$ with vertices in $U$ are oriented from $*$ to $U$. We make the following convenient notation for the set of edges going out of a vertex $a$ of $g \in \mathcal{D}_s[V]$ :

$$v_a(g) = \{(a,v) \mid (a,v) \in g\} = \{a\} \times e_a(g)$$

The weight $\mathbf{j}^{\mathcal{D}_s,q}$ depends on the real parameter $0 \leq q \leq 1$ and is defined by:

$$\mathbf{j}^{\mathcal{D}_s,q}(g_1, g_2) = \delta_{g_2\ ,\ g_1 + v_*(g_2)} \cdot (q^{|U| - |v_*(g_2)|} \cdot (1-q)^{|v_*(g_2)|})^{\frac{1}{2}}$$

In the rest of this section we prove that $(\mathcal{D}_s, \mathbf{j}^{\mathcal{D}_s,q})$ is a realization of the q-commutation relations ([26, 21, 67, 10, 45]).

**Theorem 4.7** *On* $(\mathcal{D}_s, \mathbf{j}^{\mathcal{D}_s,q})$ *we have:*

$$a(h_1)a^*(h_2) - q \cdot a^*(h_2)a(h_1) = \langle h_1, h_2 \rangle\, \mathbf{1}$$

*Proof.* We employ lemmas 4.5 and 4.4. First:

$$\begin{aligned}(\mathbf{j} \cdot \bar{\mathbf{j}})_n(f, f') &= \sum_g \mathbf{j}^{\mathcal{D}_s,q}(f,g) \cdot \overline{\mathbf{j}^{\mathcal{D}_s,q,}(f',g)} \\ &= \sum_g \delta_{g, f+v_n(g)} \cdot \delta_{g, f'+v_n(g)} \cdot q^{n-|v_n(g)|} \cdot (1-q)^{|v_n(g)|} \\ &= \delta_{f,f'} \cdot \sum_{v_n \subset n} q^{n-|v_n|} \cdot (1-q)^{|v_n|} = \delta_{f,f'} \end{aligned} \tag{4.13}$$

It remains to be proved that $(\mathbf{j} \cdot \bar{\mathbf{j}})_k(f, f') = q \cdot (\bar{\mathbf{j}} \cdot \mathbf{j})_k(f, f')$ for $k \in \{0, 1, ..n-1\}$ and $f, f' \in \mathcal{D}_s[n]$.
In the sum

$$(\mathbf{j} \cdot \bar{\mathbf{j}})_k(f, f') = \sum_g \mathbf{j}^{\mathcal{D}_s,q}(f,g) \cdot \overline{\mathbf{j}^{\mathcal{D}_s,q,}(f', \mathcal{D}_s[\tau_{n,k}](g))}$$

the only nonzero contribution comes from $g \in \mathcal{D}_s[U + \{*\}]$ such that:

$$g = f + \{n\} \times e_n(g) = (f \setminus v_k(f)) + v_k(f) + \{n\} \times e_n(g)$$

and

$$\begin{aligned}\mathcal{D}_s[\tau_{n,k}](g) &= (f \setminus v_k(f)) + \{n\} \times e_k(f) + \{k\} \times e_n(g) \\ &= f' + \{n\} \times e_k(f)\end{aligned}$$

which together imply

$$(f \setminus v_k(f)) + \{k\} \times e_n(g) = f'.$$

But this means that $e_n(g) = e_k(f')$ and $f \setminus v_k(f) = f' \setminus v_k(f')$. Then we have the expression

$$(\mathbf{j} \cdot \bar{\mathbf{j}})_k(f, f') = \delta_{f\setminus v_k(f), f'\setminus v_k(f')} \cdot q^{n-\frac{|v_k(f)|+|v_k(f')|}{2}} \cdot (1-q)^{\frac{|v_k(f)|+|v_k(f')|}{2}} \tag{4.14}$$

On the other hand in $(\bar{\mathbf{j}} \cdot \mathbf{j})_k(f, f')$ we get only the contribution from those $g$ for which:

$$g' = \mathcal{D}_s[\tau_{n-1,k}](f) \restriction_{n-1} = \mathcal{D}_s[\tau_{n-1,k}](f') \restriction_{n-1}$$

Thus we obtain

$$(\bar{\mathbf{j}} \cdot \mathbf{j})_k(f, f') = \delta_{f\setminus v_k(f), f'\setminus v_k(f')} \cdot q^{n-1-\frac{|v_k(f)|+|v_k(f')|}{2}} \cdot (1-q)^{\frac{|v_k(f)|+|v_k(f')|}{2}}. \tag{4.15}$$

Finally from (4.14) and (4.15) we have the desired expression:

$$(\mathbf{j} \cdot \bar{\mathbf{j}})_k(f, f') = q \cdot (\bar{\mathbf{j}} \cdot \mathbf{j})_k(f, f')$$

□

# 5 Fock States and Operations with Combinatorial Fock Spaces

The operations between species of structures described in Section 2 are helpful in understanding the action of creation and annihilation operators in terms of elementary ones. The guiding example is Green's representation of the operators appearing in parastatistics, as sums of bosonic (fermionic) operators with the 'wrong' commutation relations [25]. Similar ideas appear in [55] where the author considers macroscopic fields as linear combinations of basic bosonic fields with various commutation relations.

Thus, the first question we address in this section is the following: given two combinatorial Fock spaces $(F, \mathbf{j}_F)$ and $(G, \mathbf{j}_G)$, is there a natural weight associated to the species $F + G$, $F \cdot G$, $F \times G$, $F \circ G$ ? The second question is related to the notion of positive definite functions on pair partitions. A general theory of such functions has been introduced in [13] in connection with the so called generalized Brownian motion.

## 5.1 Fock States

We will start with the latter question by introducing the necessary definitions.

**Definition.**Let $S$ be a finite ordered set. We denote by $\mathcal{P}_2(S)$ the set of pair partitions of $S$, that is $\mathcal{V} \in \mathcal{P}_2(S)$ if $\mathcal{V} = \{V_1, .., V_r\}$ where each $V_i$ is an ordered set containing two elements $V_i = (k_i, l_i)$ with $k_i, l_i \in S$, $k_i < l_i$ and $\{V_1, .., V_r\}$ is a partition of $S$ ($V_i \bigcap V_j = \emptyset$ for $i \neq j$ and $\bigcup_{i=1}^r V_i = S$). The set of all pair partitions is

$$\mathcal{P}_2(\infty) = \bigcup_{r=1}^{\infty} \mathcal{P}_2(2r).$$

Let $\mathcal{K}$ be a Hilbert space. We denote by $\mathcal{C}_\mathcal{K}$ the $^*$-algebra obtained from the free algebra with generators $c(f)$ and $c^*(f)$, $(f \in \mathcal{K})$ divided by the relations:

$$c^*(\lambda f_1 + \mu f_2) = \lambda c^*(f_1) + \mu c^*(f_2),\ \lambda, \mu \in \mathbb{C},\ f_1, f_2 \in \mathcal{K},$$

and

$$c^*(f) = (c(f))^*.$$

We are interested in a particular type of positive functionals on $\mathcal{C}_\mathcal{K}$, called *Fock states* [13] which have the following expression on monomials of creation and annihilation operators:

$$\rho_t(c^{\sharp_1}(f_1) \cdot .. \cdot c^{\sharp_n}(f_n)) = \begin{cases} 0 & \text{if } n \text{ odd} \\ \sum_{\mathcal{V}=\{V_1,..,V_{\frac{n}{2}}\}} \rho_t[V_1] \cdot .. \cdot \rho_t[V_{\frac{n}{2}}] \cdot t(\mathcal{V}) & \text{if } n \text{ even} \end{cases} \tag{5.1}$$

the sum running over all pair partitions $\mathcal{V}$ in $\mathcal{P}_2(2r)$, and the symbols $\sharp_i$ standing for creation or annihilation. For $V = (k, l) \in \mathcal{V}$

$$\rho_t[V] = \langle f_k, f_l \rangle \cdot Q(\sharp_k, \sharp_l)$$

with the 2 by 2 covariance matrix

$$Q = \begin{pmatrix} \rho(c_i c_i) & \rho(c_i c_i^*) \\ \rho(c_i^* c_i) & \rho(c_i^* c_i^*) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

where $c_i = c(e_i)$ and $e_i$ is an arbitrary normalized vector in $\mathcal{K}$.
Let us consider a real subspace $\mathcal{K}_\mathbb{R}$ of $\mathcal{K}$ such that $\mathcal{K} = \mathcal{K}_\mathbb{R} \oplus i\mathcal{K}_\mathbb{R}$. The subalgebra of $\mathcal{C}_\mathcal{K}$ generated by the 'field operators' $\mathbf{j}(f) = c(f) + c^*(f)$ with $f \in \mathcal{K}_\mathbb{R}$ is denoted by $\mathcal{A}_\mathcal{K}$. If the restriction of the functional $\rho_t$ to the algebra $\mathcal{A}_\mathcal{K} \subset \mathcal{C}_\mathcal{K}$ is a state, then we call the function

$$t : \mathcal{P}_2(\infty) \to \mathbb{C}.$$

*positive definite* [13]. In particular if $\rho_t$ is a state on $\mathcal{C}_\mathcal{K}$ then $t$ is positive definite. The converse is not true in general. We will show first that the vacuum state of a symmetric Hilbert space is an example of Fock state.

**Proposition 5.1** *Let $(F, \mathbf{j}_F)$ be a combinatorial Fock space with $|F[\emptyset]| = 1$. Let $\Omega_F$ be the vacuum vector of $\mathcal{F}_F(\mathcal{K})$, $\mathcal{K}$ a Hilbert space. Then the functional $\rho_F(\cdot) = \langle \Omega_F, \cdot \Omega_F \rangle$ is a Fock state on $\mathcal{C}_{\mathcal{K}}$.*

In order to prove this proposition we need to introduce one more tool.

**Definition.**Let $\mathcal{K}$ be a Hilbert space, $(F, \mathbf{j}_F)$ a combinatorial Fock space and $A \in \mathcal{B}(\mathcal{K})$ a bounded operator on $\mathcal{K}$. The *second quantization* of $A$ is defined by

$$\begin{aligned} d\Gamma_F(A) : \mathcal{F}_F(\mathcal{K}) &\rightarrow \mathcal{F}_F(\mathcal{K}) \\ (d\Gamma_F(A)\varphi)(g) &= d\Gamma(A)(\varphi(g)) \end{aligned}$$

where the meaning of $d\Gamma(A)$ on the right side is

$$d\Gamma(A) : \mathcal{K}^{\otimes n} \rightarrow \mathcal{K}^{\otimes n}$$

$$d\Gamma(A) : \psi_0 \otimes .. \otimes \psi_{n-1} \rightarrow \sum_{k=0}^{n-1} \psi_0 \otimes .. \otimes A\psi_k \otimes .. \otimes \psi_{n-1}$$

**Remark.** The second quantization operator is a well defined operator on $\mathcal{F}_F(\mathcal{K})$. Indeed let $\varphi \in \mathcal{F}_F(\mathcal{K})$ and $\tau \in \mathrm{S}(n)$, then

$$\begin{aligned} (d\Gamma_F(A)\varphi)(F[\tau](g)) &= d\Gamma(A)\varphi(F[\tau](g)) \\ = d\Gamma(A) \cdot U(\tau)\varphi(g) &= U(\tau)(d\Gamma_F(A)\varphi)(g) \end{aligned}$$

and thus $d\Gamma_F(A)\varphi \in \mathcal{F}_F(\mathcal{K})$. We have used the invariance of $d\Gamma(A)$ under permutations:

$$d\Gamma(A)U(\tau) = U(\tau)d\Gamma(A).$$

**Lemma 5.2** *We have the following commutation relations:*

$$[a(h), d\Gamma_F(A)] = a(A^*h) \tag{5.2}$$

$$[d\Gamma_F(A), d\Gamma_F(B)] = d\Gamma_F([A, B]) \tag{5.3}$$

*Proof.* Let $\varphi \in \mathcal{F}_F(\mathcal{K}),\ f \in F[n],\ g \in F[n+1]$. Then

$$\begin{aligned}
&(a(h)d\Gamma(A)\varphi)(f) = \\
&= \sum_g \mathbf{j}(f,g) \cdot \mathrm{inp}_n(h, (d\Gamma(A)\varphi)(g)) \\
&= \sum_g \mathbf{j}(f,g) \cdot \mathrm{inp}_n(h, \sum_{k=0}^{n} \mathbf{1} \otimes .. \otimes A \otimes .. \otimes \mathbf{1}\varphi(g)) \\
&= \sum_g \mathbf{j}(f,g) \cdot \sum_{k=0}^{n-1} \mathbf{1} \otimes .. \otimes A \otimes .. \otimes \mathbf{1}\ \mathrm{inp}_n(h, \varphi(g)) + \\
&\quad + \sum_g \mathbf{j}(f,g) \cdot \mathrm{inp}_n(h, \mathbf{1} \otimes .. \otimes A\varphi(g)) = \\
&= \left(\sum_{k=0}^{n-1} \mathbf{1} \otimes .. \otimes A \otimes .. \otimes \mathbf{1}\right) \sum_g \mathbf{j}(f,g) \cdot \mathrm{inp}_n(h, \varphi(g)) + \\
&+ \sum_g \mathbf{j}(f,g) \cdot \mathrm{inp}_n(A^* h, \varphi(g)) = d\Gamma_F(A)a(h) + a(A^* h)
\end{aligned}$$

which proves (5.2). The other commutator follows directly from the definition of the second quantization operator.

□

**Lemma 5.3** *Let $\{e_j\}_{j\in J}$ be an orthonormal basis of $\mathcal{K}$ and denote $a_j^\sharp = a(e_j)^\sharp$. Then the following equation holds:*

$$a_i \prod_{k=1}^{n} a_{i_k}^{\sharp_k}\Omega = \sum_{k=1}^{n} \delta_{i,i_k} \cdot \delta_{\sharp_k,*} \cdot a_{i_0} \prod_{p=1}^{k-1} a_{i_p}^{\sharp_p} \cdot a_{i_0}^* \cdot \prod_{q=k+1}^{n} a_{i_q}^{\sharp_q}\Omega \tag{5.4}$$

*where the colors $(i_k)_{k=0,..,n}$ satisfy the property $i_k \neq i_0$ for all $k = 1, ..n$.*

*Proof.* For simplicity we denote $\Psi = \prod_{k=1}^{n} a_{i_k}^{\sharp_k}\Omega$. We notice that $a_{i_0}\Psi = 0$ due to the assumption that $i_k \neq i_0$ for all $k = 1, ..n$. Then using (5.2) we get

$$a_i\Psi = [a_{i_0}, d\Gamma(|i_0\rangle\langle i|)]\Psi = a_{i_0} d\Gamma(|i_0\rangle\langle i|)\Psi \tag{5.5}$$

By successively applying the following commutator

$$[d\Gamma(|i_0\rangle\langle i|), a_{i_k}^{\sharp_k}] = \delta_{i_k,i} \cdot \delta_{\sharp_k,*} \cdot a_{i_0}^*$$

we obtain the sum in (5.4).

□

*Proof of Proposition 5.1.* It is clear that $\rho_F$ is a positive linear functional on $\mathcal{C}_{\mathcal{K}}$. We need to prove that it has the expression (5.1). From linearity of the creation operators and anti-linearity of the annihilation operators we conclude that it is sufficient to consider the vectors $f_i$ in (5.1) belonging to the chosen orthogonal basis. From

$$\rho_F(\prod_{k=1}^{n} a_{i_k}^{\sharp_k}) = \left\langle \Omega, \prod_{k=1}^{n} a_{i_k}^{\sharp_k}\Omega \right\rangle$$

and considering the fact that the creation operator increases the level by one while the annihilation operator decreases it by one, we deduce that nonzero expectations can appear only if $n$ is even and the number of creators in the monomial $\prod_{k=1}^{n} a_{i_k}^{\sharp_k}$ is equal to that of annihilators. Furthermore $a_{i_1}^{\sharp_1}$ must be an annihilator and $a_{i_n}^{\sharp_n}$, a creator. We will thus consider that this is the case.

We put the monomial in the form $a_{i_1}\prod_{k=2}^{n} a_{i_k}^{\sharp_k}$ and apply lemma 5.3. We obtain a sum over all pairs $(a_{i_1}, a_{i_k}^*)$ of the same color ($i_1 = i_k$) and replace $i_1$ by a new color $i_0$. We go now to the next annihilator in each term of the sum and repeat the procedure, the new color which we add this time being different from all the colors used previously. After $\frac{n}{2}$ steps we obtain a sum containing all possible pairings of annihilators and creators of the same color in $\prod_{k=1}^{n} a_{i_k}^{\sharp_k}$:

$$\rho_F(\prod_{k=1}^{n} a_{i_k}^{\sharp_k}) = \sum_{\mathcal{V}=\{V_1,..,V_{\frac{n}{2}}\}} \prod_{p=1}^{\frac{n}{2}} \delta_{i_{k_p}, i_{l_p}} \cdot Q(\sharp_{k_p}, \sharp_{l_p}) \cdot t(\mathcal{V})$$

with $V_p = (k_p, l_p)$ and $t(\mathcal{V})$ is given by

$$t(\mathcal{V}) = \rho_F(\prod_{k=1}^{n} a_{j_k}^{\sharp_k})$$

such that $j_{k_p} = j_{l_{p'}}$ if and only if $p = p'$, for $p, p' \in \{1, .., \frac{n}{2}\}$, $\sharp_{k_p}$ is annihilator and $\sharp_{l_p}$ is creator.

$\square$

Thus for each combinatorial Fock space $(F, \mathbf{j}_F)$ (which has a vacuum), the vacuum state is described by a positive definite function $t_F$ on $\mathcal{P}_2(\infty)$.

**Remark.** We observe that the result can be generalized to a larger range of states and monomials. Let us partition the index set $J$ of the orthonormal basis of the Hilbert space $\mathcal{K}$

$$J = J_1 + J_2$$

and choose a state $\rho_\Phi(\cdot) = \langle \Phi, \cdot\Phi\rangle$ and monomials $\prod_{k=1}^{n} a_{j_k}^{\sharp_k}$ such that $j_k \in J_1$ and $\Phi \in \mathcal{F}_F(\mathcal{K}_2) \subset \mathcal{F}_F(\mathcal{K})$ is a normalized vector where $\mathcal{K}_2$ is the subspace of $\mathcal{K}$ with the orthogonal basis $\{e_j\}_{j\in J_2}$. Then it is easy to see that the argument used in the above proof still holds and $\rho_\Phi$ is a Fock state for $\mathcal{C}_{\mathcal{K}_1}$. In general $\rho_\Phi$ and $\rho_F$ do not coincide. When they do coincide we say that $\rho_F$ has the *pyramidal independence* property [13].

## 5.2 Operations with symmetric Hilbert Spaces

We pass now to the first question which we have posed in the beginning of this section. The various operations between species offer the opportunity of creating new symmetric Hilbert spaces which sometimes give rise to interesting interpolations between the two members. For the definitions of the operations we refer back to Section 2.

**1) Sums.** Let $(F, \mathbf{j}_F)$ and $(G, \mathbf{j}_G)$ be two combinatorial Fock spaces. From Section 3 we know that

$$\mathcal{F}_{F+G} = \mathcal{F}_F \oplus \mathcal{F}_G.$$

Note that the vacuum of $\mathcal{F}_{F+G}$ has dimension $\geq 2$ if $F[\emptyset] \neq \emptyset \neq G[\emptyset]$. The natural weight on $F+G$ is

$$\mathbf{j}_{F+G}(t_1, t_2) = \mathbf{j}_F(t_1, t_2) + \mathbf{j}_G(t_1, t_2)$$

which gives rise to operators

$$a_{F+G}(h) = a_F(h) \oplus 0 + 0 \oplus a_G(h).$$

We consider a linear combination of the two vacua (for $|F[\emptyset]| = |G[\emptyset]| = 1$)

$$\Omega_\lambda = \sqrt{\lambda}\Omega_F + \sqrt{1-\lambda}\Omega_G.$$

The corresponding state $\rho_{F+G,\lambda}(\cdot) = \langle \Omega_\lambda, \cdot \Omega_\lambda \rangle$ interpolates linearly between $\rho_F$ and $\rho_G$

$$\rho_{F+G,\lambda} = \lambda \rho_F + (1-\lambda)\rho_G$$

and the same is true for the positive definite functions

$$t_{F+G,\lambda} = \lambda t_F + (1-\lambda) t_G. \tag{5.6}$$

**2) Products.** Let $(F, \mathbf{j}_F)$ and $(G, \mathbf{j}_G)$ be two combinatorial Fock spaces. We consider the product species $F \cdot G$. As we have proved in Section 3, there is the following isomorphism

$$\mathcal{F}_{F \cdot G}(\mathcal{K}) = \mathcal{F}_F(\mathcal{K}) \otimes \mathcal{F}_G(\mathcal{K}). \tag{5.7}$$

Again there is a natural weight for the species $F \cdot G$. For $f \in F[U_1], g \in G[U_2], f' \in F'[U_1], g' \in G'[U_2]$:

$$\begin{aligned} \mathbf{j}_{F \cdot G,\lambda}((f,g),(f,g')) &= \sqrt{\lambda}\mathbf{j}_G(g,g') \\ \mathbf{j}_{F \cdot G,\lambda}((f,g),(f',g)) &= \sqrt{1-\lambda}\mathbf{j}_F(f,f') \end{aligned}$$

all other values of $\mathbf{j}_{F \cdot G,\lambda}$ being 0.
From (5.7) and the expression of $\mathbf{j}_{F \cdot G}$ we obtain

$$a^\sharp_{F \cdot G}(h) = \sqrt{\lambda} a^\sharp_F(h) \otimes \mathbf{1} + \sqrt{1-\lambda}\mathbf{1} \otimes a^\sharp_G(h)$$

If $|F[\emptyset]| = |G[\emptyset]| = 1$ then the state $\rho_{F\cdot G}(\cdot) = \langle \Omega_F \otimes \Omega_G, \cdot\Omega_F \otimes \Omega_G\rangle$ generates the positive definite function:

$$t_{F\cdot G}(\mathcal{V}) = \sum_{\mathcal{V}_1,\mathcal{V}_2} \lambda^{|\mathcal{V}_1|}(1-\lambda)^{|\mathcal{V}_2|} t_F(\mathcal{V}_1)\cdot t_G(\mathcal{V}_2)$$

where the sum runs over all partitions of $\mathcal{V}$ in two sets, $\mathcal{V}_1$ and $\mathcal{V}_2$.

**Example:** The Green representation [25] of the (Fermi) parastatistics of order p is an example of application of the product of species. We consider the p-th power $(E_\pm)^p$ of the species of oriented sets $E_\pm$. Then the annihilation operators are

$$a(h) = \frac{1}{\sqrt{p}}\sum_{k=1}^{p} a^{(k)}(h)$$

and the vacuum state is $\rho(\cdot) = \langle\Omega, \cdot\Omega\rangle$ where $a^{(k)}$ is the term corresponding to the k-th term in the product and

$$\Omega = \Omega_a^{(1)} \otimes .. \otimes \Omega_a^{(p)}$$

is the tensor product of the antisymmetric vacua of each of the species $E_\pm^{(k)}$.

**3) Cartesian Products.** Let $(F, \mathbf{j}_F)$ and $(G, \mathbf{j}_G)$ be two combinatorial Fock spaces. We consider the cartesian product species $F \times G$. The corresponding weight has the expression:

$$\mathbf{j}_{F\times G}((f,g),(f',g')) = \mathbf{j}_F(f,f')\cdot \mathbf{j}_G(g,g')$$

We note that $\mathbf{j}_{F\times G}$ satisfies the invariance condition stated in the definition of the weight.

**Proposition 5.4** *Let $(F, \mathbf{j}_F)$ and $(G, \mathbf{j}_G)$ be two combinatorial Fock spaces both having a single structure on $\emptyset$. Then the positive definite function associated to the vacuum state of $(F \times G, \mathbf{j}_{F\times G})$ satisfies:*

$$t_{F\times G}(\mathcal{V}) = t_F(\mathcal{V})\cdot t_G(\mathcal{V}) \tag{5.8}$$

*for all $\mathcal{V} \in \mathcal{P}_2(\infty)$.*

*Proof.* We construct a linear operator $T$ from $\mathcal{F}_{F\times G}(\mathcal{K})$ to $\mathcal{F}_F(\mathcal{K})\otimes\mathcal{F}_G(\mathcal{K})$ with the property that its restriction to a certain subspace $\mathcal{F}^{\text{ext}}_{F\times G}$ of $\mathcal{F}_{F\times G}(\mathcal{K})$, is an isometry. The subspace $\mathcal{F}^{\text{ext}}_{F\times G}$ is spanned by vectors $\delta_{[(f,g),c]}$ of the orthogonal basis $(F\times G)(J)$ of $\mathcal{F}_{F\times G}(\mathcal{K})$ which have all colors different from each other, i.e. $c(i) \neq c(j)$ for $i \neq j$. We refer to Section 3 for the definitions related to the orthogonal basis of $\mathcal{F}_{F\times G}(\mathcal{K})$.
The action of $T$ on the basis vectors is:

$$\begin{array}{rcl} T : \mathcal{F}^{\text{ext}}_{F\times G}(\mathcal{K}) & \rightarrow & \mathcal{F}^{\text{ext}}_F(\mathcal{K})\otimes\mathcal{F}^{\text{ext}}_G(\mathcal{K}) \\ \delta_{[(f,g),c]} & \mapsto & \delta_{[f,c]}\otimes\delta_{[g,c]} \end{array}$$

We check that the operator is well defined. Indeed the map

$$\begin{array}{rcl} i : \sum_{n=0}^{\infty} (F \times G)[n] \times J^n & \rightarrow & (\sum_{n=0}^{\infty} F[n] \times J^n) \times (\sum_{n=0}^{\infty} G[n] \times J^n) \\ ((f,g),c) & \mapsto & ((f,c),(g,c)) \end{array}$$

commutes with the action of $\mathrm{S}(n)$ on the two sides, at each level and thus projects to a well defined map on the quotient:

$$\begin{array}{rcl} i : (F \times G)(J) & \rightarrow & F(J) \times G(J) \\ [(f,g),c] & \mapsto & ([f,c],[g,c]) \end{array}$$

This means that $T$ is well defined. But as we have shown in Section 3, the vectors $\delta_{[(f,g),c]}$, $\delta_{[f,c]}$ and $\delta_{[g,c]}$ for which $c(i) \neq c(j)$ if $i \neq j$, have norm one which implies that $T$ is an isometry.
Let us now consider the vector

$$\varphi_F^{(p)} = \prod_{k=1}^{p} a_{F,i_k}^{\sharp_k} \, \Omega_F$$

the colors $(i_k)_{k=1,..,n}$ satisfying the condition that there are no three identical colors, and if there exists $k_1 < k_2$ such that $i_{k_1} = i_{k_2}$, then $a_{i_{k_1}}^{\sharp_{k_1}} = a_{i_{k_1}}$ and $a_{i_{k_2}}^{\sharp_{k_2}} = a_{i_{k_2}}^*$. It is clear that $\varphi_F^{(p)} \in \mathcal{F}_F^{\mathrm{ext}}(\mathcal{K})$. Analogously we define $\varphi_G^{(p)}$ and $\varphi_{F\times G}^{(p)}$. We want to prove by induction w.r.t. $p$ that the action of the isometry $T$ is such that

$$T : \varphi_{F\times G}^{(p)} \rightarrow \varphi_F^{(p)} \otimes \varphi_G^{(p)}. \tag{5.9}$$

This implies in particular (5.8), when the monomial $\prod_{k=1}^{p} a_{i_k}^{\sharp_k}$ contains equal number of creators and annihilators pairing each other according to color, no two pairs having the same color.
For $p = 0$ we have $T(\Omega_{F\times G}) = \Omega_F \otimes \Omega_G$. Suppose (5.9) holds for p. Then there are two possibilities for increasing the length of the monomial $\prod_{k=1}^{p} a_{i_k}^{\sharp_k}$ by one: either by adding on the first position a creation operator $a_{i_0}^*$ such that the color $i_0$ does not appear in the rest of the monomial, or by adding an annihilation operator $a_{i_0}$ such that the term $a_{i_0}^*$ appears once in the rest of the monomial. We treat the two cases separately.
1.) suppose that we have $\varphi_{F\times G}^{(p)} = \prod_{k=1}^{p} a_{i_k}^{\sharp_k} \, \Omega_{F\times G}$, $i_0 \neq i_k$ which has the decomposition

$$\varphi_{F\times G}^{(p)} = \sum_{[(f,g),c]} \varphi([(f,g),c])\delta_{[(f,g),c]}$$

with $\varphi([(f,g),c]) \in \mathbb{C}$. Then

$$a_{F\times G,i_0}^* \varphi_{F\times G}^{(p)} = \sum_{[(f,g),c],(f'g')} \varphi([(f,g),c]) \cdot \mathbf{j}_{F\times G}((f,g),(f',g')) \; \delta_{[(f',g'),c_{i_0}^+]}$$

which implies

$$\begin{aligned} &T(a^*_{F\times G,i_0}\varphi^{(p)}_{F\times G}) = \\ &= \sum_{[(f,g),c],(f'g')} \varphi([(f,g),c])\ a^*_{F,i_0}\delta_{[f,c]} \otimes a^*_{G,i_0}\delta_{[g,c]} \\ &= a^*_{F,i_0} \otimes a^*_{G,i_0} T(\varphi^{(p)}_{F\times G}) = a^*_{F,i_0}\varphi^{(p)}_F \otimes a^*_{G,i_0}\varphi^{(p)}_G . \end{aligned}$$

2.) suppose that we have $\varphi^{(p)}_{F\times G} = \prod_{k=1}^p a^{\sharp_k}_{i_k}\ \Omega_{F\times G}$ such that the term $a^*_{i_0}$ appears exactly one time in the the monomial $\prod_{k=1}^p a^{\sharp_k}_{i_k}$. We use again the Fourier decomposition

$$\varphi^{(p)}_{F\times G} = \sum_{[(f,g),c]} \varphi([(f,g),c])\delta_{[(f,g),c]}$$

and identify in each orbit $[(f,g),c] \in (F\times G)(J)$, a representant $((f,g),c) \in (F\times G)[n]\times J^n$ such that $c(n-1) = i_0$. Then

$$a_{F\times G,i_0}\varphi^{(p)}_{F\times G} = \sum_{[(f,g),c],(f'g')} \varphi([(f,g),c])\cdot \mathbf{j}_{F\times G}((f',g'),(f,g))\ \delta_{[(f',g'),c^-_{i_0}]}$$

where $c^-_{i_0}$ is the restriction of $c$ to the set $n-1$. Finally

$$\begin{aligned} &T(a_{F\times G,i_0}\varphi^{(p)}_{F\times G}) = \\ &= \sum_{[(f,g),c],(f'g')} \varphi([(f,g),c])\cdot \mathbf{j}_{F\times G}((f',g'),(f,g))\ T(\delta_{[(f',g'),c^-_{i_0}]}) \\ &= \sum_{[(f,g),c],(f'g')} \varphi([(f,g),c])\cdot \mathbf{j}_F(f,f')\cdot \mathbf{j}_G(g,g')\delta_{[f',c^-_{i_0}]} \otimes \delta_{[g',c^-_{i_0}]} \\ &= a_{F,i_0}\varphi^{(p)}_F \otimes a_{G,i_0}\varphi^{(p)}_G \end{aligned}$$

which proves the induction hypothesis for $p+1$ and the proposition.

□

**Application:** Combining the result of the previous proposition and certain variations on the species of rooted trees, we investigate more general commutation relations of the type:

$$[a(h_1), a^*(h_2)] = \langle h_1, h_2\rangle \cdot f(N)$$

with $f : \mathbf{N} \to \mathbf{R}$ and $N$ the number operator characterized by

$$[N, a^*(h)] = a^*(h).$$

**Theorem 5.5** *Let $P$ be a real polynomial with positive coefficients. Then the commutation relations*

$$[a(h_1), a^*(h_2)] = \langle h_1, h_2\rangle \cdot P(N)$$

*are realizable on a symmetric Hilbert space.*

We split the proof in a few lemmas.

**Lemma 5.6** *Let* $(F, \mathbf{j}_F)$ *and* $(G, \mathbf{j}_G)$ *be two symmetric Hilbert spaces for which the commutation relations hold*

$$\begin{aligned} [a_F(h_1), a_F^*(h_2)] &= \langle h_1, h_2 \rangle \cdot a(N) \\ [a_G(h_1), a_G^*(h_2)] &= \langle h_1, h_2 \rangle \cdot b(N) \end{aligned}$$

*where* $a, b$ *are real functions. Then on* $(F \times G, \mathbf{j}_{F\times G})$ *we have*

$$[a_{F\times G}(h_1), a_{F\times G}^*(h_2)] = \langle h_1, h_2 \rangle \cdot (a \cdot b)(N)$$

*Proof.* This is a direct application of Lemmas 4.4, 4.5 and the following equations:

$$\begin{aligned} (\mathbf{j} \cdot \bar{\mathbf{j}})_k((f,g),(f',g')) &= (\mathbf{j} \cdot \bar{\mathbf{j}})_k(f,f') \cdot (\mathbf{j} \cdot \bar{\mathbf{j}})_k(g,g') \\ (\bar{\mathbf{j}} \cdot \mathbf{j})_k((f,g),(f',g')) &= (\bar{\mathbf{j}} \cdot \mathbf{j})_k(f,f') \cdot (\bar{\mathbf{j}} \cdot \mathbf{j})_k(g,g') \end{aligned}$$

□

**Lemma 5.7** *Let* $\mathcal{A}$ *be the species of rooted trees. Let* $f \in \mathcal{A}[U]$, $g \in \mathcal{A}[U + \{*\}]$ *and*

$$\tilde{\mathbf{j}}_{\mathcal{A}}^c(f,g) = \mathbf{j}_{\mathcal{A}}(f,g) + c^{\frac{1}{2}} \delta_{f_*,g}$$

*a modification of the weight* $\mathbf{j}_{\mathcal{A}}$ *defined in section 4, with c, a positive constant. The structure* $f_* \in \mathcal{A}[U + \{*\}]$ *is defined by:*

$$f_*(u) = \begin{cases} f(u) & \text{if } u \neq root(f) \\ * & \text{if } u = root(f). \end{cases}$$

*Then on* $(\mathcal{A}, \tilde{\mathbf{j}}_{\mathcal{A}}^c)$ *we have:*

$$[a(h_1), a^*(h_2)] = \langle h_1, h_2 \rangle \cdot (N + c).$$

*Proof.* This is similar to the proof of Theorem 4.6, with an additional contribution to $(\mathbf{j} \cdot \bar{\mathbf{j}})_n(f,g)$ coming from the term $c^{\frac{1}{2}} \delta_{f_*,g}$ in $\tilde{\mathbf{j}}_{\mathcal{A}}^c$.

□

**Lemma 5.8** *Let* $\mathcal{A} \times \mathcal{A}$ *be the species of ordered pairs of rooted trees. We define the weight*

$$\mathbf{j}_{\mathcal{A}\times\mathcal{A}}^c((f,g),(f',g')) = \mathbf{j}_{\mathcal{A}}(f,g) \cdot \mathbf{j}_{\mathcal{A}}(f',g') + c^{\frac{1}{2}} \delta_{f_*,f'} \cdot \delta_{g_*,g'}.$$

*Then on* $(\mathcal{A} \times \mathcal{A}, \mathbf{j}_{\mathcal{A}\times\mathcal{A}}^c)$ *we have*

$$[a(h_1), a^*(h_2)] = \langle h_1, h_2 \rangle \cdot (N^2 + c).$$

*Proof.* Similar to the previous two lemmas.

□

Proof of Theorem 5.5. The polynomial $P(x)$ has a canonical expression as product of polynomials of the type $x + c$ and $x^2 + c$ with $c \geq 0$. The theorem follows by applying the previous 3 lemmas.

□

**Remark.** The result can be extended to power series with positive coefficients and infinite radius of convergence. In particular for $0 \leq q \leq 1$

$$s(x) = q^{-x} = \sum_{k=0}^{\infty} \frac{1}{n!} \cdot (-\log q)^n \cdot x^n$$

gives the commutation relations

$$[a_i, a_j^*] = q^{-N} \delta_{i,j}$$

which characterize the *q-deformations* [10], [26], up to a 'rescaling' of the creation and annihilation operators with a function of $N$.

**4) Compositions.** Let $(F, \mathbf{j}_F)$ and $(G, \mathbf{j}_G)$ be two combinatorial Fock spaces. We recall that the composition of $G$ in $F$ is a species whose structures are $F$-assemblies of $G$-structures:

$$F \circ G[U] = \sum_{\pi} F[\pi] \times \prod_{p \in \pi} G[p].$$

We would like to define the annihilation and creation operators for the species $F \circ G$ by making use of the available weights $\mathbf{j}_F$ and $\mathbf{j}_G$. Apart from the condition $|G[\emptyset]| = 0$ we require $|G[1]| = 1$. We consider an arbitrary structure $(f, \pi, (g_p)_{p \in \pi}) \in F \circ G[U]$ where $\pi$ is a partition of the finite set $U$. Then we note that there are two essentially different possibilities to 'add' a new point $*$, to the set $U$: one can enlarge the size of $\pi$ by creating a partition of $U + \{*\}$ of the form $\pi^+ = \pi + \{\{*\}\}$, or one can keep the size of $\pi$ constant by adding $*$ to one of the sets $p \in \pi$ and obtain the partition $\pi_p^+$. Between $\pi$ and $\pi_p^+$ there is the bijection

$$\alpha_p : p' \to \begin{cases} p' & \text{if } p' \neq p \\ p + \{*\} & \text{if } p' = p \end{cases}$$

We recognize that in the first case the weight $\mathbf{j}_F$ should play a role, while in the second, the weight $\mathbf{j}_G$. According to the properties of the species $F$, one can further distinguish among the subsets to which $*$ is added, by choosing (as we did for the creation and annihilation operators) a weight $\mathbf{j}_{F,\epsilon}$ on the cartesian product $F \times \epsilon$ where $\epsilon$ is the species of elements: $\epsilon[U] = U$. Putting together the three data $(\mathbf{j}_F, \mathbf{j}_G, \mathbf{j}_{F,\epsilon})$, we define:

$$\mathbf{j}_{F\circ G}((f,\pi,(g_p)_{p\in\pi}),(f',\pi',(g'_{p'})_{p'\in\pi'})) = \mathbf{j}_F(f,f')\cdot\prod_{p\in\pi}\delta_{g_p,g'_p}$$
$$+ \quad \sum_{p\in\pi}\delta_{f',F[\alpha_p](f)}\cdot\mathbf{j}_{F,\epsilon}(f,p)\cdot\mathbf{j}_G(g_p,g'_{p+\{*\}}).$$

where $f\in F[\pi],\ g_p\in G[p]$, etc.
**Remark.**We find this definition rather natural and broad enough to cover some interesting examples. One can easily check that $\mathbf{j}_{F\circ G}$ satisfies the invariance property characterizing the weights.
**Example:** The species Bal of ordered partitions or Ballots is the composition of $L$ (the species of linear orderings ), with $E_+$ (the species of nonempty sets). A typical structure over a finite set $U$ looks like: $s=(U_1,..,U_k)$ with $(U_p)_{p\in\{1,..,k\}}$, a partition of $U$. The vacuum is the empty sequence $\mathrm{Bal}[\emptyset]=\{\emptyset\}$. We use the weights $\mathbf{j}_E$ and $\mathbf{j}_L$ as defined in section 4. The action of the creation operator at the combinatorial level can be described as follows: we can add the point $*$ in the last subset of the sequence $s=(U_1,..,U_k)$ and obtain $s_k^+=(U_1,..,U_k+\{*\})$, or we can create a new subset $U_{k+1}=\{*\}$ and position it at the end of the sequence $s$, producing $s^+=(U_1,..,U_{k+1})$. We see that in this case the weight $\mathbf{j}_{L,\epsilon}$ is simply identifying the last element of the sequence: $\mathbf{j}_{L,\epsilon}(s,U_k)=\delta_{U_k,U_p}$. For the vacuum we set $\mathbf{j}_{\mathrm{Bal}}(\{\emptyset\},\{*\})=1$. We use $0\le q\le 1$ as an interpolation parameter:

$$\mathbf{j}_{\mathrm{Bal}}(s,s') = q^{\frac{1}{2}}\delta_{s_k^+,s'} \ + \ (1-q)^{\frac{1}{2}}\delta_{s^+,s'} \tag{5.10}$$

Let us denote by $t_{\mathrm{Bal}}$ the positive definite function associated to the vacuum state of the combinatorial Fock space $(\mathrm{Bal},\mathbf{j}_{\mathrm{Bal}})$, as defined in subsection 5. Following [13] we associate to any pair partition $\mathcal{V}\in\mathcal{P}_2(\infty)$ a set $B(\mathcal{V})=\{\mathcal{V}_1,..,\mathcal{V}_k\}$ such that $\mathcal{V}=\mathcal{V}_1\cup...\cup\mathcal{V}_k$ is the decomposition of $\mathcal{V}$ into connected sub-partitions or *blocks*.

**Theorem 5.9** *Let $\mathcal{V}\in\mathcal{P}_2(\infty)$. Then*

$$t_{\mathrm{Bal}}(\mathcal{V}) = q^{|\mathcal{V}|-|B(\mathcal{V})|} \tag{5.11}$$

*Proof.* We split the task of proving (5.11) into two simpler ones: first we prove the *strong multiplicativity* property for $t$:

$$t(\mathcal{V}) = \prod_{i=1}^{k} t(\mathcal{V}_i) \quad \text{if} \quad B(\mathcal{V})=\{\mathcal{V}_1,..,\mathcal{V}_k\}$$

and then for $\mathcal{V}$ consisting of a single block, $t_{\mathrm{Bal}}(\mathcal{V})=q^{|\mathcal{V}|-1}$. The proof of the strong multiplicativity is analogous to that of Proposition 5.4. We consider an orthogonal basis $(e_j)_{j\in J}$ of the Hilbert space $\mathcal{K}$ and a partition $J = J_1 + J_2$ of $J$ with the corresponding relation $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$. We define an isometry

$$S : \mathcal{F}^{\text{ext}}_{\text{Bal}}(\mathcal{K}_1) \otimes \mathcal{F}^{\text{ext}}_{\text{Bal}}(\mathcal{K}_2) \to \mathcal{F}^{\text{ext}}_{\text{Bal}}(\mathcal{K}_1 \oplus \mathcal{K}_2)$$

and we will prove that it has a natural action on monomials of creation and annihilation operators:

$$S(\prod_k a^{\sharp_k}_{i_k}\Omega \otimes \prod_p a^{\sharp_p}_{j_p}\Omega) = \prod_k a^{\sharp_k}_{i_k} \cdot \prod_p a^{\sharp_p}_{j_p}\Omega. \tag{5.12}$$

We recall that the two monomials satisfy certain properties which are described in Proposition 5.4. The multiplicativity of $t_{\text{Bal}}$ follows from equation (5.12) and the isometric property of $S$.
The action of $S$ on the orthogonal bases defined in Section 3 is:

$$\delta_{[s_1,c_1]} \otimes \delta_{[s_2,c_2]} \to \sum_s q^{\frac{a(s)}{2}} \cdot (1-q)^{\frac{b(s)}{2}} \cdot \delta_{[s,c]}$$

where, for arbitrary $s_1 = (U_1, .., U_k)$ and $s_2 = (V_1, .., V_p)$, the sum runs over all $s = (V_1, .., V_{p-1},\ V,\ U,\ U_2, .., U_k)$ with $V_p \subset V,\ U \subset U_1$ and $U + V = U_1 + V_p$. The coloring $c$ restricts to $c_1$ and $c_2$ on the sets $\bigcup_\alpha U_\alpha$ respectively $\bigcup_\beta V_\beta$. The coefficients appearing on the right side are $a(s) = |V| - |V_p|$ and $b(s) = |U|$. As $\|\delta_{[s,c]}\| = 1$ and $a(s) + b(s) = |U_1|$, we obtain

$$\begin{aligned} \|S(\delta_{[s_1,c_1]} \otimes \delta_{[s_2,c_2]})\|^2 &= \sum_{j=0}^{|U_1|} \binom{|U_1|}{j} \cdot q^k \cdot (1-q)^{|U_1|-k} \\ = 1 &= \|\delta_{[s_1,c_1]} \otimes \delta_{[s_2,c_2]}\|^2, \end{aligned}$$

which proves the isometry property. The equation (5.12) follows by induction w.r.t. $k$. For $k = 0$ is is obvious that

$$S\left(\Omega \otimes \prod_p a^{\sharp_p}_{j_p}\Omega\right) = \prod_p a^{\sharp_p}_{j_p}\Omega.$$

Then one can check on the basis vectors that

$$S \cdot (a^{\sharp_j}_j \otimes \mathbf{1}) = a^{\sharp_j}_j \cdot S$$

for $j \in J_1$, which provides the tool for the incrementation of $k$.

We pass now to the expression of $t_{\text{Bal}}(\mathcal{V})$ for a one block partition $\mathcal{V}$. The basic observation is that the creation and annihilation operators have the following form, stemming from that of $\mathbf{j}_{\text{Bal}}$ (see (5.10) ):

$$a^{\sharp_i}_i = q^{\frac{1}{2}} a^{\sharp_i}_{E,i} + (1-q)^{\frac{1}{2}} a^{\sharp_i}_{L,i}$$

with the choice $a^*_{E,i}\Omega = 0$. Let $M_{\mathcal{V}} = \prod_{l=1}^{2n} a^{\sharp_l}_{i_l}$ be a monomial associated to the pair partition $\mathcal{V} \in \mathcal{P}_2(2n)$. It is sufficient to prove that the only nonzero contribution to $M_{\mathcal{V}}\Omega$ is brought by the the term $a_{L,i_1}\prod_{l=2}^{2n-1} a_{E,i_l} \cdot a^*_{L,i_{2n}}\Omega = q^{n-1}\Omega$. Indeed the action of $a^*_{L,i_1}$ at the combinatorial level is to increase the number of subsets in a sequence by 1. Thus the terms which are nonzero must contain an equal number of creation and annihilation operators of type $L$. Let us consider such a term. Then there exist $1 \leq l_1 \leq l_2 \leq 2n$ such that on the positions $l_1$ and $l_2$ we have annihilation respectively creation operators of type $L$ and for $l_1 \leq l \leq l_2$ we have type $E$ operators. We have identified a submonomial

$$m = a_{L,l_1} \cdot \prod_{l=l_1+1}^{l_2-1} a^{\sharp_l}_{E,i_l} \cdot a^*_{L,l_2}$$

which is nonzero only if it corresponds to a pair partition, that is if all creation and annihilation operators pair each other according to the color. But this is possible only when $l_1 = 1$ and $l_2 = 2n$ because $\mathcal{V}$ is a one-block pair partition.

□

**4) Free Products.** Inspired by the notion of freeness introduced by Dan Voiculescu [65] we make the following:
**Definition.**Let $(F_\alpha)_{\alpha\in J}$ be a finite set of species of structures with $F_\alpha[\emptyset] = \{\emptyset\}$ for all $\alpha \in J$. The *free product* of $(F_\alpha)_{\alpha\in J}$ is the species defined by:

$$\begin{aligned} *_{\alpha\in J}(F_\alpha)[U] &= \{(\pi,(s_1,..,s_p))|\pi = (U_1,..,U_p) \in \mathrm{Bal}[U], s_i \in F_{\alpha_i}[U_i], \\ &\quad \alpha_i \neq \alpha_{i+1} \text{ for } i = 1,..p-1\} \end{aligned} \tag{5.13}$$

for $U \neq \emptyset$ and $*_{\alpha\in J}F_\alpha[\emptyset] = \{\emptyset\}$. The transport is induced from the species $(F_\alpha)_{\alpha\in J}$ and Bal. From the definition it is clear that we have the following combinatorial equation:

$$*_{\alpha\in J}(F_\alpha) = 1 + \sum_{p\geq 1}\ \sum_{\alpha_1\neq\alpha_2\neq..\neq\alpha_p} F_{\alpha_1+}\cdot F_{\alpha_2+}\cdot \ldots \cdot F_{\alpha_p+}$$

and using the property $\mathcal{F}_{F\cdot G} = \mathcal{F}_F \otimes \mathcal{F}_G$ we obtain

$$\mathcal{F}_{*_{\alpha\in J}(F_\alpha)}(\mathcal{K}) = *_{\alpha\in J}(\mathcal{F}_{F_\alpha}, \Omega_\alpha)$$

where the last object is the Hilbert space free product [65].
The corresponding weight is similar to the one used for the species Bal. For $f_i \in F_{\alpha_i}[U_i]$ and $f'_i \in F_{\alpha'_i}[V_i]$, it has the expression:

$$\begin{aligned} &\mathbf{j}_{*_{\alpha\in J(F_\alpha)}}((\pi,(f_1,..,f_p)),(\pi',(f'_1,..,f'_q))) = \\ &\delta_{p,q}\cdot\delta_{\alpha_p,\alpha'_p}\prod_{i=1}^{p-1}\delta_{f_i,f'_i}\cdot \mathbf{j}_{\alpha_p}(f_p.f'_p) + \delta_{p+1,q}\cdot\prod_{i=1}^{p}\delta_{f_i,f'_i}\cdot\mathbf{j}_{\alpha'_q}(\{\emptyset\},f'_q). \end{aligned}$$

Moreover the creation and annihilation operators can be written like

$$a^{\sharp}_{*_{\alpha \in J(F_\alpha)}, i} = \sum_{\alpha} a^{\sharp}_{F_\alpha, i}$$

with the relations [65],

$$a_{F_\alpha, i} \cdot a^{*}_{F_\beta, j} = 0$$

for $\alpha \neq \beta$.

CHAPTER III

# Generalised Brownian motion and second quantisation [2]

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ABSTRACT

A new approach to the generalised Brownian motion introduced by M. Bożejko and R. Speicher is described, based on symmetry rather than deformation. The symmetrization principle is provided by Joyal's notions of tensorial and combinatorial species. Any such species $V$ gives rise to an endofunctor $\mathcal{F}_V$ of the category of Hilbert spaces with contractions. A generalised Brownian motion is an algebra of creation and annihilation operators acting on $\mathcal{F}_V(\mathcal{H})$ for arbitrary Hilbert spaces $\mathcal{H}$ and having a prescription for the calculation of vacuum expectations in terms of a function $\mathbf{t}$ on pair partitions. The positivity is encoded by a $^*$-semigroup of 'broken pair partitions' whose representation space with respect to $\mathbf{t}$ is $V$. The existence of two second quantisation functors $\Gamma_{\mathbf{t}}^{\infty}$ and $\Delta_{\mathbf{t}}$ is discussed and connected to the multiplicativity property of the function $\mathbf{t}$ . For a certain one parameter interpolation between the fermionic and the free Brownian motion it is shown that the 'field algebras' $\Gamma(\mathcal{K})$ are type $\mathrm{II}_1$ factors when $\mathcal{K}$ is infinite dimensional.

[2] This chapter is based on reference [27].

# 1 Introduction

In non-commutative probability theory one is interested in finding generalisations of classical probabilistic concepts such as independence and processes with independent stationary increments. Motivated by a central limit theorem result and by the analogy with classical Brownian motion, M. Bożejko and R. Speicher proposed in [13] a class of operator algebras called 'generalised Brownian motions' and investigated an example of interpolation between the classical [53] and the free motion of Voiculescu [65]. A better known interpolation is provided by the '$q$-deformed commutation relations' [9, 10, 11, 21, 23, 26, 45, 67]. Such an operator algebra is obtained by performing the GNS representation of the free tensor algebra $\mathcal{A}(\mathcal{K})$ over an arbitrary infinite dimensional real Hilbert space $\mathcal{K}$, with respect to a 'Gaussian state' $\tilde{\rho}_{\mathbf{t}}$ defined by the following 'pairing prescription':

$$\tilde{\rho}_{\mathbf{t}}(\omega(f_1)\ldots\omega(f_n)) = \begin{cases} 0 & \text{if } n \text{ odd} \\ \sum\limits_{\mathcal{V}\in\mathcal{P}_2(n)} \mathbf{t}(\mathcal{V}) \prod\limits_{(k,l)\in\mathcal{V}} \langle f_k, f_l\rangle & \text{if } n \text{ even} \end{cases} \tag{1.1}$$

where $f_i \in \mathcal{K}, \omega(f_i) \in \mathcal{A}(\mathcal{K})$ and the sum runs over all pair partitions of the ordered set $\{1, 2, \ldots, n\}$. The functional is uniquely determined by the complex valued function $\mathbf{t}$ on pair partitions. Classical Brownian motion is obtained by taking $\mathcal{K} = \mathrm{L}^2(\mathbb{R}_+)$ and $B_s := \omega(\mathbf{1}_{[0,s)})$ with the constant function $\mathbf{t}(\mathcal{V}) = 1$ on all pair partitions; the free Brownian motion [65] requires $\mathbf{t}$ to be 0 on crossing partitions and 1 on non-crossing partitions.

If one considers complex Hilbert spaces, the analogue of a Gaussian state is called a Fock state. We show that the GNS representation of the free algebra $\mathcal{C}(\mathcal{H})$ of creation and annihilation operators with respect to a Fock state $\rho_{\mathbf{t}}$ can be described in a functorial way inspired by the notions of tensorial species of Joyal [35, 36]: the representation space has the form

$$\mathcal{F}_{\mathbf{t}}(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \frac{1}{n!} V_n \otimes_s \mathcal{H}^{\otimes n} \tag{1.2}$$

where $V_n$ are Hilbert spaces carrying unitary representations of the symmetric groups $\mathrm{S}(n)$ and $\otimes_s$ means the subspace of the tensor product containing vectors which are invariant under the double action of $\mathrm{S}(n)$. The creation operators have the expression:

$$a^*_{\mathbf{t}}(h)\ v \otimes_s (h_0 \otimes \ldots \otimes h_{n-1}) = (j_n v) \otimes_s (h_0 \otimes \ldots \otimes h_{n-1} \otimes h_n) \tag{1.3}$$

where $j_n : V_n \to V_{n+1}$ is an operator which intertwines the action of $\mathrm{S}(n)$ and $\mathrm{S}(n+1)$.

In Section 3 we connect these Fock representations with positive functionals on a certain algebraic object $\mathcal{BP}_2(\infty)$ which we call the *-semigroup of 'broken pair partitions'. The elements of this $^*$-semigroup can be described graphically as segments located between two vertical lines which cut through the graphical representation of a pair partition. In particular, the pair partitions are elements of $\mathcal{BP}_2(\infty)$. We show that if $\rho_{\mathbf{t}}$ is a Fock state then the function $\mathbf{t}$ has a natural extension to a positive functional $\hat{\mathbf{t}}$ on $\mathcal{BP}_2(\infty)$. The GNS-like representation with respect to $\hat{\mathbf{t}}$ provides the combinatorial data $(V_n, j_n)_{n=0}^{\infty}$ associated to $\rho_{\mathbf{t}}$.

The representation of $\mathcal{A}(\mathcal{K})$ with respect to a Gaussian state $\tilde{\rho}_{\mathbf{t}}$ is a $^*$-algebra generated by 'fields' $\omega_{\mathbf{t}}(f)$. Monomials of such fields can be seen as moments, with the corresponding cumulants being a generalisation of the Wick products known from the $q$-deformed Brownian motion [9]. Using generalised Wick products we prove that any Gaussian state $\tilde{\rho}_{\mathbf{t}}$ extends to a Fock state $\rho_{\mathbf{t}}$ on the algebra of creation and annihilation operators $\mathcal{C}(\mathcal{K}_{\mathbb{C}})$ (see section 4).

Second quantisation is a special type of *functor of white noise*, a functor from the category of real Hilbert spaces with contractions to the category of (non-commutative) probability spaces such that the zero dimensional Hilbert space $\{0\}$ is mapped into the algebra $\mathbb{C}$. The underlying idea is to use the field operators $\omega_{\mathbf{t}}(\cdot)$ to construct von Neumann algebras $\Gamma_{\mathbf{t}}(\mathcal{K})$ for any real Hilbert space $\mathcal{K}$ and a fixed positive definite functions $\mathbf{t}$. The question is for which $\mathbf{t}$ one can carry out the construction of such a functor $\Gamma_{\mathbf{t}}$. From general considerations on functors of second quantisation we obtain that the function $\mathbf{t}$ must have the multiplicative property, a form of statistical independence. Conversely, for multiplicative $\mathbf{t}$ the field operators are essentially selfadjoint, and provide a natural definition of the von Neumann algebra $\Gamma_{\mathbf{t}}(\mathcal{K})$. However two other algebras seem to be more suitable for proving functorial properties: $\Gamma_{\mathbf{t}}^{\infty}(\mathcal{K})$ and $\Delta_{\mathbf{t}}(\mathcal{K})$ both acting on $\mathcal{F}_{\mathbf{t}}(\mathcal{K}_{\mathbb{C}} \oplus \ell^2(\mathbb{Z}))$ by letting the $\ell^2_{\mathbb{R}}(\mathbb{Z})$ 'modes' passive. We show that $\Delta_{\mathbf{t}}$ is a functor of second quantisation while $\Gamma_{\mathbf{t}}^{\infty}$ is a functor from Hilbert spaces to non-commutative probability spaces for which $\Gamma^{\infty}(\mathbb{R}) = \mathbb{C}$ if and only if $\rho_{\mathbf{t}}$ is faithful for $\Gamma_{\mathbf{t}}(\ell^2_{\mathbb{R}}(\mathbb{Z}))$. In this case the two functors coincide.

In the last section we develop a useful criterion, in terms of the spectrum of a characteristic contraction, for factoriality of the algebras $\Gamma_{\mathbf{t}}(\ell^2(\mathbb{Z}))$ in the case when the vacuum state $\rho_{\mathbf{t}}$ is tracial. We then apply it to a particular example of positive definite function $\mathbf{t}_q$ where $0 \leq q < 1$, which interpolates between the bosonic and free cases and has been introduced in [13] (see chapter II for another proof of the positivity). We conclude that $\Gamma_{\mathbf{t}}(\ell^2(\mathbb{Z}))$ is a type $\mathrm{II}_1$ factor.

## 2 Definitions and description of the Fock representation

The generalised Brownian motions [13] are representations with respect to special *gaussian* states on free algebras over real Hilbert spaces. We start by giving all necessary definitions and subsequently we will analyze the structure of the *Fock representations* which are intimately connected with the generalised Brownian motion (see section 4) .

**Definition 2.1** Let $\mathcal{K}$ be a real Hilbert space. The algebra $\mathcal{A}(\mathcal{K})$ is the free unital $^*$-algebra with generators $\omega(h)$ for all $h \in \mathcal{K}$, divided by the relations:

$$\omega(af + bg) = a\omega(f) + b\omega(g), \qquad \omega(f) = \omega(f)^* \tag{2.1}$$

for all $f, g \in \mathcal{K}$ and $a, b \in \mathbb{R}$.

**Definition 2.2** Let $\mathcal{H}$ be a complex Hilbert space. The algebra $\mathcal{C}(\mathcal{H})$ is the free unital $^*$-algebra with generators $a(h)$ and $a^*(h)$ for all $h \in \mathcal{H}$, divided by the relations:

$$a^*(\lambda f + \mu g) = \lambda a^*(f) + \mu a^*(g), \qquad a^*(f) = a(f)^* \tag{2.2}$$

for all $f, g \in \mathcal{H}$ and $\lambda, \mu \in \mathbb{C}$.

We notice the existence of the canonical injection from $\mathcal{A}(\mathcal{K})$ to $\mathcal{C}(\mathcal{K}_\mathbb{C})$

$$\omega(h) \mapsto a(h) + a^*(h) \tag{2.3}$$

where $\mathcal{K}_\mathbb{C}$ is the complexification of the real Hilbert space $\mathcal{K}$. On the algebras defined above we would like to define positive linear functionals by certain pairing prescriptions for which we need some notions of pair partitions.

**Definition 2.3** Let $S$ be a finite ordered set. We denote by $\mathcal{P}_2(S)$ is the set of pair partitions of $S$, that is $\mathcal{V} \in \mathcal{P}_2(S)$ if $\mathcal{V}$ consists of $\frac{1}{2}n$ disjoint ordered pairs $(l, r)$ with $l < r$ having $S$ as their reunion. The set of all pair partitions is

$$\mathcal{P}_2(\infty) := \bigcup_{r=0}^{\infty} \mathcal{P}_2(2r). \tag{2.4}$$

Note that $\mathcal{P}_2(n) = \emptyset$ if $n$ is odd. In this paper the symbol $\mathbf{t}$ will always stand for a function $\mathbf{t} : \mathcal{P}_2(\infty) \to \mathbb{C}$. We will always choose the normalization $\mathbf{t}(p) = 1$ for $p$ the pair partition containing only one pair.

**Definition 2.4** A *Fock state* on the algebra $\mathcal{C}(\mathcal{H})$ is a positive normalized linear functional $\rho_\mathbf{t} : \mathcal{C}(\mathcal{H}) \to \mathbb{C}$ of the form

$$\rho_\mathbf{t}(a^{\sharp_1}(f_1) \dots a^{\sharp_n}(f_n)) = \sum_{\mathcal{V} \in \mathcal{P}_2(n)} \mathbf{t}(\mathcal{V}) \prod_{(k,l) \in \mathcal{V}} \langle f_k, f_l \rangle \cdot Q(\sharp_k, \sharp_l) \tag{2.5}$$

the symbols $\sharp_i$ standing for creation or annihilation and the two by two covariance matrix $Q$ is given by

$$Q = \begin{pmatrix} \rho(a_i a_i) & \rho(a_i a_i^*) \\ \rho(a_i^* a_i) & \rho(a_i^* a_i^*) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

where $a_i = a(e_i)$ and $e_i$ is an arbitrary normalized vector in $\mathcal{H}$. Note that the l.h.s. of (2.5) is zero for odd values of $n$.

**Definition 2.5** A *Gaussian state* on $\mathcal{A}(\mathcal{K})$ is a positive normalized linear functional $\tilde{\rho}_{\mathbf{t}}$ with moments

$$\tilde{\rho}_{\mathbf{t}}(\omega(f_1)\dots\omega(f_n)) = \sum_{\mathcal{V}\in\mathcal{P}_2(n)} \mathbf{t}(\mathcal{V}) \prod_{(k,l)\in\mathcal{V}} \langle f_k, f_l\rangle \tag{2.6}$$

**Remark.** The restriction of a Fock state $\rho_{\mathbf{t}}$ on $\mathcal{C}(\mathcal{K}_{\mathbb{C}})$ to the subalgebra $\mathcal{A}(\mathcal{K})$ is the Gaussian state $\tilde{\rho}_{\mathbf{t}}$. If $\rho_{\mathbf{t}}$ is a Fock state for all choices of $\mathcal{K}$ then we call the function

$$\mathbf{t} : \mathcal{P}_2(\infty) \to \mathbb{C}$$

*positive definite*.

The GNS representations associated to pairs $(\mathcal{C}(\mathcal{H}), \rho_{\mathbf{t}})$ have been studied in a number of cases. One obtains a representation $\pi_{\mathbf{t}}$ of $\mathcal{C}(\mathcal{H})$ as $^*$-algebra of creation and annihilation operators acting on a Hilbert space $\mathcal{F}_{\mathbf{t}}(\mathcal{H})$ which has a Fock-type structure

$$\mathcal{F}_{\mathbf{t}}(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n$$

with $\mathcal{H}_n$ being a (symmetric) subspace of $\mathcal{H}^{\otimes n}$ in the case of bosonic or fermionic algebras [53], the full tensor product in models of free probability [65], a deformation of it in the case of q-deformations [9, 10, 11, 21, 23, 26, 45, 67], or even 'larger' spaces containing more copies of $\mathcal{H}^{\otimes n}$ with a deformed inner product in the case of another deformation depending on a parameter $-1 \leq q \leq 1$ constructed in [13]. The action of the creation operators is $a^*(f)\Omega_{\mathbf{t}} = f \in \mathcal{H}$,

$$a^*(f) f_1 \otimes \dots \otimes f_n = f \otimes f_1 \otimes \dots \otimes f_n$$

while that of the annihilation operator is less transparent, depending on the inner product on $\mathcal{H}_n$. Proving the positivity of this inner product is in general nontrivial.

In reference [28] (chapter II of this thesis) we have followed a different, more combinatorial approach to the study of the representations $\pi_{\mathbf{t}}(\mathcal{C}(\mathcal{H}))$ for various examples of positive definite functions $\mathbf{t}$. We give here a brief description of our construction. The representation space is denoted by $\mathcal{F}_V(\mathcal{H})$ and has certain symmetry properties encoded by a sequence $(V_n)_{n=0}^{\infty}$ of (not necessarily finite dimensional) Hilbert spaces such that each $V_n$ carries a unitary representation of the symmetric group $\mathrm{S}(n)$

$$\mathrm{S}(n) \ni \pi \mapsto U(\pi) \in \mathcal{U}(V_n). \tag{2.7}$$

In concrete examples we have realized $V_n$ as $\ell^2(F[n])$ where $F[\ ]$ is a *species of structures* [5, 35, 36], i.e., a functor from the category of finite sets with bijections as morphisms to the category of finite sets with maps as morphisms. For each finite set $A$, the rule $F$ prescribes a finite set $F[A]$ whose elements are called

$F$-structures over the set $A$. Moreover for any bijection $\sigma : A \to B$ there is a map $F[\sigma] : F[A] \to F[B]$ such that $F[\sigma \circ \tau] = F[\sigma] \circ F[\tau]$ and $F[\mathrm{id}_A] = \mathrm{id}_{F[A]}$. In particular for $n := \{0, 1, \dots n-1\}$ there is an action of the symmetric group $\mathrm{S}(n)$ on the set of structures:

$$\forall \pi \in \mathrm{S}(n), \qquad F[\pi] : F[n] \to F[n]$$

which gives a unitary representation $U(\cdot)$ of $\mathrm{S}(n)$ on $V_n := \ell^2(F[n])$. Simple examples are such species as sets, ordered sequences, trees, graphs, etc.
We define

$$\mathcal{F}_V(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \frac{1}{n!} V_n \otimes_s \mathcal{H}^{\otimes n} \tag{2.8}$$

where $V_n \otimes_s \mathcal{H}^{\otimes n}$ is the subspace of $V_n \otimes \mathcal{H}^{\otimes n}$ spanned by the vectors $\psi$ invariant under the action of $\mathrm{S}(n)$:

$$\psi = (U(\pi) \otimes \tilde{U}(\pi))\psi, \qquad \text{for all } \pi \in \mathrm{S}(n)$$

with $\tilde{U}(\pi) \in \mathcal{U}(\mathcal{H}^{\otimes n})$,

$$\tilde{U}(\pi) : h_0 \otimes \dots \otimes h_{n-1} \mapsto h_{\pi^{-1}(0)} \otimes \dots \otimes h_{\pi^{-1}(n-1)}, \tag{2.9}$$

the factor $\frac{1}{n!}$ referring to the inner product. The symmetric Hilbert space $\mathcal{F}_V(\mathcal{H})$ is spanned by linear combinations of vectors of the form:

$$v \otimes_s h_0 \otimes \dots \otimes h_{n-1} := \sum_{\pi \in \mathrm{S}(n)} U(\pi) v \otimes \tilde{U}(\pi) h_0 \otimes \dots \otimes h_{n-1}. \tag{2.10}$$

The creation and annihilation operators are defined with the help of a sequence of densely defined linear maps $(j_n)_{n=0}^{\infty}$ with $j_n : V_n \to V_{n+1}$ satisfying the intertwining relations

$$j_n \cdot U(\pi) = U(\iota_n(\pi)) \cdot j_n, \qquad \forall \pi \in \mathrm{S}(n) \tag{2.11}$$

with $\iota_n : \mathrm{S}(n) \to \mathrm{S}(n+1)$ being the canonical embedding associated to the inclusion of sets

$$n := \{0, 1, \dots, n-1\} \hookrightarrow n+1 := \{0, 1, \dots, n\}. \tag{2.12}$$

In the examples using species of structures the map $j_n : \ell^2(F[n]) \to \ell^2(F[n+1])$ is constructed by giving the matrix elements $j_n(s,t) := \langle \delta_t, V_n \delta_s \rangle$ which can be seen as 'transition coefficients' between $s \in F[n]$ and $t \in F[n+1]$. For example if the species $F[\cdot]$ is that of rooted trees one can choose $j_n(s,t) = 1$ if the tree $s$ is obtained by removing the leaf with label $n$ from the tree $t$; otherwise we choose $j_n(s,t) = 0$. This is described in detail in [28], chapter II of this thesis. Notice that there is no canonical manner of defining $j_n$ but certain species of structures

offer rather natural definitions, for example the species of sets, ordered sequences, rooted trees, oriented graphs, sequences of non empty sets, etc [28].
Let $h \in \mathcal{H}$; the creation operator $a^*_{V,j}(h)$ has the action:

$$a^*_{V,j}(h)v \otimes_s (h_0 \otimes \ldots \otimes h_{n-1}) := (j_n v) \otimes_s (h_0 \otimes \ldots \otimes h_{n-1} \otimes h). \tag{2.13}$$

The annihilation operator $a_{V,j}(h)$ is the adjoint of $a^*_{V,j}(h)$. Its action on the $n+1$-th level is given by the restriction of the operator

$$\begin{array}{rcl} \tilde{a}_{V,j}(h) : V_{n+1} \otimes \mathcal{H}^{\otimes n+1} & \rightarrow & V_n \otimes \mathcal{H}^{\otimes n} \\ v \otimes (h_0 \otimes \ldots \otimes h_n) & \mapsto & \langle h, h_n \rangle \, j_n^* v \otimes (h_0 \otimes \ldots \otimes h_{n-1}). \end{array} \tag{2.14}$$

to the subspace $V_{n+1} \otimes_s \mathcal{H}^{\otimes n+1}$. Note that due to condition (2.11) the operators $a^*_{V,j}(h), a_{V,j}(h)$ are well defined. Let us denote by $\mathcal{C}_{V,j}(\mathcal{H})$ the $^*$-algebra generated by all operators $a^*_{V,j}(h), a_{V,j}(h)$ and by $\Omega_V \in V_0$ the normalized vacuum vector in $\mathcal{F}_V(\mathcal{H})$. The following theorem is a generalisation of Proposition 5.1 in chapter II:

**Theorem 2.6** *Let $(\mathcal{F}_V(\mathcal{H}), \mathcal{C}_{V,j}(\mathcal{H}), \Omega_V)$ be a representation of $\mathcal{C}(\mathcal{H})$ as described above, then the state $\rho_{V,j}(\cdot) = \langle \Omega_V, \cdot \Omega_V \rangle$ is a Fock state, i.e. there exists a positive definite function $\mathbf{t}$ on pair partitions depending on $(V_n, j_n)_{n=0}^{\infty}$ such that $\rho_{V,j} = \rho_{\mathbf{t}}$.*

*Sketch of the proof.* Let $A \in \mathcal{B}(\mathcal{H})$. On $\mathcal{F}_V(\mathcal{H})$ we define the operator

$$\mathrm{d}\Gamma_V(A) : v_n \otimes_s f_0 \otimes \ldots \otimes f_{n-1} \mapsto \sum_{k=0}^{n-1} v_n \otimes_s f_0 \otimes \ldots \otimes Af_k \otimes \ldots \otimes f_{n-1} \tag{2.15}$$

for $v_n \in V_n, f_i \in \mathcal{H}$. Then the following commutation relations hold:

$$[a_{V,j}(f), \mathrm{d}\Gamma_V(A)] = a_{V,j}(A^* f). \tag{2.16}$$

In particular by choosing an orthonormal basis $\{e_i\}_{i \in I}$ in $\mathcal{H}$ and denoting $a_i^\sharp := a^\sharp_{V,j}(e_i)$ we obtain for all $i_k \neq i_0$

$$[\mathrm{d}\Gamma_V(|e_{i_0}\rangle\langle e_i|), a_{i_k}^{\sharp_k}] = \delta_{i_k,i} \cdot \delta_{\sharp_k,*} \cdot a^*_{i_0}. \tag{2.17}$$

Let $\psi = \left(\prod_{k=1}^n a_{i_k}^{\sharp_k}\right) \Omega_V$. Then $a_{i_0}\psi = 0$ if $i_0 \neq i_k$ for all $k = 1, \ldots, n$. By using (2.16), it follows that

$$a_i \psi = [a_{i_0}, \mathrm{d}\Gamma(|e_{i_0}\rangle\langle e_i|)]\psi = a_{i_0} \mathrm{d}\Gamma(|e_{i_0}\rangle\langle e_i|)\psi. \tag{2.18}$$

We then apply (2.17) repeatedly to obtain

$$a_i \left( \prod_{k=1}^n a_{i_k}^{\sharp_k} \right) \Omega_V = \sum_{k=1}^n \delta_{i,i_k} \cdot \delta_{\sharp_k,*} \cdot a_{i_0} \left( \prod_{p=1}^{k-1} a_{i_p}^{\sharp_p} \right) \cdot a^*_{i_0} \cdot \left( \prod_{q=k+1}^n a_{i_q}^{\sharp_q} \right) \Omega_V. \tag{2.19}$$

The vacuum expectation of a monomial $\prod_{k=1}^n a_{i_k}^{\sharp_k}$ can be different from zero only if the number of creators is equal to the number of annihilators, $a_{i_1}^{\sharp_1}$ is an annihilator and $a_{i_n}^{\sharp_n}$ a creator. We will therefore assume that this is the case. We put the monomial in the form $a_{i_1}\prod_{k=2}^n a_{i_k}^{\sharp_k}$ and apply (2.19). We obtain a sum over all pairs $(a_{i_1}, a_{i_k}^*)$ of the same color ($i_1 = i_k$) and replace $i_1$ by a new color $i_0$. We pass now to the next annihilator in each term of the sum and repeat the procedure, the new color which we add this time being different from all the colors used previously. After $\frac{n}{2}$ steps we obtain a sum containing all possible pairings of annihilators and creators of the same color in $\prod_{k=1}^n a_{i_k}^{\sharp_k}$:

$$\rho_{V,j}(\prod_{k=1}^n a_{i_k}^{\sharp_k}) = \sum_{\mathcal{V}\in\mathcal{P}_2(n)} \prod_{k,l\in\mathcal{V}} \delta_{i_k,i_l}\cdot Q(\sharp_k,\sharp_l)\cdot t(\mathcal{V}) \tag{2.20}$$

with $t(\mathcal{V}) := \rho_{V,j}(\prod_{k=1}^n a_{j_k}^{\sharp_k})$, where the indices $j_k, \sharp_k$ satisfy the following conditions: if $k\neq l$ then $j_k = j_l$ if and only if $(k,l)\in\mathcal{V}$, in which case $a_{j_k}^{\sharp_k}$ is annihilator and $a_{j_l}^{\sharp_l}$ is creator.

□

We prove now that the converse is also true.

**Theorem 2.7** *Let* $\mathbf{t}$ *be a positive definite function on pair partitions. Then for any complex Hilbert space* $\mathcal{H}$ *the GNS-representation of* $(\mathcal{C}(\mathcal{H}), \rho_{\mathbf{t}})$ *is unitarily equivalent to* $(\mathcal{F}_V(\mathcal{H}), \mathcal{C}_{V,j}(\mathcal{H}), \Omega_V)$ *for a sequence* $(V_n, j_n)_{n=0}^\infty$ *dependent only up to unitary equivalence on* $\mathbf{t}$.

*Proof.* We first consider $\mathcal{H} := \ell^2(\mathbb{N}^*)$ with the orthonormal basis $(e_i)_{i=1}^\infty$. We split the proof in 3 steps.
1. Identify the spaces $V_n$ and the maps $j_n$.
Let $(\mathcal{F}_{\mathbf{t}}(\mathcal{H}), \mathcal{C}_{\mathbf{t}}(\mathcal{H}), \Omega_{\mathbf{t}})$ be the triple obtained from the GNS-construction. Let $V_n$ be the closure of the subspace of $\mathcal{F}_{\mathbf{t}}(\mathcal{H})$ spanned by vectors of the form $v_n := (\prod_{k=1}^{2p+n} a_{\mathbf{t}}^{\sharp_k}(e_{i_k}))\Omega_{\mathbf{t}}$ for which the following conditions hold:

(i) in the sequence $(a_{\mathbf{t}}^{\sharp_k}(e_{i_k}))_{k=1}^{2p+n}$ each creation operator $a_{\mathbf{t}}^*(e_j)$ appears exactly once for $1\leq j\leq n$;

(ii) the rest of the sequence contains $p$ creation operators $(a_{\mathbf{t}}^*(e_{l_q}))_{q=1}^p$ and $p$ annihilation operators $(a_{\mathbf{t}}(e_{l_q}))_{q=1}^p$ for $p$ vectors $(e_{l_q})_{q=1}^p$ different among each other and with $l_q\notin\{1,\dots,n\}$ for all $1\leq q\leq p$. The vector $v_n$ does not depend in fact on the colors $(l_q)_{q=1}^p$ but only on the positions of the creation and annihilation operators in the monomial. Thus when necessary we can consider $l_q > N$ for all $1\leq q\leq n$ and some fixed big enough $N\in\mathbb{N}$ .
The map $j_n$ is defined as the restriction of $a_{\mathbf{t}}^*(e_{n+1})$ to $V_n$:

$$j_n\prod_{k=1}^{2p+n} a_{\mathbf{t}}^{\sharp_k}(e_{i_k})\Omega_{\mathbf{t}} = a_{\mathbf{t}}^*(e_{n+1})\prod_{k=1}^{2p+n} a_{\mathbf{t}}^{\sharp_k}(e_{i_k})\Omega_{\mathbf{t}}.$$

Obviously, the image of $j_n$ lies in $V_{n+1}$.
The state $\rho_{\mathbf{t}}$ is invariant under unitary transformations $U \in \mathcal{U}(\mathcal{H})$:

$$\rho_{\mathbf{t}}(\prod_{k=1}^{n} a^{\sharp_k}(e_{i_k})) = \rho_{\mathbf{t}}(\prod_{k=1}^{n} a^{\sharp_k}(Ue_{i_k})).$$

Thus

$$\mathcal{F}_{\mathbf{t}}(U) : \prod_{k=1}^{n} a_{\mathbf{t}}^{\sharp_k}(e_{i_k})\Omega_{\mathbf{t}} \mapsto \prod_{k=1}^{n} a_{\mathbf{t}}^{\sharp_k}(Ue_{i_k})\Omega_{\mathbf{t}} \tag{2.21}$$

is unitary and $\mathcal{F}_{\mathbf{t}}(U_1)\mathcal{F}_{\mathbf{t}}(U_2) = \mathcal{F}_{\mathbf{t}}(U_1U_2)$ for two unitaries $U_1, U_2$. The action on the algebra of creation and annihilation operators is

$$\mathcal{F}_{\mathbf{t}}(U)a_{\mathbf{t}}^{\sharp}(f)\mathcal{F}_{\mathbf{t}}(U^*) = a_{\mathbf{t}}^{\sharp}(Uf). \tag{2.22}$$

Considering unitaries which act by permuting the basis vectors $\{e_1, \ldots, e_n\}$ and leave all the others invariant we obtain a unitary representation of $\mathrm{S}(n)$ on $V_n$. The intertwining property (2.11) follows immediately from the definition of $j_n$. Having the 'combinatorial data' $(V_n, j_n)$, we can construct the triple $(\mathcal{F}_V(\mathcal{H}), \mathcal{C}_{V,j}(\mathcal{H}), \Omega_V)$ according to equations (2.8, 2.13, 2.14). Similarly to $\mathcal{F}_{\mathbf{t}}(U)$ we have the unitary

$$\begin{aligned} \mathcal{F}_V(U) : \mathcal{F}_V(\mathcal{H}) &\rightarrow \mathcal{F}_V(\mathcal{H}) \\ v \otimes_s (h_0 \otimes \ldots h_{n-1}) &\mapsto v \otimes_s (Uh_0 \otimes \ldots \otimes Uh_{n-1}) \end{aligned} \tag{2.23}$$

for $U \in \mathcal{U}(\mathcal{H}), v \in V_n$. We call $F_V(U)$ the *second quantisation* of $U$ at the Hilbert space level. Its action on operators is:

$$\mathcal{F}_V(U)a_{V,j}^{\sharp}(f)\mathcal{F}_V(U^*) = a_{V,j}^{\sharp}(Uf). \tag{2.24}$$

Analogously to $V_n$ we define for any finite subset $\{i_1, \ldots, i_n\} \subset \mathbb{N}$ the linear subspace $V(i_1, \ldots, i_n)$ of $\mathcal{F}_{\mathbf{t}}(\mathcal{H})$ spanned by applying to the vacuum $\Omega_{\mathbf{t}}$ monomials $\prod_{k=1}^{2p+n} a_{\mathbf{t}}^{\sharp_k}(e_{j_k})$ for which the colors $(j_k)_{k=1}^{2p+n}$ satisfy conditions similar to i), ii) but now with $\{i_1, \ldots, i_n\}$ instead of $\{1, \ldots, n\}$. For a unitary $U$ which permutes the basis vectors, $Ue_i = e_{u(i)}$ we get

$$\mathcal{F}_{\mathbf{t}}(U)V(i_1, \ldots, i_n) = V(u(i_1), \ldots, u(i_n)). \tag{2.25}$$

One can check by calculating inner products that any two such spaces are either orthogonal or coincide. Similarly, we define the following subspaces of $\mathcal{F}_V(\mathcal{H})$

$$\tilde{V}(i_1, \ldots, i_n) := \overline{\mathrm{lin}\{v \otimes_s (e_{i_1} \otimes \ldots \otimes e_{i_n}) : \quad v \in V_n\}} \tag{2.26}$$

which are also orthogonal for different sets of 'colors' $\{i_1, \ldots, i_n\}$.
2. We proceed by proving the equality of the states $\rho_{\mathbf{t}}$ and $\rho_{V,j}$.

As $\rho_{V,j}$ is a Fock state by Theorem 2.6, we need only verify that the positive definite function $\mathbf{t}$ we have started with and the one derived from $\rho_{V,j}$ coincide. By definition there is an isometry

$$\begin{aligned} T_n : V_n &\rightarrow \mathcal{F}_{V,j}(\mathcal{H}) \\ v &\mapsto v \otimes_s (e_1 \otimes \ldots \otimes e_n). \end{aligned} \tag{2.27}$$

Furthermore for any unitary $U \in \mathcal{U}(\mathcal{H})$ which permutes the basis vectors such that $Ue_k = e_{i_k}$, the operator

$$T(i_1, \ldots, i_n) : V(i_1, \ldots, i_n) \rightarrow \tilde{V}(i_1, \ldots, i_n)$$

defined by

$$T(i_1, \ldots, i_n) := \mathcal{F}_V(U) T_n \mathcal{F}_{\mathbf{t}}(U^*) \tag{2.28}$$

depends only on the set $\{i_1, \ldots, i_n\}$. Finally, the definitions of $j_n, a^\sharp_{V,j}(f)$ amounts to the fact that the following diagram commutes

$$\begin{array}{ccc} V_n & \xrightarrow{T_n} & \tilde{V}_n \\ {\scriptstyle a^*_{\mathbf{t}}(e_{n+1})}\Big\downarrow & & \Big\downarrow{\scriptstyle a^*_{V,j}(e_{n+1})} \\ V_{n+1} & \xrightarrow{T_{n+1}} & \tilde{V}_{n+1} \end{array} \tag{2.29}$$

and by acting from the left and from the right with the appropriate second quantisation operators and using (2.28, 2.22, 2.24) we obtain

$$\begin{array}{ccc} V(i_1, \ldots, i_n) & \xrightarrow{T(i_1,\ldots,i_n)} & \tilde{V}(i_1, \ldots, i_n) \\ {\scriptstyle a^*_{\mathbf{t}}(e_{i_{n+1}})}\Big\downarrow & & \Big\downarrow{\scriptstyle a^*_{V,j}(e_{i_{n+1}})} \\ V(i_1, \ldots, i_{n+1}) & \xrightarrow{T(i_1,\ldots,i_{n+1})} & \tilde{V}(i_1, \ldots, i_{n+1}) \end{array} \tag{2.30}$$

with a similar diagram for the annihilation operators. This is sufficient for proving the equality $\rho_{\mathbf{t}}(\prod_{k=1}^{2n} a^{\sharp_k}_{\mathbf{t}}(e_{i_k})) = \rho_{V,j}(\prod_{k=1}^{2n} a^{\sharp_k}_{V,j}(e_{i_k}))$ for monomials containing $n$ pairs of creation and annihilation operators of $n$ different colors.
3. Finally we prove that $\Omega_{V,j}$ is cyclic vector for $\mathcal{C}_{V,j}(\mathcal{H})$.
The space $\mathcal{F}_V(\mathcal{H})$ has a decomposition with respect to occupation numbers

$$\mathcal{F}_V(\mathcal{H}) = \bigoplus_{\{n_1,\ldots,n_k\}} \mathcal{F}_V(n_1, \ldots, n_k)$$

with

$$\mathcal{F}_V(n_1, \ldots, n_k) = \overline{\mathrm{lin}\{v \otimes_s (\underbrace{e_1 \otimes \ldots \otimes e_1}_{n_1} \otimes \ldots \otimes \underbrace{e_k \otimes \ldots \otimes e_k}_{n_k}, v \in V_{n_1+\cdots+n_k}\}}. \tag{2.31}$$

We recall that $\tilde{V_n} = \mathcal{F}_V(\underbrace{1, \ldots, 1}_{n})$ is spanned by linear combinations of vectors of the form

$$\prod_{k=1}^{2p+n} a_{V,j}^{\sharp_k}(e_{i_k})\Omega_V = v \otimes_s (e_1 \otimes \ldots \otimes e_n)$$

with monomials satisfying the conditions i) and ii). By replacing the creation operators $(a^*(e_k))_{k=1}^n$ appearing in the monomial, with the sequence containing $n_i$ times the creator $a^*(e_i)$ for $i \in \{1, \ldots p\}$ and $\sum_{i=1}^p n_i = n$ we obtain a set of vectors which are dense in $\mathcal{F}_V(n_1, \ldots, n_p)$ and this completes the proof of the cyclicity of the vacuum. Putting together 1., 2. and 3. we conclude that the representations $(\mathcal{F}_{\mathbf{t}}(\mathcal{H}), \mathcal{C}_{\mathbf{t}}(\mathcal{H}), \Omega_{\mathbf{t}})$ and $(\mathcal{F}_V(\mathcal{H}), \mathcal{C}_{V,j}(\mathcal{H}), \Omega_V)$ are unitarily equivalent for infinite dimensional $\mathcal{H}$. The case $\mathcal{H}$ finite dimensional follows by restriction of the previous representations to the appropriate subspaces.

□

# 3 The $^*$-semigroup of broken pair partitions

The content of the last two theorems can be summarized by the following fact: there exist a bijective correspondence between positive definite functions on pair partitions $\mathbf{t}$, and 'combinatorial data' $(V_n, j_n)_{n=0}^{\infty}$. This suggests that the positivity of $\mathbf{t}$ can be characterized in a simpler way by regarding $\mathbf{t}$ as a positive functional on an algebraic object containing $\mathcal{P}_2(\infty)$ as a subset. Theorem 1 of [13] shows that a positive definite function on pair partitions $\mathbf{t}$ restricts to positive definite functions on the symmetric groups $\mathrm{S}(n)$ for all $n \in \mathbb{N}$ through the embedding

$$\mathrm{S}(n) \ni \tau \mapsto \mathcal{V}_\tau \in \mathcal{P}_2(n) \tag{3.1}$$

given by

$$\mathcal{V}_\tau := \{(i, 2n + 1 - \tau(i)) : \ i = 1, \ldots, 2n\}. \tag{3.2}$$

However $\mathbf{t}$ is not determined completely by its restriction and thus one would like to find another algebraic object which completely encodes the positivity requirement. We will show that this is the $^*$-semigroup of *broken pair partitions* which we denote by $\mathcal{BP}_2(\infty)$ and will be described below. Pictorially, the elements of the semigroup are segments obtained by sectioning pair partitions with vertical lines.

**Definition 3.1** *Let $X$ be an arbitrary finite ordered set and $(L, P, R)$ a disjoint partition of $X$. We consider all the triples $(\mathcal{V}, f_l, f_r)$ where $\mathcal{V} \in \mathcal{P}_2(P)$ and*

$$f_l : L \to \{1, \ldots, |L|\}, \qquad f_r : R \to \{1, \ldots, |R|\} \tag{3.3}$$

*are bijections. Any order preserving bijection $\alpha : X \to Y$ induces an obvious map*

$$(\mathcal{V}, f_l, f_r) \to (\alpha \circ \mathcal{V}, f_l \circ \alpha^{-1}, f_r \circ \alpha^{-1}) \tag{3.4}$$

*where* $\alpha \circ \mathcal{V} := \{(\alpha(a), \alpha(b)) : (a, b) \in \mathcal{V}\}$. *This defines an equivalence relation; an element* $d$ *of* $\mathcal{BP}_2(\infty)$ *is an equivalence class of triples* $(\mathcal{V}, f_l, f_r)$ *under this equivalence relation.*

We have the following pictorial representation: an element $d$ is given by a diagram containing a sequence of $l + r + 2n$ points displayed horizontally with $2n$ of them connected into $n$ pairs, $l$ points are connected with other $l$ points vertically ordered on the left side (left legs) and $r$ points are connected with $r$ points vertically ordered on the right (right legs). An example is given in Figure 3.1. In this case we have $X = \{1, \ldots, 5\}$, $\mathcal{V} = \{(1, 4)\}$, the left legs are connecting the points labeled 2 and 5 on the horizontal to the the points on the left side which are ordered vertically and labeled by 1 and 2. Similarly for the right legs. Usually we will label the ordered set of horizontal points will be of the form $\{n, n+1, \ldots n+m\}$.

The product of two diagrams is calculated by drawing the diagrams next to each other and joining the right legs of the left diagram with the left legs of the right diagram which are situated at the same level on the vertical. Figure 3.2 illustrates an example.

More formally if $d_i = (\mathcal{V}_i, f_{l,i}, f_{r,i})$ for $i = 1, 2$ with the notations from Definition 3.1, then $d_1 \cdot d_2 = (\mathcal{V}, f_l, f_r)$ with

$$\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2 \cup \{(f_{r,1}^{-1}(i), f_{l,2}^{-1}(i)) : i \leq \min(|R_1|, |L_2|)\}, \tag{3.5}$$

$f_l$ is defined on the disjoint union $L_1 + (L_2 \setminus f_{l,2}^{-1}(\{1, \ldots, \min(|R_1|, |L_2|)\})$ by

$$\begin{cases} f_l(a) = f_{l,1}(a) & \text{for } a \in L_1 \\ f_l(b) = f_{l,2}(b) + |L_1| & \text{for } b \in L_2 \setminus f_{l,2}^{-1}(\{1, \ldots, \min(|R_1|, |L_2|)\} \end{cases}$$

and similarly for $f_r$. The product does not depend on the chosen representatives for $d_i$ in their equivalence class and is associative. The diagrams with no legs are the pair partitions, thus $\mathcal{P}_2(\infty) \subset \mathcal{BP}_2(\infty)$.

The involution is given by mirror reflection (see Figure 3.3). If $d = (\mathcal{V}, f_l, f_r)$ then $d^* = (\mathcal{V}^*, f_r, f_l)$ with the underlying set $X^*$ obtained by reversing the order on $X$ and

$$\mathcal{V}^* := \{(b, a) : (a, b) \in \mathcal{V}\} \tag{3.6}$$

is the adjoint of $\mathcal{V}$. It is easy to check that

$$(d_1 \cdot d_2)^* = d_2^* \cdot d_1^*.$$

Let $\mathbf{t}$ be a linear functional on pair partitions. We extend it to a function $\hat{\mathbf{t}}$ on $\mathcal{BP}_2(\infty)$ defined as

$$\hat{\mathbf{t}}(d) = \begin{cases} \mathbf{t}(d) & \text{if } d \in \mathcal{P}_2(\infty) \\ 0 & \text{otherwise.} \end{cases} \tag{3.7}$$

**Theorem 3.2** *The function* $\mathbf{t}$ *on pair partitions is positive definite if and only if* $\hat{\mathbf{t}}$ *is positive on the* $^*-$*semigroup* $\mathcal{BP}_2(\infty)$.

*Proof.* The main ideas are already present in the proof of Proposition 2.7. A GNS-type of construction associates to the pair $(\mathcal{BP}_2(\infty), \hat{\mathbf{t}})$ a cyclic representation $\chi_{\mathbf{t}}$ of $\mathcal{BP}_2(\infty)$ on a Hilbert space $V$ with cyclic vector $\xi \in V$. We have $\langle \xi, \chi_{\mathbf{t}}(d)\xi \rangle = \hat{\mathbf{t}}(d)$. We denote by $\mathcal{BP}_2^{(n,0)}$ the set of diagrams with $n$ left legs and no right legs. Then using

$$\langle \chi_{\mathbf{t}}(d_1)\xi, \chi_{\mathbf{t}}(d_2)\xi \rangle_V = \hat{\mathbf{t}}(d_1^* \cdot d_2) \tag{3.8}$$

we obtain:
1. the representation space $V$ is of the form

$$V = \bigoplus_{n=0}^{\infty} V_n \qquad \text{where} \qquad V_n = \overline{\mathrm{lin}\{\chi_{\mathbf{t}}(d)\xi : d \in \mathcal{BP}_2^{(n,0)}\}} \tag{3.9}$$

2. on $\mathcal{BP}_2^{(n,0)}$ there is an obvious action of $\mathrm{S}(n)$ by permutations of the positions of the left ends of the legs. Figure 3.4 shows the action of the transposition $\tau_{1,2}$. This induces a unitary representation of $\mathrm{S}(n)$ on $V_n$ as

$$\tau(d_1)^* \cdot \tau(d_2) = d_1^* \cdot d_2 \tag{3.10}$$

for all $d_1, d_2 \in \mathcal{BP}_2^{(n,0)}$ and $\tau \in \mathrm{S}(n)$.
3. let $d_0 \in \mathcal{BP}_2^{(1,0)}$ be the 'left hook' (the diagram with no pairs). Then $j := \chi_{\mathbf{t}}(d_0)$ is an operator on $V$ whose restriction $j_n$ to $V_n$ maps it into $V_{n+1}$ and satisfies the intertwining condition (2.11) with respect ot the representations of the symmetric groups on $V_n$ and $V_{n+1}$.
Using the data $(V_n, j_n)$ we construct the triple $(\mathcal{F}_V(\mathcal{H}), \mathcal{C}_{V,j}(\mathcal{H}), \Omega_V)$. According to Proposition 2.6 there exists a positive definite function on pair partitions $\mathbf{t}'$ such that $\rho_{V,j} = \rho_{\mathbf{t}'}$. We have to prove that $\mathbf{t}$, which is the restriction of $\hat{\mathbf{t}}$ to $\mathcal{P}_2(\infty)$ coincides with $\mathbf{t}'$.
Any pair partition $\mathcal{V}$ can be written in a 'standard form' (see Figure 3.5):

$$\mathcal{V} = (d_0^*)^{p_m} \cdot \pi_{m-1}(\dots \pi_2(d_0^{k_2} \cdot (d_0^*)^{p_1} \cdot \pi_1(d_0^{k_1}))) \tag{3.11}$$

where the permutations $\pi_i$ are uniquely defined by the requirement that any two lines connecting two pairs in the associated graphic intersect minimally and at the rightmost possible position.
Let $\prod_{k=1}^{2n} a_{V,j}^{\sharp_k}(e_{i_k})$ be a monomial containing $n$ creation operators and $n$ annihilation operators such that by pairing creators with annihilators of the same color on their right side, we generate a pair partition $\mathcal{V}$. The definitions (2.13), (2.14) of the creation and annihilation operators give their expressions in terms of the operator $j, j^*$ and the unitary representations of the permutation groups

on the spaces $V_n$. By using the intertwining property (2.11) we can pass all permutations to the left of the $j$-terms and obtain:

$$\begin{aligned}\mathbf{t}'(\mathcal{V}) &= \left\langle \Omega_V, \prod_{k=1}^{2n} a_{V,j}^{\sharp_k}(e_{i_k})\Omega_V \right\rangle \\ &= \left\langle \xi, (j^*)^{p_m} \cdot U(\pi_{m-1}) \ldots U(\pi_2) \cdot j^{k_2} \cdot (j^*)^{p_1} \cdot U(\pi_1) \cdot j^{k_1}\xi \right\rangle_V \\ &= \langle \xi, \chi_{\mathbf{t}}(\mathcal{V})\xi \rangle_V = \hat{\mathbf{t}}(\mathcal{V})\end{aligned}$$

Conversely, starting from a positive definite function $\mathbf{t}$ we construct the representation $(V, \chi_{\mathbf{t}}(\mathcal{BP}_2(\infty)), \xi)$ through applying Theorem 2.7 and thus $\hat{\mathbf{t}}$ is positive on $\mathcal{BP}_2(\infty)$.

□

# 4 Generalised Wick products

As argued in the introduction, the representations of the 'field algebras' $\mathcal{A}(\mathcal{K})$ with respect to Gaussian states $\tilde{\rho}_{\mathbf{t}}$ give rise to (non-commutative) processes called generalised Brownian motions [13] for $\mathcal{K}$ (infinite dimensional) real Hilbert space. In all known examples such representations appear as restrictions to the subalgebra $\mathcal{A}(\mathcal{K})$ of Fock representations of the algebra of creation and annihilation operators $\mathcal{C}(\mathcal{K}_{\mathbb{C}})$ with respect to the state $\rho_{\mathbf{t}}$. We will prove that this is always the case, thus answering a question put in [13].
Let

$$\mathbf{t} : \mathcal{P}_2(\infty) \to \mathbb{C} \tag{4.1}$$

be such that $\tilde{\rho}_{\mathbf{t}}$ is a Gaussian state on $\mathcal{A}(\mathcal{K})$ for $\mathcal{K}$ infinite dimensional Hilbert space. Let $(\tilde{\mathcal{F}}_{\mathbf{t}}(\mathcal{K}), \tilde{\pi}_{\mathbf{t}}(\mathcal{A}(\mathcal{K})), \tilde{\Omega}_{\mathbf{t}})$ be the GNS-triple associated to $(\mathcal{A}(\mathcal{K}), \tilde{\rho}_{\mathbf{t}})$. The $^*$-algebra $\tilde{\pi}_{\mathbf{t}}(\mathcal{A}(\mathcal{K}))$ is generated by the symmetric operators $\omega_{\mathbf{t}}(f) := \tilde{\pi}_{\mathbf{t}}(\omega(f))$ for all $f \in \mathcal{K}$ with common domain $D := \tilde{\pi}_{\mathbf{t}}(\mathcal{A}(\mathcal{K}))\tilde{\Omega}_{\mathbf{t}}$. The selfadjointness of the field operators will be addressed in section 5. For the moment, all operators discussed are defined on $D$.

In analogy to (2.21) for any orthogonal operator $O \in \mathcal{O}(\mathcal{K})$ there exists a unitary

$$\tilde{\mathcal{F}}_{\mathbf{t}}(O) : \prod_{k=1}^{n} \omega_{\mathbf{t}}(f_k)\tilde{\Omega}_{\mathbf{t}} \to \prod_{k=1}^{n} \omega_{\mathbf{t}}(Of_k)\tilde{\Omega}_{\mathbf{t}} \tag{4.2}$$

and $\tilde{\mathcal{F}}_{\mathbf{t}}(O_1)\tilde{\mathcal{F}}_{\mathbf{t}}(O_2) = \tilde{\mathcal{F}}_{\mathbf{t}}(O_1 \cdot O_2)$ for $O_1, O_2 \in \mathcal{O}(\mathcal{K})$. This induces an action on the $^*$-algebra $\tilde{\pi}_{\mathbf{t}}(\mathcal{A}(\mathcal{K}))$:

$$\tilde{\Gamma}_{\mathbf{t}}(O) : X \mapsto \tilde{\mathcal{F}}_{\mathbf{t}}(O)X\tilde{\mathcal{F}}_{\mathbf{t}}(O^*). \tag{4.3}$$

Certain operators play a similar role to that of the Wick products in quantum field theory [53, 59] or for the $q$-deformed Brownian motion [9, 10].

**Definition 4.1** Let $\{P, F\}$ be a partition of the ordered set $\{1, \ldots, 2p+n\}$ with $|P| = 2p$ and $|F| = n$. Let $\mathcal{V} = \{(l_1, r_1), \ldots, (l_p, r_p)\} \in \mathcal{P}_2(P)$ and $\mathbf{f} : F \to \mathcal{K}$. For every $\mathcal{V}' = \{(l'_1, r'_1), \ldots, (l'_{p'}, r'_{p'})\} \in \mathcal{P}_2(P')$ with $P' \subset F$ we define

$$\eta_{\mathbf{f}}(\mathcal{V}') := \prod_{i=1}^{p'} \langle \mathbf{f}(l'_i), \mathbf{f}(r'_i) \rangle \,. \tag{4.4}$$

The *generalised Wick product* associated to $(\mathcal{V}, \mathbf{f})$ is the operator $\Psi(\mathcal{V}, \mathbf{f})$ determined recursively by

$$\begin{aligned} \Psi(\mathcal{V}, \mathbf{f}) \quad &+ \sum_{\emptyset \neq P' \subset F} \; \sum_{\mathcal{V}' \in \mathcal{P}_2(P')} \eta_{\mathbf{f}}(\mathcal{V}') \cdot \Psi(\mathcal{V} \cup \mathcal{V}', \mathbf{f} \restriction_{F \setminus P'}) = M(\mathcal{V}, \mathbf{f}) \\ M(\mathcal{V}, \mathbf{f}) \quad &:= \quad \underset{n \to \infty}{\mathrm{w-lim}} \prod_{k=1}^{2p+n} \omega_{\mathbf{t}}(f_{k,n}) \end{aligned} \tag{4.5}$$

where $f_{k,n} := \mathbf{f}(k)$ for $k \in F$ and $f_{l_i,n} = f_{r_i,n} = e_{np+i}$ for $i = 1, \ldots, p$ with $(e_l)_{l \in \mathbb{N}}$ a set of normalized vectors, orthogonal to each other.

**Remarks.** 1) The right side of the last equation needs some clarifications. The operator $M(\mathcal{V}, \mathbf{f})$ is defined on $D$ by its matrix elements. If $\psi_i = \prod_{a=1}^{m_i} \omega_{\mathbf{t}}(g_a^{(i)}) \tilde{\Omega}_{\mathbf{t}}$ for $i = 1, 2$ are vectors in $D$ then from the definition of the Gaussian state follows immediately that

$$\langle \psi_1, M(\mathcal{V}, \mathbf{f}) \psi_2 \rangle = \lim_{n \to \infty} \left\langle \psi_1, \prod_{k=1}^{2p+n} \omega_{\mathbf{t}}(f_{k,n}) \psi_2 \right\rangle \tag{4.6}$$

exists and does not depend on the choice of the vectors $(e_i)_{i \in \mathbb{N}}$ (as long as they are normal and orthogonal to each other) but depends only on their positions in the monomial which are determined by the pair partition $\mathcal{V}$. In the limit only those pair partitions which contain the pairs $(l_i, r_i) \in \mathcal{V}$ give a nonzero contribution. Thus $M(\mathcal{V}, \mathbf{f})$ is well defined.
2) If the vectors $(\mathbf{f}(k))_{k=1}^n$ are orthogonal on each other then $\eta_{\mathbf{f}}(\mathcal{V}') = 0$, thus $\Psi(\mathcal{V}, \mathbf{f}) = M(\mathcal{V}, \mathbf{f})$.
3) The dense domain $D$ is spanned by the vectors of the form $\Psi(\mathcal{V}, \mathbf{f}) \tilde{\Omega}_{\mathbf{t}}$. Indeed let $\psi = \prod_{k=1}^n \omega_{\mathbf{t}}(\mathbf{f}(k)) \tilde{\Omega}_{\mathbf{t}}$; then

$$\psi = \Psi(\emptyset, \mathbf{f}) \tilde{\Omega}_{\mathbf{t}} + \sum_{\emptyset \neq P' \subset F} \; \sum_{d' \in \mathcal{P}_2(P')} \eta_f(\mathcal{V}') \cdot \Psi(\mathcal{V}', \mathbf{f} \restriction_{F \setminus P'}) \tilde{\Omega}_{\mathbf{t}} \tag{4.7}$$

with $F = \{1, \ldots, n\}$.
4) The choice for $\{1, \ldots, 2p+n\}$ as the underlying ordered set is not essential. It is useful to think of $\Psi(\mathcal{V}, \mathbf{f})$ in terms of an arbitrary underlying finite ordered

set $X$, where $\mathcal{V} \in \mathcal{P}_2(A)$, $A \subset X$, $\mathbf{f} : X \setminus A \to \mathcal{K}$. For example we can consider the set $X = \{0\}$ and $\mathbf{f}(0) = h$, then $\Psi(\emptyset, \mathbf{f}) = \omega_{\mathbf{t}}(h)$.
The relation between $M(\mathcal{V}, \mathbf{f})$ and $\Psi(\mathcal{V}, \mathbf{f})$ is similar to the one between moments and cumulants.

**Lemma 4.2** *Let $\Psi(\mathcal{V}, \mathbf{f})$, $M(\mathcal{V}, \mathbf{f})$ be as in Definition 4.1. The equations (4.5) can be inverted into:*

$$\Psi(\mathcal{V}, \mathbf{f}) = M(\mathcal{V}, \mathbf{f}) + \sum_{\emptyset \neq P' \subset F} \sum_{\mathcal{V}' \in \mathcal{P}_2(P')} (-1)^{\frac{|P'|}{2}} \eta_{\mathbf{f}}(\mathcal{V}') \cdot M(\mathcal{V} \cup \mathcal{V}', \mathbf{f} \restriction_{F \setminus P'}). \tag{4.8}$$

*Proof.* Apply Möbius inversion formula.

□

Let $X$ be an ordered set. Let $\{P, F\}$ be a partition of $X$ into disjoint sets and consider a pair $(\mathcal{V} \in \mathcal{P}_2(P), \mathbf{f} : F \to \mathcal{K})$. Then for $X^*$ as underlying set we define the pair $(\mathcal{V}^*, \mathbf{f}^*)$ where $\mathcal{V}^* \in \mathcal{P}_2(X^*)$ contains the same pairs as $\mathcal{V}$ but with the reversed order and $\mathbf{f}^* = \mathbf{f}$.

**Lemma 4.3** *With the above notations the following relation holds:*

$$\Psi(\mathcal{V}, \mathbf{f})^* = \Psi(\mathcal{V}^*, \mathbf{f}^*). \tag{4.9}$$

*Proof.* Apply Lemma 4.2 and use $M(\mathcal{V}, \mathbf{f})^* = M(\mathcal{V}^*, \mathbf{f}^*)$ which follows directly from Definition 4.1.

□

For two ordered sets $X$ and $Y$ we define their concatenation $X + Y$ as the disjoint union with the original order on $X$ and $Y$ and with $x < y$ for any $x \in X, y \in Y$. If $\mathbf{f}_X : X \to \mathcal{K}$ and $\mathbf{f}_Y : Y \to \mathcal{K}$ then we denote by $\mathbf{f}_X \oplus \mathbf{f}_Y$ the function on $X + Y$ which restricts to $\mathbf{f}_X$ and $\mathbf{f}_Y$ on $X$ respectively $Y$. Finally if $|X| = |Y| = m$ we identify the subset of $\mathcal{P}_2(X + Y)$:

$$\mathcal{P}_2(X, Y) := \{\{(x_1, y_1), \ldots, (x_m, y_m)\} \ : \ x_i \in X, y_i \in Y, i = 1, \ldots, m\} \tag{4.10}$$

**Lemma 4.4** *Let $(P_i, F_i)$ be a disjoint partition of $X_i$ and $\mathcal{V}_i \in \mathcal{P}_2(P_i)$, $\mathbf{f}_i : F_i \to \mathcal{K}$ for $i = 1, 2$. Then*

$$\left\langle \Psi(\mathcal{V}_1, \mathbf{f}_1)\tilde{\Omega}_{\mathbf{t}}, \ \Psi(\mathcal{V}_2, \mathbf{f}_2)\tilde{\Omega}_{\mathbf{t}} \right\rangle = \delta_{|F_1|, |F_2|} \sum_{\mathcal{V} \in \mathcal{P}_2(F_1^*, F_2)} \eta_{\mathbf{f}_1^* \oplus \mathbf{f}_2}(\mathcal{V}) \cdot \mathbf{t}(\mathcal{V}_1^* \cup \mathcal{V}_2 \cup \mathcal{V}) \tag{4.11}$$

*with the convention $\eta_{\mathbf{f}_1^* \oplus \mathbf{f}_2}(\mathcal{V}) = 1$ for $F_1 = F_2 = \emptyset$.*

*Proof.* From Definitions 2.5, 4.1 it follows that

$$\left\langle M(\mathcal{V}_1, \mathbf{f}_1)\tilde{\Omega}_{\mathbf{t}}, \ M(\mathcal{V}_2, \mathbf{f}_2)\tilde{\Omega}_{\mathbf{t}} \right\rangle = \sum_{\mathcal{V} \in \mathcal{P}_2(F_1^* + F_2)} \eta_{\mathbf{f}_1^* \oplus \mathbf{f}_2}(\mathcal{V}) \cdot \mathbf{t}(\mathcal{V}_1^* \cup \mathcal{V}_2 \cup \mathcal{V}). \tag{4.12}$$

We apply Lemma 4.2 and obtain:

$$\Big\langle \Psi(\mathcal{V}_1, \mathbf{f}_1)\tilde{\Omega}_{\mathbf{t}},\ \Psi(\mathcal{V}_2, \mathbf{f}_2)\tilde{\Omega}_{\mathbf{t}} \Big\rangle = \sum_{\mathcal{V}_1', \mathcal{V}_2'} (-1)^{\frac{|P_1'|+|P_2'|}{2}} \cdot \eta_{\mathbf{f}_1^* \oplus \mathbf{f}_2}(\mathcal{V}_1'^* \cup \mathcal{V}_2') \cdot$$
$$\cdot \Big\langle M(\mathcal{V}_1 \cup \mathcal{V}_1', \mathbf{f}_1 \restriction_{F_1 \setminus P_1'})\tilde{\Omega}_{\mathbf{t}},\ M(\mathcal{V}_2 \cup \mathcal{V}_2', \mathbf{f}_2 \restriction_{F_2 \setminus P_2'})\tilde{\Omega}_{\mathbf{t}} \Big\rangle$$

where the sum runs over all $\mathcal{V}_i' \in \mathcal{P}_2(P_i'), P_i' \subset F_i$ for $i = 1, 2$. Substituting in the last expression the result from equation (4.12) it becomes:

$$\sum_{\mathcal{V}_1', \mathcal{V}_2'} \sum_{\mathcal{V}} (-1)^{\frac{|P_1'|+|P_2'|}{2}} \cdot \eta_{\mathbf{f}_1^* \oplus \mathbf{f}_2}(\mathcal{V}_1'^* \cup \mathcal{V}_2' \cup \mathcal{V}) \cdot \mathbf{t}((\mathcal{V}_1 \cup \mathcal{V}_1')^* \cup \mathcal{V}_2 \cup \mathcal{V}_2' \cup \mathcal{V}) \tag{4.13}$$

with the second sum running over all $\mathcal{V} \in \mathcal{P}_2((F_1 \setminus P_1')^* + (F_2 \setminus P_2'))$. We make the notation $\tilde{\mathcal{V}} := \mathcal{V}_1'^* \cup \mathcal{V}_2' \cup \mathcal{V}$ and by grouping together all terms containing $\tilde{\mathcal{V}}$ the initial expression looks like:

$$\Big\langle \Psi(\mathcal{V}_1, \mathbf{f}_1)\tilde{\Omega}_{\mathbf{t}},\ \Psi(\mathcal{V}_2, \mathbf{f}_2)\tilde{\Omega}_{\mathbf{t}} \Big\rangle = \sum_{\tilde{\mathcal{V}}} m(\tilde{\mathcal{V}}) \cdot \eta_{\mathbf{f}_1^* \oplus \mathbf{f}_2}(\tilde{\mathcal{V}}) \cdot \mathbf{t}(\mathcal{V}_1^* \cup \tilde{\mathcal{V}} \cup \mathcal{V}_2) \tag{4.14}$$

where the symbol $m(\tilde{\mathcal{V}})$ stands for total contribution from the terms of the form $(-1)^{\frac{|P_1'|+|P_2'|}{2}}$. We calculate now $m(\tilde{\mathcal{V}})$:

$$m(\tilde{\mathcal{V}}) = \sum_{\mathcal{V}_1', \mathcal{V}_2', \mathcal{V}} (-1)^{|\mathcal{V}_1'|+|\mathcal{V}_2'|}, \tag{4.15}$$

this sum running over all $\mathcal{V} \in \mathcal{P}_2((F_1 \setminus P_1')^* + (F_2 \setminus P_2'))$, $\mathcal{V}_i' \in \mathcal{P}_2(P_i')$, $P_i' \subset F_i$ for $i = 1, 2$ with the constraint $\tilde{\mathcal{V}} = \mathcal{V}_1'^* \cup \mathcal{V}_2' \cup \mathcal{V}$.
Suppose that $\tilde{\mathcal{V}} \in \mathcal{P}_2(F_1^*, F_2)$, then $\mathcal{V}_1' = \mathcal{V}_2' = \emptyset$ and $m(\tilde{\mathcal{V}}) = 1$. Otherwise $\tilde{\mathcal{V}}$ can be written in a unique way as

$$\tilde{\mathcal{V}} = \tilde{\mathcal{V}}_1^* \cup \tilde{\mathcal{V}}_c \cup \tilde{\mathcal{V}}_2 \tag{4.16}$$

where $\tilde{\mathcal{V}}_i \in \mathcal{P}_2(\tilde{P}_i)$, $\emptyset \neq \tilde{P}_i \subset X_i$ for $i = 1, 2$ and $\tilde{\mathcal{V}}_c \in \mathcal{P}_2((X_1 \setminus \tilde{P}_1)^*, X_2 \setminus \tilde{P}_2)$. Then one has the inclusions $\mathcal{V}_i' \subset \tilde{\mathcal{V}}_i$ for $i = 1, 2$ and $\mathcal{V}_c \subset \mathcal{V}$. The calculation of $m(\tilde{\mathcal{V}})$ reduces then to

$$m(\tilde{\mathcal{V}}) = \sum_{\mathcal{V}_1' \subset \tilde{\mathcal{V}}_1, \mathcal{V}_2' \subset \tilde{\mathcal{V}}_2} (-1)^{|\mathcal{V}_1'|+|\mathcal{V}_2'|} = (1-1)^{|\tilde{\mathcal{V}}_1|+|\tilde{\mathcal{V}}_2|} = 0. \tag{4.17}$$

In conclusion

$$\Big\langle \Psi(\mathcal{V}_1, \mathbf{f}_1)\tilde{\Omega}_{\mathbf{t}},\ \Psi(\mathcal{V}_2, \mathbf{f}_2)\tilde{\Omega}_{\mathbf{t}} \Big\rangle = \sum_{\tilde{\mathcal{V}} \in \mathcal{P}_2(F_1^*, F_2)} \eta_{\mathbf{f}_1^* \oplus \mathbf{f}_2}(\tilde{\mathcal{V}}) \cdot \mathbf{t}(\mathcal{V}_1^* \cup \mathcal{V}_2 \cup \tilde{\mathcal{V}}) \tag{4.18}$$

□

A similar result holds for algebras of creation and annihilation operators. Suppose that $\mathbf{t}$ is a function (not necessarily positive definite) on pair partitions. Let $P, F, \mathcal{V}, \mathbf{f}$ be as in Definition 4.1 and define in the representation space $\mathcal{F}_{\mathbf{t}}(\mathcal{K}_{\mathbb{C}})$ the vectors

$$\psi(\mathcal{V}, \mathbf{f}) = \prod_{k=1}^{2p+n} a_{\mathbf{t}}^{\sharp_k}(f_k))\Omega_{\mathbf{t}} \tag{4.19}$$

with $a^{\sharp_k}(f_k) = a^*(\mathbf{f}(k))$ for $k \in F$, $a^{\sharp_{l_i}}(f_{l_i}) = (a^{\sharp_{r_i}}(f_{r_i}))^* = a(g_i)$ for $i = 1, \dots, p$ and $(g_i)_{i=1,\dots,p}$ a set of normalized vectors, orthogonal to each other and to the vectors $(\mathbf{f}(k))_{k=1}^{n}$.

**Lemma 4.5** *Let* $\mathbf{t}$ *be a function on pair partitions. Then*

$$\langle \psi(\mathcal{V}_1, \mathbf{f}_1),\ \psi(\mathcal{V}_2, \mathbf{f}_2)\rangle_{\mathcal{F}_{\mathbf{t}}(\mathcal{K}_{\mathbb{C}})} = \sum_{\mathcal{V} \in \mathcal{P}_2(F_1^*, F_2)} \eta_{\mathbf{f}_1^* \oplus \mathbf{f}_2}(\mathcal{V}) \cdot \mathbf{t}(\mathcal{V}_1^* \cup \mathcal{V}_2 \cup \mathcal{V}) \tag{4.20}$$

*Proof.* The equation follows then directly from Definition 2.4.

□

Now we are ready for the main result of this section.

**Theorem 4.6** *Let* $\mathbf{t}$ *be a function on pair partitions. If* $\tilde{\rho}_{\mathbf{t}}$ *is a Gaussian state on* $\mathcal{A}(\mathcal{K})$ *for any real Hilbert space* $\mathcal{K}$ *then* $\rho_{\mathbf{t}}$ *is a Fock state on* $\mathcal{C}(\mathcal{K}_{\mathbb{C}})$*.*

*Proof.* Suppose that $\rho_{\mathbf{t}}$ is not a Fock state. Then in the representation space $\mathcal{F}_{\mathbf{t}}(\mathcal{K}_{\mathbb{C}})$ there exists a vector of the form

$$\psi = \sum_{a=1}^{m} c_a \cdot \psi(\mathcal{V}_a, \mathbf{f}_a) \tag{4.21}$$

with all $\mathbf{f}_a$ taking values in the real subspace $\mathcal{K}$ of $\mathcal{K}_{\mathbb{C}}$ and $c_a \in \mathbb{C}$, such that $\langle \psi, \psi \rangle < 0$. But from lemmas 4.4 and 4.5 it results that $\| \sum_{a=1}^{m} c_a \cdot \Psi(\mathcal{V}_a, \mathbf{f}_a)\tilde{\Omega}_{\mathbf{t}} \|^2 < 0$ which is a contradiction. Thus $\rho_{\mathbf{t}}$ is a positive functional and $\mathbf{t}$ is a positive definite function on pair partitions.

□

From Lemmas 4.2 and 4.3 we conclude that the generalised Wick products $\Psi(\mathcal{V}, \mathbf{f})$ acting on $\mathcal{F}_{\mathbf{t}}(\mathcal{K})$ form a $^*$-algebra of operators which contains $\pi_{\mathbf{t}}(\mathcal{A}(\mathcal{K}))$ and will be denoted by $\mathbf{W}_{\mathbf{t}}(\mathcal{K})$. Let us first note that Theorem 4.6 implies that the representations of $\mathbf{W}_{\mathbf{t}}(\mathcal{K})$ on $\mathcal{F}_{\mathbf{t}}(\mathcal{K}_{\mathbb{C}})$ and $\tilde{\mathcal{F}}_{\mathbf{t}}(\mathcal{K})$ are unitarily equivalent, thus:

**Corollary 4.7** *The vacuum vector* $\Omega_{\mathbf{t}}$ *is cyclic for the* $^*$*-algebra* $\mathbf{W}_{\mathbf{t}}(\mathcal{K})$ *for any real Hilbert space* $\mathcal{K}$*.*

# 5 Second Quantisation

This section is dedicated to the description of functorial properties of the generalised Brownian motion which go by the name of second quantisation and appear at two different levels depending on the categories with which we work.
Let $\mathcal{H}$, $\mathcal{H}'$ be Hilbert spaces and $T$ a contraction from $\mathcal{H}$ to $\mathcal{H}'$. Define the second quantisation of $T$ at the Hilbert space level by

$$\begin{aligned} \mathcal{F}_{\mathbf{t}}(T) : \mathcal{F}_{\mathbf{t}}(\mathcal{H}) &\rightarrow \mathcal{F}_{\mathbf{t}}(\mathcal{H}') \\ v \otimes_s h_0 \otimes \ldots \otimes h_{n-1} &\mapsto v \otimes_s Th_0 \otimes \ldots \otimes Th_{n-1} \end{aligned} \tag{5.1}$$

for all $v \in V_n$, $h_i \in \mathcal{H}$ when $n \geq 1$, and equal to the identity on $V_0$. Clearly $\mathcal{F}_{\mathbf{t}}(T)$ is a contraction, satisfies the equation $\mathcal{F}_{\mathbf{t}}(T_1) \cdot \mathcal{F}_{\mathbf{t}}(T_2) = \mathcal{F}_{\mathbf{t}}(T_1 \cdot T_2)$ and for $T$ unitary it coincides with the operator defined in the equations (2.23) and (2.24).

**Definition 5.1** We call $\mathcal{F}_{\mathbf{t}}$ the functor of *second quantisation at the Hilbert space level* .

**Lemma 5.2** *Let $\psi(\mathcal{V}, \mathbf{f})$ as defined in equation (4.19). Then*

$$\mathcal{F}_{\mathbf{t}}(T)\psi(\mathcal{V}, \mathbf{f}) = \psi(\mathcal{V}, T \circ \mathbf{f}). \tag{5.2}$$

*Proof.* We use the representation $\chi_{\mathbf{t}}$ of the $^*$-semigroup of broken pair partitions $\mathcal{BP}_2(\infty)$ with respect to the state $\hat{\mathbf{t}}$ (see equation 3.7). Let $\{F, P\}$ be a partition of $\{1, \ldots, 2p+n\}$ and $\mathcal{V} \in \mathcal{P}_2(P)$, $\mathbf{f} : F \rightarrow \mathcal{H}$. Then using (4.19) and the equations (2.13, 2.14) we obtain

$$\psi(\mathcal{V}, \mathbf{f}) = \chi_{\mathbf{t}}(\tilde{\mathcal{V}})\xi \otimes_s \bigotimes_{k \in F} \mathbf{f}(k) \tag{5.3}$$

for $\tilde{\mathcal{V}} \in \mathcal{BP}_2^{n,0}$ the diagram with the set of pairs $\mathcal{V}$ and $n$ legs to the left which do not intersect each other.

□

There is however a more interesting notion of second quantisation.

**Definition 5.3** [42] i) The category of *non-commutative probability spaces* has as objects pairs $(\mathcal{A}, \rho_{\mathcal{A}})$ of von Neumann algebras and normal states and as morphisms between two objects $(\mathcal{A}, \rho_{\mathcal{A}})$ and $(\mathcal{B}, \rho_{\mathcal{B}})$ all completely positive maps $T : \mathcal{A} \rightarrow \mathcal{B}$ such that $T(\mathbf{1}_{\mathcal{A}}) = \mathbf{1}_{\mathcal{A}}$ and $\rho_{\mathcal{B}}(Tx) = \rho_{\mathcal{A}}(x)$ for all $x \in \mathcal{A}$.
ii) A functor $\Gamma$ from the category of (real) Hilbert spaces with contractions to the category of non-commutative probability spaces is called *functor of white noise* if $\Gamma(\{0\}) = \mathbb{C}$ where $\{0\}$ stands for the zero dimensional Hilbert space.

In some cases we will ask for the following continuity requirement

$$\text{w}-\lim_{n \rightarrow \infty} \Gamma(T_n)(X) = \Gamma(T)(X), \tag{5.4}$$

for any sequence of contractions $T_n : \mathcal{K} \rightarrow \mathcal{K}'$ converging weakly to $T$.

For completeness we include the following standard result.

**Proposition 5.4** *If $\Gamma$ is a functor of white noise then $\Gamma(T)$ is an injective $^*$-homomorphism (automorphism) if $T$ is an (invertible) isometry , and $\Gamma(P)$ is a conditional expectation if $P$ is an orthogonal projection.*

*Proof.* For separating vacuum the proof has been given in [41]. Here we do not assume this property.
1. Let $O : \mathcal{K} \to \mathcal{K}'$ be an orthogonal operator and $X \in \Gamma(\mathcal{K})$. As $\Gamma(O^*)$ and $\Gamma(O)$ are completely positive we have the inequalities

$$\Gamma(O^*)(\Gamma(O)(X^*) \cdot \Gamma(O)(X)) \geq \Gamma(O^*O)(X^*) \cdot \Gamma(O^*O)(X) = X^*X \qquad (5.5)$$

and

$$\Gamma(O)(X^*)\Gamma(O)(X) \leq \Gamma(O)(X^*X) \qquad (5.6)$$

which by applying the positive operator $\Gamma(O^*)$ becomes

$$\Gamma(O^*)(\Gamma(O)(X^*) \cdot \Gamma(O)(X)) \leq \Gamma(O^*O)(X^*X) = X^*X \qquad (5.7)$$

From (5.5, 5.7) we get $\Gamma(O)(X^*) \cdot \Gamma(O)(X) = \Gamma(O)(X^*X)$ and by repeating the argument for $X + Y$ and $X + iY$ we obtain that $\Gamma(O)$ is a $^*$-isomorphism.
2. Let $\mathcal{K}$ be a real Hilbert space, $I : \mathcal{K} \to \mathcal{K}'$ an isometry and $P = II^*$ orthogonal projection. Then $\Gamma(P)\Gamma(P) = \Gamma(P)$ which is thus a norm one projection, and conditional expectation [63]. The map $\Gamma(I)$ is injective as $\Gamma(I^*)\Gamma(I) = \mathrm{id}_{\Gamma(\mathcal{K})}$ and its image is $\Gamma(P)\Gamma(\mathcal{K}')$. Indeed we have $\Gamma(P)\Gamma(I)(X) = \Gamma(I)(X)$ and $\Gamma(P)(Y) = \Gamma(I)\Gamma(I^*)(Y)$. This means that $\Gamma(I)$ is bijective between $\Gamma(\mathcal{K})$ and $\Gamma(P)\Gamma(\mathcal{K}')$, and by a similar argument to that used in step 1 we obtain that $\Gamma(I)$ is a $^*$-homomorphism.

□

**Corollary 5.5** *If $\Gamma$ is a functor of second quantisation such that condition 5.4 holds. Then for any real Hilbert space $\mathcal{H}$ and any infinite dimensional real Hilbert space $\mathcal{K}$ the algebras $\Gamma(\mathcal{H} \oplus \mathcal{K})^{\mathcal{O}(\mathcal{K})}$ and $\Gamma(\mathcal{H})$ are isomorphic , in particular $\Gamma(\mathcal{K})^{\mathcal{O}(\mathcal{K})} = \mathbb{C}\mathbf{1}$.*

*Proof.* We can choose $\mathcal{K} = \ell^2(\mathbb{Z})$. Let $S$ be the right shift on $\ell^2(\mathbb{Z})$ and $O = \mathbf{1} \oplus S$ orthogonal operator on $\mathcal{H} \oplus \mathcal{K}$. By taking the limit of $O^n$ for $n \to \infty$, one can obtain that $\Gamma(\mathcal{H} \oplus \mathcal{K})^{\mathcal{O}(\mathcal{K})}$ is isomorphic with $\Gamma(I)\Gamma(\mathcal{H})$ where $I$ is the natural isometry from $\mathcal{H}$ to $\mathcal{H} \oplus \mathcal{K}$. Thus $\Gamma(\mathcal{H} \oplus \mathcal{K})^{\mathcal{O}(\mathcal{K})} \simeq \Gamma(\mathcal{H})$.

□

After these general considerations we come back to our construction from the previous section: for a fixed positive definite function $\mathbf{t}$ we have associated to each Hilbert space $\mathcal{K}$ an algebra of fields $\pi_{\mathbf{t}}(\mathcal{A}(\mathcal{K}))$ and an algebra of Wick products $\mathbf{W}_{\mathbf{t}}(\mathcal{K})$ which is in general larger that the previous one, both acting on $\mathcal{F}_{\mathbf{t}}(\mathcal{K}_{\mathbb{C}})$, and a positive functional $\langle\Omega_{\mathbf{t}}, \cdot\Omega_{\mathbf{t}}\rangle$ on these algebras. We would like to transform this correspondences into functors of white noise. The natural way to do this for $\pi_{\mathbf{t}}(\mathcal{A}(\mathcal{K}))$ is to construct the von Neumann algebra generated by the spectral projections of the selfadjoint field operators $\omega_{\mathbf{t}}(f)$ for all $f \in \mathcal{K}$. However these operators are in general only symmetric and, unless bounded, one has to make sure that they are essentially selfadjoint. Let us suppose for the moment that this is the case. In the case of $\mathbf{W}_{\mathbf{t}}(\mathcal{K})$ we will look for simplicity only at the cases where all Wick products are bounded. Then we identify the following candidates for the image objects under functors of white noise associated to $\mathbf{t}$:

1) $\Gamma_{\mathbf{t}} : \mathcal{K} \mapsto (\Gamma_{\mathbf{t}}(\mathcal{K}), \langle\Omega_{\mathbf{t}}, \cdot\Omega_{\mathbf{t}}\rangle)$ where $\Gamma_{\mathbf{t}}(\mathcal{K})$ is the von Neumann algebra generated by all the spectral projections of the (closed) field operators $\omega_{\mathbf{t}}(f)$ acting on $\mathcal{F}_{\mathbf{t}}(\mathcal{K}_{\mathbb{C}})$ for all $f \in \mathcal{K}$.

2) $\Gamma^{\infty}_{\mathbf{t}} : \mathcal{K} \mapsto (\Gamma^{\infty}_{\mathbf{t}}(\mathcal{K}), \langle\Omega_{\mathbf{t}}, \cdot\Omega_{\mathbf{t}}\rangle)$ where $\Gamma^{\infty}_{\mathbf{t}}(\mathcal{K})$ is the von Neumann subalgebra of $\Gamma_{\mathbf{t}}(\mathcal{K}\oplus\ell^2(\mathbb{Z}))$ consisting of operators which commute with the unitaries $\mathcal{F}_{\mathbf{t}}(\mathbf{1}\oplus O)$ for all $O \in \mathcal{O}(\ell^2(\mathbb{Z}))$, i.e.

$$\Gamma^{\infty}_{\mathbf{t}}(\mathcal{K}) := \Gamma_{\mathbf{t}}(\mathcal{K} \oplus \ell^2(\mathbb{Z}))^{\mathcal{O}(\ell^2(\mathbb{Z}))}. \tag{5.8}$$

3) $\Delta_{\mathbf{t}} : \mathcal{K} \mapsto (\Delta_{\mathbf{t}}(\mathcal{K}), \langle\Omega_{\mathbf{t}}, \cdot\Omega_{\mathbf{t}}\rangle)$ where $\Delta_{\mathbf{t}}(\mathcal{K})$ is the von Neumann algebra generated by the Wick products $\Psi(\mathcal{V}, \mathbf{f})$ in $\mathbf{W}_{\mathbf{t}}(\mathcal{K} \oplus \ell^2(\mathbb{Z}))$ for which $\mathrm{Im}\, \mathbf{f} \subset \mathcal{K}$.

In the cases known so far $-$ the gaussian functor [53], the free white noise [65] and the $q$-deformed Brownian motion [9] $-$ the three definitions are equivalent. This is due to the fact that the vacuum state is *faithful* and the algebra of Wick products coincide with that of the fields. This will be clear later in this section. For a general treatment it appears however that the last two choices are more appropriate. For any orthogonal operator $O : \mathcal{K} \to \mathcal{K}'$, the natural choice for $\Gamma^{\infty}_{\mathbf{t}}(O)$ is

$$\Gamma^{\infty}_{\mathbf{t}}(O)(X) = \mathcal{F}_{\mathbf{t}}(O \oplus \mathbf{1})X\mathcal{F}_{\mathbf{t}}(O \oplus \mathbf{1})^* \tag{5.9}$$

where $X \in \Gamma^{\infty}_{\mathbf{t}}(\mathcal{K})$, and similarly for $\Delta_{\mathbf{t}}(O)$. Our task is now to find for which functions $\mathbf{t}$ one can construct such von Neumann algebras, i.e. the field operators are selfadjoint, and moreover the map $\mathcal{K} \to \Gamma^{\infty}_{\mathbf{t}}(\mathcal{K})$ can be enriched with the morphisms

$$\Gamma^{\infty}_{\mathbf{t}}(T) : \Big(\Gamma^{\infty}_{\mathbf{t}}(\mathcal{K}), \langle\Omega_{\mathbf{t}}, \cdot\Omega_{\mathbf{t}}\rangle\Big) \to \Big(\Gamma^{\infty}_{\mathbf{t}}(\mathcal{K}'), \langle\Omega_{\mathbf{t}}, \cdot\Omega_{\mathbf{t}}\rangle\Big)$$

for all contractions $T : \mathcal{K} \to \mathcal{K}'$ such that $\Gamma^{\infty}_{\mathbf{t}}$ is a functor of white noise. Again, the same question for $\Delta_{\mathbf{t}}$.

**Definition 5.6** A functor $\Gamma^{\infty}_{\mathbf{t}}$ (or $\Delta_{\mathbf{t}}$) with the above properties will be called *second quantisation at algebraic level* and the completely positive map $\Gamma^{\infty}_{\mathbf{t}}(T)$ (respectively $\Delta_{\mathbf{t}}$), the second quantisation of the contraction $T$.

The existence of the second quantisation at algebraic level turns out to be connected to a property of the functions on pair partitions.

**Definition 5.7** [13] A function $\mathbf{t}$ on pair partitions is called *multiplicative* if for all $k, l, n \in \mathbb{N}$ with $0 \leq k < l \leq n$ and all $\mathcal{V}_1 \in \mathcal{P}_2(\{1, \ldots, k, l+1, \ldots, n\})$ and $\mathcal{V}_2 \in \mathcal{P}_2(\{k+1, \ldots, l\})$ we have

$$\mathbf{t}(\mathcal{V}_1 \cup \mathcal{V}_2) = \mathbf{t}(\mathcal{V}_1) \cdot \mathbf{t}(\mathcal{V}_2). \tag{5.10}$$

**Lemma 5.8** *Let* $\mathbf{t}$ *be a positive definite function on pair partitions and suppose that there exists a functor of second quantisation* $\Gamma_{\mathbf{t}}^{\infty}$ *or* $\Delta_{\mathbf{t}}$ *. Then* $\mathbf{t}$ *is multiplicative.*

*Proof.* Let $\mathcal{K} = \ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})$ with the two projections $P_1$ and $P_2$ on the $\ell^2(\mathbb{Z})$ subspaces. Let $\mathcal{V}_1 \cup \mathcal{V}_2$ be a pair partition as in Definition 5.7. We consider a monomial of fields $M(\mathcal{V}_1 \cup \mathcal{V}_2) = M_{l,1} \cdot M_2 \cdot M_{r,1}$ containing $|\mathcal{V}_1| + |\mathcal{V}_2|$ pairs of different colors arranged according to the pair partition $\mathcal{V}_1 \cup \mathcal{V}_2$ and such that the colors for the pairs in $\mathcal{V}_1$ belong to the first $\ell^2(\mathbb{Z})$ in $\mathcal{K}$, and those for the pairs in $\mathcal{V}_2$ belong to the second term. Then using the fact that $\Gamma_{\mathbf{t}}^{\infty}(0)(\cdot) = \rho_{\mathbf{t}}(\cdot)\mathbf{1}$ we have:

$$\mathbf{t}(\mathcal{V}_1 \cup \mathcal{V}_2)\mathbf{1} = \Gamma_{\mathbf{t}}^{\infty}(0)(M(\mathcal{V}_1 \cup \mathcal{V}_2)) = \Gamma_{\mathbf{t}}^{\infty}(P_1 P_2)(M(\mathcal{V}_1 \cup \mathcal{V}_2)) \tag{5.11}$$
$$= \Gamma_{\mathbf{t}}^{\infty}(P_2)(M_{l,1}\ \Gamma_{\mathbf{t}}^{\infty}(P_1)(M_2)\ M_{r,1}) = \mathbf{t}(\mathcal{V}_1)\mathbf{t}(\mathcal{V}_2)\mathbf{1}. \tag{5.12}$$

All this holds for $\Delta_{\mathbf{t}}$ as well.

□

**Lemma 5.9** *Let* $\mathbf{t}$ *be multiplicative positive definite function. Then the operator* $j := \chi_{\mathbf{t}}(d_0)$ *defined in Theorem 3.2 is an isometry.*

*Proof.* We have

$$\langle \chi_{\mathbf{t}}(d_1)\xi, j^* j \chi_{\mathbf{t}}(d_1)\xi \rangle = \hat{\mathbf{t}}(d_1^* \cdot p \cdot d_2) = \hat{\mathbf{t}}(d_1^* d_2) \cdot 1 = \langle \chi_{\mathbf{t}}(d_1)\xi, \chi_{\mathbf{t}}(d_1)\xi \rangle$$

where $p = d_0^* d_0$ is the diagram consisting of one pair and $\mathbf{t}(p) = 1$ by the normalization convention in the definition of $\mathbf{t}$.

□

**Proposition 5.10** *Let* $\mathbf{t}$ *be multiplicative positive definite function and* $\psi_k \in \mathcal{F}_{\mathbf{t}}^{(k)}(\mathcal{K})$ *a* $k$*-particles vector. Then*

$$\|\omega_{\mathbf{t}}(f_1) \ldots \omega_{\mathbf{t}}(f_n)\psi_k\| \leq 2^{\frac{n}{2}} \sqrt{(k+1)\ldots(k+n)}\|\psi_k\| \prod_{i=1}^{n} \|f_i\| \tag{5.13}$$

*and* $\omega_{\mathbf{t}}(f)$ *is essentially selfadjoint for all* $f \in \mathcal{K}$.

*Proof.* Let $l(f)$ be the creation operator on the full Fock space over $\mathcal{K}$ and $j_n$ the restriction to $V_n$ of the isometry $j$. The main estimates are

$$\begin{aligned}\|a(f)\psi_k\|^2 &= \frac{1}{(k-1)!}\|j^*_{k-1}\otimes l^*(f)\psi_k\|^2_{V_{k-1}\otimes\mathcal{K}^{\otimes k-1}} \leq \\ &\leq \frac{k!}{(k-1)!}\|f\|^2\|\psi_k\|^2 = k\|f\|^2\|\psi_k\|^2, \end{aligned} \tag{5.14}$$

and similarly

$$\|a^*(f)\psi_k\|^2 \leq (k+1)\|f\|^2\|\psi_k\|^2. \tag{5.15}$$

This gives the same result as in the case of the symmetric Fock space (Theorem X.41 in [52]):

$$\|a^\sharp_{\mathbf{t}}(f_1)\dots a^\sharp_{\mathbf{t}}(f_n)\psi_k\| \leq \sqrt{(k+1)\dots(k+n)}\|\psi_k\|\prod_{i=1}^n\|f_i\|. \tag{5.16}$$

In particular the vectors with finite number of particles form a dense set $D$ of analytic vectors for the field operators $\omega_{\mathbf{t}}(f)$. By Nelson's analytic vector theorem we conclude that $\omega_{\mathbf{t}}(f)$ is essentially selfadjoint.

□

From now we will denote by the same symbol the closure of $\omega_{\mathbf{t}}(f)$. We are now in the position to construct the von Neumann algebras $\Gamma^\infty_{\mathbf{t}}(\mathcal{K})$ as described in 5.8 for any multiplicative positive definite $\mathbf{t}$. If $O:\mathcal{K}\to\mathcal{K}'$ is an orthogonal operator between two Hilbert spaces we define its second quantisation as in equation 5.9.

**Corollary 5.11** *Let $\psi_k\in\mathcal{F}_{\mathbf{t}}(\mathcal{K})$ be a $k-$particles vector and $f_1,\dots,f_n\in\mathcal{K}$. Then*

$$\prod_{p=1}^n e^{i\omega_{\mathbf{t}}(f_p)}\psi_k = \sum_{m_1,\dots,m_n=0}^\infty \frac{(i\omega_{\mathbf{t}}(f_1))^{m_1}\dots(i\omega_{\mathbf{t}}(f_n))^{m_n}}{m_1!\dots m_n!}\psi_k. \tag{5.17}$$

*Proof.* Using the previous proposition we get

$$\sum_{m_1,\dots,m_n=0}^\infty \frac{||\omega_{\mathbf{t}}(f_1)^{m_1}\dots\omega_{\mathbf{t}}(f_n)^{m_n}\psi_k||}{m_1!\dots m_n!} \leq$$
$$\frac{||\psi_k||}{\sqrt{k!}}\sum_{m_1,\dots,m_p=0}^\infty\frac{||f_1||^{m_1}\dots||f_n||^{m_n}}{m_1!\dots m_n!}\sqrt{(k+m_1+\dots+m_n)!}<\infty.$$

This means that all vectors of the form $\prod_{p=1}^n e^{i\omega_{\mathbf{t}}(f_p)}\psi_k$ are analytic for the field operators. In particular one can expand as in 5.17. We denote the space of linear combinations of such 'exponential vectors' by $D_e$.

□

**Lemma 5.12** *Let $\mathcal{K}, \mathcal{K}'$ be real Hilbert spaces and $I : \mathcal{K} \to \mathcal{K}'$ an isometry. Then there exists an injective $^*$-homomorphism $\Gamma_{\mathbf{t}}^{\infty}(I)$ from $\Gamma_{\mathbf{t}}^{\infty}(\mathcal{K})$ to $\Gamma_{\mathbf{t}}^{\infty}(\mathcal{K}')$, and similarly for $\Delta_{\mathbf{t}}$.*

*Proof.* There exists an orthogonal operator $O_I : \mathcal{K} \oplus \ell^2(\mathbb{Z}) \to \mathcal{K}' \oplus \ell^2(\mathbb{Z})$ such that the restriction to $\mathcal{K}$ coincides with $I$. Then the map

$$\Gamma_{\mathbf{t}}(\mathcal{K} \oplus \ell^2(\mathbb{Z})) \ni X \mapsto \mathcal{F}_{\mathbf{t}}(O_I) X \mathcal{F}_{\mathbf{t}}(O_I^*) \in \Gamma_{\mathbf{t}}(\mathcal{K}' \oplus \ell^2(\mathbb{Z})) \tag{5.18}$$

sends an element $X \in \Gamma_{\mathbf{t}}^{\infty}(\mathcal{K})$ into $\Gamma_{\mathbf{t}}^{\infty}(\mathcal{K}')$ when restricted to the subalgebra $\Gamma_{\mathbf{t}}^{\infty}(\mathcal{K})$ and the restriction does not depend on the choice of the orthogonal $O_I$. The Wick products in $\Delta(\mathcal{K})$ have the same behavior.

□

In the case of the algebras $\Delta_{\mathbf{t}}(\mathcal{K})$ the coisometries $P$ act 'good' as well:

$$\Delta_{\mathbf{t}}(P) : \Psi(\mathcal{V}, \mathbf{f}) \mapsto \Psi(\mathcal{V}, P \circ \mathbf{f}) = \mathcal{F}_{\mathbf{t}}(P) \Psi(\mathcal{V}, \mathbf{f}) \mathcal{F}_{\mathbf{t}}(P)^* \tag{5.19}$$

as it can be checked directly, this being in fact one of the reasons why we consider the Wick products as building blocks of the algebra of fields. As any contraction can be written as product of a coisometry and an isometry we obtain that for any $T : \mathcal{K} \to \mathcal{K}'$

$$\Delta_{\mathbf{t}}(T) : \Psi(\mathcal{V}, \mathbf{f}) \mapsto \Psi(\mathcal{V}, T \circ \mathbf{f}) \tag{5.20}$$

is a well defined completely positive map from $\Delta_{\mathbf{t}}(\mathcal{K})$ to $\Delta_{\mathbf{t}}(\mathcal{K}')$ and has the functorial properties. Moreover by definition $\Delta_{\mathbf{t}}(\{0\}) = \mathbb{C}$ for multiplicative $\mathbf{t}$ and thus $\Delta_{\mathbf{t}}$ is a functor of white noise. This ends our discussion of the functor $\Delta_{\mathbf{t}}$ which will reappear in the following chapter, but we remind the reader that we have restricted ourselves to the case of bounded Wick products. We believe that the same can be done for the unbounded case with more care when dealing with affiliated Wick products.

We return now to the case of $\Gamma_{\mathbf{t}}^{\infty}$.

**Proposition 5.13** *Let $P : \mathcal{K} \to \mathcal{K}'$ be a coisometry i.e. $PP^* = \mathbf{1}_{\mathcal{K}'}$ and $\mathbf{t}$ a positive definite multiplicative function. Then*

$$\Gamma_{\mathbf{t}}^{\infty}(P) : X \mapsto \mathcal{F}_{\mathbf{t}}(P \oplus \mathbf{1}) X \mathcal{F}_{\mathbf{t}}(P \oplus \mathbf{1})^* \tag{5.21}$$

*maps $\Gamma_{\mathbf{t}}^{\infty}(\mathcal{K})$ onto $\Gamma_{\mathbf{t}}^{\infty}(\mathcal{K}')$.*

*Proof.* We denote by $I$ the adjoint of $P$. We fix an orthonormal basis $(e_i)_{i=1}^{\infty}$ in $\mathcal{K}' \oplus \ell^2(\mathbb{Z})$ and $(f_j)_{j=1}^{M}$ in $\mathcal{K} \ominus (I\mathcal{K}')$. Let $X = \prod_{p=1}^{n} e^{i\lambda_p \omega_{\mathbf{t}}(g_p)}$ be an element of $\Gamma_{\mathbf{t}}(\mathcal{K} \oplus \ell^2(\mathbb{Z}))$ where each $g_p$ is either an $(I \oplus \mathbf{1})e_i$ or an $f_j$. We will prove that $\mathcal{F}_{\mathbf{t}}(P \oplus \mathbf{1}) X \mathcal{F}_{\mathbf{t}}(P \oplus \mathbf{1})^*$ belongs to $\Gamma_{\mathbf{t}}(\mathcal{K}' \oplus \ell^2(\mathbb{Z}))$. Let

$$Y = \Gamma_{\mathbf{t}}(O_P)(X) = \prod_{p=1}^{n} e^{i\lambda_p \omega_{\mathbf{t}}(O_P g_p)} \in \Gamma_{\mathbf{t}}(\mathcal{K}' \oplus \ell^2(\mathbb{Z})) \tag{5.22}$$

where $O_P : \mathcal{K} \oplus \ell^2(\mathbb{Z}) \to \mathcal{K}' \oplus \ell^2(\mathbb{Z})$ is an orthogonal operator which satisfies the condition that $O_P g_p = (P \oplus \mathbf{1}) g_p$ for all $g_p \in I\mathcal{K}' \oplus \ell^2(\mathbb{Z})$. We denote by $\mathcal{H}_X$ the finite dimensional subspace of $\mathcal{K}' \oplus \ell^2(\mathbb{Z})$ spanned by the vectors $(P \oplus \mathbf{1}) g_p$ for $1 \leq p \leq n$. Let $T$ be an operator which is of the form $\mathbf{1}_{\mathcal{H}_X} \oplus S$ with $S$ an arbitrary orthogonal operator which acts as a bilateral shift on the orthonormal basis of the orthogonal complement of $\mathcal{H}_X$ in $\mathcal{K}' \oplus \ell^2(\mathbb{Z})$. We claim that

$$\operatorname*{w-lim}_{l\to\infty} \Gamma_{\mathbf{t}}(T^l)(Y) = \mathcal{F}_{\mathbf{t}}(P \oplus \mathbf{1}) X \mathcal{F}_{\mathbf{t}}(P \oplus \mathbf{1})^*. \tag{5.23}$$

It is sufficient to check this for expectation values with respect to vectors of the form $\psi(\mathcal{V}, \mathbf{e}) = \Psi(\mathcal{V}, \mathbf{e})\Omega_{\mathbf{t}}$ where the components of $\mathbf{e}$ are elements of the basis $(e_i)_{i=1}^{\infty}$. The linear span of such vectors forms the dense domain $D \subset \mathcal{F}_{\mathbf{t}}(\mathcal{K}' \oplus \ell^2(\mathbb{Z}))$. We apply now corollary 5.11 and find that for $l$ large enough:

$$\begin{aligned}
&\langle \psi(\mathcal{V}, \mathbf{e}), \Gamma_{\mathbf{t}}(T^l)(Y)\psi(\mathcal{V}, \mathbf{e}) \rangle = \left\langle \psi(\mathcal{V},\ \mathbf{e}), \prod_{p=1}^{n} e^{i\lambda_p \omega_{\mathbf{t}}(T^l O_P g_p)} \psi(\mathcal{V}, \mathbf{e}) \right\rangle = \\
&\sum_{m_1,\ldots,m_n=0}^{\infty} \left\langle \psi(\mathcal{V}, \mathbf{e}),\ \prod_{q=1}^{n} \frac{(i\lambda_q \omega_{\mathbf{t}}(T^l O_P g_q))^{m_q}}{m_q!} \psi(\mathcal{V}, \mathbf{e}) \right\rangle = \\
&\sum_{m_1,\ldots,m_n=0}^{\infty} \left\langle \psi(\mathcal{V}, I\mathbf{e}),\ \prod_{q=1}^{n} \frac{(i\lambda_q \omega_{\mathbf{t}}(g_q))^{m_q}}{m_q!} \psi(\mathcal{V}, I\mathbf{e}) \right\rangle = \\
=\quad &\langle \psi(\mathcal{V}, \mathbf{e}),\ \mathcal{F}_{\mathbf{t}}(P \oplus \mathbf{1}) X \mathcal{F}_{\mathbf{t}}(P \oplus \mathbf{1})^* \psi(\mathcal{V}, \mathbf{e}) \rangle .
\end{aligned}$$

Indeed the pairing prescription of the fields of the same color insures that the terms in the two sums are equal one by one if we choose $l$ such that no vector $T^l O_P g_p$ in the orthogonal complement of $\mathcal{H}_X$ coincides with a component of $\mathbf{e}$. As the span of the operators of the form $\prod_{p=1}^{n} e^{i\lambda_p \omega_{\mathbf{t}}(g_p)}$ is weakly dense in $\Gamma_{\mathbf{t}}(\mathcal{K} \oplus \ell^2(\mathbb{Z}))$ we can extend the map

$$\Gamma_{\mathbf{t}}(P \oplus \mathbf{1})(X) = \mathcal{F}_{\mathbf{t}}(P \oplus \mathbf{1}) X \mathcal{F}_{\mathbf{t}}(P \oplus \mathbf{1})^* \tag{5.24}$$

to the whole algebra such that $\Gamma_{\mathbf{t}}(P \oplus \mathbf{1})(X) \in \Gamma_{\mathbf{t}}(\mathcal{K}' \oplus \ell^2(\mathbb{Z}))$. Now, if $X$ commutes with $\mathcal{F}_{\mathbf{t}}(1 \oplus O)$ acting on $\mathcal{F}_{\mathbf{t}}(\mathcal{K} \oplus \ell^2(\mathbb{Z}))$ for $O \in \mathcal{O}(\ell^2(Z))$ then it is easy to see that $\Gamma_{\mathbf{t}}(P \oplus \mathbf{1})(X)$ commutes with $\mathcal{F}_{\mathbf{t}}(1 \oplus O)$ acting on $\mathcal{F}_{\mathbf{t}}(\mathcal{K}' \oplus \ell^2(\mathbb{Z}))$. In other words the restriction $\Gamma^{\infty}_{\mathbf{t}}(P)$ of $\Gamma_{\mathbf{t}}(P \oplus \mathbf{1})$ to $\Gamma^{\infty}_{\mathbf{t}}(\mathcal{K})$ has the desired property:

$$\Gamma^{\infty}_{\mathbf{t}}(P) : \Gamma^{\infty}_{\mathbf{t}}(\mathcal{K}) \to \Gamma^{\infty}_{\mathbf{t}}(\mathcal{K}'). \tag{5.25}$$

□

**Corollary 5.14** *Let $I : \mathcal{K} \to \mathcal{K}'$ be an isometry. Then $\Gamma^{\infty}_{\mathbf{t}}(I^*)\Gamma^{\infty}_{\mathbf{t}}(I) = \mathrm{id}_{\Gamma^{\infty}_{\mathbf{t}}(\mathcal{K})}$. If $I' : \mathcal{K}' \to \mathcal{K}''$ is another isometry then $\Gamma^{\infty}_{\mathbf{t}}(I')\Gamma^{\infty}_{\mathbf{t}}(I) = \Gamma^{\infty}_{\mathbf{t}}(I'I)$.*

*Proof.* The map $\Gamma^\infty_{\mathbf{t}}(I^*)\Gamma^\infty_{\mathbf{t}}(I)$ is implemented by

$$\Gamma^\infty_{\mathbf{t}}(I^*)\Gamma^\infty_{\mathbf{t}}(I) : X \to \mathcal{F}_{\mathbf{t}}((I^* \oplus \mathbf{1})O_I)X\mathcal{F}_{\mathbf{t}}((I^* \oplus \mathbf{1})O_I)^*. \tag{5.26}$$

But $(I^* \oplus \mathbf{1})O_I = \mathbf{1} \oplus Q$ where $Q$ is a coisometry on $\ell^2(\mathbb{Z})$. Any such operator on $\ell^2(\mathbb{Z})$ can be obtained as a weak limit of orthogonal operators. The functor $\mathcal{F}_{\mathbf{t}}$ respects weak limits and as $X$ commutes with all $\mathcal{F}_{\mathbf{t}}(\mathbf{1} \oplus O)$ for $O$ orthogonal operator, it also commutes with $\mathcal{F}_{\mathbf{t}}(\mathbf{1} \oplus Q)$, thus we get

$$\Gamma^\infty_{\mathbf{t}}(I^*)\Gamma^\infty_{\mathbf{t}}(I)(X) = \mathcal{F}_{\mathbf{t}}(\mathbf{1} \oplus Q)X\mathcal{F}_{\mathbf{t}}(\mathbf{1} \oplus Q)^* = X. \tag{5.27}$$

The other identity follows directly from the definition of $\Gamma^\infty_{\mathbf{t}}(I)$.

□

Any contraction $T : \mathcal{K}_1 \to \mathcal{K}_2$ can be written as $T = PI$ where $I : \mathcal{K}_1 \to \mathcal{K}$ is an isometry and $P : \mathcal{K} \to \mathcal{K}_2$ is a coisometry. This decomposition is not unique. We define the second quantisation of $T$ by using the already constructed $\Gamma^\infty_{\mathbf{t}}(I)$ and $\Gamma^\infty_{\mathbf{t}}(P)$:

$$\Gamma^\infty_{\mathbf{t}}(T) := \Gamma^\infty_{\mathbf{t}}(P)\Gamma^\infty_{\mathbf{t}}(I) : \Gamma^\infty_{\mathbf{t}}(\mathcal{K}_1) \to \Gamma^\infty_{\mathbf{t}}(\mathcal{K}_2). \tag{5.28}$$

We will verify that $\Gamma^\infty_{\mathbf{t}}(T)$ does not depend on the choice of $I$ and $P$. Firstly we note that we can restrict only to 'minimal' $\mathcal{K}$, that is, $\mathcal{K}$ is spanned by $I\mathcal{K}_1$ and $P^*\mathcal{K}_2$. If this is not the case then we make the decomposition $I = I_2I_1$ and $P = P_2I_2^*$ such that $T = P_2I_1$ is minimal and we use the previous corollary,

$$\Gamma^\infty_{\mathbf{t}}(T) = \Gamma^\infty_{\mathbf{t}}(P)\Gamma^\infty_{\mathbf{t}}(I) = \Gamma^\infty_{\mathbf{t}}(P_2)\Gamma^\infty_{\mathbf{t}}(I_2^*)\Gamma^\infty_{\mathbf{t}}(I_2)\Gamma^\infty_{\mathbf{t}}(I_1) = \Gamma^\infty_{\mathbf{t}}(P_2)\Gamma^\infty_{\mathbf{t}}(I_1). \tag{5.29}$$

Secondly, we compare two minimal decompositions $T = PI = P'I'$ with $I' : K_1 \to \mathcal{K}'$. By minimality, there exists an orthogonal $O$ from $\mathcal{K}'$ and $\mathcal{K}$ defined by

$$\begin{aligned} O &: I'f \mapsto If, \qquad f \in \mathcal{K}_1 \\ O &: P'^*g \mapsto P^*g, \qquad g \in \mathcal{K}_2. \end{aligned}$$

Then $PI = P'O^*OI'$ and by applying again the previous corollary we get $\Gamma^\infty_{\mathbf{t}}(P)\Gamma^\infty_{\mathbf{t}}(I) = \Gamma^\infty_{\mathbf{t}}(P')\Gamma^\infty_{\mathbf{t}}(I')$.

**Lemma 5.15** *For any contractions $T_1 : \mathcal{K}_1 \to \mathcal{K}_2$ and $T_2 : \mathcal{K}_2 \to \mathcal{K}_3$ we have $\Gamma^\infty_{\mathbf{t}}(T_2)\Gamma^\infty_{\mathbf{t}}(T_1) = \Gamma^\infty_{\mathbf{t}}(T_2T_1)$.*

*Proof.* The completely positive maps $\Gamma^\infty_{\mathbf{t}}(T_1)$ and $\Gamma^\infty_{\mathbf{t}}(T_2)$ are implemented by

$$\Gamma^\infty_{\mathbf{t}}(T_i) : X \mapsto \mathcal{F}_{\mathbf{t}}(P_i)X\mathcal{F}_{\mathbf{t}}(P_i)^* \tag{5.30}$$

with $P_i : \mathcal{K}_i \oplus \ell^2(\mathbb{Z}) \to \mathcal{K}_{i+1} \oplus \ell^2(\mathbb{Z})$ are coisometries with the matrix expression

$$P_i = \begin{pmatrix} T_i & A_i \\ 0 & P_i' \end{pmatrix}$$

for $i = 1, 2$. Their product $P_2P_1$ is a coisometry with matrix expression of the same form

$$P_2P_1 = \begin{pmatrix} T_2T_1 & T_2A_1 + A_2P_1' \\ 0 & P_2'P_1' \end{pmatrix}.$$

This implies that $\Gamma_{\mathbf{t}}^{\infty}(T_2T_1) = \Gamma_{\mathbf{t}}^{\infty}(T_2)\Gamma_{\mathbf{t}}^{\infty}(T_1)$.

□

By putting together all the results of this section we obtain the theorem.

**Theorem 5.16** *Let* $\mathbf{t}$ *be a positive definite multiplicative function. Then there exists a functor* $\Gamma_{\mathbf{t}}^{\infty}$ *from real Hilbert spaces to non-commutative probability spaces. This is a functor of second quantisation if and only if* $\rho_{\mathbf{t}}$ *is faithful for* $\Gamma_{\mathbf{t}}(\ell^2_{\mathbb{R}}(\mathbb{Z}))$.

*Proof.* From the previous results we conclude that $\Gamma_{\mathbf{t}}^{\infty}$ is a functor from real Hilbert spaces to non-commutative probability spaces.
The upgrading to functor of white noise requires additionally $\Gamma_{\mathbf{t}}^{\infty}(\{0\}) = \mathbb{C}$ which means that $\Gamma_{\mathbf{t}}(\ell^2_{\mathbb{R}}(\mathbb{Z}))^{\mathcal{O}(\ell^2_{\mathbb{R}}(\mathbb{Z}))} = \mathbb{C}$. If the vacuum is a faithful state then indeed there can be no nontrivial vector, (and element of the von Neumann algebra) which is invariant under all $\mathcal{F}_{\mathbf{t}}(O)$, (respectively $\Gamma_{\mathbf{t}}^{\infty}(O)$).

Let $\mathcal{M}$ denote the von Neumann algebra generated by the unitaries $\{\Gamma_{\mathbf{t}}^{\infty}(\{0\})\}$ for all $O \in \mathcal{O}(\ell^2_{\mathbb{R}}(\mathbb{Z}))$. Then $\Gamma_{\mathbf{t}}(\ell^2_{\mathbb{R}}(\mathbb{Z})) \cap \mathcal{M}' = \mathbb{C}$ is equivalent to $\Gamma_{\mathbf{t}}(\ell^2_{\mathbb{R}}(\mathbb{Z}))' \vee \mathcal{M} = \mathcal{B}(\mathcal{F}_{\mathbf{t}}(\ell^2(\mathbb{Z})))$. If the vacuum state is not faithful then $\Omega_{\mathbf{t}}$ is not cyclic for $\Gamma_{\mathbf{t}}(\ell^2_{\mathbb{R}}(\mathbb{Z}))'$. This algebra has an obvious automorphism group $\Gamma_{\mathbf{t}}'(O)$ which implies that the vacuum is not cyclic for $\Gamma_{\mathbf{t}}(\ell^2_{\mathbb{R}}(\mathbb{Z}))' \vee \mathcal{M}$.

□

In the end we show that for faithful vacuum states there is essentially only one associated functor of second quantisation and we make the connection with the known cases of second quantisation arising from the $q$-deformed commutation relations algebra.

**Corollary 5.17** *Let* $\mathbf{t}$ *be a positive definite multiplicative function such that the vector* $\Omega_{\mathbf{t}}$ *is cyclic and separating for* $\Gamma_{\mathbf{t}}(\ell^2(\mathbb{Z}))$. *Then we have the following:*

*1) the cyclic representation of* $\Gamma_{\mathbf{t}}^{\infty}(\mathcal{K})$ *with respect to* $\Omega_{\mathbf{t}}$ *is faithful and the subspace of* $\mathcal{F}_{\mathbf{t}}(\mathcal{K} \oplus \ell^2(\mathbb{Z}))$ *spanned by* $\Gamma_{\mathbf{t}}^{\infty}(\mathcal{K})\Omega_{\mathbf{t}}$ *is isomorphic to* $\mathcal{F}_{\mathbf{t}}(\mathcal{K})$. *In this representation the second quantisation of a contraction* $T : \mathcal{K}_1 \to \mathcal{K}_2$ *is the completely positive map* $\Gamma_{\mathbf{t}}^{\infty}(T)$ *from* $\Gamma_{\mathbf{t}}^{\infty}(\mathcal{K}_1)$ *to* $\Gamma_{\mathbf{t}}^{\infty}(\mathcal{K}_2)$ *such that*

$$\Gamma_{\mathbf{t}}^{\infty}(T)(X)\Omega_{\mathbf{t}} = \mathcal{F}_{\mathbf{t}}(T)X\Omega_{\mathbf{t}} \tag{5.31}$$

*for* $X \in \Gamma_{\mathbf{t}}^{\infty}(\mathcal{K}_1)$.

*2) if the field operators are bounded then* $\Gamma_{\mathbf{t}}^{\infty}$ *coincides with* $\Delta_{\mathbf{t}}$.

*Proof.* If $X \in \Gamma^\infty_{\mathbf{t}}(\mathcal{K})$ then $\psi = X\Omega_{\mathbf{t}}$ is left invariant by $\mathcal{F}_{\mathbf{t}}(\mathbf{1} \oplus O)$ for all $O \in \mathcal{O}(\ell^2(\mathbb{Z}))$. This means that $\psi$ is orthogonal on all vectors of the form $\Psi(\mathcal{V}, \mathbf{e})\Omega_{\mathbf{t}}$ where $\mathbf{e}$ takes values in an orthogonal basis of $\mathcal{K} \oplus \ell^2(\mathbb{Z})$ such that at least one of the components is an element of the basis in $\ell^2(\mathbb{Z})$. By corollary 4.7 we conclude that the cyclic space of $\Gamma^\infty_{\mathbf{t}}(\mathcal{K})$ is (up to a unitary isomorphism) $\mathcal{F}_{\mathbf{t}}(\mathcal{K}) \subset \mathcal{F}_{\mathbf{t}}(\mathcal{K} \oplus \ell^2(\mathbb{Z}))$.

It is not difficult to prove that on basis of the faithfulness of the vacuum we have $\Gamma^\infty_{\mathbf{t}}(\ell^2_{\mathbb{R}}(\mathbb{Z})) \simeq \Gamma_{\mathbf{t}}(\ell^2_{\mathbb{R}}(\mathbb{Z}))$. For finite dimensional $\mathcal{K}$ it can happen that $\Gamma_{\mathbf{t}}(\mathcal{K})$ is strictly included in $\Gamma^\infty_{\mathbf{t}}(\mathcal{K})$, and this this the reason why $\Gamma^\infty$ is the right definition of the functor of white noise. In the $q$-deformed commutation relations the two algebras coincide, but this should be seen as exceptional rather than the rule.

If the fields are bounded then $\Delta_{\mathbf{t}}$ is functor of white noise and on the other hand $\Delta_{\mathbf{t}}(\ell^2_{\mathbb{R}}) \simeq \Gamma_{\mathbf{t}}(\ell^2_{\mathbb{R}}(\mathbb{Z}))$ by construction, and the same hold for finite dimensional Hilbert spaces $\mathcal{K}$. Thus $\Delta_{\mathbf{t}}$ and $\Gamma^\infty_{\mathbf{t}}$ coincide.

□

# 6 An Example

In [13] and in [28] (chapter II of this thesis) it has been proved that for all $0 \leq q \leq 1$, the following function on pair partitions is positive definite:

$$\mathbf{t}_q(\mathcal{V}) = q^{|\mathcal{V}| - |\mathrm{B}(\mathcal{V})|} \tag{6.1}$$

where $|\mathrm{B}(\mathcal{V})|$ is the number of blocks of the pair partition $\mathcal{V}$. A block is a subpartition whose graphical representation is connected and does not intersect other pairs from the rest of the partition. The corresponding vacuum state $\rho_{\mathbf{t}_q}(\cdot) = \langle \Omega_{\mathbf{t}_q}, \cdot \Omega_{\mathbf{t}_q} \rangle$ is tracial for any von Neumann algebra $\Gamma_{\mathbf{t}_q}(\mathcal{K})$ associated to a real Hilbert space $\mathcal{K}$. Indeed for any pair partition $\mathcal{V}$ we have $\mathbf{t}_q(\mathcal{V}) = \langle \Omega_{\mathbf{t}_q}, M_{\mathcal{V}} \Omega_{\mathbf{t}_q} \rangle$ with $M_{\mathcal{V}}$ a monomial of fields containing $|\mathcal{V}|$ pairs of different colors arranged according to the pair partition $\mathcal{V}$. The trace property for the vacuum is equivalent with the invariance under circular permutations of the fields in the monomial $M_{\mathcal{V}}$ which is equivalent to the invariance of $\mathbf{t}_q$ under transformations described as follows:

$$\mathcal{P}_2(\{1, \ldots, 2r\}) \ni \mathcal{V} \quad \mapsto \quad \tilde{\mathcal{V}} \in \mathcal{P}_2(\{0, \ldots, 2r-1\}) \tag{6.2}$$

$$\{p_1, \ldots, p_{r-1}\} \cup \{(l, 2r)\} \quad \mapsto \quad \{(0, l)\} \cup \{p_1, \ldots, p_{r-1}\}. \tag{6.3}$$

Under such transformations the number of blocks remains unchanged thus $\mathbf{t}_q(\mathcal{V})$ is equal to $\mathbf{t}_q(\tilde{\mathcal{V}})$ and $\rho_{\mathbf{t}_q}$ is tracial. Thus the assumption of Corollary 5.17 is satisfied and we have second quantisation at algebraic level.

The version of $\mathbf{t}_q$ for $-1 \leq q \leq 0$ is $\mathbf{t}_q := \mathbf{t}_{-q}\mathbf{t}_{-1}$ where

$$\mathbf{t}_{-1}(\mathcal{V}) = (-1)^{|I(\mathcal{V})|} \tag{6.4}$$

and $|I(\mathcal{V})|$ is the number of crossings of $\mathcal{V}$. The operators $\omega_{\mathbf{t}_q}(f)$ are bounded for $-1 \leq q \leq 0$ [13]. Thus by corollary 5.17 the generalised Wick products form a strongly dense subalgebra of $\Gamma^{\infty}_{\mathbf{t}_q}(\mathcal{K})$, faithfully represented on $\mathcal{F}_{\mathbf{t}}(\mathcal{K})$.

In the rest of this section we want to investigate the type of the von Neumann algebras $\Gamma_{\mathbf{t}_q}(\mathcal{K})$ for $\dim \mathcal{K} = \infty$ and $-1 \leq q \leq 0$. Inspired by [9], we will first find a sufficient condition for $\Gamma_{\mathbf{t}}(\mathcal{K})$ to be a type $\mathrm{II}_1$ factor, and we will apply it to $\mathbf{t}_q$.
Let $\mathbf{t}$ be a multiplicative positive definite function such that $\rho_{\mathbf{t}}$ is trace state on $\Gamma_{\mathbf{t}}(\mathcal{K})$ for $\mathcal{K}$ infinite dimensional and such that $\omega_{\mathbf{t}}(f)$ is bounded. Let $I$ be the natural isometry from $\mathcal{K}$ to $\mathcal{K} \oplus \mathbb{R}$, and $e_0$ a unit vector in the orthogonal complement of its image. The function $\mathbf{t}$ being multiplicative implies that the map

$$\phi : \mathcal{F}_{\mathbf{t}}(\mathcal{K}) \to \mathcal{F}_{\mathbf{t}}(\mathcal{K} \oplus \mathbb{R}) \tag{6.5}$$

defined by $\phi = \omega_{\mathbf{t}}(e_0)\mathcal{F}_{\mathbf{t}}(I)$ is an isometry.

**Definition 6.1** Let $(P, L, R)$ be a disjoint partition of the ordered set $\{1, \ldots, 2n+l+r\}$ and $d = (\mathcal{V}, f_l, f_r)$ an element of the $^*$-semigroup of broken pair partitions with $\mathcal{V} \in \mathcal{P}_2(P)$, $f_l : L \to \{1, \ldots l\}$ the left legs and $f_r : R \to \{1, \ldots r\}$ the right legs. We denote by $\underline{d} := (\underline{\mathcal{V}}, f_l, f_r)$ the element obtained by adding to $\mathcal{V}$ one pair which embraces all other pairs

$$\underline{\mathcal{V}} := \mathcal{V} \cup \{(0, 2n+l+r+1)\} \in \mathcal{P}_2(\{0\} \cup P \cup \{2n+l+r+1\}). \tag{6.6}$$

Then the map

$$\Phi(\cdot) := \Gamma_{\mathbf{t}}(I^*)(\omega_{\mathbf{t}}(e_0)\Gamma_{\mathbf{t}}(I)(\cdot)\omega_{\mathbf{t}}(e_0)) = \phi^*\Gamma_{\mathbf{t}}(I)(\cdot)\phi \tag{6.7}$$

has the following action on the generalised Wick products:

$$\Phi(\Psi(\mathcal{V}, \mathbf{f})) = \Psi(\underline{\mathcal{V}}, \mathbf{f}) \tag{6.8}$$

which on the level of von Neumann algebras gives the completely positive contraction from $\Gamma_{\mathbf{t}}(\mathcal{K})$ to itself. We fix an orthonormal basis $(e_n)_{n=1}^{\infty}$ in $\mathcal{K}$. Then by direct computation one can check that:

$$\Phi(X) = \operatorname*{w-lim}_{n\to\infty} \omega_{\mathbf{t}}(e_n)X\omega_{\mathbf{t}}(e_n). \tag{6.9}$$

Let now $\tau$ be an arbitrary tracial normal state on $\Gamma_{\mathbf{t}}(\mathcal{K})$. Then using the fact that $\omega_{\mathbf{t}}(e_n)^2 \to \mathbf{1}$ weakly as $n \to \infty$, we get:

$$\tau(\Phi(X)) = \lim_{n\to\infty} \tau(\omega_{\mathbf{t}}(e_n)X\omega_{\mathbf{t}}(e_n)) = \lim_{n\to\infty} \tau(\omega_{\mathbf{t}}(e_n)^2 X) = \tau(X). \tag{6.10}$$

Suppose now that

$$\operatorname*{w-lim}_{k\to\infty} \Phi^k(X) = \rho_{\mathbf{t}}(X)\mathbf{1} \tag{6.11}$$

for all $X \in \Gamma_{\mathbf{t}}(\mathcal{K})$ which by the faithfulness of the vacuum state is equivalent to $\lim_{k\to\infty} \Phi^k(X)\Omega_{\mathbf{t}} = \rho_{\mathbf{t}}(X)\Omega_{\mathbf{t}}$. Then by equation (6.10) we conclude that $\rho_{\mathbf{t}}$ is the only trace state on $\Gamma_{\mathbf{t}}(\mathcal{K})$ which is thus a type $\mathrm{II}_1$ factor. Let us take a closer look at the contraction

$$\Theta : X\Omega_{\mathbf{t}} \mapsto \Phi(X)\Omega_{\mathbf{t}}. \tag{6.12}$$

From equation (6.8) the operator $\Theta$ commutes with the orthogonal projectors on the spaces with definite 'occupation numbers' $\mathcal{F}_{\mathbf{t}}(n_1, \ldots, n_k)$ (see 2.31). Thus

$$\Theta : v \otimes_s e(\underline{n}) \mapsto \theta(v) \otimes_s e(\underline{n}) \tag{6.13}$$

where

$$e(\underline{n}) := \underbrace{e_1 \otimes \ldots \otimes e_1}_{n_1 \text{times}} \otimes \ldots \otimes \underbrace{e_k \otimes \ldots \otimes e_k}_{n_k \text{times}}, \tag{6.14}$$

$v \in V_n$ and $\theta : V \to V$ is the linear operator defined by

$$\theta : \chi_{\mathbf{t}}(d)\xi \mapsto \chi_{\mathbf{t}}(\underline{d})\xi, \qquad (d \in \mathcal{BP}_2(\infty)). \tag{6.15}$$

**Lemma 6.2** *Let* $\mathbf{t}$ *be a multiplicative positive definite function such that* $\rho_{\mathbf{t}}$ *is trace. Then the operator* $\theta : V \to V$ *defined by 6.15 is a selfadjoint contraction.*

*Proof.* Let $d_1, d_2 \in \mathcal{BP}_2^{(n,0)}$ be two diagrams with $n$ left legs and no right legs. Then

$$\langle \chi_{\mathbf{t}}(d_1)\xi,\ \theta\ \chi_{\mathbf{t}}(d_2)\xi \rangle_V = \hat{\mathbf{t}}(d_1^* \cdot \underline{d_2}). \tag{6.16}$$

But if $\rho_{\mathbf{t}}$ is a trace then

$$\hat{\mathbf{t}}(d_1^* \cdot \underline{d_2}) = \hat{\mathbf{t}}(\underline{d_1}^* \cdot d_2) \tag{6.17}$$

which implies that

$$\langle \chi_{\mathbf{t}}(d_1)\xi,\ \theta\ \chi_{\mathbf{t}}(d_2)\xi \rangle_V = \langle\ \theta\ \chi_{\mathbf{t}}(d_1)\xi, \chi_{\mathbf{t}}(d_2)\xi \rangle_V . \tag{6.18}$$

Thus $\theta$ is a selfadjoint contraction.

□

**Theorem 6.3** *If* $\xi$ *is the only eigenvector of* $\theta$ *with eigenvalue 1 then* $\Gamma_{\mathbf{t}}(\mathcal{K})$ *is a* $\mathrm{II}_1$ *factor for any infinite dimensional real Hilbert space* $\mathcal{K}$.

*Proof.* The operator $\theta$ is a selfadjoint contraction, thus

$$\operatorname*{w-lim}_{k\to\infty} \theta^k = P_\xi \tag{6.19}$$

where $P_\xi$ is the projection on the subspace $\mathbb{C}\xi$. This implies 6.11 and thus $\Gamma_{\mathbf{t}}(\mathcal{K})$ is a $\mathrm{II}_1$ factor .

□

**Corollary 6.4** *Let $\mathcal{K}$ be an infinite dimensional real Hilbert space and $\mathbf{t}_q$ the positive definite function for $-1 < q \leq 0$. Then the von Neumann algebra $\Gamma_{\mathbf{t}_q}(\mathcal{K})$ is a type* $\mathrm{II}_1$ *factor.*

*Proof.* Let $d_1, d_2 \in \mathcal{BP}_2^{(n,0)}$ be two diagrams with $n \geq 1$ left legs and no right legs. Then

$$\begin{aligned}&\left\langle \chi_{\mathbf{t}_q}(d_1)\xi,\ \theta^2\ \chi_{\mathbf{t}_q}(d_2)\xi \right\rangle_V = \hat{\mathbf{t}}_q(\underline{d_1}^* \cdot \underline{d_2}) = q(-1)^n \cdot \hat{\mathbf{t}}_q(d_1^* \cdot \underline{d_2}) \\ &= q(-1)^n \cdot \left\langle \chi_{\mathbf{t}_q}(d_1)\xi,\ \theta\ \chi_{\mathbf{t}_q}(d_2)\xi \right\rangle_V .\end{aligned} \tag{6.20}$$

where we have used the selfadjointness of $\theta$ in the first step and

$$\begin{aligned}|B(\underline{d_1}^* \cdot \underline{d_2})| &= |B(d_1^* \cdot \underline{d_2})|, \\ |\underline{d_1}^* \cdot \underline{d_2}| &= |d_1^* \cdot \underline{d_2}| + 1\end{aligned}$$

in the second equality. Thus the restriction of $\theta$ to $V \ominus \mathbb{C}\xi$ has norm $|q| < 1$ and we can apply Theorem 6.3.

□

**Remark.** The case $0 \leq q < 1$ is technically more difficult as the field operators are unbounded.

If $\rho_{\mathbf{t}}$ is a faithful, multiplicative, but non-tracial state for $\Gamma_{\mathbf{t}}(\mathcal{K})$ then the operators $\phi, \Phi, \Theta, \theta$ can still be defined in the same way. If moreover, $\xi$ is the only eigenvector with eigenvalue 1 of the operator $\theta$, then by a similar argument it can be shown that the algebra $\Gamma_{\mathbf{t}}(\mathcal{K})$ is a factor. Indeed if $X$ is an element in the center of $\Gamma_{\mathbf{t}}(\mathcal{K})$ then $\Phi(X) = \mathrm{w}{-}\lim_{k\to\infty} \omega_{\mathbf{t}}(e_n)^2 X = X$. which contradicts the assumption on $\theta$. This factor cannot be of type $II_1$ because the vacuum state is not tracial. Using this observation one could in principle construct type $III$ factors for certain positive definite multiplicative functions on pair partitions.

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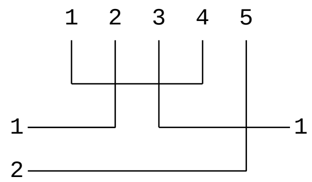


Figure 3.1: Diagram corresponding to an element of the semigroup

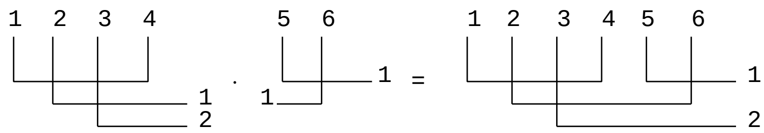


Figure 3.2: Product of two elements

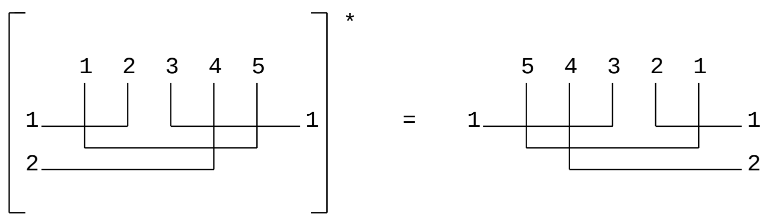


Figure 3.3: The adjoint

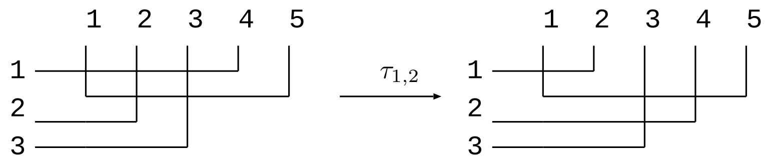


Figure 3.4: The action of a transposition

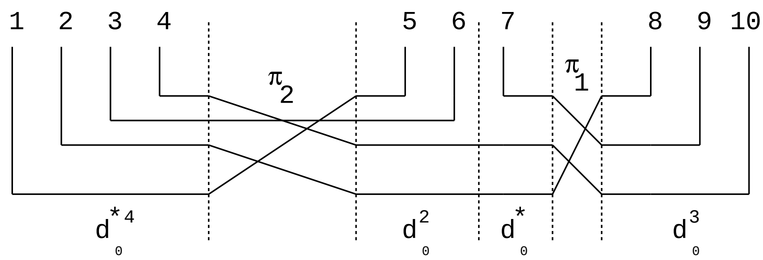


Figure 3.5: The standard form of a pair partition

CHAPTER IV

# Functors of white noise associated to characters of $S(\infty)$ [3]

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ABSTRACT

The characters $\phi_{\alpha,\beta}$ of the infinite symmetric group are extended to multiplicative positive definite functions $\mathbf{t}_{\alpha,\beta}$ on pair partitions by using an explicit representation due to Veršik and Kerov. The von Neumann algebra $\Gamma_{\alpha,\beta}(\mathcal{K})$ generated by the fields $\omega_{\alpha,\beta}(f)$ with $f$ in an infinite dimensional real Hilbert space $\mathcal{K}$ is infinite and the vacuum vector is not separating. For a family $\mathbf{t}_N$ depending on an integer $N < -1$ an 'exclusion principle' is found allowing at most $|N|$ 'identical particles' on the same state:

$$a(f)a^*(g) = \langle f, g\rangle \mathbf{1} + \frac{1}{N}\mathrm{d}\Gamma(T_{f,g}).$$

The algebras $\Gamma_N(\ell^2_{\mathbb{R}}(\mathbb{Z}))$ are type $I_\infty$ factors. Functors of white noise $\Delta_N$ are constructed and proved to be non-equivalent for different values of $N$.

[3]This chapter is based on reference [8].

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# 1 Introduction

The theory of non-commutative processes with 'independent increments' has been the object of various investigations in quantum probability the most important approaches being the Hudson-Parthasarathy calculus [50] based on tensor independence, and Voiculescu's free probability with its concept of free independence [65].

A general theory of quantum white noise, Brownian motion and Markov processes is developed by Köstler [38] in the spirit of Kümmerer's approach to quantum probability [42, 41, 39]. The white noise is described by a finite quantum probability space of $\mathcal{A}_0$-valued random variables i.e., a von Neumann algebra $\mathcal{A}$, endowed with a tracial normal state $\rho$ together with a subalgebra $\mathcal{A}_0$ and the state preserving conditional expectation $\mathrm{P}_0$ from $\mathcal{A}$ to $\mathcal{A}_0$ [61]. The triple $(\mathcal{A}, \rho, \mathcal{A}_0)$ is provided with a filtration of subalgebras $\mathcal{A}_\mathrm{I}$ of $\mathcal{A}$, for all closed intervals I of the time axis $\mathbb{R}$. A group $(S_t)_{t\in\mathbb{R}}$ of automorphisms of $(\mathcal{A}, \rho)$ acts as a shift on the local algebras $S_t(\mathcal{A}_\mathrm{I}) = \mathcal{A}_{\mathrm{I}+t}$ and lets $\mathcal{A}_0$ pointwise invariant. For disjoint intervals $\mathrm{I}, \mathrm{J}$ the local algebras $\mathcal{A}_\mathrm{I}$ and $\mathcal{A}_\mathrm{J}$ are statistically independent over $\mathcal{A}_0$ i.e., $\mathrm{P}_\mathrm{I} \circ \mathrm{P}_\mathrm{J} = \mathrm{P}_0$ a notion in which we recognize a commuting square of von Neumann algebras [24]. The quantum Brownian motion is an additive cocycle $(B_t)_{t\in\mathbb{R}}$ with respect to the white noise $(\mathcal{A}, \rho, S_t, \mathcal{A}_\mathrm{I})$ over $\mathcal{A}_0$ that is, a process which is adapted to the filtration $\mathcal{A}_{[0,t]}$, satisfying $B_{s+t} = B_t + S_t(B_s)$ and certain continuity requirements in the $\mathrm{L}^p$-norms (see definitions in chapter 3 of [38]).

A more functorial approach has been abstracted from the study of the algebra of deformed or $q$-commutation relations [9, 10, 11, 21, 23, 26, 45, 67]. The selfadjoint field operators $\omega_q(f) := a_q(f) + a_q^*(f)$ for $f \in \mathrm{L}^2_\mathbb{R}(\mathbb{R}_+)$ have vacuum expectations expressed as a sum over all possible partitions of the terms in the monomial into pairs

$$\rho_q(\omega_q(f_1)\dots\omega_q(f_n)) = \sum_{\mathcal{V}\in\mathcal{P}_2(n)} \mathbf{t}_q(\mathcal{V}) \prod_{(k,l)\in\mathcal{V}} \langle f_k, f_l\rangle \tag{1.1}$$

where $\mathbf{t}_q(\mathcal{V}) = q^{\mathrm{cr}(\mathcal{V})}$ and $\mathrm{cr}(\mathcal{V})$ is the number of crossings of the pair partition $\mathcal{V}$. The classical Brownian motion is realized for $q = 1$ while the free Brownian motion [65] for $q = 0$ by defining $B_t^{(q)} = \omega_q(\chi_{[0,t]})$. Those functions $\mathbf{t} : \mathcal{P}_2(\infty) \to \mathbb{C}$ on pair partitions which give rise to a positive 'gaussian' functionals $\rho_\mathbf{t}$ on the algebra of fields $\omega(f)$, are called positive definite [13]. In particular they restrict to positive definite functions on the infinite symmetric group $\mathrm{S}(\infty)$, by identifying the permutations with certain pair partitions [13]. The representations with respect to such functionals $\rho_\mathbf{t}$ are called generalised Brownian motions and are the object of the papers [13, 55, 28, 27].

We regard the usual (symmetric) Fock space over a Hilbert space as an endofunctor of the category of Hilbert spaces. This can be generalised to the analytic functors of Joyal [36] whose symmetries are determined by combinatorial objects like species of structures [35, 5]. In the previous two chapters it has been observed that such generalised Fock spaces are the representation spaces of generalised Brownian motions. One can define creation and annihilation operators $a^\sharp(f)$ whose sums are the fields $\omega(f)$. This framework is described in section 2.

For positive definite functions which have a certain multiplicativity property, it has been shown [27] (chapter III of this thesis) that the field operators are selfadjoint and thus one can investigate the von Neumann algebras which they generate as well as the existence of functors from the category of (real) Hilbert spaces with contractions to the category of non-commutative probability spaces.

For tracial vacuum states $\rho_{\mathbf{t}}$, the functor $\Gamma_{\mathbf{t}}$ of white noise constructed in chapter III is a concrete realization of a quantum white noise in the sense of Köstler, if we define $\mathcal{A}_0 := \mathbb{C}$, $\mathcal{A} := \Gamma_{\mathbf{t}}(\mathrm{L}^2(\mathbb{R}))$, $\mathcal{A}_{\mathrm{I}} \simeq \Gamma_{\mathbf{t}}(\mathrm{L}^2(\mathrm{I}))$ with $S_t := \Gamma(s_t)$ where $s_t$ is the shift operator on $\mathrm{L}^2(\mathbb{R})$. The Brownian motion is $\mathrm{B}_s^{(\mathbf{t})} := \omega_{\mathbf{t}}(\chi_{[0,s]})$.

No example of function $\mathbf{t}$ is yet known such that the vacuum state $\rho_{\mathbf{t}}$ is faithful but not tracial.

In this paper we treat a class of generalised Brownian motions for which the vacuum state is not faithful. They arise from the characters of the infinite symmetric group $\mathrm{S}(\infty)$. By the theorem of Thoma [62] any such character has the form

$$\phi_{\alpha,\beta}(\sigma) = \prod_{m\geq 2}\left(\sum_{i=1}^{\infty}\alpha_i^m \ + \ (-1)^{m+1}\sum_{i=1}^{\infty}\beta_i^m\right)^{\rho_m(\sigma)}$$

where $(\alpha_i)_{i=1}^{\infty}$ and $(\beta_i)_{i=1}^{\infty}$ are decreasing sequences of positive numbers with $\sum_{i=1}^{\infty}\alpha_i + \sum_{i=1}^{\infty}\beta_i \leq 1$ and $\rho_m(\sigma)$ is the number of cycles of length $m$ in the cycle decomposition of $\sigma$. An explicit construction of the representation of $\mathrm{S}(\infty)$ with respect to the positive definite function $\phi_{\alpha,\beta}$ has been presented by Veršik and Kerov in [64]. In section 3 we employ this representation to calculate the expression of a multiplicative positive definite function on pair partitions $\mathbf{t}_{\alpha,\beta}$ which extends $\phi_{\alpha,\beta}$. For an arbitrary pair partition $\mathcal{V}$ we have

$$\mathbf{t}_{\alpha,\beta}(\mathcal{V}) = \prod_{m\geq 2}\left(\sum_{i=1}^{\infty}\alpha_i^m + (-1)^{m+1}\sum_{i=1}^{\infty}\beta_i^m\right)^{\rho_m(\mathcal{V})} .$$

where $\rho_m(\mathcal{V})$ is the *number of cycles* of length $m$ in the pair partition $\mathcal{V}$, the cycle of a pair partition being a new combinatorial concept extending that from permutations. For any real Hilbert space $\mathcal{K}$ we construct the von Neumann algebra $\Gamma_{\alpha,\beta}(\mathcal{K})$ generated by selfadjoint field operators $\omega_{\alpha,\beta}(f)$ with $f \in \mathcal{K}$ and investigate its properties by using the general theory developed in chapter III. If the space $\mathcal{K}$ is infinite dimensional we find that $\Gamma_{\alpha,\beta}(\mathcal{K})$ is an infinite von

Neumann algebra and the vacuum state is not faithful. This makes the object of section 4.

In the last section we treat in more detail a particular case of positive definite functions indexed by a natural number $N < -1$,

$$\mathbf{t}_N(\mathcal{V}) = \left(\frac{1}{N}\right)^{|\mathcal{V}|-\mathrm{c}(\mathcal{V})},$$

where $|\mathcal{V}|$ is the number of pairs of $\mathcal{V}$ and $\mathrm{c}(\mathcal{V})$ the number of cycles. Alternatively to the representation inspired from Veršik and Kerov, we use the technique of deformation of the inner product on the full Fock space known from the $q$-deformed Brownian motion [9, 10, 11] and obtain an interesting example of relations between creation and annihilation operators:

$$a_N(f)a_N^*(g) = \langle f,\ g\rangle\, \mathbf{1} - \frac{1}{|N|}\mathrm{d}\Gamma(T_{f,g})$$

where the finite rank operator $T_{f,g}$ acts as $T_{f,g}h = \langle f, h\rangle\, g$. The differential second quantisation operators $\mathrm{d}\Gamma(A)$ are defined similarly to their counterparts in quantum field theory [53]. For $f = g$ this implies that the number operator $\mathbf{N}_f = \mathrm{d}\Gamma(T_{f,f})$ counting the number of one-particle $f$-states is bounded by $|N|$, an 'exclusion principle' which could have some interest also from the physics point of view. The algebra $\Gamma_N(\ell^2_{\mathbb{R}}(\mathbb{Z}))$ generated by the field operators $\omega_N(f)$ acting on $\mathcal{F}_N(\ell^2(\mathbb{Z}))$ contains all bounded operators on the Fock space.

From the functorial point of view a different algebra $\Delta_N(\mathcal{K})$ generated by the so called generalised Wick product operators $\Psi(\mathcal{V}, \mathbf{f})$ acting on $\mathcal{F}_N(\mathcal{K}_{\mathbb{C}} \oplus \ell^2(\mathbb{Z}))$ is more interesting. For any contraction $T : \mathcal{K} \to \mathcal{K}'$ between real Hilbert spaces one can define its second quantisation, a vacuum state preserving completely positive map $\Delta_N(T)$ between $\Delta_N(\mathcal{K})$ and $\Delta_N(\mathcal{K}')$. Altogether, $\Delta_N$ is a functor of white noise. For infinite dimensional $\mathcal{K}$ the von Neumann algebra $\Delta_N(\mathcal{K})$ is a discrete sum of type $I_\infty$ factors; for finite finite dimensional $\mathcal{K}$, $\Delta_N(\mathcal{K})$ is a matrix algebra. In particular

$$\Delta_N(\mathbb{R}) = \bigoplus_{k=2}^{N+1} \mathrm{M}_k(\mathbb{C}),$$

implying that the functors $\Delta_N$ are inequivalent for different values of $N$.

# 2 Theory of Generalised Brownian Motion

In this section we define the notion of generalised Brownian motion [9, 13, 10, 11], and describe some results from [28], [27] (chapters II and III of this thesis).

**Definition 2.1** Let $\mathcal{K}$ be a real separable Hilbert space. The algebra $\mathcal{A}(\mathcal{K})$ is the free unital $^*$-algebra with generators $\omega(h)$ for all $h \in \mathcal{K}$, divided by the relations:

$$\omega(af + bg) = a\omega(f) + b\omega(g), \qquad \omega(f) = \omega(f)^* \tag{2.1}$$

for all $f, g \in \mathcal{K}$ and $a, b \in \mathbb{R}$.

In this paper we will consider G.N.S.-like representations of such $^*$-algebras with respect to positive functionals called gaussian states. These states arise from a non-commutative central limit theorem (Theorem 0 in [13]) and are described by functions on pair partitions.

**Definition 2.2** Let $S$ be a finite ordered set with $n$ elements. We denote by $\mathcal{P}_2(S)$ the set of pair partitions of $S$, that is $\mathcal{V} \in \mathcal{P}_2(S)$ if $\mathcal{V}$ consists of $\frac{1}{2}n$ disjoint ordered pairs $(l, r)$ with $l < r$ having $S$ as their reunion. The set of all pair partitions is

$$\mathcal{P}_2(\infty) := \bigcup_{r=0}^{\infty} \mathcal{P}_2(2r). \tag{2.2}$$

Note that $\mathcal{P}_2(n) = \emptyset$ if $n$ is odd. We use the symbol $\mathbf{t}$ exclusively for functions $\mathbf{t} : \mathcal{P}_2(\infty) \to \mathbb{C}$. We will always choose the normalization $\mathbf{t}(p) = 1$ for $p$ the pair partition containing only one pair.

**Definition 2.3** A *Gaussian state* on $\mathcal{A}(\mathcal{K})$ is a positive normalized linear functional $\rho_{\mathbf{t}}$ with moments

$$\rho_{\mathbf{t}}(\omega(f_1) \dots \omega(f_n)) = \sum_{\mathcal{V} \in \mathcal{P}_2(n)} \mathbf{t}(\mathcal{V}) \prod_{(k,l) \in \mathcal{V}} \langle f_k, f_l \rangle \tag{2.3}$$

for even $n$, and zero for odd $n$. A function $\mathbf{t}$ is called *positive definite* if $\rho_{\mathbf{t}}$ is a Gaussian state.

**Remark 2.4** If $\mathbf{t}$ is positive definite then its restriction to the pair partitions of the form $\mathcal{V}_\pi := \{(i, 2n+1-\pi(i)) \ : \ i = 1, \dots, n)\}$ where $\pi \in \mathrm{S}(n)$, is a positive definite function on the symmetric group $\mathrm{S}(n)$ [13].

By analyzing the GNS representation associated to a Gaussian state $\rho_{\mathbf{t}}$ we have generalised [28, 27] the notion of Fock space over $\mathcal{K}_{\mathbb{C}}$ in the following way.

**Definition 2.5** Let $\mathbf{V} = (V_n)_{n=0}^{\infty}$ be a collection of Hilbert spaces such that each $V_n$ carries a unitary representation $U_n$ of the symmetric group $\mathrm{S}(n)$. Let $\mathcal{H}$ be a (complex) Hilbert space. The $\mathbf{V}$-Fock space over $\mathcal{H}$ is defined by

$$\mathcal{F}_{\mathbf{V}}(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \frac{1}{n!} V_n \otimes_s \mathcal{H}^{\otimes n}, \tag{2.4}$$

where $\otimes_s$ denotes the closed subspace of the tensor product $V_n \otimes \mathcal{H}^{\otimes n}$ whose orthogonal projection is

$$P_n = \frac{1}{n!} \sum_{\tau \in \mathrm{S}(n)} U_n(\tau) \otimes \tilde{U}_n(\tau), \tag{2.5}$$

and

$$\tilde{U}_n f_1 \otimes \ldots \otimes f_n = f_{\tau^{-1}(1)} \otimes \ldots \otimes f_{\tau^{-1}(n)} \tag{2.6}$$

for $f_i \in \mathcal{H}$. The factor $\frac{1}{n!}$ in 2.4 refers to the inner product on $V_n \otimes_s \mathcal{H}^{\otimes n}$. We note that $\mathcal{F}_{\mathbf{V}}$ is an endofunctor of the category of Hilbert spaces with contractions called *analytic functor* [36]. We use the shorter notation $v \otimes_s (h_1 \otimes \ldots \otimes h_n)$ for the vector $n! P_n\ v \otimes (h_1 \otimes \ldots \otimes h_n)$. Let $T : \mathcal{H}_1 \to \mathcal{H}_2$ be a contraction between Hilbert spaces. Then its *second quantisation* on the level of Hilbert spaces $\mathcal{F}_{\mathbf{V}}(T)$ is defined by

$$\mathcal{F}_{\mathbf{V}}(T) : v \otimes_s (h_1 \otimes \ldots \otimes h_n) \mapsto v \otimes_s (Th_1 \otimes \ldots \otimes Th_n) \tag{2.7}$$

for all $v \in V_n$, $h_i \in \mathcal{H}$ when $n \geq 1$, and equal to the identity on $V_0$.

On $\mathcal{F}_{\mathbf{V}}(\mathcal{H})$ we define creation and annihilation operators whose domain consists of vectors with 'finite number of particles'. If $\psi_{n+1} \in V_{n+1} \otimes_s \mathcal{H}^{\otimes n+1}$ then $a(f)\psi_{n+1} = (j_n^* \otimes r^*(f))\psi_{n+1}$ where $j_n : V_n \to V_{n+1}$ are densely defined linear maps having the intertwining property

$$j_n U_n(\tau) = U_n(\iota(\tau)) j_n \tag{2.8}$$

for all $\tau \in \mathrm{S}(n)$, and $\iota : \mathrm{S}(n) \hookrightarrow \mathrm{S}(n+1)$ being the natural inclusion by keeping the element $n+1$ fixed. The operator $r(f)$ is the *right* creation operator on the full Fock space in the notation of Voiculescu (see remark 2.6.7. in [65]).

The equation 2.8 insures that $(j_n^* \otimes r^*(f))P_{n+1} = P_n(j_n^* \otimes r^*(f))P_{n+1}$. The creation operator $a^*(f)$ is the adjoint of $a(f)$ and has the action

$$a^*(f) : v \otimes_s (f_1 \otimes \ldots \otimes f_n) \mapsto (j_n v) \otimes_s (f_1 \otimes \ldots \otimes f_n \otimes h) \tag{2.9}$$

for $v \in V_n, f_i \in \mathcal{H}$.
**Remark.** One can also use the left creation operator $l(f)$ by choosing another inclusion of $\mathrm{S}(n)$ into $\mathrm{S}(n+1)$.

**Theorem 2.6** *Let $\mathcal{K}$ be an infinite dimensional real Hilbert space and $\mathbf{t}$ positive definite function on pair partitions. Then there exists a unique (up to unitary equivalence) analytic functor $\mathbf{V}$ and densely defined linear maps $j_n : V_n \to V_{n+1}$ satisfying 2.8 such that the G.N.S. representation of $\mathcal{A}(\mathcal{K})$ w.r.t. $\rho_{\mathbf{t}}$ is unitarily equivalent to the $^*$-algebra of symmetric operators $\omega(f) := a(f) + a^*(f)$ for $f \in \mathcal{K}$, acting on the Hilbert space $\mathcal{F}_{\mathbf{V}}(\mathcal{K}_{\mathbb{C}})$. The state $\rho_{\mathbf{t}}$ is implemented by a unit vector $\Omega \in V_0$.*

**Remark.** In chapter III we have shown how the spaces $V_n$ and the maps $j_n$ arise through the representation of the $^*$-semigroup $\mathcal{BP}_2(\infty)$ of *broken pair partitions*, with respect to the positive functional $\hat{\mathbf{t}}$. The semigroup $\mathcal{BP}_2(\infty)$ contains $\mathcal{P}_2(\infty)$ as a sub-semigroup and $\hat{\mathbf{t}}$ is the extension of $\mathbf{t}$ to $\mathcal{BP}_2(\infty)$ by setting $\hat{\mathbf{t}}(d) = 0$ for all $d \notin \mathcal{P}_2(\infty)$. The positivity of the function $\mathbf{t}$ is linked thus with an algebraic object rather then through an indirect definition 2.3. We will therefore denote the $\mathbf{V}$-Fock space over $\mathcal{H}$ associated to the positive definite $\mathbf{t}$ by $\mathcal{F}_{\mathbf{t}}(\mathcal{H})$, and the creation and annihilation operators by $a^\sharp_{\mathbf{t}}(f)$. We denote by the same symbol $\rho_{\mathbf{t}}$ the vacuum state $\langle \cdot, \cdot \rangle$ on the algebra of creation and annihilation operators $a(f)^\sharp$ acting on $\mathcal{F}_{\mathbf{t}}(\mathcal{H})$.

# 3 Generalised Brownian Motions Associated to Characters of S($\infty$)

By S($\infty$) we denote the infinite symmetric group, i.e. the group of finitary permutations of a countable set. A finite character of S($\infty$) is a *central positive definite indecomposable* (not representable as a nontrivial convex combination of other such functions) function. The fundamental result of Thoma gives an explicit description of the finite characters.

**Theorem 3.1** *[62] All normalized finite characters of the group* S($\infty$) *are given by the formula*

$$\phi_{\alpha,\beta}(\sigma) = \prod_{m\geq 2} \left( \sum_{i=1}^{\infty} \alpha_i^m \;+\; (-1)^{m+1} \sum_{i=1}^{\infty} \beta_i^m \right)^{\rho_m(\sigma)} \tag{3.1}$$

*where* $\alpha_1 \geq \alpha_2 \geq \cdots \geq 0,\ \beta_1 \geq \beta_2 \geq \cdots \geq 0,\ \sum \alpha_i + \sum \beta_i \leq 1$, *and* $\rho_m(\sigma)$ *is the number of cycles of length* $m$ *in the permutation* $\sigma$.

We extend the character $\phi_{\alpha,\beta}$ to a positive definite function on pair partition $\mathbf{t}_{\alpha,\beta}$ having a certain multiplicative property. This extension is based on the inclusion of S($n$) in $\mathcal{P}_2(2n)$ formulated in remark 2.4.

**The representation of Veršik and Kerov** [64].

We deal first with the case $\sum_{i=1}^{\infty} \alpha_i = 1$, thus $\beta_j = 0$. We consider a fixed but arbitrary $n \in \mathbb{N} \cup \{\infty\}$. Let $\mathcal{N} = \{1, 2, \ldots\}$ and $\alpha = \alpha_1, \alpha_2, \ldots$ a measure on $\mathcal{N}$. Let $\mathcal{X}_n = \prod_1^n \mathcal{N}$ with the product measure $m_n^{(\alpha)} = \prod_1^n \alpha$. The group S($n$) acts on $\mathcal{X}_n$ by $\sigma(x_1, \ldots, x_n) = (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)})$ and preserves $m_n^{(\alpha)}$. We define $\tilde{\mathcal{X}}_n = \{(x, y) \in \mathcal{X}_n \times \mathcal{X}_n \ :\ x \sim y\}$ where $x \sim y$ means $x = \sigma y$ for some $\sigma \in$ S($n$). The Hilbert space

$$V_n^{(\alpha)} = \left\{ f : \tilde{\mathcal{X}}_n \to \mathbb{C} \mid \infty > ||f||^2 = \int_{\mathcal{X}_n} \sum_{y\sim x} |f(x, y)|^2 dm_n^{(\alpha)}(x) \right\} \tag{3.2}$$

carries a unitary representation $U_n^{(\alpha)}$ of $\mathrm{S}(n)$ given by

$$(U_n^{(\alpha)}(\sigma)h)(x,y) = h(\sigma^{-1}x, y). \tag{3.3}$$

Let $\mathbf{1}_n$ be the indicator function of the diagonal $\{(x,x) \mid x \in \mathcal{X}_n\} \subset \tilde{\mathcal{X}}_n$.

**Theorem 3.2** *On $V_n^{(\alpha)}$ we have*

$$\left\langle U_n^{(\alpha)}(\sigma)\mathbf{1}_n, \mathbf{1}_n \right\rangle = m_n^{(\alpha)}\{x \ : \ \sigma x = x\} = \phi_{\alpha,0}(\sigma). \tag{3.4}$$

*In particular for $n = \infty$ we obtain the representation of $\mathrm{S}(\infty)$ associated to the character $\phi_{\alpha,0}$ in the convex hull of the vector $\mathbf{1}_\infty$.*

For any $n \in \mathbb{N}$ there is a natural isometry $j_n$ from $V_n^{(\alpha)}$ to $V_{n+1}^{(\alpha)}$

$$(j_n h)(x,y) = \delta_{x_{n+1},y_{n+1}} h(x^{(n)}, y^{(n)}) \tag{3.5}$$

where $x = (x_1, \dots, x_n, x_{n+1}) = (x^{(n)}, x_{n+1})$. The maps $j_n$ satisfy 2.8. We have thus a representation of the $^*$-semigroup $\mathcal{BP}_2(\infty)$ on $\bigoplus_{n=0}^\infty V_n^{(\alpha)}$ (see chapter III). We denote the associated positive definite function on pair partitions by $\mathbf{t}_\alpha$.

**Definition 3.3** Let $\mathcal{V} \in \mathcal{P}_2(2n)$. There exists a unique *noncrossing* pair partitions $\hat{\mathcal{V}} \in \mathcal{P}_2(2n)$ such that the set of left points of the pairs in $\mathcal{V}$ and $\hat{\mathcal{V}}$ coincide. A cycle in $\mathcal{V}$ is a sequence $((l_1, r_1), \dots, (l_m, r_m))$ of pairs of $\mathcal{V}$ such that the pairs $(l_1, r_2)$, $(l_2, r_3), \dots, (l_m, r_1)$ belong to $\hat{\mathcal{V}}$. The length of this cycle is $m$. We denote by $\rho_m(\mathcal{V})$ the number of cycles of length $m$ in the pair partition $\mathcal{V}$.

**Theorem 3.4** *The function $\mathbf{t}_\alpha$ has the expression*

$$\mathbf{t}_\alpha(\mathcal{V}) = \prod_{m\geq 2} \left( \sum_{i=1}^\infty \alpha_i^m \right)^{\rho_m(\mathcal{V})}. \tag{3.6}$$

*Proof.* Let $\mathcal{V} \in \mathcal{P}_2(2n)$. In order o calculate $\mathbf{t}_\alpha(\mathcal{V})$ we will have to deal with the spaces $V_k^{(\alpha)}$ for $k \neq n$. The indicator functions $\delta_{(x,y)}$ for $x, y \in \mathcal{X}_k$ and $x = \sigma y$ for some $\sigma \in \mathrm{S}(k)$, form a basis in $V_k^{(\alpha)}$. As the operators $j_k$ are isometries, we identify all $V_k^{(\alpha)}$ for $0 \leq k \leq n$ with their image in $V_p^{(\alpha)}$ under $j_{k,p} := j_k \dots j_{p-1}$ for $p > k$ and denote by $P_{k,p} = j_{k,p} j_{k,p}^*$ the orthogonal projections onto these subspaces. The actions of the various operators are:

$$j_{k,p}\delta_{(x,y)} = \sum_{z\in\mathcal{X}_{p-k}} \delta_{((x,z),(y,z))} \tag{3.7}$$

$$j^*_{k,p}\delta_{(x,y)} = \prod_{i=k+1}^{p} \alpha_{x_i} \prod_{i=k+1}^{p} \delta_{x_i,y_i} \cdot \delta_{(x^{(k)},y^{(k)})} \tag{3.8}$$

$$P_{k,p}\delta_{(x,y)} = \prod_{i=k+1}^{p} \alpha_{x_i} \prod_{i=k+1}^{p} \delta_{x_i,y_i} \cdot \sum_{z\in\mathcal{X}_{p-k}} \delta_{((x^{(k)},z),(y^{(k)},z))} \tag{3.9}$$

$$U^{(\alpha)}_k(\sigma)\delta_{(x,y)} = \delta_{(\sigma x,y)}. \tag{3.10}$$

The function $\mathbf{t}_\alpha$ can be calculated (see Theorem 3.2 in chapter III) in terms of the operators $U^{(\alpha)}_p(\sigma)$ and $j_{k,p}$:

$$\mathbf{t}_\alpha(\mathcal{V}) = \left\langle \mathbf{1}_0,\ \prod_{a=1}^{r}\left( j^*_{k_{a+1},p_a} \cdot U^{(\alpha)}_{p_a}(\pi_a) \cdot j_{k_a,p_a}\right)\mathbf{1}_0 \right\rangle \tag{3.11}$$

where $k_1 = k_{r+1} = 0$ and $\mathbf{1}_0$ is a unit vector of $V^{(\alpha)}_0 = \mathbb{C}$. The numbers $k_a, p_a$ and the permutations $\pi_a$ are determined from the 'standard form' (see figure 3.5) of the pair partition $\mathcal{V}$ which consists of a repeated sequence (from right to left) of $p_a - k_a$ right legs denoted $d_0$, followed by a permutation $\pi_a$ then a sequence of $p_a - k_{a+1}$ left legs denoted $d^*_0$, in such a way that two pairs intersect at most one time at the rightmost possible permutation.

We interpret 3.11 in the following way. We begin with the characteristic function of the pair of empty sets $\delta_{\emptyset,\emptyset} = \mathbf{1}_0$. We apply the operators one by one from the right to left. Then every operator $j_{k_a,p_a}$ (equation 3.7) brings a sum over all $|\mathcal{X}_{p_a-k_a}|$ possible 'words' of $p_a - k_a$ letters to be added to both sequences of the previous pair. Next, a permutation $\pi_a$ (equation 3.10) acts on the first sequence of the pair leaving the other sequence unchanged. The operator $j^*_{k_{a+1},p_a}$ compares the last $p_a - k_{a+1}$ letters of the two seqences and if they coincide, it erases them from both sequences and produces a coefficient equal to the product of the weights $\alpha_i$ of the letters. If the letters differ in at least one position, the pair is removed. In the end we are back at the pair of empty sets and we have a coefficient which is the value of $\mathbf{t}_\alpha(\mathcal{V})$.

Now we come to the two pair partitions $\mathcal{V}$ and $\hat{\mathcal{V}}$. Let $x : \mathcal{V} \to \mathcal{X}$ be a possible sequence of letters atributed to the right legs of $\mathcal{V}$ by the steps $j_{k_a,p_a}$ of the above procedure. The right legs of $\mathcal{V}$ and $\hat{\mathcal{V}}$ coincide by definition, thus $x$ defines also a function $\hat{x} : \hat{\mathcal{V}} \to \mathcal{X}$. As the pair partition $\hat{\mathcal{V}}$ is noncrossing, it corresponds to the second sequence of the pair, on which no permutation of letters is performed. The term $x$ of the sum survives the tests of all $j^*_{k_{a+1},p_a}$ if and only if for each two pairs one from $\mathcal{V}$ and one from $\hat{\mathcal{V}}$ having the same left leg, the corresponding letters in $x$ and respectively $\hat{x}$ coincide. But this means that all the pairs in each cycle of $\mathcal{V}$ must have the same letter, which implies

$$\mathbf{t}_\alpha(\mathcal{V}) = \sum_{x:\text{ct. on cycles}} \prod_{V\in\mathcal{V}} \alpha_{x(V)} = \prod_{m\geq 2}\left(\sum_{i=1}^{\infty}\alpha_i^m\right)^{\rho_m(\mathcal{V})}. \tag{3.12}$$

□

**The general case**. Let $\gamma = 1-\sum_i \alpha_i - \sum_i \beta_i$. As previously we fix $n \in \mathbb{N}\cup\{\infty\}$. Let $\mathcal{N}_+ = \mathcal{N}_- = \mathcal{N}$ and $Q = \mathcal{N}_+ \cup \mathcal{N}_- \cup [0,\gamma]$ with the measure $\mu$ defined as the Lebesgue measure on $[0,\gamma]$, $\mu(i) = \alpha_i$ for $i \in \mathcal{N}_+$, and $\mu(j) = \beta_j$ for $j \in \mathcal{N}_-$. The measure space is $\mathcal{X}_n = \prod_1^n Q$ with measure $m_n^{(\alpha,\beta)} = \prod_1^n \mu$; $\tilde{\mathcal{X}}$ is defined as before, as well as the Hilbert space $V_n^{(\alpha,\beta)}$ (equation 3.2). The representation of S($n$) is given by

$$(U_n^{(\alpha,\beta)}(\sigma)h)(x,y) = (-1)^{i(\sigma,x)}h(\sigma^{-1}x,y) \tag{3.13}$$

where $i(\sigma,x)$ is the number of inversions in the sequence $(\sigma i_1(x), \sigma i_2(x)\dots)$ of indices $i_r(x)$ for which $x_i \in \mathcal{N}_-$. The vector $\mathbf{1}_n$ is the indicator function of the diagonal $\{(x,x)\} \subset \tilde{\mathcal{X}}$.

**Theorem 3.5** *[64] On $V_n^{(\alpha,\beta)}$ we have*

$$\left\langle U_n^{(\alpha,\beta)}(\sigma)\mathbf{1}_n, \mathbf{1}_n\right\rangle = \phi_{\alpha,\beta}(\sigma). \tag{3.14}$$

*In particular for $n = \infty$ we obtain the representation of* S($\infty$) *associated to the character $\phi_{\alpha,\beta}$ in the convex hull of the vector $\mathbf{1}_\infty$.*

We define $j_n$ as in 3.5. Then we have the general version of theorem 3.4:

**Theorem 3.6** *The function $\mathbf{t}_{\alpha,\beta}$ has the expression*

$$\mathbf{t}_{\alpha,\beta}(\mathcal{V}) = \prod_{m\geq 2}\left(\sum_{i=1}^{\infty}\alpha_i^m + (-1)^{m+1}\sum_{i=1}^{\infty}\beta_i^m\right)^{\rho_m(\mathcal{V})}. \tag{3.15}$$

**Definition 3.7** [13] A function $\mathbf{t}$ on pair partitions is called *multiplicative* if for all $k,l,n \in \mathbb{N}$ with $0 \leq k < l \leq n$ and all $\mathcal{V}_1 \in \mathcal{P}_2(\{1,\dots,k,l+1,\dots,n\})$ and $\mathcal{V}_2 \in \mathcal{P}_2(\{k+1,\dots,l\})$ we have

$$\mathbf{t}(\mathcal{V}_1 \cup \mathcal{V}_2) = \mathbf{t}(\mathcal{V}_1)\cdot\mathbf{t}(\mathcal{V}_2). \tag{3.16}$$

**Corollary 3.8** *The functions $\mathbf{t}_{\alpha,\beta}$ are multiplicative.*

*Proof.* If $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$ as in definition 3.7 then any cycle of $\mathcal{V}$ belongs either to $\mathcal{V}_1$ or to $\mathcal{V}_2$, thus

$$\rho_m(\mathcal{V}) = \rho_m(\mathcal{V}_1) + \rho_m(\mathcal{V}_2). \tag{3.17}$$

for all $m \geq 2$.

□

**Corollary 3.9** *The operators $\omega_{\alpha,\beta}(f)$ are essentially selfadjoint.*

*Proof.* This follows from the previous corollary and Proposition 5.10 in chapter III.

□

From now on we will denote by the same symbol the selfadjoint closure of the essentially selfadjoint operator $\omega_{\alpha,\beta}(f)$.

# 4 The von Neumann Algebras $\Gamma_{\alpha,\beta}(\mathcal{K})$

Using corollary 3.9 we can make a step further and construct von Neumann algebras generated by the 'field operators' $\omega_{\alpha,\beta}(f)$ for vectors $f$ in certain real Hilbert spaces.

**Definition 4.1** Let $\Gamma_{\alpha,\beta}(\mathcal{K})$ be the von Neumman algebra generated by the spectral projections of the operators $\omega_{\alpha,\beta}(f)$ acting on $\mathcal{F}_{\alpha,\beta}(\mathcal{K}_{\mathbb{C}})$, for all $f$ in $\mathcal{K}$, a real Hilbert space. On $\Gamma_{\alpha,\beta}(\mathcal{K})$ we distinguish the vacuum state $\rho_{\alpha,\beta}(X) := \langle\Omega_{\alpha,\beta}, X\Omega_{\alpha,\beta}\rangle$.

On $\mathcal{F}_{\alpha,\beta}(\mathcal{K}_{\mathbb{C}})$ we define a $^*$-algebra $\mathbf{W}_{\alpha,\beta}(\mathcal{K})$ of (not necessarily bounded) operators having $\mathcal{D} := \mathcal{F}^{(\mathrm{fin})}_{\alpha,\beta}(\mathcal{K})$ as domain and leaving this domain invariant. We call $\mathbf{W}_{\alpha,\beta}(\mathcal{K})$ *Wick algebra* and its elements generalised Wick products. Such an operator is a finite linear combination of elementary operators denoted by a symbol $\Psi(\mathcal{V}, \mathbf{f})$ where $\mathbf{f} : F \to \mathcal{K}$, $\mathcal{V} \in \mathcal{P}_2(P)$ and $\{F, P\}$ is a partition into disjoint subsets of an arbitrary ordered set. The simplest examples of Wick products are $\Psi(\mathcal{V}, \emptyset) = \mathbf{t}_{\alpha,\beta}(\mathcal{V})\mathbf{1}$, and $\Psi(\emptyset, f) = \omega_{\alpha,\beta}(f)$. For the exact definition of the Wick products we refer to section 4 of chapter III. Let $\Psi(\mathcal{V}_1, \mathbf{f}_1)$ and $\Psi(\mathcal{V}_2, \mathbf{f}_2)$ be two Wick products. Then

$$\langle\Psi(\mathcal{V}_1, \mathbf{f}_1)\Omega,\ \Psi(\mathcal{V}_2, \mathbf{f}_2)\Omega\rangle = \sum_{\tilde{\mathcal{V}} \in \mathcal{P}_2(F_1^*, F_2)} \eta_{\mathbf{f}_1^* \oplus \mathbf{f}_2}(\tilde{\mathcal{V}}) \cdot \mathbf{t}(\mathcal{V}_1^* \cup \mathcal{V}_2 \cup \tilde{\mathcal{V}}) \tag{4.1}$$

with the following notations:
1) if $A$ is a finite ordered set then $A^*$ denotes the same set with the reversed order and similarly for $\mathbf{f}^*$ and $\mathcal{V}^*$;
2) if $\mathbf{f}_i : A_i \to \mathcal{K}$ with $A_i$ finite ordered sets then $A_1 + A_2$ denotes the ordered set

obtained by concatenating $A_1$ and $A_2$, $\mathbf{f}_1 \oplus \mathbf{f}_2 : A_1 + A_2 \to \mathcal{K}$ is the map which restricts to $f_i$ on $A_i$ for $i = 1, 2$;
3) if $\{A_1, A_2\}$ is a partition into disjoint subsets of an ordered set $A$ then $\mathcal{P}_2(A_1, A_2)$ denoted the subset of $\mathcal{P}_2(A)$ whose elements contain only pairs with one element from $A_1$ and the other from $A_2$;
4) with the previous notations, for $\mathcal{V} \in \mathcal{P}_2(A_1, A_2)$

$$\eta_{\mathbf{f}_1 \oplus \mathbf{f}_2}(\mathcal{V}) := \prod_{(l,r) \in \mathcal{V}} \langle \mathbf{f}_1(l), \mathbf{f}_2(r) \rangle . \tag{4.2}$$

The adjoint of $\Psi(\mathcal{V}, \mathbf{f})$ is $\Psi(\mathcal{V}^*, \mathbf{f}^*)$. The product of two Wick products is written in terms of elementary Wick products as follows

$$\Psi(\mathcal{V}_1, \mathbf{f}_1) \cdot \Psi(\mathcal{V}_2, \mathbf{f}_2) = \sum_{\tilde{P}_1, \tilde{P}_2} \sum_{\mathcal{V} \in \mathcal{P}_2(\tilde{P}_1, \tilde{P}_2)} \eta_{\mathbf{f}_1 \oplus \mathbf{f}_2}(\mathcal{V}) \cdot \Psi(\mathcal{V}_1 \cup \mathcal{V}_2 \cup \mathcal{V}, \check{\mathbf{f}}_1 \oplus \check{\mathbf{f}}_2) \tag{4.3}$$

with $\tilde{P}_i$ denoting a subset of $P_i$, and $\check{\mathbf{f}}_i$ is the restriction of the function $\mathbf{f}_i : P_i \to \mathcal{K}$ to the complement of $\tilde{P}_i$ in $P_i$.

**Remark 4.2** In general, if the field operators $\omega_{\mathbf{t}}(f)$ are bounded then the Wick algebra $\mathbf{W}_{\mathbf{t}}(\mathcal{K})$ is weakly dense in $\Gamma_{\mathbf{t}}(\mathcal{K})$ for infinite dimensional $\mathcal{K}$. However for finite dimensional $\mathcal{K}$ the von Neumann closure of $\mathbf{W}_{\alpha,\beta}(\mathcal{K})$ can be larger than $\Gamma_{\mathbf{t}}(\mathcal{K})$. If the field operators are unbounded then the Wick operators are affiliated to $\Gamma_{\mathbf{t}}(\mathcal{K})$.

Let us take a closer look at the type of the von Neumann algebras $\Gamma_{\alpha,\beta}(\mathcal{K})$. The following cases are already known.
1) $\alpha_1 = 1$: we obtain the classical (commutative) Brownian motion $B_t := \omega(\chi_{(0,t]})$. for $\mathcal{K} = L^2_{\mathbb{R}}(\mathbb{R}_+)$, or the algebra of $n$ independent gaussian random variable for $\mathcal{K} = \mathbb{R}^n$.
2) $\beta_1 = 1$: fermionic Brownian motion, the corresponding von Neumann algebra $\Gamma_{0,1}(\mathcal{K})$ being the type $II_1$ hyperfinite factor for $\dim(\mathcal{K}) = \infty$.
3) $\alpha_i = \beta_i = 0$: free Brownian motion [65], $\Gamma_{0,0}(\mathbb{C}^n)$ is the non-hyperfinite $II_1$ factor isomorphic to the von Neumann algebra of the free group with $n$ generators ($n \geq 2$ or $n = \infty$).

In all the above cases the vacuum state $\rho_{\alpha,\beta}$ is tracial.

**Lemma 4.3** *Let $\alpha, \beta$ be as in theorem 3.1 with $\alpha_1 \neq 1$, $\beta_1 \neq 1$, $\sum_i \alpha_i + \sum_i \beta_i \neq 0$. Then $\Gamma_{\alpha,\beta}(\ell^2_{\mathbb{R}}(\mathbb{Z}))$ does not have any tracial normal state.*

*Proof.* Let us suppose that there exists a tracial state $\tau$ on $\Gamma_{\alpha,\beta}(\ell^2_{\mathbb{R}}(\mathbb{Z}))$ and consider the automorphism $\Gamma_{\alpha,\beta}(S) := \mathrm{ad}\mathcal{F}_{\alpha,\beta}(S)$ of $\Gamma_{\alpha,\beta}(\ell^2_{\mathbb{R}}(\mathbb{Z}))$ where $S$ is the right shift on $\ell^2_{\mathbb{R}}(\mathbb{Z})$. From section 5 of chapter III we have

$$\operatorname*{w-lim}_{n \to \infty} \Gamma_{\alpha,\beta}(S^n)(X) = \rho_{\alpha,\beta}(X)\mathbf{1} \tag{4.4}$$

for elements $X$ in the Wick algebra $\mathbf{W}_{\alpha,\beta}$. If the field operators $\omega_{\alpha,\beta}(f)$ are bounded then $\mathbf{W}_{\alpha,\beta}(\ell^2_{\mathbb{R}}(\mathbb{Z}))$ is dense in $\Gamma_{\alpha,\beta}(\ell^2_{\mathbb{R}}(\mathbb{Z}))$ and we can conclude that $\tau(\Gamma_{\alpha,\beta}(S^n)(XY)) \to \rho_{\alpha,\beta}(XY)$ for $X, Y \in \mathbf{W}_{\alpha,\beta}(\ell^2_{\mathbb{R}}(\mathbb{Z}))$. This would imply

$$\begin{aligned} \rho_{\alpha,\beta}(\omega_1\omega_2\omega_3\omega_4\omega_5\omega_1\omega_4\omega_3\omega_2\omega_5) &= \rho_{\alpha,\beta}(\omega_5\omega_1\omega_2\omega_3\omega_4\omega_5\omega_1\omega_4\omega_3\omega_2) \\ \rho_{\alpha,\beta}(\omega_1\omega_2\omega_3\omega_4\omega_5\omega_3\omega_6\omega_2\omega_1\omega_6\omega_5\omega_4) &= \rho_{\alpha,\beta}(\omega_4\omega_1\omega_2\omega_3\omega_4\omega_5\omega_3\omega_6\omega_2\omega_1\omega_6\omega_5) \end{aligned}$$

for $\omega_i = \omega_{\alpha,\beta}(e_i)$ with $e_i$ normalized orthogonal on each other. Thus

$$\begin{aligned} \sum_i \alpha_i^2 - \sum_i \beta_i^2 &= \left(\sum_i \alpha_i^2 - \sum_i \beta_i^2\right)\left(\sum_i \alpha_i^3 + \sum_i \beta_i^3\right), \\ \left(\sum_i \alpha_i^2 - \sum_i \beta_i^2\right)\left(\sum_i \alpha_i^4 - \sum_i \beta_i^4\right) &= \left(\sum_i \alpha_i^3 + \sum_i \beta_i^3\right)^2. \end{aligned}$$

If $\omega_{\alpha,\beta}(f)$ are unbounded operators one has to be more careful with expressions of the type $\tau(\prod_{i=1}^p \omega_{k_i})$. We consider first the cutoff fields $\omega_i^{(c)} = P_i^{(c)}\omega_i$ where $P_i^{(c)}$ is the spectral projection of $\omega_i$ associated to the interval $[-c, c]$. Then we still have

$$\operatorname*{w-lim}_{n\to\infty} \Gamma_{\alpha,\beta}(S^n)(M^{(c)}) = \rho_{\alpha,\beta}(M^{(c)})\mathbf{1} \tag{4.5}$$

for $M^{(c)} = \prod_{i=1}^p \omega_{k_i}^{(c)}$. Finally by letting $c \to \infty$ we obtain the same result as in the case of bounded fields.

□

**Lemma 4.4** *Let $\alpha, \beta$ be as in lemma 4.3. The vacuum vector $\Omega \in \mathcal{F}_{\alpha,\beta}(\ell^2_{\mathbb{R}}(\mathbb{Z}))$ is not separating for $\Gamma_{\alpha,\beta}(\ell^2_{\mathbb{R}}(\mathbb{Z}))$.*

*Proof.* Let $(e_n)_{n\in\mathbb{Z}}$ be the orthonormal basis of $\ell^2_{\mathbb{R}}(\mathbb{Z})$ and denote $a_i^\sharp = a^\sharp(e_i)$. We consider the generalised Wick products $X = \Psi(\{(1,4),(2,6)\},\mathbf{f})$ and $Y = \Psi(\{(2,4)\},\mathbf{g})$ with $\mathbf{f}(3) = \mathbf{g}(3) = e_2, \mathbf{f}(5) = \mathbf{g}(1) = e_1$. Then

$$X\Omega = Y\Omega = a_1^* a_3 a_2^* a_3^* \Omega \tag{4.6}$$

On the other hand consider

$$\psi_n = a_3 a_2^* a_4 a_3^* a_5 a_4^* \ldots a_n a_{n-1}^* a_n^* \Omega \tag{4.7}$$

then

$$\begin{aligned} \langle a_1^*\Omega, X\psi_n\rangle &= \sum_i \alpha_i^{n+2} + (-)^{n+1} \sum_i \beta_i^{n+2}, \\ \langle a_1^*\Omega, Y\psi_n\rangle &= \sum_i \alpha_i^{n} + (-)^{n+1} \sum_i \beta_i^{n} \end{aligned} \tag{4.8}$$

which cannot be equal for all $n$.

□

**Remark.** At first sight it might be surprising that the state $\rho_{\alpha,\beta}$ which is obtained by extending the characters of the infinite symmetric group is not tracial. However the trace property of $\rho_{\alpha,\beta}$ on $\Gamma_{\alpha,\beta}(\mathcal{K})$ is independent of the trace property of the characters $\phi_{\alpha,\beta}$.

# 5 An example with type $\mathrm{I}_\infty$ factors

We consider in more detail the following particular functions:

1) $\mathbf{t}_N := \mathbf{t}_{\alpha,\beta}$ with $\alpha_i = \frac{1}{N}$ for $i = 1, \ldots, N$ and $\beta_j = 0$ for all $j$;

2) $\mathbf{t}_{-N} := \mathbf{t}_{\alpha,\beta}$ with $\beta_i = \frac{1}{N}$ for $i = 1, \ldots, N$ and $\alpha_j = 0$ for all $j$.

For any $N \in \mathbb{Z} \setminus \{0\}$ we have

$$\mathbf{t}_N(\mathcal{V}) = \left(\frac{1}{N}\right)^{|\mathcal{V}| - \mathrm{c}(\mathcal{V})} \tag{5.1}$$

where $|\mathcal{V}|$ and $\mathrm{c}(\mathcal{V})$ denote the number of pairs, respectively the *number of cycles* of the pair partition $\mathcal{V}$. The corresponding character is denoted by $\phi_N$.

In the spirit of [9, 13, 10, 11] we define an alternative representation of the algebra of creation and annihilation operators using the technique of deformation of the inner product on the full Fock space. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{F}^{(\mathrm{fin})}(\mathcal{H})$ be the linear span of the vectors of the form $f_1 \otimes \ldots \otimes f_n$ with $n \geq 0$ and $f_i \in \mathcal{H}$ endowed with the usual inner product on the full Fock space over $\mathcal{H}$. We consider the new sesquilinear form $\langle \cdot, \cdot \rangle_N$ given by the sesquilinear extension of

$$\langle f_1 \otimes \ldots \otimes f_n,\ g_1 \otimes \ldots \otimes g_m \rangle_N = \delta_{n,m} \sum_{\tau \in \mathrm{S}(n)} \phi_N(\tau) \left\langle f_1, g_{\tau(1)} \right\rangle \ldots \left\langle f_n, g_{\tau(n)} \right\rangle . \tag{5.2}$$

The positivity of $\langle \cdot, \cdot \rangle_N$ follows from that of the operator $Q_N$ defined on $\mathcal{F}^{(\mathrm{fin})}(\mathcal{H})$ whose restriction to $\mathcal{H}^{\otimes n}$ is

$$Q_N^{(n)} = \sum_{\tau \in \mathrm{S}(n)} \phi_N(\tau) \tilde{U}_n(\tau). \tag{5.3}$$

We denote by $D_N$ the operator on $\mathcal{F}^{(\mathrm{fin})}(\mathcal{H})$ which restricts to

$$D_N^{(n+1)} := \mathbf{1} + \frac{1}{N} \sum_{i=2}^{n+1} \tilde{U}_{n+1}(\pi_{1,i}) \tag{5.4}$$

on $\mathcal{H}^{\otimes n+1}$ and $D_N^{(0)}\Omega = \Omega$. The permutation $\pi_{j,i}$ is the transposition of $i$ and $j$. Let $\tilde{Q}_N^{(n)}$ be the operator $\mathbf{1} \otimes Q_N^{(n)}$ acting on $\mathcal{H}^{\otimes n+1}$. The following relation is immediate

$$Q_N^{(n+1)} = D_N^{(n+1)} \tilde{Q}_N^{(n)}. \tag{5.5}$$

Let $l(f), l^*(f)$ be the left creation and annihilation operators on the full Fock space over $\mathcal{H}$. On $\mathcal{F}^{(\mathrm{fin})}(\mathcal{H})$ we define new creation operator $a_N^*(f) := l(f)$ and the annihilation operator $a_N(f) = l^*(f) D_N$. From 5.5 we get

$$\langle \eta,\ a_N^*(f)\xi\rangle_N = \langle a_N(f)\eta,\ \xi\rangle_N \tag{5.6}$$

for $\eta, \xi \in \mathcal{F}^{(\mathrm{fin})}(\mathcal{H})$. By dividing out the $||\cdot||_N$-norm zero vectors we obtain the pre-Hilbert space $\mathcal{F}_N^{(\mathrm{fin})}(\mathcal{H})$ whose completion with respect to $\langle\cdot,\cdot\rangle_N$ is denoted by $\mathcal{F}_N(\mathcal{H})$. The operators $a_N(f), a_N^*(f)$ are well defined on $\mathcal{F}_N^{(\mathrm{fin})}(\mathcal{H})$ and are each other's adjoint on $\mathcal{F}_N(\mathcal{H})$. Let $\omega_N(f) := a_N(f) + a_N^*(f)$ be the symmetric 'field operators'.

We will prove that the vacuum expectations of monomials in $\omega_N(\cdot)$ satisfy the equation 2.3 characterizing the gaussian states with $\mathbf{t} = \mathbf{t}_N$.

**Lemma 5.1** *For any $f, g \in \mathcal{H}$ the following relation holds on $\mathcal{F}_N(\mathcal{H})$:*

$$a_N(f) a_N^*(g) = \langle f,\ g\rangle\, \mathbf{1} + \frac{1}{N}\mathrm{d}\Gamma(T_{f,g}) \tag{5.7}$$

*where the differential second quantisation operator $\mathrm{d}\Gamma(A)$ is defined by*

$$\mathrm{d}\Gamma(A) f_1 \otimes \ldots \otimes f_n = \sum_{i=1}^{n} f_1 \otimes \ldots \otimes A f_i \otimes \ldots \otimes f_n \tag{5.8}$$

*for $A \in \mathcal{B}(\mathcal{H})$.*

*Proof.* Direct application of the definitions.

□

**Lemma 5.2** *Let $f_1, \ldots, f_p \in \mathcal{H}$. Then*

$$\langle \Omega, \omega_N(f_1)\ldots\omega_N(f_p)\Omega\rangle = \sum_{\mathcal{V}\in\mathcal{P}_2(p)} \mathbf{t}_N(\mathcal{V}) \prod_{(k,l)\in\mathcal{V}} \langle f_k, f_l\rangle \tag{5.9}$$

*Proof.* For $p$ odd the expectation is zero. From the definitions of the creation and annihilation operators it is clear that the vacuum state is a Fock state for a certain positive definite function $\mathbf{t}$ on pair partitions. We have to prove that $\mathbf{t} \equiv \mathbf{t}_N$ with the latter as defined in 5.1. By linearity it is enough to prove the relation for $\frac{p}{2}$ pairs of orthonormal vectors $e_1, \ldots e_{\frac{p}{2}}$ ordered such as they give rise to a certain pair partition $\mathcal{V} = \{(l_1, r_1), \ldots, (l_{\frac{p}{2}}, r_{\frac{p}{2}})\} \in \mathcal{P}_2(p)$:

$$\left\langle \Omega,\ \prod_{i=1}^{p} a_N^{\sharp_i}(f_i)\Omega \right\rangle = \mathbf{t}_N(\mathcal{V}) \tag{5.10}$$

for $a_N^{\sharp_{l_k}}(f_{l_k}) = a_N(e_k) = (a_N^{\sharp_{r_k}}(f_{r_k}))^*$.

We consider the *innermost* pairs of the non-crossing pair partition $\hat{\mathcal{V}}$ associated to $\mathcal{V}$ (see definition 3.3). Each such pair is of the form $(k, k+1)$ for some $1 \leq k < p$. The corresponding term in the monomial is $a_N(f_k)a_N^*(f_{k+1})$. We distinguish two cases:
1) if $f_k = f_{k+1} = f$ then $(k, k+1) \in \mathcal{V}$. By 5.9 we have

$$a_N(f_k)a_N^*(f_{k+1}) = \mathbf{1} + \frac{1}{N}\mathrm{d}\Gamma(P_f) \tag{5.11}$$

with $P_f$ the projection on the one dimensional space spanned by $f$. The term $\mathrm{d}\Gamma(P_f)$ brings contribution zero to the expectation because the rest of the vectors $f_i$ are orthogonal on $f$. Thus the pair $(k, k+1) \in \mathcal{V}$ can be deleted without changing the expectation.
2) if $f_k \neq f_{k+1}$ then $k$ and $k+1$ belong to different pairs $(k, a)$ and respectively $(b, k+1)$ in $\mathcal{V}$. By 5.9 we have

$$a_N(f_k)a_N^*(f_{k+1}) = \frac{1}{N}\mathrm{d}\Gamma(T_{f_k, f_{k+1}}) \tag{5.12}$$

The action of the operator $\mathrm{d}\Gamma(T_{f_k,f_{k+1}})$ on $\prod_{k+2}^{p} a^{\sharp_i}(f_i)\Omega$ is in effect to replace the vector $f_k$ which appears exactly once in any tensor product, by $f_{k+1}$. Equivalently one can delete the positions $k$ and $k+1$ from the ordered sequence and replace the pairs $(b, k+1), (k, a)$ by one pair $(b, a)$ leaving the other pairs invariant. Let us denote this pair partition by $\check{\mathcal{V}}_k$. Then

$$\left\langle \Omega,\ \prod_{i=1}^{p} a_N^{\sharp_i}(f_i)\Omega \right\rangle = \mathbf{t}(\mathcal{V}) = \frac{1}{N}\mathbf{t}(\check{\mathcal{V}}_k) \tag{5.13}$$

By repeating the procedure of reducing the number of pairs through step 1) or 2) we arrive at

$$\left\langle \Omega,\ \prod_{i=1}^{p} a_N^{\sharp_i}(f_i)\Omega \right\rangle = \mathbf{t}(\mathcal{V}) = \left(\frac{1}{N}\right)^{\frac{p}{2} - \mathrm{c}(\mathcal{V})} = \mathbf{t}_N(\mathcal{V}) \tag{5.14}$$

□

**Remark.** We thus conclude that the representation of the algebra of creation and annihilation operators constructed in this section and the the one described in section 2 are unitarily equivalent as GNS representations with respect to the Fock state $\rho_N$ associated to $\mathbf{t}_N$. As in definition 4.1, we denote by $\Gamma_N(\mathcal{K})$ the von Neumann algebra generated by the spectral projections of the selfadjoint extensions of the operators $\omega_N(f)$ acting on $\mathcal{F}_N(\mathcal{K}_{\mathbb{C}})$ for all $f \in \mathcal{K}$.

From the relations 5.9 we can conclude that $a_N^{\sharp}(f)$ is bounded for $N < 0$ and unbounded for $N > 0$. We will concentrate on the von Neumann algebra $\Gamma_N(\mathcal{K})$

for $N < -1$ and infinite dimensional $\mathcal{K}$. We have seen that except the three special cases the von Neumann algebras $\Gamma_{\alpha,\beta}(\mathcal{K})$ are infinite. In fact we will show that $\Gamma_N(\ell^2_{\mathbb{R}}(\mathbb{Z}))$ is the whole algebra of bounded operators on $\mathcal{F}_N(\ell^2(\mathbb{Z}))$ (for $N < -1$).

**Proposition 5.3** *Let $N < -1$ be an integer. Then $\Gamma_N(\ell^2_{\mathbb{R}}(\mathbb{Z})) = \mathcal{B}(\mathcal{F}_{\mathbf{t}}(\ell^2(\mathbb{Z})))$.*

*Proof.* We show that the projection $P_\Omega$ onto the one dimensional space spanned by the vacuum vector belongs to $\Gamma_N(\ell^2_{\mathbb{R}}(\mathbb{Z}))$. From this we can conclude that the algebra $\Gamma_N(\ell^2_{\mathbb{R}}(\mathbb{Z}))$ is the whole $\mathcal{B}(\mathcal{F}_N(\ell^2(\mathbb{Z})))$ because $\Omega$ is cyclic vector for $\mathbf{W}_N(\ell^2_{\mathbb{R}}(\mathbb{Z}))$ which is dense in $\Gamma_N(\ell^2_{\mathbb{R}}(\mathbb{Z}))$.

Let $\mathbf{N}_i := \mathrm{d}\Gamma(T_{e_i,e_i})$ be the number operator counting 'one-particle' $e_i$-states in $\mathcal{F}_N(\ell^2(\mathbb{Z}))$. For simplicity we make the notations $\omega_i := \omega_N(e_i)$ and similarly for $a_i, a_i^*$.
Let $\Psi(\mathcal{V}, \mathbf{f})$ be an arbitrary Wick products with $\mathbf{f} : F \to \ell^2_{\mathbb{R}}(\mathbb{Z})$, $\mathcal{V} \in \mathcal{P}_2(P)$ and $\{F, P\}$ a disjoint partition of the ordered set $\{1, \ldots, 2n+p\}$. On the Wick algebra $\mathbf{W}_N(\ell^2_{\mathbb{R}}(\mathbb{Z}))$ we define the map $\Phi$:

$$\Phi : \Psi(\mathcal{V}, \mathbf{f}) \mapsto \operatorname*{w-lim}_{n\to\infty} \ \omega_n \Psi(\mathcal{V}, \mathbf{f})\omega_n = \Psi(\underline{\mathcal{V}}, \mathbf{f}) \tag{5.15}$$

where $\underline{\mathcal{V}} = \mathcal{V} \cup \{(0, 2n + p + 1)\}$ is obtained by adding the pair $(0, 2n + p + 1)$ to $\mathcal{V}$ which embraces all other points of the set $\{1, \ldots, 2n + p\}$. Such a map has been used previously in section 6 of chapter III. The following limits are easy to check by taking expectations with respect to vectors in $\mathcal{F}_N^{(\mathrm{fin})}(\ell^2(\mathbb{Z}))$:

$$\begin{aligned}
\operatorname*{w-lim}_{n\to\infty} \omega_n a_i a_i \omega_n &= \frac{1}{N} a_i a_i, \\
\operatorname*{w-lim}_{n\to\infty} \omega_n a_i^* a_i \omega_n &= \frac{1}{N^2} \mathbf{N}_i, \\
\operatorname*{w-lim}_{n\to\infty} \omega_n \mathbf{N}_i \omega_n &= \mathbf{N}_i, \\
\operatorname*{w-lim}_{n\to\infty} \omega_n a_i a_i^* \omega_n &= \mathbf{1} + \frac{1}{N} \mathbf{N}_i.
\end{aligned}$$

These relations lead to

$$\operatorname*{w-lim}_{k\to\infty} \Phi^k(\omega_i^2) = \mathbf{1} + \frac{N+1}{N^2} \mathbf{N}_i \tag{5.16}$$

which implies that $\mathbf{N}_i \in \Gamma_N(\ell^2_{\mathbb{R}}(\mathbb{Z}))$. Let $P(i)$ denote the projections on the eigenspace of $\mathbf{N}_i$ with corresponding eigenvalue equal to zero. Then $(P(i))_{i=-\infty}^{\infty}$ form a commuting family of projections in $\Gamma_N(\ell^2_{\mathbb{R}}(\mathbb{Z}))$ and

$$P_\Omega = \operatorname*{w-lim}_{k\to\infty} \prod_{i=-k}^{k} P(i). \tag{5.17}$$

□

**Definition 5.4** i) The category of *non-commutative probability spaces* has as objects pairs $(\mathcal{A}, \rho_{\mathcal{A}})$ of von Neumann algebras and normal states and as morphisms between two objects $(\mathcal{A}, \rho_{\mathcal{A}})$ and $(\mathcal{B}, \rho_{\mathcal{B}})$ all completely positive maps $T : \mathcal{A} \to \mathcal{B}$ such that $T(\mathbf{1}_{\mathcal{A}}) = \mathbf{1}_{\mathcal{A}}$ and $\rho_{\mathcal{B}}(Tx) = \rho_{\mathcal{A}}(x)$ for all $x \in \mathcal{A}$.

ii) A functor $\Lambda$ from the category of (real) Hilbert spaces with contractions to the category of non-commutative probability spaces is called *functor of white noise* if $\Lambda(\{0\}) = \mathbb{C}$ where $\{0\}$ stands for the zero dimensional Hilbert space.

We construct for any real Hilbert space $\mathcal{K}$ a von Neumann algebra $\Delta_N(\mathcal{K})$ such that $\Delta_N$ becomes a functor of white noise.

**Definition 5.5** Let $\mathcal{K}$ be a real Hilbert space. On the Fock space $\mathcal{F}_N(\mathcal{K}_{\mathbb{C}} \oplus \ell^2(\mathbb{Z}))$ we define the von Neumann algebra $\Delta_N(\mathcal{K})$ generated by the Wick products $\Psi(\mathcal{V}, \mathbf{f})$ with $\mathrm{Im}(\mathbf{f}) \subset \mathcal{K} \oplus 0$.

**Lemma 5.6** *Let $T : \mathcal{K} \to \mathcal{K}'$ be a contraction. Then the map defined on the Wick products $\Psi(\mathcal{V}, \mathbf{f}) \in \Delta_N(\mathcal{K})$ by*

$$\Delta_N(T) : \Psi(\mathcal{V}, \mathbf{f}) \mapsto \Psi(\mathcal{V}, (T \oplus \mathbf{1}) \circ \mathbf{f}) \tag{5.18}$$

*extends to a morphism from $\Delta_N(\mathcal{K})$ to $\Delta_N(\mathcal{K}')$. Moreover $\Delta_N$ is a functor of white noise.*

*Proof.* If $TT^* = \mathbf{1}_{\mathcal{K}'}$ then

$$\Psi(\mathcal{V}, \mathbf{f}) \mapsto \mathcal{F}_N(T \oplus \mathbf{1})\Psi(\mathcal{V}, \mathbf{f})\mathcal{F}_N(T \oplus \mathbf{1})^* = \Psi(\mathcal{V}, (T \oplus \mathbf{1}) \circ \mathbf{f}) \tag{5.19}$$

restricts to the desired map on Wick products $\Psi(\mathcal{V}, \mathbf{f})$ with $\mathbf{f}(k) = f_k \oplus 0$ and subsequently extends to a completely positive map $\Delta_N(T)$ from $\Delta_N(\mathcal{K})$ to $\Delta_N(\mathcal{K}')$.

If $T^*T = \mathbf{1}_{\mathcal{K}}$ then there exists an orthogonal operator $O_T : \mathcal{K} \oplus \ell^2_{\mathbb{R}}(\mathbb{Z}) \to \mathcal{K}' \oplus \ell^2_{\mathbb{R}}(\mathbb{Z})$ whose restriction to $\mathcal{K}$ coincides with $T$ and thus

$$\mathcal{F}_N(O_T)\Psi(\mathcal{V}, \mathbf{f})\mathcal{F}_N(O_T)^* = \Psi(\mathcal{V}, O_T \circ \mathbf{f}) \tag{5.20}$$

has again the required action on Wick products with vectors from $\mathcal{K}$ and extends to a $^*$-homomorphism $\Delta_N(T)$ from $\Delta_N(\mathcal{K})$ to $\Delta_N(\mathcal{K}')$. An arbitrary contraction $T$ can be written as a product $PI$ of a co-isometry and an isometry. Then we define $\Delta_N(T) := \Delta_N(P)\Delta_N(I)$ which does not depend on the particular choice of $P$ and $I$, and apply the previous cases. By definition $\Delta_N(\emptyset) = \mathbb{C}$.

$\square$

**Theorem 5.7** *The von Neumann algebra $\Delta_N(\ell^2_{\mathbb{R}}(\mathbb{Z}))$ is isomorphic to a discrete sum of type $I_\infty$ factors.*

*Proof.* We denoted by $(e_i \oplus \check{e}_j)_{i,j\in\mathbb{Z}}$ an orthonormal basis of $\ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})$. The corresponding number operators are $\mathbf{N}_i$ and $\check{\mathbf{N}}_j$. As in the proof of proposition 5.3 one can show that $\mathbf{N}_i \in \Delta_N(\ell^2_{\mathbb{R}}(\mathbb{Z}))$ and $\check{\mathbf{N}}_j \in \Delta_N(\ell^2_{\mathbb{R}}(\mathbb{Z}))'$. The common eigenspaces of all the number operators $\mathbf{N}_i, \check{\mathbf{N}}_j$ are finite dimensional which implies that the selfadjoint elements $Z$ in the center of $\Delta_N(\ell^2_{\mathbb{R}}(\mathbb{Z}))$ have discrete spectrum and thus the center is isomorphic to $\ell^\infty(M)$ for a countable discrete set $M$. Let $\underline{n} := (n_i)_{i\in\mathbb{Z}}$ be a sequence of natural numbers such that only a finite number of them are different from zero. We denote by $\mathcal{F}_N(\underline{n})$ the joint eigenspace of the operators $\mathbf{N}_i$ with eigenvalues $n_i$. The projection $P_{\underline{n}}$ onto this space belongs to $\Delta_N(\ell^2_{\mathbb{R}}(\mathbb{Z}))$. We make a similar notation for the projections $\check{P}_{\underline{m}}$ onto the eigenspaces of $\check{\mathbf{N}}_j$. Let $Q \leq P_{\underline{n}}$ be another projection in $\Delta_N(\ell^2_{\mathbb{R}}(\mathbb{Z}))$ which is equivalent to $P_{\underline{n}}$. Then there exists a partial isometry $W \in \Delta_N(\ell^2_{\mathbb{R}}(\mathbb{Z}))$ such that $WW^* = P_{\underline{n}}$ and $W^*W = Q$. Furthermore the projections $\check{P}_{\underline{m}}Q$ and $\check{P}_{\underline{m}}P_{\underline{n}}$ have finite range for all $\underline{m}$ which implies that $Q = P_{\underline{n}}$, thus finite. As $\sum_{\underline{n}} P_{\underline{n}} = \mathbf{1}$ we conclude that each factor in the decomposition of $\Delta_N(\ell^2_{\mathbb{R}}(\mathbb{Z}))$ must be of type $I$. But just as in lemma 4.3 we can show that on $\Delta_N(\ell^2_{\mathbb{R}}(\mathbb{Z}))$ there is no tracial state and thus all factors are $I_\infty$.

□

**Theorem 5.8** *The von Neumann algebra $\Delta_N(\mathbb{R})$ is isomorphic to $\bigoplus_{p=2}^{N+1} \mathrm{M}_p(\mathbb{C})$.*

*Proof.* From theorem 5.7 we have that $\mathbf{N}_i \in \Delta_N(\ell^2_R(\mathbb{Z})) \subset \mathcal{B}(\mathcal{F}_N(\ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})))$. Let $P_{i,k}$ be the spectral projection of $\mathbf{N}_i$ corresponding to the eigenvalue $0 \leq k \leq -N$. Then the creation operator $a_i^*$ can be written as

$$a_i^* = \sum_{k=0}^{N-1} P_{i,k+1}\omega_i P_{i,k} \tag{5.21}$$

and thus all creation and annihilation operators $a_i$ belong to $\Delta_N(\ell^2_R(\mathbb{Z}))$. The Wick products $\Psi(\mathcal{V}, \mathbf{f})$ can be expressed in terms of the creation and annihilation operators by using the relations

$$a_i a_j^* = \delta_{i,j} + \frac{1}{N}\mathrm{d}\Gamma(T_{i,j}) \tag{5.22}$$

and the commutation relations

$$[a(f), \mathrm{d}\Gamma(A)] = a(A^* f) \tag{5.23}$$

for $A \in \mathcal{B}(\ell^2_R(Z))$. Thus $\Delta_N(\mathcal{K})$ is generated by the operators $a^\sharp(f \oplus 0)$ with $f \in \mathcal{K}$. In particular $\Delta_N(\mathbb{R})$ is the von Neumann algebra generated by $a^\sharp := a^\sharp(e)$ on $\mathcal{F}_N(\mathbb{C} \oplus \ell^2(\mathbb{Z}))$ with $e$ a unit vector in $\mathbb{C}$. Let $\mathbf{N}$ be the corresponding number operator and $\psi_k$ a vector with $\mathbf{N}\psi_k = k\psi_k$. Notice that $0 \leq k \leq |N|$. If moreover we have $a\psi_k = 0$ then the cyclic representation of $\psi_k$ has dimension $|N| - k + 1$ and is isomorphic to $M_{|N|-k+1}(\mathbb{C})$. By choosing the appropriate basis of the representation we obtain the matrix of $a^*$

$$a^* = \sqrt{\frac{1}{N}} \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & \sqrt{2} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \sqrt{|N|-k} \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

The space $\mathcal{F}_N(\mathbb{C} \oplus \ell^2(\mathbb{Z}))$ decomposes into an infinite number of copies of each of these representations provided that we verify that such 'pseudo-vacuum' vectors $\psi_k$ exist in $\mathcal{F}_N(\mathbb{C} \oplus \ell^2(\mathbb{Z}))$. Let $b := a(e_0)$ be another annihilation operator. Define for $0 \le k < N$

$$\psi_k = b^* a^{*k}\Omega + \frac{1}{|N| - k + 1} a^* \mathrm{d}\Gamma(T_{e,e_0}) a^{*k}\Omega. \tag{5.24}$$

Then $\mathbf{N}\psi_k = k\psi_k$ and

$$a\psi_k = \frac{1}{N}\mathrm{d}\Gamma(T_{e,e_0})a^{*k}\Omega + \frac{1}{|N| - k + 1}(1 + \frac{k-1}{N})\Gamma(T_{e,e_0})a^{*k}\Omega = 0. \tag{5.25}$$

We show that $\psi_k \neq 0$. Making use of the previous equality we have

$$\begin{aligned} \langle \psi_k, \psi_k \rangle &= \left\langle \Omega,\ a^k b \psi_k \right\rangle = ||a^{*k}\Omega||^2 - \\ &- \frac{1}{|N|(|N| - k + 1)} \left\langle \Omega, a^k \mathrm{d}\Gamma(T_{e_0,e})\mathrm{d}\Gamma(T_{e,e_0})a^{*k}\Omega \right\rangle = \\ &= ||a^{*k}\Omega||^2 - \frac{k}{|N|(|N| - k + 1)}||a^{*k}\Omega||^2 = \\ &= \frac{(|N| - k)(|N| + 1)}{|N|(|N| - k + 1)}||a^{*k}\Omega||^2 \neq 0. \end{aligned} \tag{5.26}$$

Finally let $\psi_N$ be a vector with $\mathbf{N}\psi_N = N\psi_N$. The monomials in creation operators $a^*$ and $(a_k^*)_{k\in\mathbb{Z}}$ applied to the vacuum form a total set in the Fock space $\mathcal{F}_N(\mathbb{C} \oplus \ell^2(\mathbb{Z}))$. Let us write

$$\psi_N = \sum_{k=-p}^{p} a_k^* \hat{\psi}_k + a^* \tilde{\psi}, \tag{5.27}$$

for some vectors $\hat{\psi}_k$ with $\mathbf{N}\hat{\psi}_k = N\hat{\psi}_k$ and $\mathbf{N}\tilde{\psi} = (N-1)\tilde{\psi}$. We show that

$||\psi_N||^2 = |N| \cdot ||a\psi_N||^2$:

$$||\psi_N||^2 = \left\langle \psi_N,\ a^*a(\sum_{k=-p}^{p} a_k^*\hat{\psi}_k + a^*\tilde{\psi}) \right\rangle =$$

$$= \quad \frac{1}{|N|} \left\langle \psi_N,\ \sum_{k=-p}^{p} a_k^*\hat{\psi}_k + a^*\tilde{\psi} \right\rangle = \frac{1}{|N|}||\psi_N||^2 \qquad (5.28)$$

where we have used that $aa^*\tilde{\psi} = \frac{1}{|N|}\tilde{\psi}$, and

$$a^*aa_k^*\hat{\psi}_k = \frac{1}{N}a^*\mathrm{d}\Gamma(T_{e,e_k})\hat{\psi}_k = \frac{1}{N}[a^*, \mathrm{d}\Gamma(T_{e,e_k})]\hat{\psi}_k = \frac{1}{|N|}a_k^*\hat{\psi}_k.$$

□

**Corollary 5.9** *The functors $\Delta_{N_1}$ and $\Delta_{N_2}$ are not isomorphic for $N_1 \neq N_2$.*

CHAPTER V

# The $q$-product of generalised Brownian motions [4]

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ABSTRACT

The notion of generalised Brownian motion is extended to multiple processes indexed by a set $\mathcal{I}$. For $-1 \leq q \leq 1$ the $q$-product of positive definite functions on pair partitions having the multiplicative property is defined, and shown to be a positive definite function on $\mathcal{I}$-colored pair partitions. The resulting $\mathcal{I}$-indexed generalised Brownian motion interpolates between graded tensor product ($q = -1$), reduced free product ($q = 0$) and tensor product ($q = 1$) of the given Brownian motions.

[4] This chapter is based on reference [29].

# 1 Introduction

Two different notions of independence stand out in non-commutative probability: tensor and free independence. The quantum stochastic calculus of Hudson and Parhasarathy [50] generalises the classical independence to tensor independence, and the free probability of Voiculescu [65] relies on free independence.

Beyond the well established frameworks of these two theories, an investigation concerning the notions of quantum white noise, Brownian motion and Markov processes is developed by Köstler [38] in the spirit of Kümmerer's approach to quantum probability [42, 41, 39]. Related to this, is the theory of generalised Brownian motion initiated by Bożejko and Speicher which provides concrete examples of quantum white noises but also raises interesting operator algebraic questions.

The generalised Brownian motions [13] are non-commutative processes $\omega(f)$ indexed by elements $f$ of a real Hilbert space $\mathcal{K}$ and endowed with a distinguished 'gaussian' positive linear functional $\rho_{\mathbf{t}}$ given by the following pair prescription:

$$\rho_{\mathbf{t}}(\omega(f_1)\dots\omega(f_n)) = \sum_{\mathcal{V}\in\mathcal{P}_2(n)} \mathbf{t}(\mathcal{V}) \prod_{(k,l)\in\mathcal{V}} \langle f_k, f_l\rangle \tag{1.1}$$

where the sum runs over all pair partitions of the ordered set $\{1,\dots,n\}$ and $\mathbf{t} : \bigcup_{k\in\mathbb{N}} \mathcal{P}_2(2k) \to \mathbb{C}$ is a function defined on all possible pair partitions called positive definite. We regard $\omega(f)$ as symmetric operators on the Hilbert space arising from the GNS construction.

The most important examples of positive definite functions are $\mathbf{t}(\mathcal{V}) = 1$ for all $\mathcal{V}$ which characterizes the classical Brownian motion [53] and, $\mathbf{t}(\mathcal{V}) = 0$ for crossing partitions and $\mathbf{t}(\mathcal{V}) = 1$ for non-crossing partitions which characterizes the free Brownian motion [65].

A remarkable interpolation between these cases arises from the algebra of $q$-deformed commutation relations $a(f)a^*(g)-qa^*(g)a(f) = \langle f, g\rangle\,\mathbf{1}$ for $-1 \le q \le 1$ investigated in a number of papers [10, 11, 12, 21, 23, 26, 33, 34, 45, 55, 67]. The monomials in fields $\omega(f) := a(f) + a^*(f)$ have vacuum expectations as in 1.1 with $\mathbf{t}_q(\mathcal{V}) = q^{\mathrm{cr}(\mathcal{V})}$ and $\mathrm{cr}(\mathcal{V})$ being the number of crossings of the pair partition $\mathcal{V}$. Further functorial and operator algebraic properties of this Brownian motion are studied in [9]. It turns out that we have a functor of white noise [42, 41] that is, a functor from the category of real Hilbert spaces with contractions to the category of non-commutative probability spaces.
Another interpolation between the classical and free Brownian motion is found in [13] : $\mathbf{t}_p(\mathcal{V}) = p^{|\mathcal{V}|-\mathrm{B}(\mathcal{V})}$ where $|\mathcal{V}|$ is the number of pairs of $\mathcal{V}$ and $\mathrm{B}(\mathcal{V})$ is the number of connected components or blocks of $\mathcal{V}$, and $0 \le p \le 1$.

Inspired by ideas from combinatorics, such as that of species of structures [5, 35] and analytic functors [36], the author together with Hans Maassen [28, 27] (chapters II and III of this thesis) have found a more functorial approach to the study of generalised Brownian motion. The GNS representation space of the algebra of fields $\omega(f)$ over a real Hilbert space $\mathcal{K}$, with respect to the gaussian state $\rho_{\mathbf{t}}$ has the Fock-like form

$$\mathcal{F}_{\mathbf{t}}(\mathcal{K}_{\mathbb{C}}) := \bigoplus_{n=0}^{\infty} \frac{1}{n!} V_n \otimes_s \mathcal{K}_{\mathbb{C}}^{\otimes n}, \tag{1.2}$$

where $V_n$ is a Hilbert space carrying a unitary representation $U_n$ of the symmetric group $\mathrm{S}(n)$, and $\otimes_s$ denotes the closed subspace of the tensor product $V_n \otimes \mathcal{K}_{\mathbb{C}}^{\otimes n}$ whose orthogonal projection is

$$P_n = \frac{1}{n!} \sum_{\tau \in \mathrm{S}(n)} U_n(\tau) \otimes \tilde{U}_n(\tau), \tag{1.3}$$

and

$$\tilde{U}_n f_1 \otimes \ldots \otimes f_n = f_{\tau^{-1}(1)} \otimes \ldots \otimes f_{\tau^{-1}(n)} \tag{1.4}$$

for $f_i \in \mathcal{K}_{\mathbb{C}}$. The factor $\frac{1}{n!}$ in 1.2 refers to the inner product on $V_n \otimes_s \mathcal{K}_{\mathbb{C}}^{\otimes n}$. We note that $\mathcal{F}_{\mathbf{t}}$ is an endofunctor of the category of Hilbert spaces with contractions called *analytic functor* [36]. The fields $\omega(f)$ can be represented as the sum of creation and annihilation operators $a(f)+a^*(f)$ which are defined with the help of a sequence of operators $j_n : V_n \to V_{n+1}$ satisfying the intertwining relations with respect to the representation of the symmetric groups $U_n$ and $U_{n+1}$: $j_n U_n(\tau) = U_{n+1}(\iota(\tau)) j_n$, with $\iota$ the natural embedding of $U_n$ into $U_{n+1}$ by keeping the last letter fixed. Concretely,

$$a^*(f) : P_n(v_n \otimes \psi_n) \mapsto (n+1) P_{n+1}(j_n v_n \otimes r(f) \psi_n). \tag{1.5}$$

where $v_n \in V_n$, $\psi_n \in \mathcal{K}_C^{\otimes n}$ and $r(f)$ is the right creation operator as defined on the full Fock space over $\mathcal{K}_{\mathbb{C}}$. For more details we refer to the first 3 sections of chapter III. Using this insight, a new class of positive functions on pair partitions has been found [8], which extend the indecomposable characters of the infinite symmetric group and for which the gaussian state is not tracial.

In the present work we answer a question of Roland Speicher concerning the existence of the $q$-product of generlised Brownian motions. In his paper [57] Speicher has analyzed the existence of universal products on the category of unital algebras with normalized linear functionals. The universal product should satisfy some natural requirements such as associativity and universal calculation rule for mixed moments. The result is that the only possibilities are the tensor product and the reduced free product. In the case of $q$-deformations, a concrete result in this direction has been proved in [45]. However one can still define a $q$-product for the algebras of generalised Brownian motions with gaussian states

$\rho_{\mathbf{t}_a}$ characterized by the positive definite functions $\mathbf{t}_a$ with $a$ in a index set $\mathcal{I}$, by the calculational rule:

$$(\mathop{*}_{a\in\mathcal{I}}{}^{(q)}\rho_{\mathbf{t}_a})\left(\prod_{i=1}^{n}\omega_{c(i)}(f_i)\right) = \sum_{\mathcal{V}} q^{\mathrm{cr}(\mathcal{V},c)} \prod_{a\in\mathcal{I}} \mathbf{t}_a(\mathcal{V}_a) \prod_{(i,j)\in\mathcal{V}} \langle f_i, f_j\rangle \,. \tag{1.6}$$

The sum is taken over those pair partitions $\mathcal{V}$ such that if $(i,j)\in\mathcal{V}$ then $c(i) = c(j) \in \mathcal{I}$, which we call the 'color' of the pair $(i,j)$, the pair partition $\mathcal{V}_a$ is the subset of pairs in $\mathcal{V}$ which are colored in the color $a$, and the coefficient $\mathrm{cr}(\mathcal{V},c)$ counts the number of crossings between pairs of different colors. The main objective of the paper is to prove the positivity of this functional for functions $\mathbf{t}_a$ which have a certain multiplicativity property. This is done in section 3, but as a preparation for that we develop the framework for $\mathcal{I}$-indexed generalised Brownian motion which is a fairly straightforward extension of the usual theory to the case of more than one processes 'coexisting', this time the key notion being that of positive definite function on $\mathcal{I}$-colored pair partitions.

The $q$-product interpolates between graded tensor product [47] (q=-1), reduced free product [65] (q=0) and tensor product [53] (q=1).

Finally, a central limit theorem is proved showing that as $n\to\infty$, the functional obtained by taking the $q$-product of a function $\mathbf{t}$ a number $n$ of times and restricting it to the algebra of fields $\omega(f) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\omega_i(f)$, converges in law to that of the $q$-deformed fields given by $\mathbf{t}_q(\mathcal{V}) = q^{\mathrm{cr}(\mathcal{V})}$.

## 2 Generalised Brownian motions and the $^*$-semigroup $\mathcal{BP}_2^{\mathcal{I}}(\infty)$

Let $\mathcal{I}$ be an arbitrary index set. In this section we will extend the notions of generalised Brownian motion and the associated $^*$-semigroup of broken pair partitions defined in chapter III to the case of multidimensional processes indexed by the set $\mathcal{I}$.

**Definition 2.1** *Let $P$ be a finite ordered set. By $\mathcal{P}_2^I(P)$ we denote the set of $I$-colored pair partitions, that is pairs $(\mathcal{V},c)$ with $\mathcal{V}\in\mathcal{P}_2(P)$ and $c:\mathcal{V}\to I$.*

**Definition 2.2** *Let $X$ be an arbitrary finite ordered set and $(L_a,P_a,R_a)_{a\in\mathcal{I}}$ a disjoint partition of $X$ into triples of subsets indexed by elements of $\mathcal{I}$. For each $a\in\mathcal{I}$ we consider a triple $(\mathcal{V}_a, f_a^{(l)}, f_a^{(r)})$ where $\mathcal{V}_a\in\mathcal{P}_2(P_a)$ and*

$$f_a^{(l)}: L_a\to\{1,\ldots,|L_a|\}, \qquad f_a^{(r)}: R_a\to\{1,\ldots,|R_a|\} \tag{2.1}$$

*are bijections. Any order preserving bijection $\alpha: X\to Y$ induces a map*

$$\alpha_a : (\mathcal{V}_a, f_a^{(l)}, f_a^{(r)}) \to (\alpha\circ\mathcal{V}_a,\ f_a^{(l)}\circ\alpha^{-1},\ f_a^{(r)}\circ\alpha^{-1}) \tag{2.2}$$

*where* $\alpha \circ \mathcal{V} := \{(\alpha(i), \alpha(j)) : (i,j) \in \mathcal{V}\}$. *This defines an equivalence relation. The corresponding element* $\mathbf{d}$ *of* $\mathcal{BP}_2^{\mathcal{I}}(\infty)$ *is such an equivalence class of collections of triples* $(\mathcal{V}_a, f_a^{(l)}, f_a^{(r)})$ *for* $a \in \mathcal{I}$.

In other words the elements of $\mathcal{BP}_2^{\mathcal{I}}(\infty)$ are broken pair partitions as in definition 3.1 in chapter III, with additional labeling with indices from $\mathcal{I}$ of the pairs and legs. For each $a \in \mathcal{I}$ there is a broken pair partition $\mathbf{d}_a$, however only a finite number of them are not empty. The product and involution are defined as for the $^*$-semigroup $\mathcal{BP}_2(\infty)$ with the additional condition that the left legs and right legs which are joined must be indexed by the same element of $\mathcal{I}$.
Let $\mathbf{d}_1 = (\mathcal{V}_{a,1},\ f_{a,1}^{(l)},\ f_{a,1}^{(r)})_{a\in\mathcal{I}}$ and $\mathbf{d}_2 = (\mathcal{V}_{a,2},\ f_{a,2}^{(l)},\ f_{a,2}^{(r)})_{a\in\mathcal{I}}$ be two elements of $\mathcal{BP}_2^{\mathcal{I}}(\infty)$ with the notations from definition 2.2. Let $\mathrm{M} = \min(|R_{a,1}|, |L_{a,2}|)$ be the number of legs which 'join' by taking the product of $\mathbf{d}_1$ and $\mathbf{d}_2$. Then we define

$$\mathbf{d}_1 \cdot \mathbf{d}_2 = (\mathcal{V}_a, f_a^{(l)}, f_a^{(r)})_{a\in\mathcal{I}} \tag{2.3}$$

with

$$\mathcal{V}_a = \mathcal{V}_{a,1} \cup \mathcal{V}_{a,2} \cup \Big\{ \Big( (f_{a,1}^{(r)})^{-1}(i),\ (f_{a,2}^{(l)})^{-1}(i) \Big)\ :\ i \leq \mathrm{M} \Big\}, \tag{2.4}$$

and the map $f_a^{(l)}$ defined on the disjoint union of the sets $L_{a,1}$ of left $a$-colored legs from the left diagram $\mathbf{d}_1$, and $L_{a,2} \setminus (f_{a,2}^{(l)})^{-1}(\{1,\ldots,\mathrm{M}\})$ consisting of the unpaired left $a$-colored legs of the diagram $\mathbf{d}_2$ by

$$\begin{cases} f_a^{(l)}(i) = f_{a,1}^{(l)}(i) & \text{for } i \in L_{a,1} \\ f_a^{(l)}(j) = f_{a,2}^{(l)}(j) - \mathrm{M} + |L_{a,1}| & \text{for } j \in L_{a,2} \setminus (f_{a,2}^{(l)})^{-1}(\{1,\ldots,\mathrm{M}\}) \end{cases}.$$

The function $f_a^{(r)}$ is defined similarly. The product does not depend on the chosen representatives for $\mathbf{d}_i$ in their equivalence class and is associative. The diagrams with no legs are the $\mathcal{I}$-colored pair partitions, thus $\mathcal{P}_2^{\mathcal{I}}(\infty) \subset \mathcal{BP}_2^{\mathcal{I}}(\infty)$.

The involution is given by mirror reflection. If $\mathbf{d} = (\mathcal{V}_a, f_a^{(l)}, f_a^{(r)})$ then $\mathbf{d}^* = (\mathcal{V}_a^*, f_a^{(r)}, f_a^{(l)})$ with the underlying set $X^*$ obtained by reversing the order on $X$ and

$$\mathcal{V}_a^* := \{(i,j) : (j,i) \in \mathcal{V}_a\} \tag{2.5}$$

is the adjoint of $\mathcal{V}_a$. It can be checked that

$$(\mathbf{d}_1 \cdot \mathbf{d}_2)^* = \mathbf{d}_2^* \cdot \mathbf{d}_1^*. \tag{2.6}$$

We are interested in positive functionals $\hat{\mathbf{t}}$ on the $^*$-semigroup $\mathcal{BP}_2^{\mathcal{I}}(\infty)$ which have the form

$$\hat{\mathbf{t}}(\mathbf{d}) = \begin{cases} \mathbf{t}(\mathbf{d}) & \text{if } \mathbf{d} \in \mathcal{P}_2^{\mathcal{I}}(\infty) \\ 0 & \text{otherwise.} \end{cases} \tag{2.7}$$

The function $\mathbf{t} : \mathcal{P}_2^{\mathcal{I}}(\infty) \to \mathbb{C}$ will be called *positive definite* on $\mathcal{I}$-colored pair partitions and will be shown to characterize generalised Brownian motions indexed by $\mathcal{I}$.
Let $\mathbf{n} : \mathcal{I} \to \mathbb{N}$ be a function which is equal to zero except a finite number of elements. We denote by $\mathcal{BP}_2^{\mathcal{I}}(\mathbf{n}, \mathbf{0})$ the set of diagrams $\mathbf{d}$ for which $|R_a| = 0$ and $|L_a| = \mathbf{n}(a)$. The GNS-representation $(\chi_{\mathbf{t}}, V, \xi_{\mathbf{t}})$ with respect to $\hat{\mathbf{t}}$ is characterized by

$$\langle \chi_{\mathbf{t}}(\mathbf{d}_1)\xi_{\mathbf{t}}, \chi_{\mathbf{t}}(\mathbf{d}_2)\xi_{\mathbf{t}} \rangle_V = \hat{\mathbf{t}}(\mathbf{d}_1^* \cdot \mathbf{d}_2) \tag{2.8}$$

which implies

$$V = \bigoplus_{\mathbf{n}} V_{\mathbf{n}} \qquad \text{where} \qquad V_{\mathbf{n}} = \overline{\operatorname{lin}\{\chi_{\mathbf{t}}(\mathbf{d})\xi_{\mathbf{t}} : \mathbf{d} \in \mathcal{BP}_2^{\mathcal{I}}(\mathbf{n}, \mathbf{0})\}}. \tag{2.9}$$

On $V_{\mathbf{n}}$ there is a unitary representation of the direct product group

$$\mathrm{S}(\mathbf{n}) := \mathsf{X}_{a \in \mathcal{I}} \mathrm{S}(\mathbf{n}(a)) \tag{2.10}$$

each of the terms $\mathrm{S}(\mathbf{n}(a))$ permuting the $a$-colored left legs of the diagrams in $\mathcal{BP}_2^{\mathcal{I}}(\mathbf{n}, \mathbf{0})$. We denote by $\pi(\mathbf{d})$ the diagram obtained by applying the permutation $\pi$ to the element $\mathbf{d}$ and the representation by $U_{\mathbf{n}}$. We distinguish the operators $j_a := \chi_{\mathbf{t}}(\mathbf{d}_{a,0})$ where $\mathbf{d}_{a,0}$ is the diagram containing one left leg indexed by $a \in \mathcal{I}$. The following intertwining relation will be important in later constructions:

$$\begin{aligned} & j_a : V_{\mathbf{n}} \to V_{\mathbf{n}+\delta_a} \\ & j_a \cdot U_{\mathbf{n}}(\tau) = U_{\mathbf{n}+\delta_a}(\iota_{\mathbf{n}}^{(a)}(\tau)) \cdot j_a \end{aligned} \tag{2.11}$$

with $(\mathbf{n} + \delta_a)(b) := \mathbf{n}(b) + \delta_{a,b}$ and $\iota_{\mathbf{n}}^{(a)}$ the natural embedding of $\mathrm{S}(\mathbf{n})$ into $\mathrm{S}(\mathbf{n} + \delta_a)$.

The function $\mathbf{t}$ on a $\mathcal{I}$-colored pair partition $(\mathcal{V}, c)$ can be calculated by putting the pair partition in the 'standard form', as a sequence (from the right to left) consisting of left legs, followed by permutations acting on the legs of the same index in $\mathcal{I}$ and right legs which connect with the left legs having the same index, etc. The permutations can be fixed by asking that if two pairs of the same color cross, then the crossing should be performed once and only once by the right-most permutation possible.

For example, let $X = \{1, \dots, 6\}$ and $\mathcal{V} = \{(1,4), (2,5), (3,6)\}$ with the coloring $c((1,4)) = c((3,6)) = a$ and $c((2,5)) = b$. Then

$$(\mathcal{V}, c) = \mathbf{d}_{a,0}^* \cdot \mathbf{d}_{b,0}^* \cdot \mathbf{d}_{a,0}^* \cdot (\pi_{1,2},\ \mathrm{e})(\mathbf{d}_{a,0} \cdot \mathbf{d}_{b,0} \cdot \mathbf{d}_{a,0}) \tag{2.12}$$

where $(\pi_{1,2}, e)$ is an element of $\mathrm{S}(2) \mathsf{X} \mathrm{S}(1)$. The function $\mathbf{t}$ can be calculated as follows

$$\mathbf{t}((\mathcal{V}, c)) = \langle \xi_{\mathbf{t}},\ j_a^*\ j_b^*\ j_a^*\ U_{2,1}(\pi_{1,2},\ \mathrm{e})\ j_a\ j_b\ j_a\ \xi_{\mathbf{t}} \rangle\,. \tag{2.13}$$

In general, let $\mathcal{V} \in \mathcal{P}_2(2n)$ be a pair partition and $c : \mathcal{V} \to \mathcal{I}$ a coloring. Then $c$ can be also seen as defined on $\{1, \ldots, 2n\}$ with the condition that it takes the same value in points belonging to the same pair of $\mathcal{V}$. We split the ordered set $\{1, \ldots, 2n\}$ in a number $2m$ of disjoint subsets $B_i^{(r)} := \{k_{i-1}, \ldots, p_i\}$ and $B_j^{(l)} := \{p_i + 1, \ldots, k_i\}$, with $k_0 = 1$ and $k_r = 2n$ such that the blocks $B_j^{(l)}$ contain left legs of pairs in $\mathcal{V}$ and $B_j^{(r)}$ contain right legs. Finally the value of $\mathbf{t}$ can be written as the expectation

$$\mathbf{t}((\mathcal{V}, c)) = \left\langle \xi_{\mathbf{t}},\ \prod_{l=1}^{p_1} j^*_{c(l)}\ U_{\mathbf{n}_1}(\pi_1) \prod_{l=p_1+1}^{k_1} j_{c(l)}\ \ldots U_{\mathbf{n}_r}(\pi_r) \prod_{l=p_m+1}^{2n} j_{c(l)}\ \xi_{\mathbf{t}} \right\rangle. \tag{2.14}$$

On the basis of this structure we pass now to the construction of the generalised Brownian motion indexed by $\mathcal{I}$. Let $\mathbf{t} : \mathcal{P}_2^{\mathcal{I}}(\infty) \to \mathbb{C}$ be a positive definite function on $\mathcal{I}$-colored pair partitions, and $\mathcal{H}$ a Hilbert space. We define the Fock-like space

$$\mathcal{F}_{\mathbf{t}}(\mathcal{H}) := \bigoplus_{\mathbf{n}} \frac{1}{\mathbf{n}!} V_{\mathbf{n}} \otimes_s \bigotimes_{a \in \mathcal{I}} \mathcal{H}^{\otimes \mathbf{n}(a)} \tag{2.15}$$

with the factor $\frac{1}{\mathbf{n}!} := \prod_a \frac{1}{\mathbf{n}(a)!}$ referring to the inner product. The symbol $\otimes_s$ is a short notation for the subspace consisting of vectors $\psi$ which lie in the range of the projection $P_{\mathbf{n}}$

$$P_{\mathbf{n}} := \frac{1}{\mathbf{n}!} \sum_{\tau \in \mathrm{S}(\mathbf{n})} U(\tau) \otimes \tilde{U}(\tau). \tag{2.16}$$

The operators $\tilde{U}(\tau)$ act on the space $\bigotimes_{a \in \mathcal{I}} \mathcal{H}^{\otimes \mathbf{n}(a)}$ by permuting the vectors in each term $\mathcal{H}^{\otimes \mathbf{n}(a)}$.

On $\mathcal{F}_{\mathbf{t}}(\mathcal{H})$ we define creation and annihilation operators for each index $a \in \mathcal{I}$ and vector $f \in \mathcal{H}$. Let $r_b(f)$ be the creation operator which acts on the tensor product

$$r_b(f) : \bigotimes_{a \in \mathcal{I}} \mathcal{H}^{\otimes \mathbf{n}(a)} \to \bigotimes_{a \in \mathcal{I}} \mathcal{H}^{\otimes \mathbf{n}(a) + \delta_{a,b}} \tag{2.17}$$

as identity on $\mathcal{H}^{\otimes \mathbf{n}(a)}$ for $a \neq b$ and as *right* creation operator on $\mathcal{H}^{\otimes \mathbf{n}(b)}$.
The annihilation operator $a_b(f)$ acts on a vector of the level $\mathbf{n}$ of the Fock space as follows:

$$\begin{aligned} a_b(f) : V_{\mathbf{n}} \otimes_s \bigotimes_{a \in \mathcal{I}} \mathcal{H}^{\otimes \mathbf{n}(a)} \quad &\to \quad V_{\mathbf{n} - \delta_b} \otimes_s \bigotimes_{a \in \mathcal{I}} \mathcal{H}^{\otimes \mathbf{n}(a) - \delta_{a,b}} \\ a_b(f) : \psi_{\mathbf{n}} \quad &\mapsto \quad (j_b^* \otimes r_b^*(f)) \psi_{\mathbf{n}} \end{aligned} \tag{2.18}$$

As a consequence of 2.11 we have

$$P_{\mathbf{n} - \delta_b} (j_b^* \otimes r_b^*(f)) P_{\mathbf{n}} = (j_b^* \otimes r_b^*(f)) P_{\mathbf{n}} \tag{2.19}$$

which means that $a_b(f)$ is a well defined operator on the dense domain $\mathcal{F}_{\mathbf{t}}^{(\mathrm{fin})}(\mathcal{H})$ of the Fock space $\mathcal{F}_{\mathbf{t}}(\mathcal{H})$ consisting of 'finite number of particles' states. The creation operator acts on a simple vector $v_{\mathbf{n}} \otimes_s \mathbf{f} := \mathbf{n}! P_{\mathbf{n}} v_{\mathbf{n}} \otimes \mathbf{f}$ as

$$a_b^*(f)\ v_{\mathbf{n}} \otimes_s \mathbf{f} = (j_b v_{\mathbf{n}}) \otimes_s (r_b(f)\mathbf{f}). \tag{2.20}$$

We give without proof the following result which is a straightforward extension of the similar one in the case of one generalised Brownian motion [27]:

**Theorem 2.3** *Let $f_1, \ldots, f_n$ be vectors in a Hilbert space $\mathcal{H}$. Then the expectation values with respect to the vacuum state $\rho_{\mathbf{t}}$ of the monomials in creation and annihilation operators have the expression*

$$\rho_{\mathbf{t}}(\prod_{i=1}^{n} a_{b_i}^{\sharp_i}(f_i)) = \sum_{(\mathcal{V},c)\in\mathcal{P}_2^{\mathcal{I}}(n)} \mathbf{t}((\mathcal{V},c)) \prod_{(i,j)\in\mathcal{V}} \langle f_i, f_j\rangle \cdot \delta_{b_i,b_j} \cdot Q(\sharp_i, \sharp_j) \tag{2.21}$$

*where*

$$Q = \begin{pmatrix} \rho_{\mathbf{t}}(a_b a_b) & \rho_{\mathbf{t}}(a_b a_b^*) \\ \rho_{\mathbf{t}}(a_b^* a_b) & \rho_{\mathbf{t}}(a_b^* a_b^*) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Furthermore consider $\mathcal{H} = \mathcal{K}_{\mathbb{C}}$ with $\mathcal{K}$ a real Hilbert space and define the fields $\omega_b(f) := a_b(f) + a_b^*(f)$ for vectors $f$ in $\mathcal{K}$. Then the restriction of $\rho_{\mathbf{t}}$ to the $^*$-algebra generated by the fields is the gaussian state characterizing the $\mathcal{I}$-indexed generalised Brownian motion.

**Corollary 2.4** *Let $f_1, \ldots, f_n$ be vectors in the real Hilbert space $\mathcal{K}$. Then*

$$\rho_{\mathbf{t}}(\prod_{i=1}^{n} \omega_{b_i}(f_i)) = \sum_{(\mathcal{V},c)\in\mathcal{P}_2^{\mathcal{I}}(n)} \mathbf{t}((\mathcal{V},c)) \prod_{(i,j)\in\mathcal{V}} \langle f_i, f_j\rangle \cdot \delta_{b_i,b_j}. \tag{2.22}$$

**Remark 2.5** *The restriction of the state $\rho_{\mathbf{t}}$ to the algebra generated by the fields $\omega_b(f)$ for a fixed $b \in \mathcal{I}$ is a generalised Brownian motion in the usual sense. For different indices $b$ one obtains in general different Brownian motions.*

An important class of positive definite functions on pair partitions are those which have the *multiplicativity* property

$$\mathbf{t}(\mathcal{V}) = \mathbf{t}(\mathcal{V}_1) \cdot \mathbf{t}(\mathcal{V}_2) \tag{2.23}$$

for any pair partition $\mathcal{V}$ which is the reunion of two subpartitions $\mathcal{V}_1$ and $\mathcal{V}_2$ not crossing each other. For a multiplicative function $\mathbf{t}$, the operators $j_a$ are isometric (see lemma 5.9 in chapter III). Furthermore by proposition 5.10 of chapter III, the field operators $\omega_b(f)$ are in this case essentially selfadjoint.

# 3 The $q$-product of generalised Brownian motions

**Definition 3.1** *Let $(\mathcal{V}, c) \in \mathcal{P}_2^{\mathcal{I}}(\infty)$ be a $\mathcal{I}$-colored pair partition. The number of inter-crossings of $(\mathcal{V}, c)$ is defined by the total number of crossings between the pairs of different colors in $\mathcal{V}$*

$$\mathrm{cr}(\mathcal{V}, c) = \frac{1}{2}\natural\{(p,q) \mid p, q \in \mathcal{V} \text{ crossing}, \ c(p) \neq c(q)\}. \tag{3.1}$$

The main result of this work is the existence of the $q$-product of generalised Brownian motions which satisfy the multiplicativity property.

**Theorem 3.2** *Let $\mathcal{I}$ be an index set and $-1 \leq q \leq 1$. Let $\mathbf{t}_a$ be a given multiplicative positive definite function on $\mathcal{P}_2(\infty)$ for every $a \in \mathcal{I}$. Then the function*

$$(\underset{a\in\mathcal{I}}{*}^{(q)} \mathbf{t}_a)((\mathcal{V}, c)) := q^{\mathrm{cr}(\mathcal{V},c)} \prod_{a\in\mathcal{I}} \mathbf{t}_a(c^{-1}(a)). \tag{3.2}$$

*is positive definite on $\mathcal{P}_2^{\mathcal{I}}(\infty)$.*

*Proof.* We will firstly prove that for each $\mathbf{n} : \mathcal{I} \to \mathbb{N}$ the kernel $k_{\mathbf{n}}$ defined on the set of diagrams in $\mathcal{BP}_2^{\mathcal{I}}(\mathbf{n}, \mathbf{0})$ is positive definite. Let $\mathbf{d}_i = (\mathcal{V}_{a,i}, f_{a,i}^{(l)})_{a\in\mathcal{I}}$ for $i = 1, 2$, be two such diagrams with legs only to the left. Then we have

$$k_{\mathbf{n}}(\mathbf{d}_1, \mathbf{d}_2) = (\underset{a\in\mathcal{I}}{*}^{(q)} \mathbf{t}_a)(\mathbf{d}_1^* \cdot \mathbf{d}_2). \tag{3.3}$$

The kernel $k_{\mathbf{n}}$ can be written as a product of three kernels

$$k_{\mathbf{n}}(\mathbf{d}_1, \mathbf{d}_2) = \prod_{a\in\mathcal{I}} \mathbf{t}_a((\mathbf{d}_1^* \cdot \mathbf{d}_2)_a) \cdot q^{\mathrm{cr}(\mathbf{d}_1)+\mathrm{cr}(\mathbf{d}_2)} \cdot q^{\mathrm{cr}(\mathbf{d}_1,\mathbf{d}_2)}. \tag{3.4}$$

where by $(\mathbf{d}_1^* \cdot \mathbf{d}_2)_a$ we denote the $a$-colored component of $\mathbf{d}_1^* \cdot \mathbf{d}_2$. The first product is a positive kernel by the positivity of each of the functions $\mathbf{t}_a$. The second product is also a positive definite kernel. The exponent $\mathrm{cr}(\mathbf{d}_i)$ stands for the number of inter-crossings of $\mathbf{d}_i$, that is number of crossings between pairs $p, q$ with different indices - i.e., $p \in \mathcal{V}_{a,i}$ and $q \in \mathcal{V}_{b,i}$ with $a \neq b$ - plus the number of crossings between pairs and left legs with different indices - i.e., $p = (l, r) \in \mathcal{V}_{a,i}$ and $k \in L_{a,i} = \mathrm{Dom}(f_b^{(l)})$ such that $l < k < r$ and $a \neq b$. To obtain the total number of inter-crossings $\mathrm{cr}(\mathbf{d}_1^* \cdot \mathbf{d}_2)$ of the $\mathcal{I}$-indexed pair partition $\mathbf{d}_1^* \cdot \mathbf{d}_2$, we need to add the number of crossings of right legs of $\mathbf{d}_1^*$ and left legs of $\mathbf{d}_2$ which have different colors. This is the exponent $\mathrm{cr}(\mathbf{d}_1, \mathbf{d}_2)$ of the last term of the product 3.4. The factor $q^{\mathrm{cr}(\mathbf{d}_1,\mathbf{d}_2)}$ depends only on the functions $f_{a,i}^{(l)}$ which determine the positions of the left legs, and does not depend on the pair partitions $\mathcal{V}_{a,i}$. The positivity of $\mathrm{cr}(\mathbf{d}_1, \mathbf{d}_2)$ is thus equivalent to that of the vacuum representation of an algebra with commutation relations which is described below.

Consider the algebra generated by the operators $a_{b,i}$ with $i = 1, \ldots, \mathbf{n}(b)$ satisfying the commutation relations

$$a_{b,i}a^*_{c,j} - q_{a,b}\ a^*_{c,j}a_{b,i} = \delta_{a,b}\delta_{i,j}\mathbf{1}, \tag{3.5}$$

with $q_{a,b} = 1$ if $a = b$ and $q_{a,b} = q$ if $a \neq b$. Such algebras have been investigated in [33, 34, 55] and more generally in [12]. There it is proved that for $|q| \leq 1$ the algebra can be represented on a Hilbert space with vacuum vector $\tilde{\Omega}$ satisfying $a_{b,i}\tilde{\Omega} = 0$. In particular this implies that the third kernel in 3.4 is positive definite and thus $k_{\mathbf{n}}$ as well.

We denote the Hilbert spaces generated by the kernels $k_{\mathbf{n}}$ by $V_{\mathbf{n}}$. Let

$$\lambda_{\mathbf{n}} : \mathcal{BP}_2^{\mathcal{I}}(\mathbf{n}, \mathbf{0}) \to V_{\mathbf{n}} \tag{3.6}$$

be the Gelfand map, i.e. $\langle \lambda_{\mathbf{n}}(\mathbf{d}_1), \lambda_{\mathbf{n}}(\mathbf{d}_2)\rangle = k_{\mathbf{n}}(\mathbf{d}_1,\ \mathbf{d}_2)$. On $\mathcal{BP}_2^{\mathcal{I}}(\mathbf{n}, \mathbf{0})$ there is an action of the group $\mathrm{S}(\mathbf{n})$ and $k_{\mathbf{n}}$ is invariant under this action, thus it gives rise to the unitary representation $U_{\mathbf{n}}$ on $V_{\mathbf{n}}$. On the Hilbert space $V := \bigoplus_{\mathbf{n}} V_{\mathbf{n}}$ we define the operators $j_a$ by

$$j_a\lambda_{\mathbf{n}}(\mathbf{d}_1) = \lambda_{\mathbf{n}+\delta_a}(\mathbf{d}_{a,0} \cdot \mathbf{d}_1). \tag{3.7}$$

The multiplicative property of the function $\mathbf{t}_a$ implies that

$$k_{\mathbf{n}}(\mathbf{d}_{a,0} \cdot \mathbf{d}_1,\ \mathbf{d}_{a,0} \cdot \mathbf{d}_2) = k_{\mathbf{n}}(\mathbf{d}_1,\ \mathbf{d}_2) \tag{3.8}$$

which means that $j_a$ is a well defined isometry. Moreover $j_a$ satisfies the intertwining property 2.11. We have thus constructed a representation of the $^*$-semigroup $\mathcal{BP}_2^{\mathcal{I}}(\infty)$ on the Hilbert space $V$, with respect to the positive functional $\hat{\mathbf{t}}$ where $\mathbf{t} = *^{(q)}_{a\in\mathcal{I}}\mathbf{t}_a$.

□

As described in the previous section we define now the creation and annihilation operators $a^\sharp_b(f)$ on the Fock space $\mathcal{F}_{*^{(q)}_{a\in\mathcal{I}}\mathbf{t}_a}(\mathcal{H})$ for $f \in \mathcal{H}$ in an arbitrary Hilbert space $\mathcal{H}$ and $b \in \mathcal{I}$. Similarly, the fields are $\omega_b(f) := a^*_b(f) + a_b(f)$.

**Definition 3.3** *Let $\mathbf{t}$ be a multiplicative positive definite function on ($\mathcal{I}$-indexed) pair partitions. We denote by $\Gamma_{\mathbf{t}}(\mathcal{K})$ the von Neumann algebra on $\mathcal{F}_{\mathbf{t}}(\mathcal{K}_{\mathbb{C}})$ generated by the selfadjoint operators $\omega_b(f)$ with $f$ in a real Hilbert space $\mathcal{K}$ and $b \in \mathcal{I}$. If the state $\rho_{\mathbf{t}}$ on $\Gamma_{\mathbf{t}}(\mathcal{K})$ is tracial then we call the function $\mathbf{t}$ tracial. For a complex Hilbert space $\mathcal{H}$ we denote the von Neumann algebra on $\mathcal{F}_{\mathbf{t}}(\mathcal{H})$ generated by all the fields $\omega_b(f)$ with $f \in \mathcal{H}$ by $\Gamma_{\mathbf{t}}(\mathcal{H})$.*

**Graded tensor product for $q = -1$.** We take a closer look at the case $q = -1$. Let $F$ be the unitary operator $Ff = -f$ on $\mathcal{H}$. Then

$$\Gamma_{\mathbf{t}}(F) : X \mapsto \mathcal{F}_{\mathbf{t}}(F)X\mathcal{F}_{\mathbf{t}}(F)^* \tag{3.9}$$

is an order two $^*$-automorphism of $\Gamma_{\mathbf{t}}(\mathcal{H})$ which we call $\mathbb{Z}_2$-grading. The vacuum state $\rho_{\mathbf{t}}$ is invariant under $\Gamma_{\mathbf{t}}(F)$. This makes $(\Gamma_{\mathbf{t}}(\mathcal{H}), \rho_{\mathbf{t}})$ a $\mathbb{Z}_2$-graded non-commutative probability space [47].

**Definition 3.4** *[47] Let $(\mathcal{A}, \phi)$ be a $\mathbb{Z}_2$-graded probability space with grading $\gamma$. Two von Neumann subalgebras $\mathcal{A}_1$ and $\mathcal{A}_2$ of $\mathcal{A}$ which are invariant under $\gamma$, are called graded independent if they gradely commute, i.e. $a_1 a_2 = (-1)^{\partial a_1 \partial a_2} a_2 a_1$ for all $a_i \in \mathcal{A}_i$ which satisfy $\gamma a_i = (-1)^{\partial a_i} a_i$, where $\partial a_i \in \{0, 1\}$ is called the grading of $a_i$, and moreover $\phi(a_1 a_2) = \phi(a_1)\phi(a_2)$. If $\mathcal{A} = \mathcal{A}_1 \bigvee \mathcal{A}_2$ then we call $\mathcal{A}$ the graded tensor product of $\mathcal{A}_1$ and $\mathcal{A}_2$.*

From the definition of the $-1$-product function $*^{(-1)}_{a\in\mathcal{I}}\mathbf{t}_a$ we can conclude that the creation and annihilation operators of different index anticommute, i.e.

$$a^{\sharp_1}_b(f_1) a^{\sharp_2}_c(f_2) = -a^{\sharp_2}_c(f_2) a^{\sharp_1}_b(f_1) \tag{3.10}$$

for $b \neq c$. This implies that the algebra $\Gamma_{*^{(q)}_{a\in\mathcal{I}}\mathbf{t}_a}(\mathcal{H})$ is the graded tensor product of the non-commutative probability spaces $\Gamma_{\mathbf{t}_a}(\mathcal{H})$ for $a \in \mathcal{I}$.

**Corollary 3.5** *The $q$-product $*^{(q)}_{a\in\mathcal{I}}\mathbf{t}_a$ of multiplicative positive definite functions $\mathbf{t}_a$ is a positive definite multiplicative function on $\mathcal{P}^{\mathcal{I}}_2(\infty)$ and interpolates between the graded tensor product (q=-1), reduced free product (q=0) and the tensor product (q=1). If all $\mathbf{t}_a$ are tracial then $*^{(q)}_{a\in\mathcal{I}}\mathbf{t}_a$ is tracial.*

The $q$-product provides a method for obtaining new positive definite functions on pair partitions by taking the product of known ones and restricting to a subalgebra generated by the sums of creation operators $a_b(f)$ over the same vector. Let $\mathcal{I}$ be a finite index set and $\mathbf{t}_a$ be positive multiplicative functions for each $a \in \mathcal{I}$. On $\mathcal{F}_{*^{(q)}_{b\in\mathcal{I}}\mathbf{t}_b}(\mathcal{K})$ we define the new creation operators $a^*(f) := \frac{1}{\sqrt{|\mathcal{I}|}} \sum_{b\in\mathcal{I}} a^*_b(f)$. The restriction of the state $*^{(q)}_{b\in\mathcal{I}}\mathbf{t}_b$ to the algebra generated by $a^\sharp(f)$ is a Fock state and the associated positive definite function on pair partitions $(*^{(q)}_{b\in\mathcal{I}}\mathbf{t}_b)^{(\mathrm{r})}$ has the following expression in terms of $\mathbf{t}_a$ and $q$:

$$(\underset{b\in\mathcal{I}}{*}{}^{(q)}\mathbf{t}_b)^{(\mathrm{r})}(\mathcal{V}) = \left(\frac{1}{|\mathcal{I}|}\right)^{|\mathcal{V}|} \sum_{c:\mathcal{I}\to\mathcal{V}} q^{\mathrm{cr}(c,\mathcal{V})} \prod_{a\in\mathcal{I}} \mathbf{t}_a(c^{-1}(a)) \tag{3.11}$$

The restriction of a $q$-product of $n$ functions which are equal to $\mathbf{t}$ will be denoted by $\mathbf{t}^{*n}_q$. We denote by $\mathbf{t}_q$ the positive definite function arising from the algebra of $q$-commutation relations:

$$\mathbf{t}_q(\mathcal{V}) = q^{\mathrm{cr}(\mathcal{V})}. \tag{3.12}$$

The following central limit theorem states that for any positive definite multiplicative $\mathbf{t}$, the $q$-product $\mathbf{t}^{*n}_q$ converges to $\mathbf{t}_q$ as $n$ goes to infinity.

**Theorem 3.6 (Central Limit)** *Let* $\mathbf{t}$ *be a positive definite multiplicative function on pair partitions. Then* $\mathbf{t}_q^{*n}$ *converges pointwise to* $\mathbf{t}_q$ *when* $n \to \infty$*.*

*Proof.* Let $\mathcal{V}$ be a pair partition. Then

$$\mathbf{t}_q^{*n}(\mathcal{V}) = \left(\frac{1}{n}\right)^{|\mathcal{V}|} \sum_{c:\underline{n}\to\mathcal{V}} q^{\mathrm{cr}(c,\mathcal{V})} \prod_{a\in\mathcal{I}} \mathbf{t}(c^{-1}(a)). \tag{3.13}$$

for $\underline{n} := \{1,\dots,n\}$. We consider $n$ big enough such that $n > |\mathcal{V}|$. From the sum we isolate the terms which give to each pair in $\mathcal{V}$ a different color, $c(p_1) \neq c(p_2)$ for $p_1 \neq p_2$. Any such term brings a contribution equal to $\left(\frac{1}{n}\right)^{|\mathcal{V}|} q^{\mathrm{cr}(\mathcal{V})}$ which in total gives $q^{\mathrm{cr}(\mathcal{V})}$.
The rest of the terms can be grouped according to the partition of $\mathcal{V}$ in subpartitions determined by the color of the pairs $\mathcal{V}_a = \{c^{-1}(a)\}$ for $a = 1\dots,n$. A fixed such partitioning of $\mathcal{V}$ has at most $|\mathcal{V}|-1$ sets and thus the number of possibilities of attributing one of the $n$ colors to each set is smaller that $n^{|\mathcal{V}|-1}$. In the limit $n\to\infty$ the contribution of this group of terms in the sum 3.13 tends to zero.

$\square$

**Example.** Consider the free Brownian motion, i.e. $\mathbf{t}(\mathcal{V}) = 0$ if $\mathcal{V}$ is crossing and $\mathbf{t}(\mathcal{V}) = 1$ if $\mathcal{V}$ is non-crossing. Then the product of $n$ such functions gives

$$\mathbf{t}_q^{*n}(\mathcal{V}) = \left(\frac{1}{n}\right)^{|\mathcal{V}|} \cdot q^{\mathrm{cr}(\mathcal{V})} \cdot \sharp\{c : \mathcal{V} \to \underline{n} \mid \mathrm{cr}(c^{-1}(i)) = 0,\ \forall i \in \underline{n}\}. \tag{3.14}$$

**Remark.** At the moment is it not clear to us in what extent the $q$-product defined here for generalised Brownian motions can be extended to other algebras. For example let $\mathcal{A}, \mathcal{B}$ be two algebras as considered in [12] with generators satisfying commutations relations of the type

$$a_i a_j^* - \sum_{k,l} t_{j,l}^{i,k} a_k^* a_l = \delta_{i,j} \tag{3.15}$$

for $\mathcal{A}$ and similarly for $\mathcal{B}$. Then the $q$-product could be defined by adding the commutation relations $a_i b_m^* = q b_m^* a_i$ for all generators $a_i$ of $\mathcal{A}$ and $b_m$ of $\mathcal{B}$. The new algebra will satisfy the conditions formulated in corollary 3.2 of [12] if the given algebras $\mathcal{A}, \mathcal{B}$ do satisfy them.

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# Bibliography


[1] Accardi, L., Bożejko, M., Interacting Fock spaces and gaussianisation of probability measures, *Infinite Dimensional Analysis Quantum Prob. Related Topics*, **1**, (1998), 663–670.

[2] Anshelevich M., Partition-dependent stochastic measures and q-deformed cumulants, *Doc. Math.*, **6**, (2001), 343–384.

[3] Appelbaum, D.B., Hudson, R.L., Fermion Itô formula and stochastic evolutions, *Commun. Math. Phys.*, **96**, (1984), 473–496.

[4] Baez J., Dolan J., From finite sets to Feynman diagrams, in B. Engquist, W. Schmid, editors, *Mathematics Unlimited - 2001 and Beyond*, Springer, Berlin (2001), 29–50.

[5] Bergeron, F., Labelle, G., Leroux, P., *Combinatorial Species and Tree-like Structures*, Cambridge University Press (1998).

[6] Biane P., Calcul stochastique non-commutatif, in P. Bernard, editor, *Lectures in Probability Theory, Ecole d'été de Probabilités de Saint-Flour XXIII-1993*, volume 1608 of *Lect. Notes Math.*, Springer, Berlin (1995), 1–96.

[7] Biane, P., Speicher, R., Stochastic calculus with respect to free Brownian motion and analysis on the Wigner space, *Prob. Theory Relat. Fields*, **112**, (1998), 373–409.

[8] Bożejko, M., Guţă, M., Functors of white noise associated to characters of the infinite symmetric group, *Commun. Math. Phys.*, to appear.

[9] Bożejko, M., Kümmerer, B., Speicher, R., q-Gaussian processes: non-commutative and classical aspects, *Commun. Math. Phys.*, **185**, (1997), 129–154.

[10] Bożejko, M., Speicher, R., An example of a generalized Brownian motion, *Commun. Math. Phys.*, **137**, (1991), 519–531.

[11] Bożejko, M., Speicher, R., An example of a generalized Brownian motion II, in *Quantum Probability and Related Topics VII*, World Scientific, Singapore (1992), 67–77.

[12] Bożejko, M., Speicher, R., Completely positive maps on Coxeter groups, deformed commutation relations, and operator spaces, *Math. Ann.*, **300**, (1994), 97–120.

[13] Bożejko, M., Speicher, R., Interpolations between bosonic and fermionic relations given by generalized Brownian motion, *Math. Z.*, **222**, (1996), 135–160.

[14] Bratteli, O., Robinson, D., *Operator Algebras and Quantum Statistical Mechanics I*, Springer, New York (1979).

[15] Bratteli, O., Robinson, D., *Operator Algebras and Quantum Statistical Mechanics II*, Springer, New York, second edition (1996).

[16] Chihara T., *An Introduction to Orthogonal Polynomials*, Gordon & Breach (1978).

[17] Connes A., *Noncommutative Geometry*, Academic Press, San Diego (1994).

[18] Cuntz J., Simple $C^*$-algebras generated by isometries, *Commun. Math. Phys.*, **57**, (1977), 173–185.

[19] Davies E., *Quantum Theory of Open Systems*, Academic Press, London (1976).

[20] Dykema K., Nica A., On the Fock representation of the $q$-commutation relations, *J. reine angew. Math.*, **440**, (1993), 201–212.

[21] Fivel D., Interpolation between Fermi and Bose statistics using generalized commutators, *Phys. Rev. Lett.*, **65**, (1990), 3361–3364.

[22] Franz, U., Schott, R., *Stochastic Processes and Operator Calculus on Quantum Groups*, Kluwer Academic Publishers, Dordrecht (1999).

[23] Frisch, U., Bourret, R., Parastochastics, *J. Math. Phys.*, **11**, (1970), 364–390.

[24] Goodman, F.H., de la Harpe, P., Jones, V.F.R., *Coxeter Graphs and Towers of Algebras*, Springer, Berlin-Heidelberg-New York (1989).

[25] Green H., A generalized method of field quantisation, *Phys. Rev.*, **90**, (1953), 270–273.

[26] Greenberg O., Particles with small violations of Fermi or Bose statistics, *Phys. Rev. D*, **43**, (1953), 270–273.

[27] Guţă, M., Maassen H., Generalised Brownian motion and second quantisation, *J. Funct. Anal.*, to appear.

[28] Guţă, M., Maassen, H., Symmetric Hilbert spaces arising from species of structures, *Math. Z.*, online first: DOI 10.1007/s002090100316.

[29] Guţă M., The $q$-product of generalised Brownian motions, submitted to the proceedings of *Quantum Probability and Infinite Dimensional Analysis*, Burg/Spreewald, 2001.

[30] Haag R., *Local Quantum Physics*, Springer, Berlin (1996).

[31] Holevo A., *Probabilistic and Statistical Aspects of Quantum Theory*, North-Holland (1982).

[32] Hudson, R.L., Parthasarathy, K.R., Quantum Itô's formula and stochastic evolutions, *Commun. Math. Phys.*, **93**, (1984), 301–323.

[33] Jørgensen, P.E.T., Schmitt, L.M., Werner, R.F., q-Canonical commutation relations and stability of the Cuntz algebra, *Pac. J. Math.*, **165**, (1994), 131–151.

[34] Jørgensen, P.E.T., Schmitt, L.M., Werner R.F., Positive representations of general commutation relations allowing Wick ordering, *J. Funct. Anal.*, **134**, (1995), 33–99.

[35] Joyal A., Une Théorie Combinatoire des Séries Formelles, *Adv. Math.*, **42**, (1981), 1–82.

[36] Joyal A., Foncteurs analytiques et especes de structures, in *Combinatoire enumerative, Proc. Colloq., Montreal/Can.*, volume 1234, Lect. Notes Math. (1986).

[37] Kadison R., Ringrose J., *Fundamentals of the Theory of Operator Algebras I, II*, Academic Press (1983).

[38] Köstler C., *Quanten-Markoff-Prozesse und Quanten-Brownsche Bewegungen*, Ph.D. thesis, Stuttgart (2000).

[39] Kümmerer B., *A Dilation Theory for Completely Positive Operators on $W^*$-Algebras*, Ph.D. thesis, Tübingen (1982).

[40] Kümmerer B., Markov dilations on $W^*$-algebras, *J. Funct. Anal.*, **63**, (1985), 139–177.

[41] Kümmerer B., Survey on a theory of non-commutative Markov processes, in L. Accardi, W. von Waldeenfels, editors, *Quantum Probability and Applications III*, volume 1303 of *Lect. Notes Math.*, Springer, Heidelberg (1988), 154–182.

[42] Kümmerer B., Quantum white noise, in H. Heyer, editor, *Infinite dimensional harmonic analysis*, Bamberg: D. u. M. Graebner (1996), 156–168.

[43] Kümmerer B., Maassen H., The essentially commutative dilations of dynamical semigroups on $M_n$, *Commun. Math. Phys.*, **109**, (1987), 1–22.

[44] Kümmerer B., Speicher R., Stochastic integration on the Cuntz algebra $O_\infty$, *J. Funct. Anal.*, **103**, (1992), 372–408.

[45] Maassen, H., van Leeuwen, H., A q-deformation of the gaussian distribution, *J. Math. Phys.*, **36**, (1995), 4743–4756.

[46] Meyer P.A., *Quantum Probability for Probabilists*, volume 1538 of *Lect. Notes Math.*, Springer (1993).

[47] Mingo, J.A., Nica, A., Crossings of set-partitions and addition of graded independent random variables, *International Journal of Mathematics*, **8**, (1997), 645–664.

[48] Nica A., A one-parameter family of transforms, linearizing convolution laws for probability distributions, *Commun. Math. Phys.*, **168**, (1995), 187–207.

[49] Ohnuki, Y., Kamefuchi, S., *Quantum Field Theory and Parastatistics*, Springer, Berlin-Heidelberg-New York (1982).

[50] Parthasarathy K., *An Introduction to Quantum Stochastic Calculus*, Birkhäuser, Basel (1992).

[51] Powers R., Representations of uniformly hyperfinite algebras and their associated von Neumann rings, *Ann. Math.*, **86**, (1967), 138–171.

[52] Reed, M., Simon, B., *Methods of Modern Mathematical Physics II*, Academic Press (1972).

[53] Simon B., *The $P(\Phi)_2$ Quantum Euclidian Field Theory*, Princeton Univ.Press (1974).

[54] Simon B., *Functional Integration and Quantum Physics*, Academic Press (1979).

[55] Speicher R., Generalized statistics of macroscopic fields, *Lett. Math. Phys.*, **27**, (1993), 97–104.

[56] Speicher R., Free probability theory and non-crossing cumulants, *Sém. Lothar. Combin.*, **B39c**, (1997), 38.

[57] Speicher R., On universal products, in D. Voiculescu, editor, *Free Probability theory*, Fields Institute Communications, Providence: AMS (1997), 257–266.

[58] Srinivas M., Davies E., Photon counting probabilities in quantum optics, *Optica Acta*, **28**, (1981), 981–996.

[59] Streater, R.F., Wightman, A.S., *PCT, Spin and Statistics, and all that*, Benjamin, New York (1964).

[60] Takesaki M., Conditional expectations in von Neumann algebras, *J. Funct. Anal.*, **9**, (1972), 306–321.

[61] Takesaki M., *Theory of Operator Algebras I*, Springer Verlag, New York (1979).

[62] Thoma E., Die Unzerlegbaren positiv-definiten Klassenfunktionen der abzëhlabar unendlichen, symmetrischen Gruppe, *Math. Z.*, **85**, (1964), 40–61.

[63] Tomiyama J., On the projection of norm one in $W^*$-algebras, *Proc. Japan Acad.*, **33**, (1957), 608–612.

[64] Veršik, A. M., Kerov, S. V., Characters and factor representations of the infinite symmetric group, *Soviet Math. Dokl.*, **23**, (1981), 389–392.

[65] Voiculescu, D., Dykema, K., Nica, A., *Free Random Variables*, Providence RI:AMS (1992).

[66] von Neumann J., *Mathematische Grundlagen der Quantenmechanik*, reprinted by Springer, Berlin-Heidelberg-New York (1981).

[67] Zagier D., Realizability of a model in infinite statistics, *Commun. Math. Phys.*, **147**, (1992), 199–210.

# Samenvatting

## Gaussische processen in niet-commutatieve kanstheorie

Zij $\omega(\cdot)$ een lineaire afbeelding van een reële Hilbertruimte $\mathcal{K} \simeq \ell^2_{\mathbb{R}}(\mathbb{Z})$ naar de ruimte van stochasten op een kansruimte met eindige momenten. Stel dat voor een orthonormale basis $(e_i)_{i=1}^{\infty}$ van $\mathcal{K}$ de stochasten $(\omega(e_i))_{i=1}^{\infty}$ onafhankelijk zijn en dezelfde verdeling hebben met variantie 1. Dan volgt uit de centrale limietstelling dat elke familie $(\omega(g_i))_{i=1}^{n}$ gezamelijk Gaussisch is met als covariantiematrix $(\langle g_i, g_j\rangle)_{i,j=1}^{n}$. We noemen $(\omega(f))_{f\in\mathcal{K}}$ een Gaussisch proces over $\mathcal{K}$. Zo'n proces heeft dan de volgende invariantie-eigenschap

$$\mathbb{E}(\omega(f_1)\omega(f_2)\ldots\omega(f_p)) = \mathbb{E}(\omega(Of_1)\omega(Of_2)\ldots\omega(Of_p)), \tag{1}$$

waarbij $O$ een willekeurig orthogonale transformatie van $\mathcal{K}$ is en $(f_i)_{i=1}^{p}$ vectoren zijn in $\mathcal{K}$.

In dit proefschrift wordt het volgende probleem onderzocht: wat gebeurt er als we niet van de stochasten eisen dat ze met elkaar commuteren? Het juiste kader wordt dan dat van de niet-commutatatieve kanstheorie. Hierin wordt de kansruimte vervangen door het paar $(\mathcal{A}, \rho)$, waarbij $\mathcal{A}$ een $^*$-algebra met eenheid is en $\rho$ een positieve genormeerde lineaire functionaal op $\mathcal{A}$. De elementen van $\mathcal{A}$ stellen binnen dit kader de stochasten voor en $\rho$ de verwachtingswaarde. Een niet-commutatief Gaussisch proces over een reële Hilbertruimte $\mathcal{K}$ voldoet per definitie aan (1) en wordt gekarakteriseerd door een functionaal

$$\rho_{\mathbf{t}}(\omega(f_1)\ldots\omega(f_n)) = \sum_{\mathcal{V}\in\mathcal{P}_2(n)} \mathbf{t}(\mathcal{V}) \prod_{(l,r)\in\mathcal{V}} \langle f_l, f_r\rangle \,.$$

Hierbij is $\mathbf{t} : \mathcal{P}_2(\infty) \to \mathbb{C}$ een afbeelding gedefinieerd op de verzameling $\mathcal{P}_2(\infty) := \cup_{n=0}^{\infty}\mathcal{P}_2(2n)$, de partities in paren van de eindige geordende verzamelingen met even aantal elementen $\{1, 2, \ldots, 2n\}$. Het bekendste voorbeeld [9, 10] van zo'n afbeelding is $\mathbf{t}_q(\mathcal{V}) = q^{\mathrm{cr}(\mathcal{V})}$, waarbij $\mathrm{cr}(\mathcal{V})$ staat voor het aantal overkruisingen tussen verschillende paren in de grafische voorstelling van $\mathcal{V}$ en waarbij $-1 \leq q \leq 1$. Deze $\mathbf{t}_q$ leidt tot de algebra van de $q$-commutatierelaties $a(f)a^*(g) - qa^*(g)a(f) = \langle f, g\rangle\, \mathbf{1}$, die interpoleren tussen het bosonische ($q = 1$), vrije ($q = 0$) en het fermionische ($q = -1$) geval. De $q$-Gaussische stochasten zijn $\omega_q(f) := a(f)+a^*(f)$. Bożejko en Speicher [13] onderzochten een ander voorbeeld van deze zogenaamde 'gegeneraliseerde Brownse beweging' en gaven hiermee de aanzet tot nieuw onderzoek naar deze processen.

In hoofdstuk III wordt aangetoond dat de cyclische representatieruimte van elk Gaussisch proces van de volgende vorm is:

$$\mathcal{F}_V(\mathcal{K}_{\mathbb{C}}) := \bigoplus_{n=0}^{\infty} V_n \otimes_s \mathcal{K}_{\mathbb{C}}^{\otimes n}.$$

Hierin is $(V_n)_{n=0}^{\infty}$ een rij van Hilbertruimten die elk een unitaire representatie $U_n$ van de symmetrische groep $\mathrm{S}(n)$ dragen. Het symbool '$\otimes_s$' staat voor de lineaire deelruimte van $V_n \otimes \mathcal{K}_{\mathbb{C}}^{\otimes n}$ behorend bij de orthogonale projectie

$$P_n := \frac{1}{n!} \sum_{\tau \in \mathrm{S}(n)} U_n(\tau) \otimes \tilde{U}_n(\tau),$$

waarbij $\tilde{U}_n$ de unitaire representatie van $\mathrm{S}(n)$ is die de factoren in het tensorproduct $\mathcal{K}_{\mathbb{C}}^{\otimes n}$ verwisselt. Voor elke $n \in \mathbb{N}$ bestaat er een lineaire operator $j_n : V_n \to V_{n+1}$ met de vervlechtingseigenschap: $U_{n+1}(\tau) j_n = j_n U_n(\tau)$ voor alle $\tau \in \mathrm{S}(n) \subset \mathrm{S}(n+1)$. De 'Gaussische' variabelen $\omega(f)$ kunnen dan worden voorgesteld als de som van de creatie- en annihilatieoperatoren $a^*(f) + a(f)$. Voor elke $v_n \in V_n, h \in \mathcal{K}_{\mathbb{C}}$, en $\psi_n \in \mathcal{K}_{\mathbb{C}}^{\otimes n}$ wordt $a^*(h)$ gegeven door

$$a^*(h) : \; v_n \otimes_s \psi_n \mapsto (j_n v_n) \otimes_s (\psi_n \otimes h).$$

De rij $(V_n, U_n, j_n)_{n=0}^{\infty}$ wordt verkregen via de representatie van de $^*$-halfgroep $\mathcal{BP}_2(\infty)$ met betrekking tot de functionaal $\hat{\mathbf{t}}$, de uitbreiding van $\mathbf{t}$ van $\mathcal{P}_2(\infty)$ tot $\mathcal{BP}_2(\infty)$.

Onze aanpak verschilt van die van Bożejko and Speicher, en is geïnspireerd door Joyal's theorie van combinatorische soorten [5, 35], die geïntroduceerd wordt in hoofdstuk II. Een *soort* $F$ is een familie van eindige verzamelingen $(F[n])_{n=0}^{\infty}$, waarbij $F[n]$ staat voor de '$F$-structuren' genummerd door elementen van $n := \{0, 1, \ldots, n-1\}$. Verder is er een werking van $\mathrm{S}(n)$ op $F[n]$: voor elke $\tau \in \mathrm{S}(n)$ is $F[\tau]$ een bijectie op $F[n]$ en $F[\tau]F[\sigma] = F[\tau \circ \sigma]$. Voor elke soort $F[\cdot]$ definiëren we een endofunctor $F(\cdot)$ van de categorie van verzamelingen met als morphismen de afbeeldingen. Gegeven een verzameling $J$ van 'kleuren' definiëren we

$$F(J) := \bigcup_{n=0}^{\infty} (F[n] \times J^n)/\mathrm{S}(n),$$

de verzameling van banen onder de symmetrische groep van $J$-gekleurde $F$-structuren. $F(\cdot)$ wordt *analytische functor* [36] genoemd. De Hilbertruimte $\mathcal{F}_F(\mathcal{K})$ is de lineaire versie van $F(J)$ als we voor $V_n$ de ruimte $\ell^2(F[n])$ nemen en voor $\mathcal{K}$ de Hilbertruimte met als basis $(e_i)_{i \in J}$. De afbeeldingen $j_n : V_n \to$

$V_{n+1}$ kunnen worden gecodeerd in een complexwaardig *gewicht* op het cartesisch produkt $F \times F'$ van $F$ met zijn afgeleide $F'$. Gebruikmakend van de combinatorische constructies vinden we opnieuw de bestaande voorbeelden [13, 10] en bovendien ook een nieuwe algebra met commutatierelaties $a(f)a^*(g) - a^*(g)a(f) = \mathbf{N}\langle f, g\rangle$. Deze algebra wordt verkregen door te kijken naar de symmetrisce Hilbertruimte van de gewortelde bomen, waarbij $\mathbf{N}$ dan de operator is die het aantal 'takken' aan een boom telt.

Als $\mathbf{t}$ een bepaalde *vermenigvuldigingseigenschap* heeft, dan zijn de operatoren $j_n$ isometrisch en de velden $\omega(f)$ essentieel zelfgeadjungeerd. Het tweede gedeelte van hoofdstuk III gaat over de von Neumann algebras $\Gamma_{\mathbf{t}}(\mathcal{K})$ die worden voortgebracht door de velden $\{\omega(f) : f \in \mathcal{K}\}$ op de Fockruimte $\mathcal{F}_{\mathbf{t}}(\mathcal{K}_{\mathbb{C}})$. In het geval van de $q$-gedeformeerde commutatierelaties is de afbeelding $\mathcal{K} \to \Gamma_{\mathbf{t}}(\mathcal{K})$ een functor van witte ruis [42, 9], dat wil zeggen een functor van de categorie van Hilbertruimten met als morphismen contracties, naar de categorie van niet-commutatieve kansruimten met volledig positieve afbeeldingen zodanig dat $\Gamma(\{0\}) = \mathbb{C}$. Als de toestand $\rho_{\mathbf{t}}$ trouw is dat bestaat er een witte ruis functor $\Gamma_{\mathbf{t}}^{\infty}$. Voor niet trouwe toestanden $\rho_{\mathbf{t}}$ maar begrendsde operatoren $\omega(f)$ construeren we een functor $\Delta_{\mathbf{t}}$ waarbij de von Neumann algebra $\Delta_{\mathbf{t}}(\mathcal{K})$ voortgebracht wordt door de zogenaamde *gegeneraliseerde Wickproduct*-operatoren op de Fockruimte $\mathcal{F}_{\mathbf{t}}(\mathcal{K}_{\mathbb{C}} \oplus \ell^2(\mathbb{Z}))$.

In de laatste sectie wordt geanalyseerd voor welke afbeeldingen $\mathbf{t}$ de algebra $\Gamma_{\mathbf{t}}(\ell^2_{\mathbb{R}}(\mathbb{Z}))$ een factor is. Op basis van een algmeen 'contractiecriterium' wordt aangetoond dat $\Gamma_{\mathbf{t}_q}(\ell^2_{\mathbb{R}}(\mathbb{Z}))$ een factor van type $II_1$ is. De afbeelding $\mathbf{t}_q$ wordt in dit geval gegeven door [10]:

$$\mathbf{t}_q(\mathcal{V}) := (-1)^{\mathrm{cr}(\mathcal{V})} q^{|\mathcal{V}| - \mathrm{B}(\mathcal{V})},$$

waarbij $0 \leq q < 1$ en $\mathrm{B}(\mathcal{V})$ het aantal *samenhangscomponenten* van $\mathcal{V}$ telt.

In hoofdstuk IV wordt er een nieuwe familie van positief definiete afbeeldingen $\mathbf{t}_{\alpha,\beta}$ onderzocht. De afbeelding $\mathbf{t}_{\alpha,\beta}$ is een uitbreiding tot $\mathcal{P}_2(\infty)$ van het *karakter* $\phi_{\alpha,\beta}$ van de oneindige symmetrische groep [62]:

$$\mathbf{t}_{\alpha,\beta}(\mathcal{V}) = \prod_{m \geq 2} \left( \sum_{i=1}^{\infty} \alpha_i^m + (-1)^{m+1} \sum_{i=1}^{\infty} \beta_i^m \right)^{\rho_m(\mathcal{V})}.$$

Het essentiële concept is dat van *cykel* van een paarpartitie $\mathcal{V}$. De exponent $\rho_m(\mathcal{V})$ telt het aantal cykels van lengte $m$ van $\mathcal{V}$. De von Neumann algebra $\Gamma_{\alpha,\beta}(\ell^2_{\mathbb{R}}(\mathbb{Z}))$ is *oneindig* en de vacuümtoestand $\rho_{\alpha,\beta}$ is niet trouw. Een deelfamilie van positieve afbeeldingen $\mathbf{t}_N(\mathcal{V}) := N^{\mathrm{c}(\mathcal{V}) - |\mathcal{V}|}$ wordt in groter detail geanalyseerd. Hierbij is $N$ een positief natuurlijke getal en $\mathrm{c}(\mathcal{V})$ het totale aantal cykels van $\mathcal{V}$. De creatie- en annihilatieoperatoren voldoen aan de relaties

$$a_N(f)a_N^*(g) = \langle f, g\rangle \mathbf{1} + \frac{1}{N} \mathrm{d}\Gamma(T_{f,g}),$$

waarbij $\mathrm{d}\Gamma(T_{f,g})$ de differentiële tweede quantisatie van de compacte operator $T_{f,g} : h \to \langle g, h\rangle\, f$ is. In het geval van negatieve $N$ leidt dit tot het volgende 'exclusiepricipe': een vectortoestand kan hoogstens $|N|$ deeltjes met dezelfde eendeeltjestoestand bevatten.

De algebras $\Delta_N(\ell^2_{\mathbb{R}}(\mathbb{Z}))$ zijn discrete sommen van factoren van type $I_\infty$. Voor de een-dimensionele Hilbertruimte $\mathbb{R}$ krijgen we de matrixalgebras

$$\Delta_N(\mathbb{R}) = \bigoplus_{p=2}^{N+1} \mathrm{M}_p(\mathbb{C}).$$

Als gevolg daarvan zijn de functoren $\Delta_N$ niet met elkaar equivalent voor verschillende waarden van $N$.

In hoofdstuk V wordt de notie van gegeneraliseerde Brownse beweging uitgebreid tot multipele processen geïndiceerd door een verzameling $\mathcal{I}$. Voor elke $-1 \leq q \leq 1$ wordt het *$q$-product van positief definiete afbeeldingen op paarpartities* gedefineerd. Dit is een positief definiete afbeelding op $\mathcal{I}$-gekleurde paarpartities en heeft bovendien de vermenigvuldigingseigenschap. Het bijbehorende proces interpoleert tussen het tensorproduct ($q = 1$), het gereduceerde vrije product ($q = 0$) en het gegradeerde tensorproduct ($q = -1$) van de gegeven Brownse bewegingen.

# Curriculum Vitae

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The author was born on August 29th 1973 in Craiova, Romania. After secondary school he studied theoretical physics at the University of Craiova. In 1998 he started his Ph.D. work at the University of Nijmegen in the department of mathematics under the supervision of Hans Maassen.

**Secondary School**, *Colegiul Nicolae Balcescu* Craiova:

Period: September 1988 to June 1992.
March 1992: second prize National Physics Olympiad (Romania),

**Theoretical Physics Study**, University of Craiova:

Period: September 1992 to June 1997.
September 1992: first position admission examen (9,97).
1995: won Tempus mobility grant.
Graduation work: Algebraic Quantum Field Theory.
Advisors: Prof. Dr. M. Winnink (Groningen) and Dr. G. Steinbrecher (Craiova).

**PhD Thesis**, University of Nijmegen:

Period: January 1998 to January 2002.
Supervisor: Prof. Dr. A .C. M. van Rooij.
Advisor: Dr. J. D. M. Maassen.
Subject: Non-commutative Gaussian processes.

**Exchange**, University of Groningen:

Period: November 1995 to May 1996 and December 1996 to May 1997.
Grant: Tempus.

**Post-doctoral position**, EURANDOM Eindhoven:

Period: January 2002 to January 2004.
Research group: Complex Statistical Models, Prof. Dr. R. D. Gill.