QUANTUM CONTINUAL MEASUREMENTS AND A POSTERIORI COLLAPSE ON CCR

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Abstract. A quantum theory for the Markovian dynamics of an open system under the unsharp observation which is continuous in time, is developed within the CCR stochastic approach. A stochastic classical equation for the posterior evolution of quantum continuously observed system is derived and the spontaneous collapse (stochastically continuous reduction of the wave packet) is described. The quantum Langevin evolution equation is solved for the general linear case of a quasi–free Hamiltonian in the initial CCR algebra with a fixed output observable field, and the posterior Kalman dynamics coresponding to an initial Gaussian state is found. It is shown for an example of the posterior dynamics of quantum unstable open system that any mixed state under a complete nondemolition measurement collapses exponentially to a pure Gaussian one.

1. Introduction

The time evolution of quantum system under an observation which is continuous in time cannot be described by any Schrödinger equation due to the stochastic irreversible nature of von Neumann reduction of the wave packet at any instant of measurement. An adequate model of the quantum unitary evolution giving a continuous collapse by a conditioning with respect to the measurements can be obtained in the framework of quantum stochastic (QS) calculus [15], firstly introduced for output nondemolition processes in [2, 3] and recently developed in a quite general form in [1, 4, 5]. A stochastic wave equation for an observed quantum system derived in [5] by using the quantum filtering method [4], provides an explanation of pure quantum relaxation of an atom under a complete observation [6] (Zeno paradox) and a Watch–Dog effect [9] for the reduced wave function of a quantum particle under the continuous observation.

In this paper we develop a regorous quantum stochastic theory of unsharp nondemolition measurements of continual families of arbitrary noncommuting observables $R_{t,\mathbf{x}}$ given sequentially in the real space-time $(t, \mathbf{x}) \in \mathbb{R}^{1+d}$. In the case $d = 0$ this defines the standard unitary dilation of an instrumental process for the quantum measurements, which are continuous in time, considered within an operational approach by Barchielli and Lupieri [1]. We give the direct proof of stochastic evolution equation for the posterior states of a general quantum system under a continuous indirect measurement of a noncommutative field-process $\boldsymbol{R}_t = \{R_{t,\mathbf{x}} \mid \mathbf{x} \in \mathbb{R}^d\}$. The observed process $Y(t)$ is supposed to be nondemolition in the sense [4] – [6] of the

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commutativity $[Y(r),\ X(t)] = 0$ of the past observables $Y^t = \{Y(r)|r \leq t\}$ with the Heisenberg operators $X(t)$ of the system for every t and self-nondemolition (commutative) $[Y(r),\ Y(t)] = 0$ for all $r,\ t$. In that case a posteriori state can be found [9] for any initial prior state by the Takesaki conditional expectations $\epsilon\{X(t)|Y^t\}$ on $\{Y(r) \mid r \leq t\}'$ restricted tothe future von Neumann algebras $\mathcal{L}_t = \{X(s)|s \geq t\}''$. We shall show that it is possible to represent the open quantum system under observation within a class of quantum stochastic evolutions in such a way that the observed commutative process $Y(t)$ for the sequential unsharp measurements of a noncommutative process R_t is described as the sum of noncommutative Heisenberg operators $R(t) = U^*(t)R_tU(t)$ of the subsystem under the measurement and a classical (commutative) while noise (error) $e(t)$. The unitary evolution $U(t)$ of such systems perturbed by a singular interaction with a meter is described in a 'Bose reservoir' by a quantum stochastic Schrödinger equation [15], driven by a white noise (force) $f(t)$. Note that the force $f(t)$ responsible for the perturbation of the system due to the measurements, may appear in the quantum Langevin equation as well as the classical (commutative) white noise [4]–[6]. But the pair $(e,\ f)$ cannot be described within the classical theory of generalized processes any more because the error $e(t)$ does not commute with $f(t)$ given the nondemolition condition for $R(t)$ and $Y(t) = R(t) + e(t)$.

It is interesting to note that stochastic equations of the particular diffusive type of (3.3) and (3.9), in their normalized nonlinear version [4, 5, 6], have appeared in the physical literature also in connection with phenomenological dynamical theories of quantum reduction and spontaneous collapse [17] — [11]. The idea is that the wave-function reduction associated to a continual measurement is some kind of diffusion process and some particular equations of this type are postulated. Our approach shows that this diffusion postulate as well as the continual counting reduction [8] can be derived in the natural general form from the unitary stochastic evolution of a big quantum system by the conditioning with respect to a chosen nondemolition process under the continual measurement. The unsharp self-nondemolition measurements and the objectification problem are discussed now intensively in the physical literature [12, 13] within the Davies-Lewis-Ludwigs operational approach, but real progress in clarifying the connection between the operational theory of continual measurements [1] and the spontaneous reduction theories [17] – [10] can be done only by using the quantum stochastic and nonlinear filtering methods [4] – [6],[8] which are considered in this paper.

2. A quantum stochastic model with continual unsharp measurements

Let us consider the dynamical problem of a sequential observation in continuous time $t \geq 0$ of a measurable family $\boldsymbol{L}_t = \{L_{t,\mathbf{x}}|\mathbf{x} \in \Lambda\}$ of operators $L_x = L_{t,\mathbf{x}},\ x = (t, \mathbf{x})$ in a Hilbert space $\mathcal{H}$, where Λ is a Borel space with a σ-algebra $\mathcal{A}$. We do not suppose that the operators L_x are pairwise commutative or even self–adjoint or normal. But we at first assume that they are bounded, $L_x \in \mathcal{L}(\mathcal{H})$, almost everywhere on the space $\mathbb{R}_+ \times \Lambda$ with respect to the product $\lambda(\mathrm{d}x) = \mathrm{d}t\lambda(\mathrm{d}\mathbf{x})$ of a positive measure $\lambda(\mathrm{d}\mathbf{x})$ on the Borel space Λ and the standard Lebesque measure $\mathrm{d}t$ on $\mathbb{R}_+$. Here $\mathcal{L}(\mathcal{H})$ denotes the space of continuous (bounded) operators in $\mathcal{H}$.

One can consider for example the problem of the (indirect) measurement of spin momenta $L_{t,\mathbf{x}} = L_{\mathbf{x}}$, described in the Schrödinger picture by the operators in

$\mathcal{H} = \mathbb{C}^2$ of spin projections $L_{\mathbf{x}} = \frac{1}{2}\, M(\mathrm{d}\mathbf{x})/\mathrm{d}\mathbf{x} + L^*_{\mathbf{x}}$, where $M(\Delta) = \int_\Delta R_{\mathbf{x}} \mathrm{d}\mathbf{x}$ is an operator–valued measure $M(\Delta) \in \mathcal{L}(\mathbb{C}^2)$ of the momentum in a solid angle $\Delta \subset \Lambda$ with $R_{\mathbf{x}} = L_{\mathbf{x}} + L^*_{\mathbf{x}}$ having the eigen–values ± 1, and $\lambda(\mathrm{d}\mathbf{x}) = \mathrm{d}\mathbf{x}$ is the standard solid angle measure on the sphere $\Lambda = \{\mathbf{x} \in \mathbb{R}^3 || \mathbf{x}| = 1\}$, normalized to 4π.

Due to the absence of a joint spectral resolution for the noncommutative family $\{L_{\mathbf{x}} | \mathbf{x} \in \Lambda\}$, there is no possibility of measuring the corresponding physical quantities in the usual (direct) sense. Moreover there is no way within orthodox quantum mechanics and measurement theory to describe an observation which is continuous in time even for a single self–adjoint operator $L_{\mathbf{x}} = L$ with a simple spectrum or to predict the dynamics of the quantum system under such an observation due to the absence of nontrivial mathematical models for noninstantaneous measurements.

We shall show that these difficulties can be removed within the quantum theory of open systems and indirect measurements, based on the quantum stochastic approach [15, 3]. The basic idea is that the quantum system under an observation must be described as the subsystem of a big system, including a Boson field $\boldsymbol{A}$ as a model of an observation channel coupled to the system by a singular interaction. The measurement information about the physical quantities L_x under such coupling can be continuously extracted in a nondemolition way from the continually-sequential unsharp observation of the output field $\boldsymbol{B} = U_\infty \boldsymbol{A} U^*_\infty$ given by the direct measurements of the compartible complex observables $\boldsymbol{Z} = \boldsymbol{B} + \boldsymbol{A}^*_\circ$.

Let $A(\mathrm{d}x)$ be the Bose-field annihilation measure on $\Gamma_1 = \mathbb{R}_+ \times \Lambda$, satisfying the canonical commutation relations (CCR)

$$(2.1) \qquad [A(\Delta'),\, A^*(\Delta)] = \lambda(\Delta \cap \Delta'), \quad \forall \Delta,\, \Delta' \in \mathcal{A}(\Gamma_1)$$

in the Fock space $\mathcal{F}$ over the Hilbert space $L^2(\Gamma_1)$ of square integrable functions of $x \in \Gamma_1$. One can realize [7] $\mathcal{F}$ as the space $L^2(\Gamma) = \oplus_{n=0}^\infty L^2(\Gamma_n)$ of functions f, square integrable in the sense that

$$(2.2) \quad \int |f(\chi)|^2 \lambda(\mathrm{d}\chi) = \sum_{n=0}^\infty \int \cdots \int_{0 \le t_n < \ldots < t_n < \infty} |f(x_1, \ldots, x_n)|^2 \prod_{i=1}^n \lambda(\mathrm{d}x_i) < \infty,$$

of chains $\chi = (x_1, \ldots, x_n)$, $x_i = (t_i, \mathbf{x}_i)$, $t_1 < \ldots < t_n$ of all finite lengths $|\chi| = n = 0, 1, \ldots$ with respect to the natural measure $\lambda(\mathrm{d}\chi) = \prod_{x \in \chi} \lambda(\mathrm{d}x)$. We identify the chains $\chi \in \Gamma$ as subsets $\{x_1, \ldots, x_n\} \subset \Gamma_1$, $t_1 < \ldots < t_n$ and the time ordered elements $(x_1, \ldots, x_n) \in \Gamma_1^n$ of the n-cube Γ_1^n, so that $\Gamma = \bigcup_{n=o}^\infty \Gamma_n$ is considered as the direct union of the sets $\Gamma_n = \{(x_1, \ldots, x_n) | t_1 < \ldots < t_n\}$. Then the annihilation operator $A(\Delta)$ of the Boson quanta in a measurable region $\Delta \in \Gamma_1$ is

$$(2.3) \qquad [A(\Delta)\, f](\chi) = \int_\Delta f(\chi \sqcup x) \lambda(\mathrm{d}x),$$

where $\chi \sqcup x$ is defined as the chain $(x_1, \ldots, x_i, x, x_{i+1}, \ldots, x_n)$ of length $n+1$ for almost all $x = (t, \mathbf{x})$, namely if $t \notin \{t_1, \ldots, t_n\}$.

One can easily find that the operator (2.3) is adjoint to the creation operator $A^*(\Delta)$ of the quanta in $\Delta \in \mathcal{A}(\Gamma_1)$

$$(2.4) \qquad (A^*(\Delta) f)(\chi) = \sum_{x \in \chi(\Delta)} f(\chi \backslash x), \quad \chi(\Delta) = \chi \cap \Delta,$$

with respect to the scalar product (2.2) and satisfies the CCR (2.1), where $\chi \backslash x = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ is the complement of the elementary chain $x \in \Gamma_1$ in the chain $\chi \in \Gamma_n$ with $x_i = x \in \chi$. In the following we shall regard the operators

$A(\Delta)$, $A^*(\Delta)$ acting as (2.3), (2.4) in the Hilbert space $\mathcal{H}\otimes\mathcal{F}$ of square integrable vector–functions $h:\Gamma\to\mathcal{H}$ with the invariant domain $\mathcal{D}=\bigcup_{\xi>1}\mathcal{D}(\xi)$, where

$$\mathcal{D}(\xi)=\{h\in\mathcal{H}\otimes\mathcal{F}|\ \int\xi^{|\chi|}\|h(\chi)\|^2\lambda(\mathrm{d}\chi)<\infty\}.$$

Let us consider the quantum stochastic evolution $U_t,\ t\in\mathbb{R}_+$ in $\mathcal{H}\otimes\mathcal{F}$, given by the Hudson–Parthasarathy operator equation [15, 7] $\mathrm{d}U+KU\mathrm{d}t=(\boldsymbol{L}\mathrm{d}\boldsymbol{A}^*-\boldsymbol{L}^*\mathrm{d}\boldsymbol{A})U$ for $U(t)=U_t^*$ having in our (nonstationary) case the form

$$\mathrm{d}U_t^*+K_tU_t^*\mathrm{d}t=\int_\Lambda(\mathrm{d}A^*(t,\mathrm{d}\mathbf{x})L_x-L_x^*\mathrm{d}A(t,\mathrm{d}\mathbf{x}))U_t^*,\ \ U_0^*=I, \tag{2.5}$$

where $K_t=iH_t+\frac{1}{2}\int L_{t,\mathbf{x}}^*L_{t,\mathbf{x}}\lambda(\mathrm{d}\mathbf{x})$, the integral is taken over $\mathbf{x}\in\Lambda$, $A(t,\mathrm{E})=A([0,t)\times\mathrm{E})$, and $\mathrm{d}A(t,\mathrm{E})=A(t+\mathrm{d}t,\mathrm{E})-A(t,\mathrm{E})$ is the forward differential of the process $A(t,\mathrm{E})$ for fixed $\mathrm{E}\in\mathcal{A}$. The necessary condition for the unitarity $U_t^*=U_t^{-1}$ of the family $U_t,\ t>0$ satisfying the quantum stochastic differential equation (2.5) is [15] the self–adjointness of the operators H_t (Hamiltonian) in $\mathcal{H}$ and that the integrals $\int_\Lambda L_{t,\mathbf{x}}^*L_{t,\mathbf{x}}\lambda(\mathrm{d}\mathbf{x})$ exist and equal $K_t+K_t^*$.

The solution of equation (2.5) can be described [7] explicitly in terms of the quantum stochastic multiple integral in Fock scale provided the conditions

$$\int_{t<s}\|H_t\|\mathrm{d}t<\infty,\ \int_{t<s}\int_\Lambda\|L_{t,\mathbf{x}}\|^2\mathrm{d}t\lambda(\mathrm{d}\mathbf{x})<\infty,\ \ \forall s\in\mathbb{R}_+ \tag{2.6}$$

hold which are sufficient for the existence and uniqueness of the unitary solution U_t of equation (2.5) with $H_t=H_t^*$.

Let us define the output observed process $\boldsymbol{Y}(t)$ of unsharp measurements of the continual family $\{L_x+L_x^*\mid x\in\Gamma_1\}$ as the time dependent selfadjoint operator–valued measure $Y(t,\mathrm{E}),\ \mathrm{E}\in\mathcal{B}$ on some σ-semi-ring $\mathcal{B}\subseteq\mathcal{A}$ with $\lambda(\mathrm{E})<\infty$ given by the quantum stochastic (forward) differential

$$\mathrm{d}Y(t,\mathrm{E})=M(t,\mathrm{E})\mathrm{d}t+\mathrm{d}Q(t,\mathrm{E}),\ \ Y(0,\mathrm{E})=0, \tag{2.7}$$

where $M(t,\mathrm{E})=\int_\mathrm{E}U_t(L_{t,\mathbf{x}}+L_{t,\mathbf{x}}^*)U_t^*\lambda(\mathrm{d}\mathbf{x})$, $Q(t,\mathrm{E})=A(t,\mathrm{E})+A^*(t,\mathrm{E})$. In the case of the initial vacuum state $|0\rangle\in\mathcal{F}$ of the Bose field and $\mathcal{B}$ generating $\mathcal{A}$, the generalized processes $\dot{Y}_\mathbf{x}(t)=\mathrm{d}Y(t,\mathrm{d}\mathbf{x})/\mathrm{d}t\lambda(\mathrm{d}\mathbf{x})$ can be regarded as a complete indirect observation of noncommuting operators $R_x=L_x+L_x^*,\ x\in\Gamma_1$ given by the instantaneous sequential measurements of the commuting operators $Y(\mathrm{d}x)=\mathrm{d}Y(t,\mathrm{d}\mathbf{x})$. Indeed the differentials $\mathrm{d}Q(t,\mathrm{E})$ for all measurable $\mathrm{E}\in\Lambda$ in that case are statisticaly equivalent to Wiener increments with zero mean values $\langle 0|\mathrm{d}Q(t,\mathrm{E})|0\rangle=0$ and minimal covariances $\langle 0|\mathrm{d}Q(t,\mathrm{E}')\mathrm{d}Q(t,\mathrm{E})|0\rangle=\mathrm{d}t\lambda(\mathrm{E}\cap\mathrm{E}')$ compatible with the CCR (2.1). They are independent of the operators $M(t,\mathrm{E})=\int_\mathrm{E}R_\mathbf{x}(t)\lambda(\mathrm{d}\mathbf{x})$, defined at the infinitesimal volume $\mathrm{E}=\mathrm{d}\mathbf{x}$ by the Heisenberg operators $R_\mathbf{x}(t)=U_tR_xU_t^*$ as $M(t,\mathrm{d}\mathbf{x})=R_\mathbf{x}(t)\lambda(\mathrm{d}\mathbf{x})$. Hence the differences between the increments $\mathrm{d}Y(t,\mathrm{d}\mathbf{x})=Y(t+\mathrm{d}t,\mathrm{d}\mathbf{x})-Y(t,\mathrm{d}\mathbf{x})$ of the form (2.7) and the operators $R_\mathbf{x}(t)\mathrm{d}t\lambda(\mathrm{d}\mathbf{x})$ are just independent Gaussian variables $\mathrm{d}Q(t,\mathrm{d}\mathbf{x})$, defining the minimal random error of the measurement of the noncommutative family $\boldsymbol{R}_t=\{R_{t,\mathbf{x}}|\mathbf{x}\in\Lambda\}$ in the continuous time $t\in\mathbb{R}_+$ as white noise $\dot{\boldsymbol{Q}}(t)=\{\dot{Q}_\mathbf{x}(t)|\mathbf{x}\in\Lambda\}$. One can consider $Y(t,\mathrm{E}),\ \mathrm{E}\in\mathcal{B}$ as a coarse–graining $Y_i(t)=Y(t,\mathrm{E}_i)$ of the family $Y(t,\mathrm{E}),\ \mathrm{E}\in\mathcal{A}$, corresponding to a σ-partition $\mathcal{B}=\{\mathrm{E}_i\in\mathcal{A}\mid i\in I\}$ of a measurable subset $\mathrm{M}=\sum\mathrm{E}_i\subseteq\Lambda$.

The following theorem shows that the QS equation (2.5) up to the Hamiltonian H_t corresponds to the unique Evans–Hudson diffusion $j(t,X) = U_t X U_t^*$, satisfying the nondemolition principle

$$[X(s), B(t,\mathrm{E})], \ \ \forall t \le s \in \mathbb{R}_+, \ X \in \mathcal{L}, \ \mathrm{E} \in \mathcal{A}$$

for all $X(t) = j(t,X)$ over a von-Neumann initial subalgebra $\mathcal{L} \subseteq \mathcal{L}(\mathcal{H})$ respectively to the given output field

$$B(t,\mathrm{E}) = \int_0^t \int_{\mathrm{E}} L_x(r)\,\lambda(\mathrm{d}\mathbf{x})\mathrm{d}r + A(t,\mathrm{E}), \ \ \mathrm{E} \in \mathcal{A}.$$

Theorem 1. *Let $j(t) : \mathcal{L} \to \mathcal{L}(\mathcal{H} \otimes \mathcal{F})$, $t \in \mathbb{R}_+$ be a quantum diffusion over a unital $*$-algebra $\mathcal{L} \subseteq \mathcal{L}(\mathcal{H})$ having the QS-differential*

$$\mathrm{d}j(t,X) = \gamma(t,X)\mathrm{d}t + \boldsymbol{\Delta}(t,X)\mathrm{d}\boldsymbol{A}^*(t) + \boldsymbol{\Delta}^*(t,X)\mathrm{d}\boldsymbol{A}(t),$$

where $\mathrm{d}\boldsymbol{A}^*\boldsymbol{\Delta} = \int_\Lambda \mathrm{d}A^*(\mathrm{d}\mathbf{x})\delta_x$, $\boldsymbol{\Delta}^*\mathrm{d}\boldsymbol{A} = \int_\Lambda \delta_x \mathrm{d}A(\mathrm{d}\mathbf{x})$, and

$$\delta_x(t,X^*) = \delta_x^*(t,X)^*, \gamma(t,X^*) = \gamma(t,X)^*$$

are the linear structural maps $\mathcal{L} \to \mathcal{L}(\mathcal{H} \otimes \mathcal{F})$, necessary satisfying the conditions

$$\begin{array}{ll} (i) & \delta_x(t,X^*X) = j(t,X^*)\delta_x(t,X) + \delta_x^*(t,X^*)j(t,X), \\ (ii) & \delta_x^*(t,X^*X) = j(t,X^*)\delta_x^*(t,X) + \delta_x(t,X^*)j(t,X), \\ (iii) & \gamma(t,X^*X) = \boldsymbol{\Delta}(t,X)^*\boldsymbol{\Delta}(t,X) - j(t,X)^*\gamma(t,X) - \gamma(t,X)^*j(t,X), \end{array}$$

with $\boldsymbol{\Delta}(X)^*\boldsymbol{\Delta}(X) = \int_\Lambda \delta_x(X)^*\delta_x(X)\lambda(\mathrm{d}\mathbf{x})$, $\delta_x(t,I) = 0 = \delta_x^*(t,I)$, $\gamma(t,I) = 0$. The family $\{X(t) = j(t,X) \mid X \in \mathcal{L}\}$ satisfies the nondemolition condition

$$[X(s), Y(t,\mathrm{E})] = 0, \ \ \forall s \le t \tag{2.8}$$

with respect to the output fields $Y(t) \in \{B(t,\mathrm{E}), B^*(t,\mathrm{E})\}$,

$$\begin{aligned} \mathrm{d}B(t,\mathrm{E}) &= \int_{\mathrm{E}} j(t,L_x)\lambda(\mathrm{d}\mathbf{x})\mathrm{d}t + \mathrm{d}A(t,\mathrm{E}), \ \ \mathrm{E} \in \mathcal{A}, \\ \mathrm{d}B^*(t,\mathrm{E}) &= \int_{\mathrm{E}} j(t,L_x^*)\lambda(\mathrm{d}\mathbf{x})\mathrm{d}t + \mathrm{d}A^*(t,\mathrm{E}), \ \ t \in \mathbb{R}_+ \end{aligned}$$

if and only if $\boldsymbol{\Delta}$, $\boldsymbol{\Delta}^*$ are the inner differentiations:

$$\begin{aligned} \delta_x(t,X) &= j(t,[X,L_{t,\mathbf{x}}]), \ \ \delta_x^*(t,X) = j(t,[L_{t,\mathbf{x}}^*,X]), \\ \gamma(t,X) &= \beta(t,X) + \frac{1}{2}\int_\Lambda j(t, L_x^*[X,L_x] + [L_x^*,X]L_x)\lambda(\mathrm{d}\mathbf{x}), \end{aligned}$$

where $\beta(t,X^*) = \beta(t,X)^*$ is a $j(t)$-differentiation $\mathcal{L} \to \mathcal{L}(\mathcal{H} \otimes \mathcal{F})$:

$$\beta(t,X^*X) = j(t,X^*)\beta(t,X) + \beta(t,X^*)j(t,X), \ \ \beta(t,I) = 0.$$

In the inner case $\beta(t,X) = j(t,i[H_t,X])$ these conditions together with (2.6) uniquelly define the quantum Markov spatial flow $j(t,X) = U_t X U_t^*$ given by the Hudson–Parthasarathy equatuion (2.5). Moreover, the output fields $\boldsymbol{B}(t)$, $\boldsymbol{B}^*(t)$ and, hence, the nondemolition process $\mathbf{Y}(t) = \boldsymbol{B}(t) + \boldsymbol{B}^*(t)$ are locally unitary equivalent to the input fields $\boldsymbol{A}(t)$, $\boldsymbol{A}^*(t)$ and to the commutative process $\boldsymbol{Q}(t) = \{Q(t,\mathrm{E}) \mid \mathrm{E} \in \mathcal{B}\}$: $\boldsymbol{B}(t) = U_\infty \boldsymbol{A}(t) U_\infty^*$, $\boldsymbol{B}^*(t) = U_\infty \boldsymbol{A}^*(t) U_\infty^*$ in the sense

$$Y(t,\mathrm{E}) = U_s Q(t,\mathrm{E}) U_s^*, \ \ \ \forall s \ge t. \tag{2.9}$$

In particular, $Y(t,\mathrm{E}) = U_t Y_t(\mathrm{E}) U_t^*$ for all $\mathrm{E} \in \mathcal{B}$, where $Y_t(\mathrm{E}) = Q([0,t) \times \mathrm{E}) = Q(t,\mathrm{E})$, $Q(\mathrm{d}x) = A(\mathrm{d}x) + A^*(\mathrm{d}x)$, $x \in \Gamma_1$.

Proof. The increments $\mathrm{d}X(t)=X(t+\mathrm{d}t)-X(t)$ of the linear $*$-maps $j(t):X\mapsto X(t)$, $j(t,X)^*=j(t,X)^*$ uniquelly define the linear $*$-map $\gamma(t):\mathcal{L}\to\mathcal{L}(\mathcal{H}\otimes\mathcal{F})$ and the adjoint maps $\boldsymbol{\Delta}(t)$, $\boldsymbol{\Delta}^*(t)$ due to the linear independence of the differentials $\mathrm{d}t$ and $\mathrm{d}\boldsymbol{A}^*(t)$, $\mathrm{d}\boldsymbol{A}(t)$. An application of the QS Ito formula [15] to the conditions $j(t,I)=I$, $j(t,X^*X)=j(t,X)^*j(t,X)$ gives $\gamma(t,I)=0$, $\boldsymbol{\Delta}(t,I)=0=\boldsymbol{\Delta}^*(t,I)$, and

$$\begin{aligned}\mathrm{d}(X(t)^*X(t)) &= \mathrm{d}X(t)^*\mathrm{d}X(t)+\mathrm{d}X(t)^*X(t)+X(t)^*\mathrm{d}X(t)\\ &= [\boldsymbol{\Delta}(t,X)^*\boldsymbol{\Delta}(t,X)-\gamma(t,X)^*j(t,X)-j(t,X)^*\gamma(t,X)]\mathrm{d}t\\ &\quad+(\boldsymbol{\Delta}^*(X)j(X)+j(X)^*\boldsymbol{\Delta}^*(X))\mathrm{d}\boldsymbol{A}\\ &\quad+(\boldsymbol{\Delta}(X^*)j(X)+j(X^*)\boldsymbol{\Delta}(X))\mathrm{d}\boldsymbol{A}^*.\end{aligned}$$

Comparing this with the QS differential

$$\mathrm{d}j(t,X^*X)=\boldsymbol{\Delta}^*(X^*X)\mathrm{d}\boldsymbol{A}+\boldsymbol{\Delta}(X^*X)\mathrm{d}\boldsymbol{A}^*-\gamma(X^*X)\mathrm{d}t,$$

we obtain the conditions (i), (ii), (iii), found in [**?**] for the Markovian case.

If $Y(t)$ is a nondemolition process respectively to $X(t)$, then

$$[\mathrm{d}X(t),Y(s)]=[X(t+\mathrm{d}t),Y(s)]-[X(t),Y(s)]=0,\ \ \forall t\geq s$$

for $t\geq s$ and hence

$$[\gamma(t,X),Y(s)]=[\delta_x(t,X),Y(s)]=[\delta_x^*(t,X),Y(s)]=0,\ \ \forall t\geq s$$

due to the commutativity of $\mathrm{d}t$, $\mathrm{d}A^*(t,\mathrm{E})$, $\mathrm{d}A(t,\mathrm{E})$ with $Y(s)$, $s\leq t$. Applying the QS Ito formula to the condition $[X(t),Y(t)]=0$ for $Y(t)\in\{B(t,\mathrm{E}),B^*(t,\mathrm{E})\}$ we obtain

$$\begin{aligned}\mathrm{d}[X(t),B(t,\mathrm{E})] &= \int_{\mathrm{E}}([X(t),L_x(t)]-\delta_x(t,X))\lambda(\mathrm{d}\mathbf{x})=0\\ \mathrm{d}[X(t),B^*(t,\mathrm{E})] &= \int_{\mathrm{E}}([X(t),L_x^*(t)]+\delta_x^*(t,X))\lambda(\mathrm{d}\mathbf{x})=0,\end{aligned}$$

i. e. $\delta_x(X)=[X,L_x]$, $\delta_x^*(X)=[L_x^*,X]$ due to $[\mathrm{d}X(t),Y(t)]=0$, $L_x(t)=j(t,L_{t,\mathbf{x}})$, $L_x^*(t)=j(t,L_{t,\mathbf{x}}^*)$. This together with $\beta(X)=j(i[H,X])$ gives $\gamma(t,X)=j(t,\gamma_t(X))$, where

$$\gamma_t(X)=i[H_t,X]+\frac{1}{2}\int_\Lambda(L_x^*[X,L_x]+[L_x^*,X]L_x)\lambda(\mathrm{d}\mathbf{x})$$

is the solution of the equation

$$\gamma_t(X^*X)-X^*\gamma_t(X)-\gamma_t(X)^*X=\int[L_x^*,X][X,L_x]\lambda(\mathrm{d}\mathbf{x}),$$

uniquely defined up to a $*$-differentiation $\beta_t(X)=i[H_t,X]$, $H_t=H_t^*$. The unique solution $j(t,X)=U_tXU_t^*$ of the derived nonstationary Langevin equation under the boundness conditions (2.6) was found in [7], Corollary 4.

Let us denote by $U(s,t)$, $s\geq t$ the solution of the quantum stochastic evolution equation (2.5) on the interval $(t,s]$ with $U(t,t)=I$ under the integrability conditions (2.6). The operators $U(s,r)$ commute with $\boldsymbol{Y}_r$, $r\leq s$, due to commutativity of $\boldsymbol{Y}_r\in\boldsymbol{A}(r),\boldsymbol{A}^*(r)$ and the operators $\boldsymbol{L}_t,\boldsymbol{L}_t^*$, $\mathrm{d}\boldsymbol{A}(t)$, $\mathrm{d}\boldsymbol{A}^*(t)$, $t\in[r,s)$ generating $U(s,r)$. Hence $U_s\boldsymbol{Y}_tU_s^*=U_t\boldsymbol{Y}_tU_t^*$ because $U_s^*=U(s,t)^*U_t^*$ for any $s>t$ and because of unitarity of $U(s,t)$. Using the quantum Ito formula [15] one can easily find

$$\begin{aligned}
\mathrm{d}Y(t,\mathrm{E}) &= \mathrm{d}(U_tY_t(\mathrm{E})U_t^*) = \mathrm{d}U_tY_t(\mathrm{E})U_t^* + U_t\mathrm{d}Y_t(\mathrm{E})U_t^* + U_tY_t(\mathrm{E})\mathrm{d}U_t^* \\
&\quad + \mathrm{d}U_t\mathrm{d}Y_t(\mathrm{E})U_t^* + \mathrm{d}U_tY_t(\mathrm{E})\mathrm{d}U_t^* + U_t\mathrm{d}Y_t(\mathrm{E})\mathrm{d}U_t^* + \mathrm{d}U_t\mathrm{d}Y_t(\mathrm{E})\mathrm{d}U_t^*, \\
\mathrm{d}\boldsymbol{B}(t) &= \mathrm{d}\boldsymbol{A}(t) + U_t\boldsymbol{L}_tU_t^*\mathrm{d}\lambda, \ \ \mathrm{d}\boldsymbol{B}^*(t) = \mathrm{d}\boldsymbol{A}^*(t) + U_t\boldsymbol{L}_t^*U_t^*\mathrm{d}\lambda, \\
\mathrm{d}Y(\mathrm{E}) &= \mathrm{d}Q(\mathrm{E}) + \int_{\mathrm{E}} U(L_x + L_x^*)U^*\lambda(\mathrm{d}\mathbf{x}) = \mathrm{d}Q(\mathrm{E}) + M(\mathrm{E})\mathrm{d}t
\end{aligned}$$

for $Y(t,\mathrm{E}) = B(t,\mathrm{E}) + B^*(t,\mathrm{E}),\ \mathrm{E} \in \mathcal{B}$ due to the only nonzero infinitesimal multiplication $\mathrm{d}A(t,\mathrm{E}')\mathrm{d}A^*(t,\mathrm{E}) = \mathrm{d}t\lambda(\mathrm{E}\cap\mathrm{E}')$, where M is defined by (2.7). The relation (2.8) for the process (2.7) is a simple consequence of (2.9) and $[X, Q(\mathrm{E})] = 0$ for any $\mathrm{E} \in \mathcal{A}(\Gamma_1)$ and $X \in \mathcal{L}(\mathcal{H}) \otimes I_{\mathcal{F}}$:

$$[X(s), Y(t,\mathrm{E})] = [U_sXU_s^*,\ U_sY_t(\mathrm{E})U_s^*] = U_s[X, Y_t(\mathrm{E})]U_s^* = 0.$$

Remark 1. Considering instead of $\boldsymbol{Y}(t) = U_s\boldsymbol{Q}(t)U_s^*$ the sequential measurements of the output momentum process $\boldsymbol{Y}(t) = U_s\boldsymbol{V}(t)U_s^*,\ s \geq t$, defined by $V(t,\mathrm{E}) = \frac{1}{i}\ (A(t,\mathrm{E}) - A^*(t,\mathrm{E}))$ as

$$\mathrm{d}Y(t,\mathrm{E}) = N(t,\mathrm{E})\mathrm{d}t + \mathrm{d}V(t,\mathrm{E}), \quad \mathrm{E} \in \mathcal{B},$$

where $N(t,\mathrm{E}) = \frac{1}{i}\ \int_{\mathrm{E}} U(L_{t,\mathbf{x}} - L_{t,\mathbf{x}}^*)U_t^*\lambda(\mathrm{d}\mathbf{x})$, one can extract the information about the noncommuting self–adjoint operators $S_x = \frac{1}{i}\ (L_x - L_x^*)$. Moreover, by doubling $\Lambda \to \Lambda \times \{-,+\}$ the space Λ and considering the family $\{L_{t,\mathbf{x},-},\ L_{t,\mathbf{x},+}\}$ with $L_{t,\mathbf{x},\mp} = L_{t,\mathbf{x}}/\sqrt{\mp 2}$ instead of $\{L_{t,\mathbf{x}}\}$ one can realize the continuous time–sequential indirect observation of the pairs of operators

$$R_{x,+} = \frac{1}{\sqrt{2}}\ (L_x + L_x^*), \ \ R_{x,-} = \frac{1}{\sqrt{2}i}\ (L_x - L_x^*), \ \ x \in \mathbb{R}_+ \times \Lambda$$

by the measurement of the two commutative output processes $\boldsymbol{Y}_\mp(t) = U_\infty\boldsymbol{Q}_\mp(t)U_\infty^*$. Here $\boldsymbol{Q}_\mp(t) = \boldsymbol{A}_\mp(t) + \boldsymbol{A}_\mp^*(t)$ are given by the independent Boson measures $A_\mp$ on $\mathcal{A}(\mathbb{R}_+ \times \Lambda)$ as $A_\mp(t,\mathrm{E}) = A_\mp([0,t) \times \mathrm{E}),\ \mathrm{E} \in \mathcal{B}$, and U_t satisfies the equation (2.5) with two-fold quantum stochastic integral over $\Lambda \times \{-,+\}$ instead of Λ which can be written again as (2.5) in terms of $A = \frac{1}{\sqrt{2}}\ (A_+ + iA_-)$. The complexified observable process $\boldsymbol{Z} = \frac{1}{\sqrt{2}}\ (\boldsymbol{Y}_+ + i\boldsymbol{Y}_-)$ defines the unsharp observation $Z(t,\mathrm{E}) = B(t,\mathrm{E}) + A_\circ^*(t,\mathrm{E}),\ \mathrm{E} \in \mathcal{B},\ A_\circ^* = \frac{1}{\sqrt{2}}\ (A_+^* + iA_-^*)$ of the nondemolition output field $\boldsymbol{B}(t) = U_\infty\boldsymbol{A}(t)U_\infty^*$.

In the case $\mathcal{B} = \mathcal{A}$ such the continuous measurement gives a complete nondemolition sequential observation [3] of the non–Hermitian operators L_x in terms of the complexified output process $\boldsymbol{Z}(t) = U_\infty\boldsymbol{W}(t)U_\infty^*$ having the stochastic differential

$$\mathrm{d}Z(t,\mathrm{E}) = \mathrm{d}t\int_{\mathrm{E}} L_{\mathbf{x}}(t)\lambda(\mathrm{d}\mathbf{x}) + \mathrm{d}W(t,\mathrm{E}), \ \ \mathrm{E} \in \mathcal{B}, \tag{2.10}$$

where $L_{\mathbf{x}}(t) = U_tL_{t,\mathbf{x}}U_t^*$ and $\boldsymbol{W}(t) = \frac{1}{\sqrt{2}}\ (\boldsymbol{Q}_+(t) + i\boldsymbol{Q}_-(t)) = \boldsymbol{A}(t) + \boldsymbol{A}_\circ^*(t)$ is the complex Wiener process in Fock space over $L^2(\mathbb{R}_+ \times \Lambda \times \{-,+\})$ with multiplication table

$$\begin{aligned}
\mathrm{d}W^*(t,\mathrm{E})\mathrm{d}W(t,\mathrm{E}') &= \mathrm{d}t\lambda(\mathrm{E}\cap\mathrm{E}') = \mathrm{d}W(t,\mathrm{E}')\mathrm{d}W^*(t,\mathrm{E}), \\
\mathrm{d}W(t,\mathrm{E})\mathrm{d}W(t,\mathrm{E}') &= 0, \ \ \mathrm{d}W^*(t,\mathrm{E})\mathrm{d}W^*(t,\mathrm{E}') = 0.
\end{aligned}$$

3. A posteriori quantum dynamics under the continual measurements.

Let us consider the quantum diffusion $j(t) : \mathcal{L} \to \mathcal{L}(\mathcal{H} \otimes \mathcal{F})$ of the system over a unital $*$–algebra $\mathcal{L}$ in $\mathcal{H}$, together with the given nondemolition output fields $\mathrm{d}\boldsymbol{B} = \boldsymbol{L}\mathrm{d}\lambda + \mathrm{d}\boldsymbol{A}$, $\mathrm{d}\boldsymbol{B}^* = \boldsymbol{L}^*\mathrm{d}\lambda + \mathrm{d}\boldsymbol{A}^*$. The operators $j(t, X) = X(t)$ under the conditions of Theorem 1 satisfy the quantum Langevin equation

$$\mathrm{d}X(t) - \mathrm{d}t \int_\Lambda \frac{1}{2}\ \left(L^*_{\mathbf{x}}(t)[X(t),\, L_{\mathbf{x}}(t)] + [L^*_{\mathbf{x}}(t),\, X(t)]L_{\mathbf{x}}(t)\right) \lambda(\mathrm{d}\mathbf{x})$$

$$(3.1)= \quad i[H(t),\, X(t)]\mathrm{d}t + \int_\Lambda \left(\mathrm{d}A^*(t, \mathrm{d}\mathbf{x})[X(t),\, L_{\mathbf{x}}(t)] + [L^*_{\mathbf{x}}(t),\, X(t)]\mathrm{d}A(t, \mathrm{d}\mathbf{x})\right),$$

having the unique solution $X(t) = U_t X U^*_t$, where U^*_t, $t \in \mathbb{R}_+$ are the unitary operators defined by the QS equation (2.5), and

$$K(t) = U_t K_t U^*_t,\ \ K^*(t) = U_t K^*_t U^*_t,\ \ L_{\mathbf{x}}(t) = U_t L_{\mathbf{x},t} U^*_t,\ \ L^*_{\mathbf{x}}(t) = U_t L^*_{\mathbf{x},t} U^*_t.$$

The equation (3.1) can be obtained from (2.5) by using the QS Ito formula

$$\mathrm{d}(U_t X U^*_t) = \mathrm{d}U_t X U^*_t + U_t X \mathrm{d}U^*_t + \mathrm{d}U_t X \mathrm{d}U^*_t$$

and the Hudson–Parthasarathy multiplication table [15]

$$\begin{aligned} \mathrm{d}A^*(t, \mathrm{E})\mathrm{d}A(t, \mathrm{E}') &= 0,\ \ \mathrm{d}A(t, \mathrm{E}')\mathrm{d}A^*(t, \mathrm{E}) = \mathrm{d}t\lambda(\mathrm{E} \cap \mathrm{E}'), \\ \mathrm{d}A(t, \mathrm{E})\mathrm{d}A(t, \mathrm{E}') &= 0,\ \mathrm{d}A^*(t, \mathrm{E})\mathrm{d}A^*(t, \mathrm{E}') = 0,\ \forall \mathrm{E}, \mathrm{E}' \in \mathcal{A} \end{aligned}$$

The *a posterior dynamics* of the system under the observation (2.7) with a given initial state ϕ_0 is the dynamics $\phi_0 \mapsto \hat{\pi}_t$, $t \in \mathbb{R}_+$ of the a posterior state $\hat{\pi}_t$ on $\mathcal{L}$, giving posterior mean values $\hat{x}_t = \hat{\pi}_t\{X\}$ of $X \in \mathcal{L}$ as stochastic functions of the trajectories of the observed process $\boldsymbol{Y}^t = \{\boldsymbol{Y}(r)|r \leq t\}$. According to [4] the *a posterior state* is defined by the conditional expectation $\epsilon\{X\}(t) = \epsilon_t\{X(t)|\boldsymbol{Y}^t\}$ on the commutant $\mathcal{N}_t = \{\boldsymbol{Y}(r)|r \leq t\}'$ in $\mathcal{L}(\mathcal{H} \otimes \mathcal{F})$, which contains $\boldsymbol{Y}^t$ and $X(t)$ due to the nondemolition property (2.8). By Theorem 1 the operators $\epsilon\{X\}(t) \in \mathcal{N}'_t$ have in the Schrödinger picture the form

$$(3.2) \qquad U^*_t \epsilon\{X\}(t) U_t = U^*_t \epsilon_t\{U_t X U^*_t|\boldsymbol{Y}^t\} U_t = I \otimes \hat{\pi}_t\{X\},\ \ \forall X \in \mathcal{L}$$

since $U^*_t \mathcal{N}'_t U_t$ commutes with $\mathcal{L}(\mathcal{H}) \otimes I$. As a map $\hat{\pi}_t :\ \mathcal{L} \to \mathcal{M}_t$ into the Abelian algebra $\mathcal{M}_t = U^*_t \mathcal{N}'_t U_t \subset \mathcal{L}(\mathcal{F})$ generated by $\{\boldsymbol{Y}_r|r \leq t\}$ on $\mathcal{F}$, the a posterior state satisfies a nonlinear stochastic equation , obtained for the first time with respect to $\boldsymbol{Y}(t)$ as the quantum filtering equation in [4, 5]. Here we shall derive a linear quantum stochastic equation for a nonnormalized posterior state $\hat{\phi}_t\{X\} = \hat{\rho}_t\hat{\pi}_t\{X\}$, where $\hat{\rho}_t$ is a positive stochastic functional $\hat{\rho}_t = \hat{\rho}(\boldsymbol{Y}^t)$ of $\boldsymbol{Y}_r = \boldsymbol{Q}(r)$, $r \leq t$. Moreover, we shall prove that the stochastic normalization factor $\hat{\rho}_t$ can be taken as the probability density $\hat{\rho}(\boldsymbol{v}^t)$ of the trajectories $\boldsymbol{v}^t = \{\boldsymbol{v}(r)|r \leq t\}$ of the observed process $\boldsymbol{Y}^t$ with respect to the standard probability measure ν of a Wiener process $\boldsymbol{w}$, represented in the Fock space $\mathcal{F}$ as $\boldsymbol{Q}$ with respect to the vacuum state $|0\rangle \in \mathcal{F}$. Once the density operator $\hat{\rho}_t = \hat{\phi}_t\{I\}$ is found by the solution of the linear posterior evolution equation, the density function $\hat{\rho}(\boldsymbol{v}^t) = \rho^{\boldsymbol{w}}_t$ is given by the Segal (duality) transformation $\boldsymbol{Q} \mapsto \boldsymbol{w}$ of the observable process $\boldsymbol{Q}^t = U^*_t \boldsymbol{Y}^t U_t$ in the Schrödinger picture.

We shall say the nondemolition observation is *complete* for the quantum diffusion, described by the stochastic evolution equation (3.1), if the subsets $\mathrm{E} \in \mathcal{B}$ in (2.7) generate the σ-algebra $\mathcal{A}$. Let us see now in that case the posterior dynamics is not mixing: $\hat{\pi}_t = \hat{T}_t \phi_0 \hat{T}_t^*$, i. e. it is defined as $\phi_t^{\boldsymbol{w}}\{X\} = (\varphi_t^{\boldsymbol{w}} | X \varphi_t^{\boldsymbol{w}})$, for $\phi_0\{X\} = (\psi | X\psi)$ by a posterior stochastic propagator $T_t^{\boldsymbol{w}} : \psi \in \mathcal{H} \mapsto \varphi_t^{\boldsymbol{w}} = T_t^{\boldsymbol{w}} \psi$. We show the renormalized propagator $\hat{F}_t^{\boldsymbol{w}} = \sqrt{\rho_t^{\boldsymbol{w}}} T_t^{\boldsymbol{w}}$ also satisfies a linear stochastic wave equation $\hat{F} + K\hat{F}\mathrm{d}t = \boldsymbol{L}\mathrm{d}\boldsymbol{w}\hat{F}$ in $\mathcal{H}$, given in the Fock space representation by the operator evolution equation in $\mathcal{H} \otimes \mathcal{F}$,

$$\mathrm{d}\hat{F}_t + K_t \hat{F}_t \mathrm{d}t = \int_\Lambda L_{t,\mathbf{x}} \hat{F}_t \mathrm{d}Y_t(\mathrm{d}\mathbf{x}), \quad \hat{F}_0 = I, \tag{3.3}$$

where $\boldsymbol{L}_t \mathrm{d}\boldsymbol{Y}_t := \int_\Lambda L_{t,\mathbf{x}} \mathrm{d}Y_t(\mathrm{d}\mathbf{x}) = \boldsymbol{L}_t \mathrm{d}\boldsymbol{Q}(t)$, $(\hat{F}_t(\boldsymbol{w})\psi \mid \hat{F}_t(\boldsymbol{w})\psi) = \rho_t^{\boldsymbol{w}}$. The proof is given in Lemma 1 and Lemma 2 in terms of $\hat{\Phi}_t\{X\} = \hat{F}_t^* X \hat{F}_t$.

Firstly let us note that the wave propagator $\hat{F}_t : \mathcal{H} \to \mathcal{H} \otimes \mathcal{M}_t$ as any other adapted Wiener functional of Y is defined in the Fock representation $F^t = \hat{F}_t|0\rangle$ by the generating functional $F_g^t = \int_\Gamma F^t(\chi) \prod_{x\in\chi} g(x)\lambda(\mathrm{d}x)$ coinsiding with the Wick symbol $\langle f|\hat{F}_t|f\rangle = F_g^t$ for $g = f + f^*$, where $|f\rangle \in \mathcal{F}$ is the coherent state

$$|f\rangle(\chi) = e^{-\|f\|^2/2} \prod_{x\in\chi} f(x), \ \|f\|^2 = \int |f(x)|^2 \lambda(\mathrm{d}x)$$

for a $f : \Gamma_1 \to \mathbb{C}$ with $\|f\|^2 < \infty$, denoted as $f^2 = \|f\|^2$, if $f^* = f$. It helps to prove the

Lemma 1. *The solution $\hat{F}_t$ of the stochastic equation (3.3) satisfies the equivalency condition $\hat{F}_t|0\rangle = U_t^*|0\rangle$ respectively to the vacuum $|0\rangle \in \mathcal{F}$ with the unitary propagator U_t^* defined by the equation (2.5), i.e. $\hat{F}_t h = U_t^* h$ for all $h = \psi \otimes |0\rangle$, where $\psi \in \mathcal{H}$, $t \geq 0$.*

Proof. To this end we note that $\mathcal{A}$-measurability coincides with $\mathcal{B}$-measurability in this case and the equation for $F_t^* = U_t^*|0\rangle$ with $F_0^* = I$ can be simply obtained by allowing the right hand side of (\ref{eq:ccr1.5}) to act on the Fock vacuum $|0\rangle$. This gives

$$(\boldsymbol{L}_t \mathrm{d}\boldsymbol{A}^*(t) - \boldsymbol{L}_t^* \mathrm{d}\boldsymbol{A}(t))U_t^*|0\rangle = (\boldsymbol{L}_t \mathrm{d}\boldsymbol{A}^*(t) + \boldsymbol{L}_t \mathrm{d}\boldsymbol{A}(t))U_t^*|0\rangle = \boldsymbol{L}_t \mathrm{d}\mathbf{Y}_t F_t^*,$$

where $\boldsymbol{L}_t \mathrm{d}\boldsymbol{A}^*(t) = \int_\Lambda L_{t,\mathbf{x}} \mathrm{d}A^*(t, \mathrm{d}\mathbf{x})$ due to $\mathcal{B}$-measurability of the map $\boldsymbol{L}_t : \mathbf{x} \mapsto L_{t,\mathbf{x}}$ for almost all t, the commutativity of the increments $\mathrm{d}B_t(\mathrm{E}) = \mathrm{d}A(t, \mathrm{E})$ with U_t^* and with $\hat{F}_t$ and the annihilation property $\mathrm{d}B_t(\mathrm{E})|0\rangle = 0 = \mathrm{d}A(t,\mathrm{E})|0\rangle$ for all $\mathrm{E} \in \mathcal{A}$. The equation for the $\mathcal{L}(\mathcal{H})$-valued symbol $F_g^t = \langle g|\hat{F}_t|0\rangle e^{g^2/2}$ of the nonunitary classical stochastic evolution $\hat{F}_t$ defined by $(h_g|\hat{F}_t h_0') = (\psi|F_g^t \psi')$ for all $h_g = \psi \otimes e^{g^2/2}|g\rangle$, $h_0' = \psi' \otimes |0\rangle$, is given by

$$\frac{\mathrm{d}}{\mathrm{d}t} F_g^t + K_t F_g^t = \int_\Lambda L_{t,\mathbf{x}} F_g^t g(t,\mathbf{x}) \lambda(\mathrm{d}\mathbf{x}), \quad g = g^* \in L^2(\Gamma_1). \tag{3.4}$$

This coinsides with the equation for $\langle g|U_t^*|0\rangle e^{g^2/2} = F_{g,t}^*$ having the same form as (3.3) with $F_{g,t}^*$ instead of $\hat{F}_t$ and $\boldsymbol{G}_t = \int_0^t \boldsymbol{g}(r)\mathrm{d}r$, $\boldsymbol{g}(t) = \boldsymbol{f}(t) + \boldsymbol{f}^*(t)$ instead of $\boldsymbol{Y}_t = \boldsymbol{B}_t + \boldsymbol{B}_t^*$, $B_t(\mathrm{E}) = A(t, \mathrm{E})$, $\mathrm{E} \in \mathcal{B}$, and the initial operator $F_{g,0}^* = I$. It means that $F_{g,t}^* = F_t^g$ and $U_t^*|0\rangle = \hat{F}_t|0\rangle$ due to the uniqueness of the solution of the equation (3.3) proved in [7] under the conditions (2.6). $\square$

Secondly, let us find the QS Langevin equation for the process $X_g(t) = U_t X_g^t U_t^*$, $X_g^t = \hat{e}_g^t X$, where $g \in L^2_{\mathcal{B}}(\Gamma_1)$ is a $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}$–measurable square integrable function and $\hat{e}_g^t =: e^{q(g_t)} := \hat{e}_{g_t}$ is the Wick ordered exponential

$$\hat{e}_{g_t} =: \exp \int_0^t \int_\Lambda g(r, \mathbf{x}) \mathrm{d}Q(r, \mathrm{d}\mathbf{x}) := e^{a^*(g_t)} e^{a(g_t)}$$

of the observable $y_t(g) = \int_0^t \boldsymbol{g} \mathrm{d}\boldsymbol{Y} = q(g_t)$ in the Schrödinger picture, corresponding to the product $e_g(\chi) = \prod_{x \in \chi} g(x)$, $\chi \in \Gamma$ in the Fock representation of $e_g = \hat{e}_g|0\rangle$. Here and below $g_t \in L^2_{\mathcal{B}}(\Gamma_1)$ denotes the projection of g with $g_t(x) = 0$, if $x \notin [0,t) \times \mathrm{E}$ for every $\mathrm{E} \in \mathcal{B}$, otherwise $g_t = g$, and $a^*(g_t) = \int g_t(x) A^*(\mathrm{d}x) = a(g_t^*)^*$, $q(g_t) = (a + a^*)(g_t)$. Taking into account that this exponential is defined by the equation $\mathrm{d}\hat{e}_g^t = \hat{e}_g^t \boldsymbol{g}(t) \mathrm{d}\boldsymbol{Y}_t$ with $\hat{e}_0 = 1$, we can obtain for $G(t) = U_t X_g^t U_t^* = \hat{e}_g(t) X(t)$

$$\begin{aligned} \mathrm{d}G + (GK + K^*G)\mathrm{d}t &= \mathrm{d}t \int_\Lambda \{L_{\mathbf{x}}^* G L_{\mathbf{x}} + (L_{\mathbf{x}}^* G + G L_{\mathbf{x}}) g(\mathbf{x})\} \lambda(\mathrm{d}\mathbf{x}) \\ &\quad + \int_\Lambda \{[L_{\mathbf{x}}^*, G] \mathrm{d}B(\mathrm{d}\mathbf{x}) + \mathrm{d}B^*(\mathrm{d}\mathbf{x})[G, L_{\mathbf{x}}] \\ &\quad + g(\mathbf{x})(\mathrm{d}B(\mathrm{d}\mathbf{x}) + \mathrm{d}B^*(\mathrm{d}\mathbf{x}))G\} \end{aligned}$$

using the quantum Ito formula $\mathrm{d}(\hat{e}X) = \mathrm{d}\hat{e}X + \hat{e}\mathrm{d}X + \mathrm{d}\hat{e}\mathrm{d}X$ for $\hat{e}_g(t) = U_t \hat{e}_g^t U_t^*$ and (3.1). It helps to write the equation for the vacuum expectation operator

$$\Phi_g^t\{X\} = \langle 0|G(t)|0\rangle = F_t \hat{e}_g^t X F_t^*, \;\; \Phi_g^t\{I\} = \mathrm{P}_{g_t} := \Phi_{g_t}^s\{I\}, \; \forall s \geq t$$

as $\langle 0|\{\mathrm{d}G + (GK + K^*G)\mathrm{d}t\}(t)|0\rangle = \mathrm{d}t\langle 0|\{\boldsymbol{L}^* G \boldsymbol{L} + (\boldsymbol{L}^* G + G\boldsymbol{L})\boldsymbol{g}\}(t)|0\rangle$, or equivalently

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \; \Phi_g^t\{X\} + \Phi_g^t\{K_t^* X + X K_t\} \\ &= \int_\Lambda \Phi_g^t\{L_{t,\mathbf{x}}^* X L_{t,\mathbf{x}} + (X L_{t,\mathbf{x}} + L_{t,\mathbf{x}}^* X) g(t, \mathbf{x})\} \lambda(\mathrm{d}\mathbf{x}). \end{aligned} \tag{3.5}$$

The equation (3.5), with $\Phi_g^0\{X\} = X$, defines both the prior quantum Markovian dynamics [16] $\mathrm{M}^t : X \mapsto F_t X F_t^*$ as $\mathrm{M}^t = \Phi_0^t$ and an operator–valued generating functional $\mathrm{P}_g = F_\infty \hat{e}_g F_\infty^* = \lim_{t\to\infty} \Phi_g^t\{I\}$ of factorial (normal ordered) moment operators

$$\langle 0| : \dot{Y}(x_1) \ldots \dot{Y}(x_n) : |0\rangle = \delta^n \mathrm{P}_g / \delta g(x_1) \ldots \delta g(x_n) \mid_{g=0}$$

for the measurements at $t_m < t$, $\mathbf{x}_m \in \Lambda$, $m = 1, \ldots, n$ of generalized derivatives $\dot{Y}(x) = Y(\mathrm{d}x)/\lambda(\mathrm{d}x) \equiv \dot{Y}_{\mathbf{x}}(t)$ of the measure $\boldsymbol{Y}(\mathrm{d}x)$ on $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}$. It follows from the Weyl representation

$$\hat{e}_g^t(q) = \exp \int_0^t \{ \int_\Lambda g(r, \mathbf{x}) \mathrm{d}Y_r(\mathrm{d}\mathbf{x}) - \frac{1}{2} \; \int_\Lambda g(r, \mathbf{x})^2 \lambda(\mathrm{d}\mathbf{x}) \mathrm{d}r\} = e^{q(g_t) - g_t^2/2} \tag{3.6}$$

of the Wick exponent $\hat{e}_g^t =: e^{q(g_t)}:$, that equation (3.5) defines the characteristic operator $\Theta_g^t\{X\} = \langle 0|e^{iy(g_t)} X(t)|0\rangle$ of $y(g_t) = \int g_t(x) Y(\mathrm{d}x) = U_t q_t(g_t) U_t^*$:

$$\Theta_g^t\{X\} = F_t e^{iq_t(g_t)} X F_t^* = e^{-g_t^2/2} \Phi_{ig}^t\{X\}.$$

Let us denote by $\boldsymbol{v}_t = \{v_t(\mathrm{E})|\mathrm{E} \in \mathcal{B}\}$ a stochastic trajectory $v_t(\mathrm{E}) : \Omega \to \mathbb{R}$ of the process $\boldsymbol{Y}_t$ in the Wiener representation $v_t(\mathrm{E}) = Y_t(\mathrm{E}, \omega)$ and by $v_t(g) = \int_0^t \boldsymbol{g} \mathrm{d}\boldsymbol{v}$, the Wiener integral of $g(x)$, $x \in \Gamma_1$. Now we prove the absolute continuity $\mathrm{I}_t\{X\}(\mathrm{d}\omega) = \hat{\Phi}_t\{X\}(v_t)\mathrm{d}\mu(\omega)$ of the corresponding instrument $\mathrm{I}_t\{X\}(\mathrm{E})$, $\mathrm{E} \in \mathcal{B}$ with respect to the standard Wiener measure $\mathrm{d}\mu(\omega)$.

Lemma 2. *The solution of the equation (3.5) is given by the expectation*

$$\Phi_g^t\{X\} = \int_\Omega e^{\upsilon_t(g)-g_t^2/2}\hat{\Phi}_t\{X\}(\upsilon_t)\mathrm{d}\mu(\omega) \tag{3.7}$$

of a stochastic map $\Phi_t^\omega : X \mapsto \hat{\Phi}_t\{X\}(\upsilon_t)$ *as the nonanticipating function* $\Phi_t^\omega = \hat{\Phi}_t(\upsilon_t)$ *of* $\boldsymbol{v}_r$, $r < t$, *normalized by a stochastic operator –function* $\mathrm{P}_t^\omega = \hat{\mathrm{P}}(\upsilon_t)$ *and the factorial exponent (3.6) of the representation* $q(g_t) \mapsto \upsilon_t(g)$.

Proof. Let us take Ω as the spectrum of the commutative field measure $Q(\mathrm{d}x)$, denoted as $w(\mathrm{d}x)$ in the standard Wiener representation $\omega(f) = \int f(x)w(\mathrm{d}x)$, $f \in L^2(\Gamma_1)$ and μ as the Gaussian probability measure on Ω with the correlations

$$\int_\Omega w(\Delta)w(\Delta')\mathrm{d}\mu(\omega) = \lambda(\Delta \cap \Delta') = \langle 0|Q(\Delta)Q(\Delta')|0\rangle$$

induced by the Fock vacuum state. Then $\omega(g_t) = \upsilon_t(g)$ as $q(g_t) = y_t(g)$ for every $\mathcal{B}$-measurable $g : \Gamma_1 \to \mathbb{R}$, and due to $U_t^*|0\rangle = \hat{F}_t|0\rangle$ and the commutativity $\hat{e}_g^t\hat{F}_t = \hat{F}_t\hat{e}_g^t$ one can obtain

$$\begin{aligned}
\Phi_g^t\{X\} &= \langle 0|U_t\hat{e}_g^t X U_t^*|0\rangle = \langle 0|\hat{e}_g^t\hat{F}_t^* X\hat{F}_t|0\rangle \\
&= \int_\Omega \hat{e}_g^t(\omega)\hat{F}_t^*(\omega)X\hat{F}_t(\omega)\mathrm{d}\mu(\omega) \\
&= \int_\Omega e^{\upsilon_t(g)-g_t^2/2}\hat{F}_t^*(\omega)X\hat{F}_t(\omega)\mathrm{d}\mu(\omega).
\end{aligned}$$

Here $\hat{F}_t(\omega) = F_t^\omega$ is the solution $\hat{F}_t = F_t^q$ of the equation (3.3) as the functional of $Y_r(\mathrm{E}) = q(1_r(\mathrm{E}))$, $r < t$, $\mathrm{E} \in \mathcal{B}$ in the Wiener representation, where $1_r(\mathrm{E})$ is the indicator of $[0, r) \times \mathrm{E}$, and $\hat{e}_g^t(\omega) = e^{\upsilon_t(g)-g_t^2/2}$ is the Wick exponent (3.6). Due to arbitrariness of $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}$–measurable g, it defines the posterior map $\hat{\Phi}_t = \Phi_t(y_t)$ in (3.7) as the classical conditional expectation

$$\hat{\Phi}_t\{X\}(\upsilon_t) = \int_\Omega \hat{F}_t^*(\omega)X\hat{F}_t(\omega)\mathrm{d}\mu(\omega|\upsilon_t) \tag{3.8}$$

with respect to the σ–algebra on Ω, generated by the data $v_r(\mathrm{E}) = w([0, r) \times \mathrm{E})$, $r \in [0, t)$, $\mathrm{E} \in \mathcal{B}$. It is given by integrating on Ω with the Gaussian conditional measure $\mathrm{d}\mu(\omega|\upsilon_t) = \mathrm{d}\mu(\omega)/\mathrm{d}\mu(\upsilon_t)$, where $\mathrm{d}\mu(\upsilon_t)$ is the induced Gaussian probability measure on the trajectories $\boldsymbol{v}^t = \{\boldsymbol{v}(r)|r < t\} = \boldsymbol{w}^t|\mathcal{B}$ of the standard Wiener measure $\boldsymbol{w}(t, \mathrm{E}) = \boldsymbol{w}([0, t) \times \mathrm{E})$ on $\mathcal{B} \ni \mathrm{E}$. Hence the probability measure of the data $\boldsymbol{v}^t$ for the nondemolition observation (2.7) with a given initial wave function $\psi \in \mathcal{H}$ has the density

$$\rho_t^\omega = \int \|F_t(\omega)\psi\|^2\mathrm{d}\mu(\omega|\upsilon_t) = (\psi|\hat{\mathrm{P}}(\upsilon_t)\psi) \equiv \hat{\rho}(\upsilon_t),$$

where $\hat{\mathrm{P}}(\upsilon_t) = \hat{\Phi}_t\{I\}(\upsilon) = \mathrm{P}_t^\omega$. The non-Gaussian measure $\mathrm{d}\nu = \rho\mathrm{d}\mu$ defines the factorial generating functionals $\rho_g^t = \langle\hat{e}_g^t(y)\rangle$ for the process $\boldsymbol{Y}^t$ as $(\psi|\Phi_g^t\{I\}\psi)$ and the mean values $\langle X(t)\rangle$ of the operators $X(t)$ at the initial states $\psi \in \mathcal{H}$ as $(\psi|\Phi_t^{(0)}\{X\}\psi)$ by the averaging

$$(\psi|\Phi_g^t\{X\}\psi) = \int e^{\upsilon_t(g)-g_t^2/2}\hat{\pi}_t\{X\}(\upsilon_t)\mathrm{d}\nu(\upsilon_t) = \phi_g^t\{X\}$$

of the product $\hat{e}_g^t(\upsilon_t)\hat{\pi}_t\{X\}(\upsilon_t)$, where $\hat{\pi}_t\{X\}(\upsilon_t) = (\psi|\hat{\Phi}_t\{X\}(\upsilon_t)\psi)/\hat{\rho}(\upsilon_t)$, over all the observed in the past trajectories $\boldsymbol{\upsilon}^t$. □

Let us derive the corresponding linear stochastic equation for the non–normalized posterior map (3.8) $X \mapsto \hat{\Phi}_t\{X\}$ defining the posterior transformation $\phi_0 \mapsto \phi_0 \circ \hat{\Pi}$ for any initial ϕ_0 by $\hat{\Pi}_t\{X\} = \hat{\Phi}_t\{X\}/\hat{\rho}_t$, $\hat{\rho}_t = \phi_0\{\hat{\mathrm{P}}_t\}$. In the case of a complete nondemolition observation it can be obtained in the Schrödinger picture from (3.3) in the same way as (3.1) from (2.5) by using the Ito's formula for $\hat{F}_t^* X \hat{F}_t = \hat{\Phi}_t\{X\}$:

$$\begin{aligned} &\mathrm{d}(\hat{F}_t^* X \hat{F}_t) + \hat{F}_t(XK_t + K_t^* X - \int_\Lambda L_{t,\mathbf{x}}^* X L_{t,\mathbf{x}} \lambda(\mathrm{d}\mathbf{x}))\hat{F}_t \mathrm{d}t \\ = \quad &\int_\Lambda \hat{F}_t^*(XL_{t,\mathbf{x}} + L_{t,\mathbf{x}}^* X)\hat{F}_t \mathrm{d}Y_t(\mathrm{d}\mathbf{x}). \end{aligned}$$

In the general case the stochastic differential equation for (3.8) gives the following theorem.

Theorem 2. *The conditional expectation (3.8), defining in (3.7) the absolutely continuous operational measure $\hat{\Phi}_t\{X\}(\upsilon_t)\mathrm{d}\mu(\omega)$ with respect to the Wiener process $\upsilon_t(\omega)$, represented in Fock space by $\mathbf{Y}_t = \{\mathrm{Y}_t(\mathrm{E})|\mathrm{E} \in \mathcal{B}\}$, satisfies the linear stochastic equation*

$$\begin{aligned} &\mathrm{d}\hat{\Phi}_t\{X\} + \hat{\Phi}_t\{XK_t + K_t^* X - \int_\Lambda L_{t,\mathbf{x}}^* X L_{t,\mathbf{x}} \lambda(\mathrm{d}\mathbf{x})\}\mathrm{d}t \\ = \quad &\int_\Lambda \hat{\Phi}_t\{X\bar{L}_{t,\mathbf{x}} + \bar{L}_{t,\mathbf{x}}^* X\}\mathrm{d}Y_t(\mathrm{d}\mathbf{x}) \end{aligned} \tag{3.9}$$

corresponding to the equation (3.5) for the Wick symbol $\Phi_g^t\{X\} = \langle f|\hat{\Phi}_t|f\rangle$, $g = 2\Re\bar{f}_t$. Here $\bar{L}_{t,\mathbf{x}}$ are $\mathcal{B}$–measurable operator–valued functions of $\mathbf{x} \in \Lambda$, $\bar{L}_{t,\mathbf{x}} = 0$, if $\mathbf{x} \notin \mathrm{E}$ for any $\mathrm{E} \in \mathcal{B}$, defined for almost all t as a conditional averaging of $L_{t,\mathbf{x}}$ with respect to $\mathcal{B} \subseteq \mathcal{A}$ by

$$\int_\Lambda \bar{L}_{t,\mathbf{x}} g(\mathbf{x})\lambda(\mathrm{d}\mathbf{x}) = \int_\Lambda L_{t,\mathbf{x}} g(\mathbf{x})\lambda(\mathrm{d}\mathbf{x})$$

for any $\mathcal{B}$–measurable square–integrable $g : \Lambda \mapsto \mathbb{R}$ and $\bar{f}(t,\mathbf{x})$ is defined similary by the averaging of $f(t,\mathbf{x})$. In particular, $\bar{L}_{t,\mathbf{x}} = \frac{1}{\lambda(\mathrm{E}_i)}\int_{\mathrm{E}_i} L_{t,\mathbf{x}}\lambda(\mathrm{d}\mathbf{x})$ for all $\mathbf{x} \in \mathrm{E}_i$, if $\mathcal{B} = \{\mathrm{E}_i \in \mathcal{A}|i \in I\}$ is a σ–partition $\mathrm{M} = \sum_{i\in I}\mathrm{E}_i$ of $\mathrm{M} \subseteq \Lambda$ and $\lambda(\mathrm{E}_i) \neq 0$.

Proof. By the classical Ito's formula

$$\begin{aligned} \mathrm{d}(\hat{e}_g^t\hat{\Phi}_t\{X\}) &= \mathrm{d}\hat{e}_g^t\hat{\Phi}_t\{X\} + \hat{e}_g^t\mathrm{d}\hat{\Phi}_t\{X\} + \mathrm{d}\hat{e}_g^t\mathrm{d}\hat{\Phi}_t\{X\} \\ &= \int_\Lambda g(t,\mathbf{x})\hat{e}_g^t\hat{\Phi}_t\{X + X\bar{L}_{t,\mathbf{x}} + \bar{L}_{t,\mathbf{x}}^* X\}\mathrm{d}Y_t(\mathrm{d}\mathbf{x}) \\ &\quad -\hat{e}_g^t\hat{\Phi}_t\{XK_t + K_t^* X - L_{t,\mathbf{x}}^* X L_{t,\mathbf{x}}\}\mathrm{d}t \\ &\quad +\hat{e}_g^t\hat{\Phi}_t\left\{\int_\Lambda (X\bar{L}_{t,\mathbf{x}} + \bar{L}_{t,\mathbf{x}}^* X)g(t,\mathbf{x})\lambda(\mathrm{d}\mathbf{x})\right\}\mathrm{d}t \end{aligned}$$

we obtain from (3.9) equation (3.5) for $\Phi_g^t\{X\} = \langle 0|\hat{e}_g^t\hat{\Phi}_t\{X\}|0\rangle$, if we take into account that $\langle 0|\mathrm{d}Y_t(\mathrm{E})|0\rangle = 0$, $\forall \mathrm{E} \in \mathcal{B}$ and

$$\int_\Lambda (X\bar{L}_{t,\mathbf{x}} + \bar{L}_{t,\mathbf{x}}^* X)g(t,\mathbf{x})\lambda(\mathrm{d}\mathbf{x}) = \int_\Lambda (XL_{t,\mathbf{x}} + L_{t,\mathbf{x}}^* X)g(t,\mathbf{x})\lambda(\mathrm{d}\mathbf{x})$$

due to $\mathcal{B}$–measurability of $g(t,\cdot)$. Hence equation (3.9) describes the conditional mean value (3.8) in the Fock representation $\boldsymbol{w}^t \mapsto \boldsymbol{Q}^t$ with respect to the probability measure $\mathrm{d}\mu(\boldsymbol{w}^t)$ induced on $\mathcal{M}_t$ by the vacuum state:

$$\int \hat{e}^t_g(\boldsymbol{w}^t)\hat{\Phi}\{X\}(\boldsymbol{w}^t)\mathrm{d}\mu(\boldsymbol{w}^t) = \langle 0|\hat{e}^t_g\hat{\Phi}_t\{X\}|0\rangle. \qquad \square$$

Remark 2. In the case of the output momentum process, described in the Schrödinger picture by $Y_t(\mathrm{E}) = V(t,\mathrm{E})$, $\mathrm{E}\in\mathcal{B}$, one can obtain in the same way the posterior equation for the non-normalized linear stochastic map $\hat{\Phi}_t$ in the form

$$\begin{aligned} &\mathrm{d}\hat{\Phi}_t\{X\} + \hat{\Phi}_t\{XK_t + K_t^* X - \int_\Lambda L^*_{t,\mathbf{x}} X L_{t,\mathbf{x}} \lambda(\mathrm{d}\mathbf{x})\}\mathrm{d}t \\ = \ &\frac{1}{i}\int_\Lambda \hat{\Phi}_t\{X\bar{L}_{t,\mathbf{x}} - \bar{L}^*_{t,\mathbf{x}}X\}\mathrm{d}Y_t(\mathrm{d}\mathbf{x}). \end{aligned}$$

Then by doubling the space Λ and considering the time-continuous measurement of the commutative family $\mathbf{Y}_{t,\mp} = \boldsymbol{Q}_{\mp}(t)$ as in section **??**, one can obtain the posterior equation, corresponding to the complex observation $Z_t(\mathrm{E}) = W(t,\mathrm{E})$, $\mathrm{E}\in\mathcal{B}$ of $\boldsymbol{L}_t = \{L_{t,\mathbf{x}}|\mathbf{x}\in\Lambda\}$:

$$\begin{aligned} &\mathrm{d}\hat{\Phi}_t\{X\} + \hat{\Phi}_t\{XK_t + K_t^* X - \int_\Lambda L^*_{t,\mathbf{x}} X L_{t,\mathbf{x}} \lambda(\mathrm{d}\mathbf{x})\}\mathrm{d}t \\ = \ &\int_\Lambda \hat{\Phi}_t\{X\bar{L}_{t,\mathbf{x}}\}\mathrm{d}Z^*_t(\mathrm{d}\mathbf{x}) + \int_\Lambda \hat{\Phi}_t\{\bar{L}^*_{t,\mathbf{x}}X\}\mathrm{d}Z_t(\mathrm{d}\mathbf{x}). \end{aligned} \tag{3.10}$$

In the case $\mathcal{B}=\mathcal{A}$ of complete complex observation this equation has a factorizable solution $\hat{\Phi}_t\{X\} = \hat{F}^*_t X\hat{F}_t$, $\forall X\in\mathcal{L}$, where $\hat{F}_t$, satisfies the stochastic equation (3.3) in the complexified version

$$\mathrm{d}\hat{F}_t + K_t\hat{F}_t\mathrm{d}t = \int_\Lambda L_{t,\mathbf{x}}\hat{F}_t\mathrm{d}Z^*_t(\mathrm{d}\mathbf{x}),\ \boldsymbol{Z}_t = \frac{1}{\sqrt{2}}\ (\mathbf{Y}_{t,+} + i\mathbf{Y}_{t,-}).$$

4. A continual observation of CCR quasifree diffusion

Let Ξ be a symplectic $\sharp$-space, i.e. a complex space with involution

$$\xi\in\Xi\mapsto\xi^\sharp\in\Xi\ ,\xi^{\sharp\sharp}=\xi\ ,\quad (\sum\lambda_i\xi_i)^\sharp = \sum\lambda_i^*\xi_i^\sharp\ ,\forall\lambda_i\in\mathbb{C}$$

and skew-symmetric bilinear $\sharp$-form $s:\Xi\times\Xi\to\mathbb{C}$, such that $s(\xi^\sharp,\xi)$ is purely imaginary for all $\xi\in\Xi$:

$$s(\xi,\xi^\sharp) = -s(\xi^\sharp,\xi)\ ,s(\xi^\sharp\,\xi)^* = s(\xi,\xi^\sharp).$$

We denote by $\Re\Xi$ the real space of the vectors $\xi=\xi^\sharp$, by Θ a separating space of complex–valued linear functionals $\vartheta:\xi\mapsto\vartheta(\xi)$, on $\Re\Xi$ enquiped with the weak* topology, $\Im\Theta = \{\vartheta\in\Theta\mid\vartheta+\vartheta^\sharp=0\}$, where $\vartheta^\sharp(\xi)=\vartheta(\xi)^*$, $\forall\xi\in\Re\Xi$ and by $R(\xi),\xi\in\Xi$ an operator $\sharp$-representation $R(\xi)^* = R(\xi^\sharp)$ of the canonical commutation relations (CCR)

$$[R(\xi),R(\xi^\sharp)] = \frac{1}{i}s(\xi,\xi^\sharp),\ \forall\xi\in\Xi \tag{4.1}$$

in a Hilbert space $\mathcal{H}$ associated with a Gaussian state

$$\phi_0\{e^{iR(\xi)}\} = e^{i\vartheta_0(\xi)-\frac{1}{2}\xi^2} \equiv \phi_0(\xi). \tag{4.2}$$

Here $i\vartheta_0 \in \Im\Theta$ is defined by the expectation $\vartheta_0(\xi) = \phi_0\{R(\xi)\}$ of R and the quadratic form $\xi^2 = \langle \xi, \xi \rangle$, satisfying the Heisenberg inequality

$$\xi^2\eta^2 - \langle \xi, \eta \rangle^2 \geq \frac{1}{4}\ s(\xi, \eta)^2, \qquad \forall \xi, \eta \in \Re\Xi,$$

is defined by the symmetric covariance form

$$\langle \xi, \eta \rangle = \frac{1}{2}\ \phi_0\{R(\xi)R(\eta) + R(\eta)R(\xi)\} - \vartheta_0(\xi)\vartheta_0(\eta).$$

One can realise $R(\xi) - \vartheta_0(\xi)$ as double real part $2\Re A_0 = (A_0 + A_0^*)(\xi)$ of the creation operator $A_0^*(\xi) = A_0(\xi^\sharp)^*$ with the vacuum state $\phi_0\{X\} = (\psi_0|X\psi_0)$ in an initial Fock space $\mathcal{H} = \mathcal{F}_0$ over the Hilbert space $\mathrm{H} = \Xi^*$, associated with the scalar product

$$(\xi|\eta) = \langle \eta, \xi^\sharp \rangle + \frac{i}{2}\ s(\eta, \xi^\sharp), \qquad \forall \xi, \eta \in \Xi.$$

Indeed, the adjoint operators $A_0(\xi^\sharp)$, $A_0^*(\xi)$ satisfying the CCR

$$[A_0(\xi^\sharp), A_0^*(\xi)] = (\xi|\xi),$$

generate $\mathcal{F}_0$ by the unitary representation

$$X(\xi) = e^{i\vartheta_0(\xi) - \xi^2/2} e^{iA_0^*(\xi)} e^{iA_0(\xi)}\ , \xi \in \Re\Xi \tag{4.3}$$

of the Weyl operators $X(\xi) = \exp\{iR(\xi)\}$ on ψ_0:

$$\begin{aligned} X(\xi)\psi_0 &= \phi_0(\xi) e^{iA_0^*(\xi)}\psi_0\ , \\ X(\eta)X(\xi) &= e^{is(\eta,\xi)} X(\eta + \xi)\ , \end{aligned}$$

and $(\psi_0|X(\xi)\psi_0) = \phi_0(\xi)$.

We shall identify the dual space Θ with the completion of Ξ in the (nondegenerate) norm $|\xi| = \langle \xi^\sharp, \xi \rangle^{1/2} = \sqrt{(\Re\xi)^2 + (\Im\xi)^2}$ on Ξ, such that $\vartheta(\xi) = \langle \xi, \vartheta \rangle$. Denoting $\mathbf{j} : \xi \mapsto \mathbf{j}\xi = \xi$ the canonical bounded map $\Xi \to \mathrm{H}$,

$$\begin{aligned} \|\mathbf{j}\xi\|^2 = (\xi|\xi) = |\xi|^2 + \frac{i}{2}\ s(\xi, \xi^\sharp) = |\xi|^2 + s(\Re\xi, \Im\xi) \\ \leq |\xi|^2 + |s(\Re\xi, \Im\xi)| \leq |\xi|^2 + 2|\Re\xi||\Im\xi| \leq 2|\xi|^2, \end{aligned}$$

one can express the scalar product $(\xi|\eta)$ through the complex metric bounded operator $\mathbf{g} = \mathbf{j}^*\mathbf{j}$ as $(\xi|\eta) = \langle \xi^\sharp, \mathbf{g}\eta \rangle$. Here $\vartheta = \mathbf{g}\eta \in \Theta$, $\forall \eta \in \Xi$ is the complex functional $\vartheta(\xi) = (\xi^\sharp|\eta) = \langle \xi, \mathbf{g}\eta \rangle$ defining together with $\vartheta^\sharp(\xi) = \langle \xi^\sharp, \mathbf{g}\eta \rangle^* = (\eta|\xi)$ the $\sharp$-functional $2\Re\vartheta = \vartheta + \vartheta^\sharp = 2\Re\eta + \mathbf{s}\Im\eta$, where $\mathbf{s} : \Re\Xi \to \Theta$ is the skew-symmetric operator $\langle \xi, \mathbf{s}\eta \rangle = s(\eta, \xi)$, $|\mathbf{s}\eta| \leq 2|\eta|$ due to the Heisenberg inequality.

Let us consider the quantum diffusion of CCR algebra under the continuous measurement of the unbounded operators $L_x = R(\zeta_x)$, $x \in \mathbb{R}_+ \times \Lambda$, defined by a family $\{\zeta_x | x \in \mathbb{R}_+ \times \Lambda\}$ of vectors in Ξ, weakly square integrable: $\int_0^t \varepsilon_\tau(\vartheta^\sharp, \vartheta) \mathrm{d}\tau < \infty$ for all $t \in \mathbb{R}_+$ and $\vartheta \in \Theta$, where

$$\varepsilon_t(\vartheta^\sharp, \vartheta) = \int_\Lambda \vartheta^\sharp(\zeta_{t,\mathbf{x}}) \vartheta(\zeta_{t,\mathbf{x}}^\sharp) \lambda(\mathrm{d}\mathbf{x}) = \int_\Lambda |\langle \zeta_{t,\mathbf{x}}^\sharp, \vartheta \rangle|^2 \lambda(\mathrm{d}\mathbf{x}), \tag{4.4}$$

$\vartheta^\sharp(\zeta) = \vartheta^\sharp(\Re\zeta) + i\vartheta^\sharp(\Im\zeta) = \vartheta(\zeta^\sharp)^*$ for all $\zeta \in \Xi$. Moreover, we shall suppouse that the integral (4.4) is a weak* continuous function of $\vartheta \in \Theta$ such that

$$\int_\Lambda f(\mathbf{x}) \vartheta(\zeta_{t,\mathbf{x}}^\sharp) \lambda(\mathrm{d}\mathbf{x}) = \langle \boldsymbol{\zeta}_t^\sharp \boldsymbol{f}, \vartheta \rangle$$

for every square integrable function $\boldsymbol{f} : \Lambda \to \mathbb{C}$, where $\boldsymbol{\zeta}_t^\sharp \boldsymbol{f}$ is an element of Ξ denoted as $\int_\Lambda \zeta_{t,\mathbf{x}}^\sharp f(\mathbf{x})\lambda(\mathrm{d}\mathbf{x})$. We shall suppose also that the Hamiltonian H_t of the system under the observation is given in the Fock space $\mathcal{H}$ by the normal ordering $H_t =: h_t(R) :$ of a quadratic form $\upsilon_t(\vartheta) + \frac{1}{2}\ \omega_t(\vartheta, \vartheta) = h_t(\vartheta)$. This means

$$(\psi_\eta | H_t \psi_\eta) = \upsilon_t(\vartheta) + \frac{1}{2}\ \omega_t(\vartheta, \vartheta),\ \vartheta = \vartheta_0 + 2\Re(\mathbf{g}\eta),$$

where $\psi_\eta = \exp\{-\frac{1}{2}\ (\eta|\eta) + A_0^*(\eta)\}\psi_0,\ \psi_0 \in \mathcal{H}$ is the normalized vacuum: $A_0\psi_0 = 0$, $(\psi_0 \mid \psi_0) = 1$ in the initial space $\mathcal{H} = \mathcal{F}_0$.

Let us suppose that $\{\upsilon_t | t \in \mathbb{R}_+\}$ is a locally integrable family of $\sharp$–linear forms $\upsilon_t(\vartheta) = \langle \vartheta, \upsilon_t \rangle$ and $\{\omega_t | t \in \mathbb{R}_+\}$ is a locally integrable family of real symmetric bilinear forms on $\Im\Theta$ such that

$$|\upsilon|_1^t = \int_0^t |\upsilon_r| \mathrm{d}r < \infty,\ \ |\omega|_1^t = \int_0^t |\omega_r| \mathrm{d}r < \infty \quad \forall t < \infty,$$

where $|\upsilon_t| = \sqrt{\upsilon_t^2}$, $\upsilon_t^2 = \langle \upsilon_t, \upsilon_t \rangle$, $|\omega_t| = \sup\{\omega_t(\vartheta', \vartheta) \mid |\vartheta'| < 1, |\vartheta| < 1\}$. Assuming the weak* continuity of the linear functions $\upsilon_t(\vartheta)$ and $\omega_t(\vartheta', \vartheta)$ on $\Theta \ni \vartheta$, $\forall \vartheta' \in \Theta$, we identify $\upsilon_t(\vartheta)$ with $\vartheta(\upsilon_t)$, $\upsilon_t \in \Re\Xi$ and $\omega_t(\vartheta', \vartheta)$, with $\vartheta'(\boldsymbol{\omega}_t\vartheta) = \langle \vartheta', \boldsymbol{\omega}_t \vartheta \rangle$, where $\boldsymbol{\omega}_t$ is a symmetric and hence bounded operator on the Hilbert space Θ. The quadratic form of H_t, corresponding to

$$i[H_t, R(\xi)] = \upsilon_t(\mathbf{s}\xi) + R(\boldsymbol{\omega}_t \mathbf{s}\xi)\ ,\ \ \forall \xi \in \Xi$$

gives together with $i[R(\xi), L_{t,\mathbf{x}}] = s(\xi, \zeta_{t,\mathbf{x}})$, $i[L_{t,x}^*, R(\xi)] = s(\zeta_{t,\mathbf{x}}^\sharp, \xi)$ the linear quantum Langevin equation (3.1) for $X(t) = j(t, R(\xi))$:

$$\mathrm{d}R(t,\xi) + R(t, i\boldsymbol{\kappa}_t \mathbf{s}\xi)\mathrm{d}t = \mathrm{d}P(t,\xi) + \upsilon_t(\mathbf{s}\xi)\mathrm{d}t. \tag{4.5}$$

Here $\mathrm{d}P(t,\xi) = i \int_\Lambda \{s(\xi, \zeta_{t,\mathbf{x}}^\sharp)\mathrm{d}A(t, \mathrm{d}\mathbf{x}) - s(\xi, \zeta_{t,\mathbf{x}})\mathrm{d}A^*(t, \mathrm{d}\mathbf{x})\}$, $\boldsymbol{\kappa}_t : \Im\Theta \to \Re\Xi$ is the linear imaginary operator $\boldsymbol{\kappa}_t = \frac{1}{2}\ \boldsymbol{\gamma}_t + i\boldsymbol{\omega}_t$, where $\boldsymbol{\gamma}_t = \boldsymbol{\varepsilon}_t - \boldsymbol{\varepsilon}_t^\sharp$ is given by the weak* continuous function $\gamma_t(\vartheta', \vartheta) = 2i\Im\varepsilon_t(\vartheta', \vartheta)$ of $\vartheta, \vartheta' \in \Im\Theta$, $\vartheta'(\boldsymbol{\kappa}_t\vartheta) = \kappa_t(\vartheta', \vartheta) = \langle \vartheta', \boldsymbol{\kappa}_t \vartheta \rangle$,

$$\kappa_t(\vartheta', \vartheta) = i\Im\varepsilon_t(\vartheta', \vartheta) + i\omega_t(\vartheta', \vartheta),\ \forall \vartheta, \vartheta' \in \Im\Theta\ .$$

The following theorem gives the solution of the operator equation (4.5) together with an integral of a $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}$-measurable locally squareintegrable function $g : \Gamma_1 \to \mathbb{R}$ over the differential

$$\mathrm{d}Y(t, \mathrm{E}) = R(t, \zeta_t(\mathrm{E}) + \zeta_t^\sharp(\mathrm{E})) + \mathrm{d}Q(\mathrm{E}). \tag{4.6}$$

Here $\zeta_t(\mathrm{E}) \in \Xi$, $\zeta_t^\sharp(\mathrm{E}) = \int_\mathrm{E} \zeta_{t,\mathbf{x}}^\sharp \lambda(\mathrm{d}\mathbf{x}) = \zeta_t(\mathrm{E})^\sharp$ is defined for any E for which $\lambda(\mathrm{E}) < \infty$ due to weak* continuity on $\Theta \ni \vartheta$ of the integral $\int_\mathrm{E} \vartheta(\zeta_{t,\mathbf{x}})\lambda(\mathrm{d}\mathbf{x})$. Note, that the corresponding unitary quantum stochastic evolution (1.5) with unbounded operator

$$K_t = \frac{1}{2} \int_\Lambda R(\zeta_{t,\mathbf{x}}^\sharp) R(\zeta_{t,\mathbf{x}})\lambda(\mathrm{d}\mathbf{x}) + iH_t$$

exists only if $\langle \psi_0 | K_t | \psi_0 \rangle = \frac{1}{2}\ \int_\Lambda \|\zeta_{t,\mathbf{x}}\|^2 \lambda(\mathrm{d}\mathbf{x}) \equiv k_t(0) < \infty$ for almost all t. The Wick symbol $k_t(\vartheta) = (\psi_\eta | K_t \psi_\eta)$ is defined in this case as

$$k_t(\vartheta) = k_t(0) + i\upsilon_t(\vartheta) + \frac{1}{2}\ \left(\varepsilon_t(\vartheta, \vartheta) + i\omega_t(\vartheta, \vartheta)\right),\ \forall \vartheta = \vartheta_0 + 2\Re(\mathbf{g}\eta).$$

In this theorem we use the notations $|\cdot|_1^t$, $|\cdot|_2^t$ for the norms

$$|\boldsymbol{\kappa}|_1^t = \int_0^t |\boldsymbol{\kappa}_r|\mathrm{d}r, \quad |\xi|_2^t = \left(\int_0^t \int_\Lambda |\xi_{r,\mathbf{x}}|^2 \lambda(\mathrm{d}\mathbf{x})\mathrm{d}r\right)^{1/2},$$

where $|\xi| = \langle \xi, \xi\rangle^{1/2}$ for a $\xi \in \Re\Xi$, and $|\boldsymbol{\kappa}_t| = |i\boldsymbol{\kappa}_t|$ means the norm $|\boldsymbol{\kappa}_t| = \sup\{|\kappa_t\vartheta|/|\vartheta|\}$ of the real operator $i\boldsymbol{\kappa}_t$ on the Hilbert space Θ. Let us also denote $g_t(r) = g(r)$, $r < t$, $g_t(r) = 0$, $f_t(r, \mathbf{x}) = 0 = f_t^*(r, \mathbf{x})$, $\forall r \geq t$, and

$$f_t^*(r, \mathbf{x}, \xi) = g(r, \mathbf{x}) + is(\xi_r, \zeta_{r,\mathbf{x}}^\sharp) = f(r, \mathbf{x}, \xi^\sharp)^*, \ r < t.$$

Theorem 3. *Let the equations (4.5,4.6) for the quantum diffusion on the CCR algebra be defined by $\upsilon_t \in \Re\Xi$, $\boldsymbol{\omega}_t : \Theta \to \Re\Xi$ and $\zeta_x \in \Theta$ such that*

$$|\upsilon|_1^t < \infty, \ |i\boldsymbol{\kappa}|_1^t < \infty, \ |\bar{\zeta} + \bar{\zeta}^\sharp|_2^t < \infty, \ \forall t \in \mathbb{R}_+$$

where $x \mapsto \bar{\zeta}_x \in \Xi$ is a weakly $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}$-measurable function of $x = (t, \mathbf{x})$, defined by $\bar{\boldsymbol{\zeta}}_t \boldsymbol{g} = \boldsymbol{\zeta}_t \boldsymbol{g}$ for every $\boldsymbol{g} \in L^2_{\mathcal{B}}(\Lambda)$. Then the equation (4.5) has a unique solution, defined in the Hilbert space $\mathcal{H} \otimes \mathcal{F} = \mathcal{F}_0 \otimes L^2(\Gamma)$ by the quantum stochastic integral

$$R(t, \xi) + y(g_t) = \int_0^t s(\upsilon_r, \xi_r)\mathrm{d}r + R(\xi(t)) + a(f_t^*) + a^*(f_t) \tag{4.7}$$

along the trajectories $\xi_r = \varphi_r^{(g)}(t, \xi)$, $r \in [0, t)$ of the backward predual differential equation

$$-\dot{\xi}_r + i\boldsymbol{\kappa}_r \mathrm{s}\xi_r = \int_\Lambda g(r, \mathbf{x})(\bar{\zeta}_{r,\mathbf{x}} + \bar{\zeta}_{r,\mathbf{x}}^\sharp)\lambda(\mathrm{d}\mathbf{x}), \tag{4.8}$$

$$\xi_t = \xi \in \Xi, \ \xi(t) = \varphi_0^{(g)}(t, \xi) = \xi_0, \tag{4.9}$$

with $g = 0$ corresponding to $y(g_t) = 0$. The output integral

$$y(g_t) = \int_0^t \int_\Lambda g(t, \mathbf{x})\mathrm{d}Y(r, \mathrm{d}\mathbf{x})$$

of a $\mathcal{B}(\mathbb{R}_+) \times \mathcal{B}$-measurable function $g \in L^2_{\mathcal{B}}(\Gamma_1)$ over the differntial (4.6) is given also by the quantum stochastic integral (4.7) along the trajectory $\xi_r = \boldsymbol{\varphi}_r^{(g)}(t, 0)$ of the equation (4.8) with $\xi = 0$, corresponding to $R(t, \xi) = 0$.

Proof. First we write the weak solution of the equation (4.8) in the standard form

$$\xi_r = \boldsymbol{\varphi}_r(t)\xi + \int_r^t \int_\Lambda g(s, \mathbf{x})\boldsymbol{\varphi}_r(s)(\bar{\zeta}_{s,\mathbf{x}} + \bar{\zeta}_{s,\mathbf{x}}^\sharp)\mathrm{d}s\lambda(\mathrm{d}\mathbf{x}),$$

where $\boldsymbol{\varphi}_r(t)\xi = \varphi_r^{(0)}(t, \xi)$ is the solution of the equation (4.8) with $g = 0$. The resolving operator $\boldsymbol{\varphi}_r(t)$ exists as the chronologically ordered exponential

$$\boldsymbol{\varphi}_r(t) = \sum_{n=0}^\infty \int \dots \int_{r \leq t_1 < \dots < t_n < t} \boldsymbol{\kappa}_{t_1}\mathrm{s} \dots \boldsymbol{\kappa}_{t_n}\mathrm{s}\mathrm{d}t_1 \dots \mathrm{d}t_n$$

due to the estimate $|\boldsymbol{\varphi}_r(t)| = \sup\{|\boldsymbol{\varphi}_r(t)\xi| \mid |\xi| < 1\} \leq$

$$\leq \sum_{n=0}^\infty |\mathrm{s}|^n \int \dots \int_{0 \leq t_1 \dots t_n < t} |\boldsymbol{\kappa}_{t_1}| \dots |\boldsymbol{\kappa}_{t_n}|\mathrm{d}t_1 \dots \mathrm{d}t_n \leq \exp\{|2\boldsymbol{\omega} - i\boldsymbol{\gamma}|_1^t\}$$

because $|\mathbf{s}| \leq 2$. Hence one can obtain the existence of $\langle \xi_r, \vartheta \rangle$ for every $r \in [0, t)$ and $\vartheta \in \Theta$ due to the estimate

$$\begin{aligned} |\langle \xi_r, \vartheta \rangle| &\leq |\langle \boldsymbol{\varphi}_r(t)\xi, \vartheta \rangle| + \int_r^t \int_\Lambda |g(s, \mathbf{x}) \langle \boldsymbol{\varphi}_r(s) 2\Re \bar{\zeta}_{s,\mathbf{x}}, \vartheta \rangle| \mathrm{d}s \lambda(\mathrm{d}\mathbf{x}) \\ &\leq |\xi||\boldsymbol{\varphi}_r^\top(t)\vartheta| + |g|_2^t |\bar{\zeta} + \bar{\zeta}^\sharp|_2^t \Big(\int_r^t |\boldsymbol{\varphi}_r^\top(s)\vartheta|^2 \mathrm{d}s \Big)^{1/2} \\ &\leq \left(|\xi| + |g|_2^t |\bar{\zeta} + \bar{\zeta}^\sharp|_2^t \sqrt{t - r} \right) |\vartheta| \exp\{ |2\boldsymbol{\omega} - i\boldsymbol{\gamma}|_1^t \}. \end{aligned}$$

Now we integrate the left hand side in (4.7), taking into account (4.6) and (4.8):

$$\begin{aligned} & R(t, \xi) + \int_0^t \int_\Lambda g(r, \mathbf{x}) \left(R(r, 2\Re \zeta_r(\mathrm{d}\mathbf{x}))\mathrm{d}r + \mathrm{d}Q(r, \mathrm{d}\mathbf{x}) \right) \\ = \; & R(t, \xi) + \int_0^t \left(2\Re \left\{ \int_\Lambda g(r, \mathbf{x}) \mathrm{d}A(r, \mathrm{d}\mathbf{x}) \right\} - R(r, \dot{\xi}_r - i\boldsymbol{\kappa}_r \mathbf{s}\xi_r)\mathrm{d}r \right) \\ = \; & R(0, \xi_0) + \int_0^t (2\Re \left\{ \int_\Lambda g(r, \mathbf{x}) \mathrm{d}A(r, \mathrm{d}\mathbf{x}) \right\} + \mathrm{d}R(r, \xi_r) + R(r, i\boldsymbol{\kappa}_r \mathbf{s}\xi_r)\mathrm{d}r) \\ = \; & R(\varphi_0^{(g)}(t, \xi)) + \int_0^t (2\Re \left\{ \int_\Lambda (g(r, \mathbf{x}) + is(\xi_r, \zeta_{r,\mathbf{x}}^\sharp))\mathrm{d}A(r, \mathrm{d}\mathbf{x}) \right\} + \upsilon_r(\mathbf{s}\xi_r)\mathrm{d}r), \end{aligned}$$

where $\mathrm{d}R(r, \xi_r)$ is the quantum stochastic differential $\mathrm{d}R(r, \xi)|_{\xi = \xi_r}$, satisfying (4.5) for $t = r$. This proves Theorem (3). □

Note that the solution $R(t, \xi)$ of the equation (4.5) given by (4.7) for $g = 0$ preserves the CCR (4.1) and satisfies the nondemolition principle

$$[R(t, \xi), Y_g(t)] = 0 \ , \ \forall \xi \in \Xi, \ g \in L^2_{\mathcal{B}}(\Gamma_1), \tag{4.10}$$

where $Y_g(t) = \int_0^t \int_\Lambda g(r, \mathbf{x}) \mathrm{d}Y(r, \mathbf{x})$.

It can be proved by using the quantum Ito's formula and

$$\begin{aligned} [\mathrm{d}R(t, \xi), \mathrm{d}R(t, \xi^\sharp)] &= [\mathrm{d}P(t, \xi), \mathrm{d}P(t, \xi^\sharp)] = \gamma_t(\mathbf{s}\xi, \mathbf{s}\xi^\sharp)\mathrm{d}t, \\ [\mathrm{d}R(t, \xi), \mathrm{d}Y_g(t)] &= [\mathrm{d}P(t\xi), \mathrm{d}Q_g(t)] = is(\xi, (\boldsymbol{\zeta}_t + \boldsymbol{\zeta}_t^\sharp)\boldsymbol{g}(t))\mathrm{d}t. \end{aligned}$$

Indeed, if $[R(t, \xi), \ R(t, \xi^\sharp)] = \frac{1}{i} \ s(\xi, \xi^\sharp)$, then from (4.5) it follows

$$[\mathrm{d}R(t, \xi), R(t, \xi^\sharp)] = \kappa_t^\top(\mathbf{s}\xi, \mathbf{s}\xi^\sharp)\mathrm{d}t, \ [R(t, \xi), \mathrm{d}R(t\xi^\sharp)] = -\kappa_t(\mathbf{s}\xi, \mathbf{s}\xi^\sharp)\mathrm{d}t,$$

and $\mathrm{d}[R(t, \xi), R(t, \xi^\sharp)] = (\kappa_t^\top - \kappa_t + \gamma_t)(\mathbf{s}\xi, \mathbf{s}\xi^\sharp)\mathrm{d}t = 0$;

$$\begin{aligned} \mathrm{d}[R(t, \xi), Y_g(t)] &= [\mathrm{d}R(t, \xi), Y_g(t)] + [R(t, \xi), \mathrm{d}Y_g(t)] + [\mathrm{d}R(t, \xi), \mathrm{d}Y_g(t)] \\ &= [R(t, \xi), R(t, (\boldsymbol{\zeta}_t + \boldsymbol{\zeta}_t^\sharp)\boldsymbol{g}(t)]\mathrm{d}t + is(\xi, \boldsymbol{\zeta}_t + \boldsymbol{\zeta}_t^\sharp)\boldsymbol{g}(t))\mathrm{d}t \\ &= 0 \end{aligned}$$

if $[R(t, \xi), Y_g(t)] = 0$ and, hence, $[\mathrm{d}R(t, \xi), Y_g(t)] = 0$.

Remark 3. Let $\Xi = \Xi^- \oplus \Xi^+$ be an orthogonal decomposition of Ξ with respect to the complex scalar product $\langle \xi^\sharp, \eta \rangle$, such that

$$(\xi \oplus \eta)^\sharp = \eta^\sharp \oplus \xi^\sharp, \ \ s(\xi^\sharp, \eta) = 0 = s(\eta^\sharp, \xi), \ \forall \xi \in \Xi^-, \ \eta \in \Xi^+.$$

One can take $\Xi^\mp$ correspondingly as the negative and positive subspaces of the Hermitian form $2s(\Re\xi, \Im\xi) = is(\xi, \xi^\sharp)$:

$$c(\xi^\sharp, \xi) := is(\xi^\sharp, \xi) > 0, \ \ \forall \xi \in \Xi^-,$$

which is uniquely defined in the case of nondegeneracy: $s(\xi,\eta)=0,\ \forall\eta\in\Xi\Rightarrow\ \xi=0$, but it is not obligatory. If $\zeta_x\in\Xi^-$ for almost all $x\in\Gamma_1$, and $\omega(\xi^\sharp,\eta)=0$ for $\xi\in\Xi^-,\ \eta\in\Xi^+$, then the quantum Langevin equation (4.5) can be written in the complex linear form

$$\mathrm{d}L(t,\xi)+L(t,\boldsymbol{\kappa}_t\mathbf{c}\xi)\mathrm{d}t=\mathrm{d}A(t,\xi)+\eta_t(\mathbf{c}\xi)\mathrm{d}t, \tag{4.11}$$

where $\mathrm{d}A(t,\xi)=i\int_\Lambda s(\xi,\zeta^\sharp_{t,\mathbf{x}})\mathrm{d}A(t,\mathrm{d}\mathbf{x})$, $L(t,\xi)=R(t,\xi)$, $\mathbf{c}\xi=i\boldsymbol{s}\xi$, $\eta_t(\mathbf{c}\xi)=\upsilon_t(\mathbf{s}\xi)$, $\forall\xi\in\Xi^-$ and $L(t,\xi)=0$, $\mathrm{d}A(t,\xi)=0$, $\mathbf{c}\xi=0$, $\eta_t(\mathbf{c}\xi)=0$, $\forall\xi\in\Xi^+$. The equation (2.10) for a complex observation can be written in these terms as

$$\mathrm{d}Z(t,\mathrm{E})=L(\zeta(\mathrm{E}))\mathrm{d}t+\mathrm{d}W(t,\mathrm{E}),\quad \mathrm{E}\in\mathcal{B}.$$

5. A CCR quasi-free posterior dynamics and continuous collapse

The solution (4.7) obtained for the quasi-free diffusion (4.5) with the continuous observation (4.6) of CCR gives the possibility to solve easily the equation (3.5) at least for the initial Weyl operators $\Phi^0_g\{X\}=e^{iR(\xi)}=X(\xi)$. To this end let us represent the product $X(\xi)\otimes\hat{e}^t_g$ of the operator (4.3) on $\mathcal{H}=\mathcal{F}_0$ and the Wick exponent $\hat{e}^t_g=e^{q(g_t)-g_t^2/2}$ of the integral $y_t(g)=q(g_t)$ on $\mathcal{F}=L^2(\Gamma)$ as the exponent of an operator in Heisenberg picture :

$$G(t,\xi)=e^{b^*(g_t)}X(t,\xi)e^{b(g_t)}=e^{R(t,i\xi)+y(g_t)-g_t^2/2},$$

where $y(g_t)=\int_0^t\int_\Lambda g(r,\mathbf{x})\mathrm{d}Y(r,\mathrm{d}\mathbf{x})$. Due to (4.7) the exponent

$$R(t,i\xi)+y(g_t)=\left\{\int_0^t s(\upsilon_r,\varphi^{(g)}_r(t))\mathrm{d}r+R(\varphi^{(g)}_0(t))+a(f^*_t)+a^*(f_t)\right\}(i\xi)$$

can be written in normally ordered form with respect to

$$a^*(f(i\xi))=a^*(g-is(\varphi^{(g)}(t,i\xi),\zeta)),\ a(f^*(i\xi))=a(g-is(\zeta^\sharp,\varphi^{(g)}(t,i\xi)))$$

as

$$\begin{aligned} G(t,\xi) &= c_g(t)\exp\{R(\varphi^{(g)}_0(t))+a(f^*_t)+a^*(f_t)\}(i\xi)= \\ &= e^{I^t_g(i\xi)}e^{a^*(f_t(i\xi))}X(\frac{1}{i}\ \varphi^{(g)}_0(t,i\xi))e^{a(f^*_t(i\xi))}\ . \end{aligned}$$

Here $c_g(t)=\exp\{\int_0^t s(\upsilon_r,\xi_r\mathrm{d}r-g_t^2/2\}$, $g_t^2=\int_0^t\int_\Lambda g(r,\mathbf{x})^2\mathrm{d}r\lambda(\mathrm{d}\mathbf{x})$ and

$$I^t_g(\xi)=\ln c_g(t)+\frac{1}{2}\ |f_t|^2(\xi),\ |f_t|^2(\xi)=\int_0^t\int_\Lambda f^*(r,\mathbf{x},\xi)f(r,\mathbf{x},\xi)\mathrm{d}r\lambda(\mathrm{d}\mathbf{x})$$

is given by an integral over the trajectories $\xi_r=\varphi^{(g)}_r(t,\xi)$:

$$I^t_g(\xi)=\int_0^t\{s\left(\upsilon_r+\Im\bar{\boldsymbol{\zeta}}^\sharp_r\boldsymbol{g}(r),\ \xi_r\right)+\frac{1}{2}\ \varepsilon_r(\boldsymbol{s}\xi_r,\boldsymbol{s}\xi_r)\}\mathrm{d}r\ , \tag{5.1}$$

where $\bar{\boldsymbol{\zeta}}^\sharp_t\boldsymbol{g}(t)=\int_\Lambda\bar{\zeta}^\sharp_{t,\mathbf{x}}g(t,\mathbf{x})\lambda(\mathrm{d}\mathbf{x})=\boldsymbol{\zeta}^\sharp_t\boldsymbol{g}(t)$ for every $\mathcal{B}$-measurable function $\boldsymbol{g}(t):\mathbf{x}\in\Lambda\mapsto g(t,\mathbf{x})$. Hence the operator-function $\Phi^t_g(\xi)=\Phi^t_g\{X(\xi)\}$, being the vacuum expectation $\langle 0|G(t,\xi)|0\rangle$ is defined in $\mathcal{H}$ by

$$\Phi^t_g(\xi)=\exp\{I^t_g(i\xi)+R(\varphi^{(g)}_0(t,i\xi))\}, \tag{5.2}$$

since $e^{a(f_t^*)}|0\rangle = |0\rangle$ for every f, where $f_t^*(x) = f^*(x)$ on $x \in [0,t) \times \Lambda$ and $f_t(r, \mathbf{x}) = 0$, if $r > t$. One can easily verify that (5.2) satisfies the equation (3.5), written in the CCR quasi-free case for $X = X(\xi)$ in the differential form

$$\begin{aligned} & i\frac{\mathrm{d}}{\mathrm{d}t}\,\Phi_g^t(\xi) + \{s(\upsilon_r, \xi) - \langle \boldsymbol{\kappa}_t \mathbf{s}\xi, \partial\rangle + \frac{i}{2}\,\varepsilon_t(\mathbf{s}\xi, \mathbf{s}\xi)\}\Phi_g^t(\xi) \\ = \;& 2\Im\langle \bar{\boldsymbol{\zeta}}_t \boldsymbol{g}(t), i\partial + \frac{1}{2}\,\mathbf{s}\xi\rangle \Phi_g^t(\xi). \end{aligned} \tag{5.3}$$

This form of the main equation follows from the relations

$$[R(\zeta), X(\xi)] = s(\zeta, \xi)X(\xi),\; \frac{i}{2}\,(X(\xi)R(\zeta) + R(\zeta)X(\xi)) = \langle \zeta, \partial\rangle X(\xi)$$

defining the derivative $\langle \zeta, \partial\rangle$ of $X(\xi) = e^{iR(\xi)}$ and the right hand side in (3.5) by

$$R(\zeta)^* X(\xi) + X(\xi)R(\zeta) = \frac{2}{i}\Im\langle \zeta, i\partial + \frac{1}{2}\,\mathbf{s}\xi\rangle X(\xi),$$

and also from the definition of the quasi-free Hamiltonian evolution in terms of the Weyl operators (4.3):

$$[H, X(\xi)] = \{s(\upsilon_t, \xi) - \langle \boldsymbol{\omega}_t \mathbf{s}\xi, i\partial\rangle\}X(\xi)\;.$$

Thus we obtain the solution $\mathrm{M}^t(\xi) = \Phi_0^t(\xi)$ of the Lindblad equation for the CCR quasi-free case in term of the characteristic operator $\mathrm{M}^t\{X(\xi)\}$ of a prior dynamical map $\phi_0 \mapsto \phi_0 \mathrm{M}^t$, having the differential form (5.3) with zero right hand side , as well as the operator-valued generating functional $\mathrm{P}_g = \Phi_g^\infty(0)$ for the factorial moments of the observable process (4.6)

A posterior quasi-free dynamics of the CCR-algebra under the continual observation (4.6) is described by the characteristic a posteriori function $\hat{\Phi}_t(\xi) = \hat{\Phi}_t\{X(\xi)\}$ with the Wick symbol $\langle f|\hat{\Phi}_t(\xi)|f\rangle = \Phi_g^t(\xi),\; g = \bar{f} + \bar{f}^*$ of the Gaussian form (5.2). Hence the operator-valued function $\hat{\Phi}_t(\xi)$, normalised on the probability density operator $\hat{\mathrm{P}}_t = \hat{\Phi}_t(0)$, in the CCR quasi-free case can be represented as the normal ordered functional (5.4) of $y_t = b_t + b_t^*$ instead of $g = g_t$, where $\hat{\xi}_r = \xi_r(y)$, instead of $\xi_r = \varphi_r^{(g)}(t, \xi)$, defined by the solution $\xi_r(y) = \varphi_r^{(y)}(t, \xi)$ of the backward stochastic equation

$$-\mathrm{d}_- \hat{\xi}_r + i\boldsymbol{\kappa}_r s\hat{\xi}_r \mathrm{d}r = \int_\Lambda (\bar{\zeta}_{r,\mathbf{x}} + \bar{\zeta}^\sharp_{r,\mathbf{x}})\mathrm{d}Y_r(\mathrm{d}\mathbf{x}),\; \hat{\xi}_t = \xi. \tag{5.4}$$

In order to find the operator $\hat{\Phi}_t(\xi) = \Phi_t(\xi, y_t)$ in the form of a function $\Phi_t(\xi, \upsilon_t)$ of the trajectories $\upsilon_t(g) = \omega(g_t)$ of the observable process $y_t(g) = q(g_t)$, let us solve the equation (3.9) with $X = X(\xi)$, having in the quasi-free case the Wick symbol (5.3) in $\mathcal{H} = \mathcal{F}_0$. It can be done in terms of

$$\hat{\phi}_t^\vartheta(\xi) = (\psi_\eta|\hat{\Phi}_t(\xi)\psi_\eta),\;\; \vartheta = \vartheta_0 + 2\Re(\mathbf{g}\eta)\;,$$

by solving the linear stochastic differential equation, coresponding to (5.3)

$$\begin{aligned} & i\mathrm{d}\hat{\phi}_t(\xi) + \{s(\upsilon_t, \xi) - \langle \boldsymbol{\kappa}_t \mathbf{s}\xi, \partial\rangle + \frac{i}{2}\,\varepsilon_t(\mathbf{s}\xi, \mathbf{s}\xi)\}\hat{\phi}_t(\xi)\mathrm{d}t \\ = \;& \int_\Lambda 2\Im\langle \bar{\zeta}_{t,\mathbf{x}}, i\partial + \frac{\mathbf{s}}{2}\,\xi\rangle \hat{\phi}_t(\xi)\mathrm{d}Y_t(\mathrm{d}\mathbf{x})\;, \end{aligned} \tag{5.5}$$

as the equation for a posterior characteristic functuon $\hat{\phi}_t(\xi) = \phi^\vartheta\{\hat{\Phi}_t(\xi)\} \equiv \hat{\phi}_t^\vartheta(\xi)$ with a Gaussian $\hat{\phi}_0(\xi) = \exp\{i\vartheta(\xi) - \xi^2/2\} \equiv \phi^\vartheta(\xi)$. The stochastic function $\hat{\phi}_t^\vartheta(\xi)$

defines the operator-valued function $\hat{\Phi}_t(\xi)$ as the normal ordered form $:\hat{\phi}_t^R(\xi):$ of the initial operators $R-\vartheta_0 = A_0 + A_0^*$ in $\mathcal{H} = \mathcal{F}_0$.

Theorem 4. *Let the initial state ϕ_0 of the CCR (4.1) with a linear quantum stochastic evolution (4.5) have the Gaussian characteristic function (4.2). Then a posteriori nonnormalised state $\hat{\phi}_t(\xi) = \phi_0\{\hat{\Phi}_t(\xi)\} = \hat{\phi}_t\{X(\xi)\}$ under the continuous nondemolition observation (4.6) also has a Gaussian form*

$$\hat{\phi}_t(\xi) = \hat{\rho}_t \exp\{i\hat{\vartheta}_t(\xi) - \frac{1}{2}\ p_t(\xi,\xi)\}\ . \tag{5.6}$$

Here

$$\hat{\rho}_t = \exp\int_0^t \int_\Lambda \{\hat{\vartheta}_r(2\Re\bar{\zeta}_{r,\mathbf{x}})\mathrm{d}Y_r(\mathrm{d}\mathbf{x}) - \frac{1}{2}\ \hat{\vartheta}_r(2\Re\bar{\zeta}_{r,\mathbf{x}})^2 \mathrm{d}r\lambda(\mathrm{d}\mathbf{x})\}$$

is the probability density $\hat{\rho}_t = \hat{\phi}_t(0) = \rho(y_t)$ of the observation up to time t, $\hat{\vartheta}_t(\xi) = \langle \xi, \hat{\vartheta}_t\rangle$ is the linear stochastic functional $\hat{\vartheta}_t = \vartheta_t(y_t)$ of the posterior mean value of $R(t,\xi)$, satisfying the linear filtering equation

$$\mathrm{d}\hat{\vartheta}_t(\xi) + \hat{\vartheta}_t(i\boldsymbol{\kappa}_t\mathbf{s}\xi)\mathrm{d}t = \int_\Lambda 2\Re\langle\xi, \mathbf{k}_t\bar{\zeta}^\sharp_{t,\mathbf{x}}\rangle \mathrm{d}\tilde{Y}_t(\mathrm{d}\mathbf{x}) + \upsilon_t(\mathbf{s}\xi)\mathrm{d}t \tag{5.7}$$

with $\hat{\vartheta}_0 = \vartheta_0$, $\mathrm{d}\tilde{Y}_t(\mathrm{d}x) = \mathrm{d}Y_t(\mathrm{d}x) - \mathrm{d}t\int_\Lambda \hat{\vartheta}_t\left(2\Re\bar{\zeta}_{t,\mathbf{x}}\right)\lambda(\mathrm{d}\mathbf{x})$, $\mathbf{k}_t = \mathbf{p}_t + \frac{i}{2}\mathbf{s}$, and $p_t(\xi,\xi) = \langle\xi, \mathbf{p}_t\xi\rangle$ is the quadratic form of the posterior covariance of $R(t,\xi)$, satisfying the Riccati equation with $p_0(\xi,\xi) = \langle\xi,\xi\rangle$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\, p_t(\xi,\xi) + 2p_t(\xi, i\boldsymbol{\kappa}_t\mathbf{s}\xi) = \varepsilon_t(\mathbf{s}\xi,\mathbf{s}\xi) - \int_\Lambda |2\Re\langle\xi,\mathbf{k}_t\bar{\zeta}^\sharp_{t,\mathbf{x}}\rangle|^2\lambda(\mathrm{d}\mathbf{x}). \tag{5.8}$$

Proof. Let us find from (5.5) a stochastic equation for $\hat{\lambda}_t(i\xi) = \ln\hat{\phi}_t(\xi)$, using the Ito's formula $\mathrm{d}\hat{\lambda}_t(i\xi) = \hat{\phi}_t^{-1}\mathrm{d}\hat{\phi}_t - \frac{1}{2}\ (\mathrm{d}\hat{\lambda}_t(i\xi))^2$, and

$$\begin{aligned}(\mathrm{d}\hat{\lambda}_t(i\xi))^2 &= (\hat{\phi}_t^{-1}\mathrm{d}\hat{\phi}_t)^2 = \mathrm{d}t\int_\Lambda\{\langle 2\Re\bar{\zeta}_{t,\mathbf{x}}, \hat{\lambda}'_t(i\xi)\rangle + \langle i\Im\bar{\zeta}^\sharp_{t,\mathbf{x}},\mathbf{s}\xi\rangle\}^2\lambda(\mathrm{d}\mathbf{x})\\ &= \{\bar{\mu}_t(\hat{\lambda}'_t(i\xi),\hat{\lambda}'_t(i\xi)) + 2\bar{\kappa}_t(\hat{\lambda}'_t(i\xi),\mathbf{s}\xi) - \bar{\nu}(\mathbf{s}\xi,\mathbf{s}\xi)\}\mathrm{d}t\end{aligned}$$

due to $(\mathrm{d}Y_t(\mathrm{d}\mathbf{x}))^2 = \mathrm{d}t\lambda(\mathrm{d}\mathbf{x})$, where $\hat{\lambda}'_t(\xi) = \partial\hat{\lambda}_t(\xi)$,

$$\begin{aligned}\bar{\mu}_t(\vartheta,\vartheta) &= \int_\Lambda \langle 2\Re\bar{\zeta}_{t,x},\vartheta\rangle^2\lambda(\mathrm{d}\mathbf{x}),\ \bar{\nu}_t(\mathbf{s}\xi,\mathbf{s}\xi) = \int_\Lambda\langle\Im\bar{\zeta}_{t,\mathbf{x}},\mathbf{s}\xi\rangle^2\lambda(\mathrm{d}\mathbf{x}),\\ \bar{\kappa}_t(\vartheta,\mathbf{s}\xi) &= \int_\Lambda\langle 2\Re\bar{\zeta}_{t,\mathbf{x}},\vartheta\rangle\langle i\Im\bar{\zeta}^\sharp_{t,\mathbf{x}},\mathbf{s}\xi\rangle\lambda(\mathrm{d}\mathbf{x}) = \langle\bar{\boldsymbol{\kappa}}_t\mathbf{s}\xi,\vartheta\rangle.\end{aligned}$$

It gives a quasilinear stochastic equation of the first order for $\hat{\lambda}_t(\xi)$:

$$\begin{aligned}&\mathrm{d}\hat{\lambda}_t + \{s(\xi,\upsilon_t) + \langle i\tilde{\boldsymbol{\kappa}}_t\mathbf{s}\xi,\hat{\lambda}'_t\rangle - \frac{1}{2}\ \tilde{\varepsilon}_t(\mathbf{s}\xi,\mathbf{s}\xi)\}\mathrm{d}t\\ =\ &\int_\Lambda\{\langle 2\Re\bar{\zeta}_{t,\mathbf{x}},\hat{\lambda}'_t\rangle + \langle\Im\bar{\zeta}^\sharp_{t,\mathbf{x}},\mathbf{s}\xi\rangle\}\mathrm{d}Y_t(\mathrm{d}\mathbf{x}) - \frac{1}{2}\ \bar{\mu}_t(\hat{\lambda}'_t,\hat{\lambda}'_t)\mathrm{d}t\ ,\end{aligned}$$

where $\tilde{\boldsymbol{\kappa}}_t = \boldsymbol{\kappa}_t - \bar{\boldsymbol{\kappa}}_t$ and $\tilde{\boldsymbol{\varepsilon}}_t = \boldsymbol{\varepsilon}_t - \bar{\boldsymbol{\varepsilon}}_t$. This equation has a quadratic form solution

$$\hat{\lambda}_t(\xi) = \ln\hat{\rho}_t + \hat{\vartheta}_t(\xi) + \frac{1}{2}\ p_t(\xi,\xi),\ \hat{\lambda}'_t(\xi) = \hat{\vartheta}_t + \mathbf{p}_t\xi,\ \hat{\lambda}''_t = \mathbf{p}_t,$$

where

$$
\begin{aligned}
\mathrm{d}\ln\hat{\rho}_t &= \mathrm{d}\hat{\lambda}_t(0) = \int_\Lambda \langle 2\Re\bar{\zeta}_{t,\mathbf{x}}, \hat{\vartheta}_t\rangle \mathrm{d}Y_t(\mathrm{d}\mathbf{x}) - \frac{1}{2}\,\bar{\mu}_t(\hat{\vartheta}_t,\hat{\vartheta}_t)\mathrm{d}t,\\
\mathrm{d}\hat{\vartheta}_t &= \mathrm{d}\hat{\lambda}_t'(0) = \int_\Lambda (2\mathbf{p}_t\Re\bar{\zeta}_{t,\mathbf{x}} + \mathbf{s}\Im\bar{\zeta}_{t,\mathbf{x}})\mathrm{d}Y_t(\mathrm{d}\mathbf{x}) - \{(\mathbf{p}_t\bar{\boldsymbol{\mu}}_t - i\mathbf{s}\boldsymbol{\kappa}_t^\top)\hat{\vartheta}_t - \mathbf{s}\upsilon_t\}\mathrm{d}t,\\
\mathrm{d}\mathbf{p}_t &= \mathrm{d}\hat{\lambda}_t'' = -\{\mathbf{p}_t\bar{\boldsymbol{\mu}}_t\mathbf{p}_t + \mathbf{s}\tilde{\boldsymbol{\varepsilon}}_t\mathbf{s} + i(\mathbf{p}_t\tilde{\boldsymbol{\kappa}}_t\mathbf{s} - \mathbf{s}\tilde{\boldsymbol{\kappa}}_t^\top\mathbf{p}_t)\}\mathrm{d}t\,,
\end{aligned}
$$

where $\tilde{\kappa}^\top(\mathbf{s}\xi,\mathbf{p}\xi) = \tilde{\kappa}(\mathbf{p}\xi,\mathbf{s}\xi)$. Using the integral form of the symmetric $*$-weakly continuous operators $\bar{\boldsymbol{\mu}}_t,\ \bar{\boldsymbol{\nu}}_t : \Theta \to \Xi$, one can obtain the stochastic integral representation of $\ln\hat{\rho}_t = \int_0^t \mathrm{d}\hat{\lambda}_t(0)$ in Theorem 4 as well as the equation (5.7) and (5.8) for $\hat{\vartheta}_t(\xi) = \langle\xi,\hat{\vartheta}_t\rangle$ and $p_t(\xi,\xi) = \langle\xi,\mathbf{p}_t\xi\rangle$ with $\mathbf{k}_t = \mathbf{p}_t + \frac{i}{2}\,\mathbf{s}$ due to $\mathbf{s}^\top = -\mathbf{s}$,

$$
\begin{aligned}
\bar{\mu}_t(\mathbf{p}_t\xi,\mathbf{p}_t\xi) - 2i\bar{\kappa}_t(\mathbf{p}_t\xi,\mathbf{s}\xi) + \bar{\nu}_t(\mathbf{s}\xi,\mathbf{s}\xi) &= \int_\Lambda (2\Re\langle\xi,(\mathbf{p}_t + \frac{i}{2}\,\mathbf{s})\bar{\zeta}^\sharp_{t,\mathbf{x}}\rangle)^2\lambda(\mathrm{d}\mathbf{x});\\
2\Re\langle\xi,(\mathbf{p}_t + \frac{i}{2}\,\mathbf{s})\bar{\zeta}^\sharp_{t,\mathbf{x}}\rangle &= \langle\xi, 2\mathbf{p}_t\Re\bar{\zeta}_{t,\mathbf{x}} + \mathbf{s}\Im\bar{\zeta}_{t,\mathbf{x}}\rangle,\ \xi\in\Re\Xi.
\end{aligned}
$$

Let's point out that quantum filtering equations (5.7), (5.8), represented in the short form

$$
\begin{aligned}
\mathrm{d}\hat{\vartheta}_t(\xi) + \hat{\vartheta}_t(\boldsymbol{\alpha}_t\xi)\mathrm{d}t &= \int_\Lambda 2\Re\langle\xi,\mathbf{k}_t\bar{\zeta}^\sharp_{t,\mathbf{x}}\rangle\mathrm{d}Y_t(\mathrm{d}\mathbf{x}) + \upsilon_t(\mathbf{s}\xi)\mathrm{d}t,\\
\frac{\mathrm{d}}{\mathrm{d}t}\,p_t(\xi,\xi) + 2p_t(\xi,\boldsymbol{\alpha}_t\xi) &= \tilde{\varepsilon}_t(\mathbf{s}\xi,\mathbf{s}\xi) + \bar{\mu}_t(\mathbf{p}_t\xi,\mathbf{p}_t\xi),\ \ \mathbf{p}_0 = \mathbf{1},
\end{aligned}
\tag{5.9}
$$

where $\boldsymbol{\alpha}_t = \bar{\boldsymbol{\mu}}_t\mathbf{p}_t + i\tilde{\boldsymbol{\kappa}}_t\mathbf{s}$, give $\hat{\lambda}_t(\xi) = \ln\hat{\phi}_t(\frac{1}{i}\,\xi)$ in the form of the integral

$$
\hat{\lambda}_t(\xi) = \vartheta_0(\hat{\xi}_0) + \frac{1}{2}\,\hat{\xi}_0^2 + \int_0^t \{\langle\mathbf{s}\hat{\xi},\mathrm{d}\Upsilon_r\rangle + \frac{1}{2}\left(\tilde{\varepsilon}_r(\mathbf{s}\hat{\xi},\mathbf{s}\hat{\xi}) - \bar{\mu}_r(\hat{p}_r(\hat{\xi}),\hat{p}_r(\hat{\xi}))\right)\mathrm{d}r\}
\tag{5.10}
$$

over the stochastic trajectories $\hat{\xi}_r = \hat{\varphi}_r(t,\xi)$ of the backward linear equation (5.4) with $\hat{p}_r(\hat{\xi}) = \hat{\vartheta}_r + \mathbf{p}_r\hat{\xi}_r,\ \hat{\xi}_0 = \hat{\varphi}_0(t,\xi)$ and

$$
\mathrm{d}\Upsilon_t = \upsilon_t\mathrm{d}t + \Im\int_\Lambda \bar{\zeta}^\sharp_{t,\mathbf{x}}\mathrm{d}Y_t(\mathrm{d}\mathbf{x}).
$$

Indeed, if $\mathrm{d}_-\hat{\xi}_r = \hat{\xi}_r - \hat{\xi}_{r-\mathrm{d}r}$ is the backward stochastic differential in (5.4), then $\mathrm{d}\langle\hat{\vartheta}_r,\hat{\xi}_r\rangle = \langle\hat{\xi}_r,\mathrm{d}\hat{\vartheta}_r\rangle + \langle\hat{\vartheta}_r,\mathrm{d}_-\hat{\xi}_r\rangle$ and

$$
\mathrm{d}\Big(p_r(\hat{\xi}_r,\hat{\xi}_r)\Big) = 2p_r(\hat{\xi}_r,\mathrm{d}_-\hat{\xi}_r) + \dot{p}_r(\hat{\xi}_r,\hat{\xi}_r)\mathrm{d}r.
$$

Using the equations (5.9) and writting the equation (5.4) in the form $\mathrm{d}_-\hat{\xi}_r = \boldsymbol{\alpha}_r\hat{\xi}_r\mathrm{d}r - 2\Re\bar{\boldsymbol{\zeta}}_r\mathrm{d}\hat{\mathbf{Y}}_r$ with respect to

$$
\mathrm{d}\hat{Y}_r(\mathrm{d}\mathbf{x}) = \mathrm{d}Y_r(\mathrm{d}\mathbf{x}) + \langle\mathbf{p}_r\hat{\xi}_r, 2\Re\bar{\zeta}_{r,\mathbf{x}}\rangle\mathrm{d}r\lambda(\mathrm{d}\mathbf{x})
$$

one can obtain by integrating by parts of the difference $\hat{\lambda}_t(\xi) - \ln\hat{\rho}_t - \vartheta_0(\hat{\xi}_0) - \frac{1}{2}\,\hat{\xi}_0^2$:

$$\begin{aligned}
&\hat{\vartheta}_t(\xi) - \vartheta_0(\hat{\xi}_0) + \frac{1}{2}\,p_t(\xi,\xi) - \frac{1}{2}\,\hat{\xi}_0^2 \\
= &\int_0^t \{\langle\hat{\xi}, \mathrm{d}\hat{\vartheta} + \frac{1}{2}\,\dot{\mathbf{p}}\hat{\xi}\mathrm{d}r\rangle + \langle \mathrm{d}_-\hat{\xi}, \hat{p}(\hat{\xi})\rangle\} \\
= &\int_0^t \{\langle \mathbf{s}\hat{\xi}, \mathrm{d}\Upsilon\rangle - \langle 2\Re\bar{\boldsymbol{\zeta}}\mathrm{d}\hat{\boldsymbol{Y}}, \hat{\vartheta}\rangle + (\langle\boldsymbol{\alpha}\hat{\xi}, \mathbf{p}\hat{\xi}\rangle + \frac{1}{2}\,\dot{p}(\hat{\xi},\hat{\xi}) - \bar{\mu}(\mathbf{p}\hat{\xi}, \mathbf{p}\hat{\xi}))\mathrm{d}r\} \\
= &\int_0^t \{\langle \mathbf{s}\hat{\xi}, \mathrm{d}\Upsilon\rangle - \hat{\vartheta}(2\Re\bar{\boldsymbol{\zeta}})\mathrm{d}\boldsymbol{Y} + \frac{1}{2}\,\Big(\tilde{\varepsilon}(\mathbf{s}\hat{\xi}, \mathbf{s}\hat{\xi}) + \bar{\mu}(\hat{\vartheta},\hat{\vartheta}) - \bar{\mu}(\hat{p}(\hat{\xi}), \hat{p}(\hat{\xi}))\Big)\,\mathrm{d}r\} \\
= &\int_0^t \left\{\langle \mathbf{s}\hat{\xi}, \mathrm{d}\Upsilon\rangle + \frac{1}{2}\,\Big(\tilde{\varepsilon}(\mathbf{s}\hat{\xi}, \mathbf{s}\hat{\xi}) - \bar{\mu}(\hat{p}(\hat{\xi}), \hat{p}(\hat{\xi}))\Big)\,\mathrm{d}r\right\} - \ln\hat{\rho}_t,
\end{aligned}$$

which gives (5.10) with $\ln\hat{\rho}_t = \int_0^t(\hat{\vartheta}(2\Re\bar{\boldsymbol{\zeta}})\mathrm{d}\boldsymbol{Y} - \frac{1}{2}\,\bar{\mu}(\hat{\vartheta},\hat{\vartheta})\mathrm{d}r)$.

Remark 4. Let us consider the case of the complex observation (2.10). Then the stochastic differential equation

$$\begin{aligned}
&i\mathrm{d}\hat{\phi}_t(\xi) + \{s(\upsilon_t,\xi) - \langle\boldsymbol{\kappa}_t\mathbf{s}\xi, \partial\rangle + \frac{i}{2}\,\varepsilon_t(\mathbf{s}\xi,\mathbf{s}\xi)\}\hat{\phi}_t(\xi)\}\mathrm{d}t \\
= \;&2\Re\int_\Lambda\langle\bar{\zeta}^\sharp_{t,\mathbf{x}}, \partial\hat{\phi}_t(\xi)\rangle\mathrm{d}Z_t(\mathrm{d}\mathbf{x}) + 2\Im\int_\Lambda s(\xi,\bar{\zeta}^\sharp_{t,\mathbf{x}})\hat{\phi}_t(\xi)\mathrm{d}Z_t(\mathrm{d}\mathbf{x}),
\end{aligned}$$

corresponding to (3.10), has the Gaussian solution (5.6) defined by the density

$$\hat{\rho}_t = \exp\int_0^t\{2\Re\int_\Lambda\hat{\vartheta}_r(\bar{\zeta}^\sharp_{r,\mathbf{x}})\mathrm{d}Z_r(\mathrm{d}\mathbf{x}) - \bar{\varepsilon}_r(\hat{\vartheta}_r,\hat{\vartheta}_r)\mathrm{d}r\},$$

where $\bar{\varepsilon}_t(\vartheta,\vartheta) = \int_\Lambda|\vartheta(\bar{\zeta}^\sharp_{t,\mathbf{x}})|^2\lambda(\mathrm{d}\mathbf{x})$, and by the filtering equations

$$\mathrm{d}\hat{\vartheta}_t(\xi) + \hat{\vartheta}_t(i\boldsymbol{\kappa}_t\,\mathbf{s}\xi)\mathrm{d}t = 2\Re\int_\Lambda\langle\xi, \mathbf{k}_t\bar{\zeta}^\sharp_{t,\mathbf{x}}\rangle\mathrm{d}\tilde{Z}_t(\mathrm{d}\mathbf{x}) + \upsilon_t(\mathbf{s}\xi)\mathrm{d}t,$$

where $\mathrm{d}\tilde{Z}_t(\mathrm{E}) = \mathrm{d}Z_t(\mathrm{E}) - \mathrm{d}t\int_\mathrm{E}\hat{\vartheta}_t(\bar{\zeta}_{t,\mathbf{x}})\lambda(\mathrm{d}\mathbf{x})$. The corresponding complex form of Riccati equation is

$$\frac{\mathrm{d}}{\mathrm{d}t}\,p_t(\xi,\xi) + 2p_t(\xi, i\boldsymbol{\kappa}_t\mathbf{s}\xi) = \varepsilon_t(\mathbf{s}\xi,\mathbf{s}\xi) - 2\bar{\varepsilon}_t(\mathbf{k}_t\xi, \mathbf{k}^\sharp_t\xi),$$

where $\mathbf{k}_t = \mathbf{p}_t + \frac{i}{2}\,\mathbf{s}$, $\mathbf{k}^\sharp_t = \mathbf{p}_t - \frac{i}{2}\,\mathbf{s}$.

In the case of the complex Langevin equation (4.11), corresponding to $i\boldsymbol{\kappa}_t\mathbf{s} = \boldsymbol{\kappa}_t\mathbf{c}$ on the invariant subspace Ξ^-, the posterior quasi-free dynamics with complex observation can be described in terms of the complex parameters $\boldsymbol{\kappa}_t = \frac{1}{2}\,\boldsymbol{\varepsilon}_t + i\boldsymbol{\omega}_t$, $\boldsymbol{\kappa}^\dagger_t = \frac{1}{2}\,\boldsymbol{\varepsilon}_t - i\boldsymbol{\omega}_t$,

$$\hat{\chi}_t = \hat{\vartheta}_t|\Xi^-,\ \mathbf{l}_t = \mathbf{k}^\sharp_t|\Xi^-;\ \ \hat{\chi}_t(\xi) = 0,\ \mathbf{l}_t\xi = 0,\ \ \forall\xi\in\Xi^+.$$

In this case $\hat{\phi}_t(\xi\oplus\xi^\sharp) = \hat{\rho}_t\exp\{i(\hat{\chi}^\sharp_t(\xi) + \hat{\chi}_t(\xi^\sharp)) - \langle\xi^\sharp, \mathbf{p}_t\xi\rangle\}$,

$$\begin{aligned}
\hat{\rho}_t &= \exp\left\{\int_0^t\{2\Re\int_\Lambda\hat{\chi}_r(\bar{\zeta}^\sharp_{r,\mathbf{x}})\mathrm{d}Z_r(\mathrm{d}\mathbf{x}) - \bar{\varepsilon}_r(\hat{\chi}^\sharp_r,\hat{\chi}_r)\}\mathrm{d}r\right\}, \\
\mathrm{d}\hat{\chi}_t(\xi) + \hat{\chi}_t(\boldsymbol{\kappa}_t\mathbf{c}\xi)\mathrm{d}t &= \int_\Lambda\langle\mathbf{l}_t\xi, \bar{\zeta}^\sharp_{t,\mathbf{x}}\rangle\mathrm{d}\tilde{Z}_t(\mathrm{d}\mathbf{x}) + \eta_t(\mathbf{c}\xi)\mathrm{d}t, \\
\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbf{p}_t + \mathbf{p}_t\boldsymbol{\kappa}_t\mathbf{c} + \mathbf{c}\boldsymbol{\kappa}^\dagger_t\mathbf{p}_t &= \frac{1}{2}\,\mathbf{c}\boldsymbol{\varepsilon}_t\mathbf{c} - (\mathbf{p}_t - \frac{1}{2}\,\mathbf{c})\bar{\varepsilon}_t(\mathbf{p}_t - \frac{1}{2}\,\mathbf{c}),
\end{aligned}$$

where $\bar{\boldsymbol{\varepsilon}}_t = \bar{\boldsymbol{\kappa}}_t + \bar{\boldsymbol{\kappa}}_t^\dagger$ on Ξ^-, $\mathbf{p}_t = \mathbf{l}_t + \frac{1}{2}\ \mathbf{c}$ and

$$\mathrm{d}\tilde{Z}_t(\mathrm{E}) = \mathrm{d}Z_t(\mathrm{E}) - \mathrm{d}t \int_{\mathrm{E}} \hat{\chi}_t(\bar{\zeta}_{t,\mathbf{x}})\lambda(\mathrm{d}\mathbf{x}).$$

Example. Let us consider the stationary case $\boldsymbol{\varepsilon}_t = \boldsymbol{\varepsilon}$, $\boldsymbol{\omega}_t = \boldsymbol{\omega}$ and a complete observation when $\mathcal{A} = \mathcal{B}$, $\bar{\boldsymbol{\varepsilon}}_t = \boldsymbol{\varepsilon}$ is invertible and $\boldsymbol{\omega}$ satisfies the $\mathbf{c}$-commutativity condition $\boldsymbol{\omega}\mathbf{c}\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}\mathbf{c}\boldsymbol{\omega}$ with $\boldsymbol{\varepsilon}$. One can easily prove that the last equation describes the continuous collapse of a posterior state to a Gaussian pure (coherent) state ϕ_∞ with the minimal uncertainty $\mathbf{p}_\infty = \frac{1}{2}\ \boldsymbol{\varepsilon}^{-1}|\boldsymbol{\varepsilon}\mathbf{c}|$, where

$$\boldsymbol{\varepsilon}^{-1}|\boldsymbol{\varepsilon}\mathbf{c}| = \boldsymbol{\varepsilon}^{-1/2}|\tilde{\mathbf{c}}|\boldsymbol{\varepsilon}^{-1/2} = |\mathbf{c}\boldsymbol{\varepsilon}|\boldsymbol{\varepsilon}^{-1},\ \tilde{\mathbf{c}} = \boldsymbol{\varepsilon}^{1/2}\mathbf{c}\boldsymbol{\varepsilon}^{1/2},\ |\tilde{\mathbf{c}}| = (\tilde{\mathbf{c}}^2)^{1/2}.$$

Indeed, if $\boldsymbol{\omega}\mathbf{c}\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}\mathbf{c}\boldsymbol{\omega}$, then $\boldsymbol{\omega}|\mathbf{c}\boldsymbol{\varepsilon}| = |\boldsymbol{\varepsilon}\mathbf{c}|\boldsymbol{\omega}$, because from the commutativity of $\tilde{\mathbf{c}}$ with $\boldsymbol{\varepsilon}^{-1/2}\boldsymbol{\omega}\boldsymbol{\varepsilon}^{-1/2}$ there follows the commutativity of $|\tilde{\mathbf{c}}|$ with $\boldsymbol{\varepsilon}^{-1/2}\boldsymbol{\omega}\boldsymbol{\varepsilon}^{-1/2}$ and

$$\boldsymbol{\omega}|\mathbf{c}\boldsymbol{\varepsilon}| = \boldsymbol{\omega}\boldsymbol{\varepsilon}^{-1/2}|\boldsymbol{\varepsilon}^{1/2}\mathbf{c}\boldsymbol{\varepsilon}^{1/2}|\boldsymbol{\varepsilon}^{1/2} = \boldsymbol{\varepsilon}^{1/2}|\boldsymbol{\varepsilon}^{1/2}\mathbf{c}\boldsymbol{\varepsilon}^{1/2}|\boldsymbol{\varepsilon}^{-1/2}\boldsymbol{\omega} = |\boldsymbol{\varepsilon}\mathbf{c}|\boldsymbol{\omega}.$$

Hence the Riccati equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbf{p}_t + i(\mathbf{p}_t\boldsymbol{\omega}\mathbf{c} - \mathbf{c}\boldsymbol{\omega}\mathbf{p}_t) + \mathbf{p}_t\boldsymbol{\varepsilon}\mathbf{p}_t = \frac{1}{4}\ \mathbf{c}\boldsymbol{\varepsilon}\mathbf{c},$$

corresponding to $\tilde{\varepsilon} = \boldsymbol{\varepsilon} - \bar{\boldsymbol{\varepsilon}} = 0$, has the unique stationary positive solution $\mathbf{p}_\infty$: $\mathbf{p}_\infty\boldsymbol{\omega}\mathbf{c} = \mathbf{c}\boldsymbol{\omega}\mathbf{p}_\infty,\ \mathbf{p}_\infty\boldsymbol{\varepsilon}\mathbf{p}_\infty = \frac{1}{4}\,\boldsymbol{\varepsilon}^{-1/2}|\tilde{\mathbf{c}}|^2\boldsymbol{\varepsilon}^{-1/2} = \frac{1}{4}\ \mathbf{c}\boldsymbol{\varepsilon}\mathbf{c}$. The convergence $\mathbf{p}_t \to \mathbf{p}_\infty$ follows from properties of the Riccati equation with unique stationary solution $\mathbf{p}_\infty > 0$. Thus in the case $\boldsymbol{\varepsilon} = \varepsilon\mathbf{1}$ and $\boldsymbol{\omega} = \omega\mathbf{1}$ the positive solution $\mathbf{p}_t$ corresponding to $\mathbf{p}_0 = \mathbf{1}$ has the form

$$\mathbf{p}_t = \frac{|\mathbf{c}|}{2}\ \frac{1+\mathbf{q}_t}{1-\mathbf{q}_t}\ ,\ \mathbf{q}_t = \mathbf{q}_0\exp\{-\varepsilon|\mathbf{c}|t\},\ \mathbf{q}_0 = \frac{2-|\mathbf{c}|}{2+|\mathbf{c}|}\ ,$$

and $\mathbf{p}_\infty = \frac{1}{2}\ |\mathbf{c}|$, $\mathbf{p}_t \approx \mathbf{p}_\infty + |\mathbf{c}|\mathbf{q}_0 e^{-\varepsilon|\mathbf{c}|t}$ for $t \gg \frac{1}{\varepsilon}$, if $|\mathbf{c}| > 0$. Hence $\mathbf{p}_t = \mathbf{p}_\infty$ only in the purely quantum case $|\mathbf{c}| = 2$, and $\mathbf{p}_t \to 0$ only in the purely classical case $\mathbf{c} = 0$ when $\mathbf{p}_t = \mathbf{1}/(1+\varepsilon t)$.

This result was obtained in [3] for the case of positive definite $\mathbf{c} > 0$ (a stable quantum system), when the stationary quantum linear filter

$$\mathrm{d}\hat{\chi}_t(\xi) + \hat{\chi}_t(\kappa\mathbf{c}\xi)\mathrm{d}t = \int_\Lambda \langle \mathbf{l}\xi, \zeta_x^\sharp\rangle(\mathrm{d}Z_t(\mathrm{d}\mathbf{x}) - \hat{\chi}_t(\zeta_x)\lambda(\mathrm{d}\mathbf{x})\mathrm{d}t),$$

corresponding to $\zeta_{t,\mathbf{x}} = \zeta_x$, $\kappa = \frac{1}{2}\ \varepsilon + i\omega$, $\mathbf{l} = \frac{1}{2}\ (|\mathbf{c}| - \mathbf{c})$ and $\upsilon_t = 0$, does not depend on the observable process $Z_t : \mathbf{l} = 0$, if $\mathbf{c} \geq 0$. The a posterior state for the Gaussian initial wave function ψ_0 tends asymptotically to the ground state even without observation as the coherent a priori state: $\hat{\chi}_t(\xi) = \chi_0(e^{-\boldsymbol{\kappa}\mathbf{c}t}\xi) \to 0$.

In the contrary case $\mathbf{c} < 0$, $\mathbf{l} = |\mathbf{c}|$, corresponding to the unstable system, the complete nondemolition observation is needed to keep the system in a state with the minimal uncertainty relation. This explains why the quantum open (unstable) oscillator, corresponding [2] to the case $\Xi^- = \mathbb{C} = \Xi^+$ with $|\mathbf{c}| = 2$, tends asymptotically to the pure Gaussian state under the continuous observation of its amplitude $L = R(\zeta)$, given by the measurement of the complex nondemolition process $Z(t) = L(t) + W(t)$.

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6. Conclusion

The developed time-continuous quantum measurement theory based on the nondemolition principle as a superselection rule for the output observable processes abandons the von Neumann projection postulate as a redundancy in the stochastic extension of the quantum theory. It treats the wave packet reduction not as a real dynamical process but rather as the statistical inference of the posterior state described by the conditional wave-function for the prediction of the probabilities of the future measurements conditioned by the past observations.

There is no need to postulate in quantum stochastic theory a nonunitary stochastic linear or nonlinear evolution for the continuous state-vector reduction as it is done in the phenomenological quantum theories of quantum trajectories, state diffusion or spontaneous localization [17, 14, 10, 18, 11]. The nonunitary classical stochastic evolution giving the continuous reduction and localization of the posterior state can be rigorously derived [4, 5, 6, 9] within the quantum stochastic unitary evolution of the correspondent compound system, the object of the measurement and the input Bose field in the vacuum state as it is shown here.

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