ON STOCHASTIC GENERATORS OF POSITIVE DEFINITE EXPONENTS.

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Abstract. A characterisation of quantum stochastic positive definite (PD) exponent is given in terms of the conditional positive definiteness (CPD) of their form-generator. The pseudo-Hilbert dilation of the stochastic form-generator and the pre-Hilbert dilation of the corresponding dissipator is found. The structure of quasi-Poisson stochastic generators giving rise to a quantum stochastic birth processes is studied.

1. Introduction

Quantum probability theory provides examples of positive-definite (PD) infinitely-divisible functions on non-Abelian groups which serve as characteristic functions of quantum chaotic states, generalizing the characteristic functions of classical stochastic processes with independent increments. The simplest examples are given by quantum point processes [1] which are characterized by analytical functions on the unit ball $B = \{y \in \mathcal{B} : ||y|| \leq 1\}$ of a non-commutative group C*-algebra. Such processes generate Markov quantum dynamics by one-parameter families $\phi = (\phi_t)_{t>0}$ of nonlinear completely positive maps $\phi_t : B \to \mathcal{A}$ on the unit ball of a C*-algebra $\mathcal{B}$, into an operator algebra $\mathcal{A}$ of a Hilbert space $\mathcal{H}$. As in the linear case, an analytical map ϕ_t is completely positive iff it is positive definite (PD),

$$\sum_{x,z \in B} \langle \eta^x | \phi (x^\star z) \eta^z \rangle := \sum_{i.k} \langle \eta_i | \phi (y_i^\star y_k) \eta_k \rangle \geq 0, \quad \forall \eta_j \in \mathcal{H}, y_j \in B, \tag{1.1}$$

where $\eta^y = \eta_j \neq 0$ only for $y = y_j, j = 1, 2, \ldots$. The simplest quantum point dynamics of this kind is given by the quantum Markov birth process which is described by the one-parameter semigroup

$$\phi_s (y) \phi_r (y) = \phi_{s+r} (y), \quad \phi_0 (y) = 1, \quad y \in B$$

of infinitely divisible bounded PD functions $\phi_t : B \to \mathbb{C}$ with the normalization property $\phi_t (1) = 1$, where $1 \in B$ is (approximative) identity of $\mathcal{B}$. The continuity of the semigroup ϕ suggests the exponential form $\phi_t (y) = \exp [t\lambda (y)]$ of the functions ϕ_t . The corresponding analytic generator

$$\lambda (y) = \frac{1}{t} \ln \phi_t (y) := \lim_{t \searrow 0} \frac{1}{t} (\phi_t (y) - 1)$$

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of such semigroup is conditionally completely definite (CPD), and this is equivalent to the PD property (1.1) for $\phi = \lambda$ under the condition $\sum_j \eta^j = 0$ and $\lambda(1) = 0$. The CPD functions have been studied in [2] and the corresponding dilations $\phi_t(y) = \langle \pi_t(y) \rangle$ to the multiplicative stochastic exponents $\pi_t(y) =: \exp \Lambda(t, y)$: of a quantum process $\Lambda(t, y)$ with independent increments and the vacuum mean $\langle \Lambda(t, y) \rangle = t\lambda(y)$ in Fock space were obtained in [3, 4]. The unital $\star$ -multiplicative property

$$\pi_t(x^\star z) = \pi_t(x)^\dagger \pi_t(z), \quad \pi_t(1) = I,$$

obviously implies the PD (1.1) of $\phi = \pi_t$, and the stationarity of the increments $\Lambda^s(t) = \Lambda(t+s) - \Lambda(s)$ implies the cocycle exponential property

$$\pi_s(y) \pi_r^s(y) = \pi_{r+s}(y), \quad \forall r, s > 0,$$

with respect to the natural time-shift $\pi \mapsto \pi^s$ in the Fock space of the representation π . The dilation of the CPD generators λ over the suggests their general form $\lambda(y) = \varphi(y) - \kappa$, where φ is a PD function on B with $\varphi(0) = 0$ and $\kappa = \varphi(1)$.

Here we shall extend this dilation theorem to the stochastic PD families ϕ satisfying the cocycle exponential property

$$\phi_s(y) \phi_r^s(y) = \phi_{r+s}(y), \quad \forall r, s > 0,$$

but not yet the unital multiplicative property. In particular, we shall obtain the structure of the stochastic form-generator for a family ϕ of PD functions $\phi_t(\omega)$: $B \to \mathbb{C}$, given as the adapted stochastic process $\phi_t(\omega, y)$ for each $y \in B$ with respect to a classical process $\omega = \{\omega(t)\}$ with independent increments, and having the cocycle exponential property with respect to the time-shift $\phi_t^s(\omega) = \phi_t(\omega^s)$, $\omega^s = \{\omega(t+s)\}$. Such stochastic functions can be unbounded, but they are usually normalized, $\phi_t(\omega, 1) = m_t(\omega)$, to a positive-valued process $m_t \geq 0$, having the martingale property

$$m_t(\omega) = \epsilon_t[m_s](\omega), \quad \forall s > t, \quad m_0(\omega) = 1,$$

where ϵ_t is the conditional expectation with respect to the history of the process ω up to time t . As follows from our dilation theorem, for example the stochastic exponent

$$\phi_t(y) = (1 + \alpha(y))^{p(t)} \exp[t\lambda(y)]$$

with respect to the standard Poisson process $p(t, \omega)$ is PD and normalized in the mean iff $1 + \alpha$ and $\kappa + \lambda$ are PD for a $\kappa \geq 0$, and $\alpha(1) + \lambda(1) = 0$.

2. The Generators of Quantum Stochastic PD Exponents.

Let us consider a (noncommutative) Itô $\flat$ -algebra $\mathfrak{a}$ [4, 5], i.e. an associative $\star$ -algebra, identified with the algebra of quadruples $\boldsymbol{a} = (a_\nu^\mu)_{\nu=+,\bullet}^{\mu=-,\bullet}$,

$$a_\bullet^\bullet = i(a), \quad a_+^\bullet = k(a), \quad a_\bullet^- = k^*(a), \quad a_+^- = l(a),$$

under the product $\boldsymbol{ba} = (b_\bullet^\mu a_\nu^\bullet)$ and the involution $a \mapsto b = a^\star \in \mathfrak{a}$, $b^\star = a$, represented by the quadruples $\boldsymbol{b} = \boldsymbol{a}^\flat$ with $b_{-\nu}^\mu = a_{-\mu}^{\nu\dagger}$, where $-\pm = \mp$, $-\bullet = \bullet$. Here $i(b) k(a) = k(ba)$ is the GNS $\star$ -representation $i(a^\star) = i(a)^\dagger$ associated with a linear positive $\star$ -functional $l : \mathfrak{a} \mapsto \mathbb{C}$, $l(a^\star) = l(a)^*$, and $k^*(a^\star) = k(a)^\dagger$ is the linear functional on the pre-Hilbert space $\mathcal{K}$ of the Kolmogorov decomposition $l(a^\star a) = k(a)^\dagger k(a)$ of the functional l , separating $\mathfrak{a}$ in the sense $a = 0 \Leftrightarrow i(a) = k(a) = l(a) = 0$.

Let B denote a (noncommutative) semigroup with identity $1 \in B$ and involution $y \mapsto y^\star \in B$, $(x^\star z)^\star = z^\star x, \forall x, y, z \in B$, say, a (noncommutative) group with $y^\star = y^{-1}$, or the unital semigroup $B = 1 \oplus \mathfrak{b}$ of a $\star$ -algebra $\mathfrak{b}$ with $(1 \oplus a)^\star (1 \oplus c) = 1 \oplus a \star c$, where $a \star c = c + a^\star c + a^\star$ for $a, c \in \mathfrak{b}$. The stochastically differentiable operator-valued exponent $\phi_t(y)$ over B with respect to a quantum stationary process, with independent increments $\Lambda^s(t) = \Lambda(t+s) - \Lambda(s)$ generated by a separable Itô algebra $\mathfrak{a}$ is described by the quantum stochastic equation

$$\mathrm{d}\phi_t(y) = \phi_t(y)\,\boldsymbol{\alpha}(y)\,\mathrm{d}\boldsymbol{A}(t) := \phi_t(y)\sum_{\mu,\nu}\alpha^\mu_\nu(y)\,\mathrm{d}A^\nu_\mu, \qquad y \in B \tag{2.1}$$

with the initial condition $\phi_0(y) = I$, for all $y \in B$. Here $\boldsymbol{\alpha}(y) \in \mathfrak{a}$ is given by the quadruple $\alpha^\bullet_\bullet = [\alpha^m_n]$, $\alpha^\bullet_- = [\alpha^m_+]$, $\alpha^-_\bullet = [\alpha^-_n]$, α^-_+ of complex functions $\alpha^\mu_\nu : B \to \mathbb{C}$, $\mu \in \{-, 1, 2, ...\}$, $\quad \nu \in \{+, 1, 2, ...\}$ and $\boldsymbol{A} = \left(A^\nu_\mu\right)^{\nu=+,\bullet}_{\mu=-,\bullet}$ is the quadruple of the canonical integrators given by the standard time $A^+_-(t) = tI$, annihilation $A^n_-(t)$, creation $A^+_m(t)$ and exchange $A^n_m(t)$ operators in Fock space over $L^2(\mathbb{R}_+ \times \mathbb{N})$ with $m, n \in \mathbb{N} = \{1, 2, ...\}$. The infinitesimal increments $\mathrm{d}A^\nu_\mu = A^{t\nu}_\mu(\mathrm{d}t)$ are formally defined by the Hudson-Parthasarathy multiplication table [6] and the $\flat$ -property [4],

$$\mathrm{d}A^\beta_\mu \mathrm{d}A^\nu_\gamma = \delta^\beta_\gamma \mathrm{d}A^\nu_\mu, \qquad \boldsymbol{A}^\flat = \boldsymbol{A}, \tag{2.2}$$

where δ^β_γ is the usual Kronecker delta restricted to the indices $\beta \in \{-, 1, 2, ...\}$, $\quad \gamma \in \{+, 1, 2, ...\}$ and $A^{\flat\mu}_{-\nu} = A^{\nu\dagger}_{-\mu}$ with respect to the reflection of the indices $(-, +)$ only. The structural functions α^μ_ν for the $*$ -cocycles $\phi^*_t = \phi_t$, where $\phi^*_t(y) = \phi_t(y^\star)^\dagger$ should obviously satisfy the $\flat$ -property $\boldsymbol{\alpha}^\flat = \boldsymbol{\alpha}$, where $\alpha^{\flat\nu}_{-\mu} = \alpha^{\mu *}_{-\nu}$, $\alpha^{\mu *}_\nu(y) = \alpha^\mu_\nu(y^\star)^\dagger$ even in the case of nonlinear α^μ_ν . The summation in (2.1) is defined as a quantum stochastic differential [4] if $\sum_{n=1}^\infty \alpha^-_n(y^\star)\,\alpha^n_+(y) < \infty$ and the matrix $[\alpha^m_n(y)], m, n \in \mathbb{N}$ represents a bounded operator in the Hilbert space $\ell^2_\mathbb{N} = \left\{\zeta^\bullet : \mathbb{N} \to \mathbb{C} | \sum_{n=1}^\infty |\zeta^n|^2 < \infty\right\}$ for each $y \in B$. If the coefficients α^μ_ν are independent of t , ϕ satisfies the cocycle property $\phi_s(y)\,\phi^s_r(y) = \phi_{s+r}(y)$, where ϕ^s_t is the solution to (1) with $A^\mu_\nu(t)$ replaced by $A^{s\mu}_\nu(t)$. Define the tensors $a^\mu_\nu = \alpha^\mu_\nu(y)$ also for $\mu = +$ and $\nu = -$, by

$$\alpha^+_\nu(y) = 0 = \alpha^\mu_-(y), \qquad \forall y \in B,$$

and then one can extend the summation in (2.1) to the trace of the quadratic matrices $\mathbf{a} = [a^\mu_\nu]$ so it is also over $\mu = +$, and $\nu = -$. By such an extension the multiplication table for $\mathrm{d}A(\mathbf{a}) = \mathrm{d}A^\nu_\mu a^\mu_\nu = \boldsymbol{a}\mathrm{d}\boldsymbol{A}$ can be written as

$$\mathrm{d}A(\mathbf{b})\,\mathrm{d}A(\mathbf{a}) = \mathrm{d}A(\mathbf{ba}), \quad \mathbf{ba} = \left[b^\mu_\lambda a^\lambda_\nu\right]$$

in terms of the usual matrix product $b^\mu_\lambda a^\lambda_\nu = b^\mu_\bullet a^\bullet_\nu$ and the involution $\mathbf{a} \mapsto \mathbf{a}^\flat$ can be obtained by the pseudo-Hermitian conjugation $a^{\flat\nu}_\beta = g^{\nu\kappa} a^{\mu *}_\kappa g_{\mu\beta}$ respectively to the indefinite (Minkowski) metric tensor $\mathbf{g} = [g_{\mu\nu}]$ and its inverse $\mathbf{g}^{-1} = [g^{\mu\nu}]$, given by $g_{\mu\nu} = \delta^\mu_{-\nu} = g^{\mu\nu}$.

Let us prove that the "spatial" part $\boldsymbol{\lambda} = (\lambda^\mu_\nu)^{\mu\neq+}_{\nu\neq-}$ of the quantum stochastic germ $\lambda^\mu_\nu(y) = \delta^\mu_\nu + \alpha^\mu_\nu(y)$ for a PD cocycle exponent ϕ must be conditionally PD in the following sense.

Theorem 1. *Suppose that the quantum stochastic equation (2.1) with $\phi_0(y) = y$ has a PD solution in the sense of positive definiteness (1.1) of the matrix $[\phi_t(y^\star_i y_k)]$,*

$\forall t>0$. Then the germ-matrix $\boldsymbol{\lambda}=\boldsymbol{p}+\boldsymbol{\alpha}$ to $\boldsymbol{p}=(\delta^\mu_\nu)^{\mu\neq+}_{\nu\neq-}$ satisfies the CPD property

$$\sum_j \boldsymbol{e}\boldsymbol{\zeta}_j=0\Rightarrow\sum_{i,k}\langle\boldsymbol{\zeta}_i|\boldsymbol{\lambda}\left(y_i^\star y_k\right)\boldsymbol{\zeta}_k\rangle\geq 0.$$

Here $\boldsymbol{\zeta}\in\mathbb{C}\oplus\ell^2_{\mathbb{N}}$, $\boldsymbol{e}=(e^\mu_\nu)^{\mu\neq+}_{\nu\neq-}$, $e^\mu_\nu=\delta^+_\nu\delta^\mu_-$ is the one-dimensional projector, written both with $\boldsymbol{\lambda}$ in the matrix form as

$$\boldsymbol{\lambda}=\begin{pmatrix}\lambda & \lambda_\bullet\\ \lambda^\bullet & \lambda^\bullet_\bullet\end{pmatrix},\qquad \boldsymbol{e}=\begin{pmatrix}1&0\\0&0\end{pmatrix},\tag{2.3}$$

where $\lambda=\alpha^-_+,\quad \lambda^m=\alpha^m_+,\quad \lambda_n=\alpha^-_n,\quad \lambda^m_n=\delta^m_n+\alpha^m_n$, with $\delta^m_n(y)=\delta^m_n$ such that $\lambda\left(y^\star\right)=\lambda\left(y\right)^\dagger,\qquad \lambda^n\left(y^\star\right)=\lambda_n\left(y\right)^\dagger,\qquad \lambda^m_n\left(y^\star\right)=\lambda^n_m\left(y\right)^\dagger$.

Proof. Let us denote by $\mathcal{D}$ the $\mathbb{C}$ -span $\left\{\sum_f\xi^f\otimes f^\otimes:\xi^f\in\mathbb{C},f^\bullet\in\ell^2_{\mathbb{N}}\otimes L^2\left(\mathbb{R}_+\right)\right\}$ of coherent (exponential) functions $f^\otimes t\left(\tau\right)=\bigotimes_{t\in\tau}f^\bullet\left(t\right)$, given for each finite subset $\tau=\{t_1,...,t_n\}\subseteq\mathbb{R}_+$ by tensors $f^\otimes\left(\tau\right)=f^{n_1}\left(t_1\right)...f^{n_N}\left(t_N\right)$, where $f^n,n=\mathbb{N}$ are square-integrable complex functions on $\mathbb{R}_+$ and $\xi^f=0$ for almost all $f^\bullet=\left(f^n\right)$. The co-isometric shift T_s intertwining $A^s\left(t\right)$ with $A\left(t\right)=T_sA^s\left(t\right)T_s^\dagger$ is defined on $\mathcal{D}$ by $T_s\left(f^\otimes\right)\left(\tau\right)=f^\otimes\left(\tau+s\right)$. The PD property (1.1) of the quantum stochastic adapted map ϕ_t into the $\mathcal{D}$ -forms $\langle\eta|\phi_t\left(y\right)\eta\rangle$, for $\eta\in\mathcal{D}$ can be obviously written as

$$\sum_{i,k}\sum_{f,h}\overline{\xi}^f_i\phi_t\left(f^\bullet,y_i^\star y_k,h^\bullet\right)\xi^h_k\geq 0,\tag{2.4}$$

for any sequence $y_j\in B,j=1,2,...$, where

$$\phi_t\left(f^\bullet,y,h^\bullet\right)=\left\langle f^\otimes|\phi_t\left(y\right)h^\otimes\right\rangle e^{-\int_t^\infty f^\bullet(s)^\dagger h^\bullet(s)\mathrm{d}s},$$

$\xi^f\neq 0$ only for a finite subset of $f^\bullet\in\{f^\bullet_i,i=1,2,...\}$. If the $\mathcal{D}$ -form $\phi_t\left(y\right)$ satisfies the stochastic equation (2.1), the complex function $\phi_t\left(f^\bullet,y,h^\bullet\right)$ satisfies the differential equation

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\ln\phi_t\left(f^\bullet,y,h^\bullet\right)&=f^\bullet\left(t\right)^\dagger h^\bullet\left(t\right)+\sum_{m,n=1}^\infty f^m\left(t\right)^*\alpha^m_n\left(y\right)h^n\left(t\right)\\&\quad+\sum_{m=1}^\infty f^m\left(t\right)^*\alpha^m_+\left(y\right)+\sum_{n=1}^\infty\alpha^-_n\left(y\right)h^n\left(t\right)\phi+\alpha^-_+\left(y\right)\end{aligned}$$

where $f^\bullet\left(t\right)^\dagger h^\bullet\left(t\right)=\sum_{n=1}^\infty f^n\left(t\right)^*h^n\left(t\right)$. The positive definiteness, (2.4), ensures the conditional positivity

$$\sum_j\sum_f\xi^f_j=0\Rightarrow\sum_{i,k}\sum_{f,h}\overline{\xi}^f_i\lambda_t\left(f^\bullet,y_i^\star y_k,h^\bullet\right)\xi^h_k\geq 0$$

of the form $\lambda_t\left(f^\bullet,y,h^\bullet\right)=\frac{1}{t}\left(\phi_t\left(f^\bullet,y,h^\bullet\right)-1\right)$ for each $t>0$ and any $y_j\in B$. This applies also for the limit λ_0 at $t\downarrow 0$, coinciding with the quadratic form

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi_t\left(f^\bullet,y,h^\bullet\right)|_{t=0}=\sum_{m,n}\bar{a}^m\lambda^m_n\left(y\right)c^n+\sum_m\bar{a}^m\lambda^m\left(y\right)+\sum_n\lambda_n\left(y\right)c^n+\lambda\left(y\right),$$

where $a^\bullet=f^\bullet\left(0\right),\quad c^\bullet=h^\bullet\left(0\right)$, and the λ 's are defined in (2.3). Hence the form

$$\sum_{i,k}\sum_{\mu,\nu}\overline{\zeta}^\mu_i\lambda^\mu_\nu\left(y_i^\star y_k\right)\zeta^\nu_k:=\sum_{i,k}\overline{\zeta}_i\lambda\left(y_i^\star y_k\right)\zeta_k$$

$$+\sum_{i,k}\left(\sum_n \bar{\zeta}_i \lambda_n \left(y_i^\star y_k\right)\zeta_k^n + \sum_m \bar{\zeta}_i^m \lambda^m \left(y_i^\star y_k\right)\zeta_k + \sum_{m,n}\bar{\zeta}_i^m \lambda_n^m \left(y_i^\star y_k\right)\zeta_k^n\right)$$

with $\zeta = \sum_f \xi^f, \quad \zeta^\bullet = \sum_f \xi^f a_f^\bullet$, where $a_f^\bullet = f^\bullet(0)$, is positive if $\sum_j \zeta_j = 0$. The components ζ and $\zeta^\bullet$ of these vectors are independent because for any $\zeta \in \mathbb{C}$ and $\zeta^\bullet = \left(\zeta^1, \zeta^2, ...\right) \in \ell^2_{\mathbb{N}}$ there exists such a function $a^\bullet \mapsto \xi^a$ on $\ell^2_{\mathbb{N}}$ with a finite support, that $\sum_a \xi^a = \zeta, \quad \sum_a \xi^a a^\bullet = \zeta^\bullet$, namely, $\xi^a = 0$ for all $a^\bullet \in \ell^2_{\mathbb{N}}$ except $a^\bullet = 0$, for which $\xi^a = \zeta - \sum_{n=1}^\infty \zeta^n$ and $a^\bullet = e_n^\bullet$, the n -th basis element in $\ell^2_{\mathbb{N}}$, for which $\xi^a = \zeta^n$. This proves the complete positivity of the matrix form $\boldsymbol{\lambda}$, with respect to the matrix orthoprojector $\boldsymbol{p}_0$ defined in (2.3) on the ket-vectors $\boldsymbol{\zeta} = (\zeta^\mu)$ □

3. A Dilation Theorem for the Form-Generator.

The CPD property of the germ-matrix $\boldsymbol{\lambda}$ with respect to the projective matrix $\boldsymbol{p}_0$ (2.3) obviously implies the positivity of the dissipation form

$$\sum_{x,z}\langle \boldsymbol{\zeta}^x | \boldsymbol{\Delta}(x,z)\,\boldsymbol{\zeta}^z\rangle := \sum_{k,l}\sum_{\mu,\nu}\langle \zeta_k^\mu | \Delta_\nu^\mu (y_k, y_l)\,\zeta_l^\nu\rangle , \tag{3.1}$$

where $\zeta^- = \zeta = \zeta^+$ and $\zeta_j = \zeta^{y_j}$ for any (finite) sequence $y_j \in B$, $j = 1, 2, ...$, corresponding to non-zero $\boldsymbol{\zeta}_y \in \mathbb{C} \oplus \ell^2_{\mathbb{N}}$. Here $\boldsymbol{\Delta} = (\Delta_\nu^\mu)_{\nu=+,\bullet}^{\mu=-,\bullet}$ is the stochastic dissipator

$$\boldsymbol{\Delta}(x,z) = \boldsymbol{\lambda}(x^\star z) - \boldsymbol{e}\boldsymbol{\lambda}(z) - \boldsymbol{\lambda}(x^\star)\,\boldsymbol{e} + \boldsymbol{e}\boldsymbol{\lambda}(1)\,\boldsymbol{e}$$

with the elements

$$\begin{aligned} \Delta_n^m (x,z) &= \alpha_n^m (x^\star z) + \delta_n^m, \\ \Delta_n^- (x,z) &= \alpha_n^- (x^\star z) - \alpha_n^- (z) = \Delta_+^n (z,x)^\dagger , \\ \Delta_+^- (x,z) &= \alpha_+^- (x^\star z) - \alpha_+^- (z) - \alpha_+^- (x^\star) + d, \end{aligned} \tag{3.2}$$

where $d = \alpha_+^-(1) \le 0$ ($d = 0$ for the case of the martingale $M_t = \phi_t(1)$). In particular the matrix-valued map $\lambda_\bullet^\bullet = [\lambda_n^m]$ is PD. If the functions λ^m , λ_n, λ have the form

$$\lambda^m(y) = \varphi^m(y) - c^m, \quad \lambda_n(y) = \varphi_n(y) - c_n, \quad \lambda(y) = \varphi(y) - c \tag{3.3}$$

such that $\boldsymbol{\varphi} = \boldsymbol{\lambda} - \boldsymbol{c}$, is a PD map for a constant Hermitian matrix $\boldsymbol{c} = (c_\nu^\mu)_{\nu\ne-}^{\mu\ne+}$, the CPD condition is fulfilled for $\boldsymbol{\lambda}$.

In order to make the formulation of the following dilation theorem as concise as possible, we need the notion of the $\flat$ -representation of B in a pseudo-Hilbert space $\mathcal{E} = \mathbb{C} \oplus \mathcal{K} \oplus \mathbb{C}$ with respect to the indefinite metric

$$(\xi|\xi) = 2\,\mathrm{Re}\,\bar{\xi}^-\xi^+ + \|\xi^\circ\|^2 + |\xi^+|^2 d \tag{3.4}$$

for the triples $\xi = (\xi^-, \xi^\circ, \xi^+) \in \mathcal{E}$, where $\xi^-, \xi^+ \in \mathbb{C}, \quad \xi^\circ \in \mathcal{K}, \quad \mathcal{K}$ is a pre-Hilbert space. The operators A in this space are given by the 3×3 -block-matrices $\mathbf{A} = [A_\nu^\mu]_{\nu=+,\circ,+}^{\mu=-,\circ,+}$, and the pseudo-Hermitian conjugation $\left(A^\flat \xi|\xi\right) = (\xi|A\xi)$ is given by the usual Hermitian conjugation $A_\nu^{\dagger\mu} = A_\mu^{\nu *}$ as $\mathbf{A}^\flat = \mathbf{G}^{-1}\mathbf{A}^\dagger\mathbf{G}$ respectively to the indefinite metric tensor $\mathbf{G} = [G_{\mu\nu}]$ and its inverse $\mathbf{G}^{-1} = [G^{\mu\nu}]$, given by

$$\mathbf{G} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & I_\circ^\circ & 0 \\ 1 & 0 & d \end{bmatrix}, \qquad \mathbf{G}^{-1} = \begin{bmatrix} -d & 0 & 1 \\ 0 & I_\circ^\circ & 0 \\ 1 & 0 & 0 \end{bmatrix} \tag{3.5}$$

with a real d , where $I_\circ^\circ$ is the identity operator in $\mathcal{K}$. The algebras of all operators A on $\mathcal{K}$ and $\mathcal{E}$ with $A^\dagger\mathcal{K}\subseteq\mathcal{K}$ and $A^\flat\mathcal{E}\subseteq\mathcal{E}$ are denoted by $\mathcal{A}(\mathcal{K})$ and $\mathcal{A}(\mathcal{E})$.

Theorem 2. *The following are equivalent:*

(1) *The dissipator (3.2), defined by the* $\flat$ *-map* α *with* $\alpha_+^-(1)=d$ *, is positive definite:*

$$\sum_{x,z}\langle\boldsymbol{\zeta}_x|\boldsymbol{\Delta}(x,z)\,\boldsymbol{\zeta}_z\rangle\geq 0$$

(2) *There exist: a pre-Hilbert space* $\mathcal{K}$ *, a unital* $\dagger$ *- representation* j *in* $\mathcal{A}(\mathcal{K})$ *,*

$$j(x^\star z)=j(x)^\dagger j(z),\quad j(1)=I, \tag{3.6}$$

of the $\star$ *-multiplication structure of* B *, a* j *-cocycle on* B *,*

$$k(x^\star z)=j(x)^\dagger k(z)+k(x^\star), \tag{3.7}$$

having values in $\mathcal{K}$ *, and a function* $l:B\to\mathbb{C}$ *, having the coboundary property*

$$l(x^\star z)=l(z)+l(x^\star)+k^*(x^\star)\,k(z), \tag{3.8}$$

with $k^*(y^\star)=k(y)^*,l(y^\star)=l(y)^*$ *, such that* $\lambda(y)=l(y)+d$ *,*

$$\lambda_n(y^\star)=k(y)^\dagger L_n^\circ+L_n^-=\lambda^n(y)^\dagger,$$

and $\lambda_n^m(y)=L_m^{\circ *}j(y)\,L_n^\circ$ *for some elements* $L_n^\circ\in\mathcal{K}$ *with the adjoints* $L_n^{\circ *}=L_\circ^n:\mathcal{K}\to\mathbb{C}$ *and* $L_n^-\in\mathbb{C}$ *.*

(3) *There exist a pseudo-Hilbert space,* $\mathcal{E}$ *, namely,* $\mathbb{C}\oplus\mathcal{K}\oplus\mathbb{C}$ *with the indefinite metric tensor* $\mathbf{G}=[G_{\mu\nu}]$ *given above for* $\mu,\nu=-,\circ,+$ *, and* $d=\lambda(1)$ *, a unital* $\flat$ *-representation* $\jmath=[\jmath_\nu^\mu]_{\nu=-,\circ,+}^{\mu=-,\circ,+}$ *of the* $\star$ *-multiplication structure of* B *on* $\mathcal{E}$ *:*

$$\jmath(x^\star z)=\jmath(x)^\flat\,\jmath(z),\quad \jmath(1)=\mathbf{I} \tag{3.9}$$

with $\jmath(y)^\flat=\mathbf{G}^{-1}\jmath(y)^\dagger\,\mathbf{G}$ *, given by the matrix elements*

$$\jmath_\circ^\circ=j,\quad \jmath_+^\circ=k,\quad \jmath_\circ^-=k^*,\quad \jmath_+^-=l,\quad \jmath_-^-=1=\jmath_+^+$$

and all other $\jmath_\nu^\mu=0$ *, and a linear operator* $\mathbf{L}:\mathbb{C}\oplus\ell^2_{\mathbb{N}}\to\mathcal{E}$ *, with the components* $[L^\mu,L^\mu_\bullet]$ *, where*

$$L^-=0,\quad L^\circ=0,\quad L^+=1,\quad L_\bullet^-=\left(L_n^-\right),\quad L_\bullet^\circ=\left(L_n^\circ\right),\quad L_\bullet^+=0,$$

and $\mathbf{L}^\flat=\begin{pmatrix}1&0&\delta\\0&L_\circ^\bullet&L_+^\bullet\end{pmatrix}=\mathbf{L}^\dagger\mathbf{G}$ *, where* $L_\circ^\bullet=L_\bullet^{\circ\dagger},L_+^\bullet=L_\bullet^{-\dagger}$ *, such that*

$$\mathbf{L}^\flat\jmath(y)\,\mathbf{L}=\boldsymbol{\lambda}(y),\qquad\forall y\in B. \tag{3.10}$$

(4) *The germ-matrix* $\boldsymbol{\lambda}(y)=(\alpha_\nu^\mu(y)+\delta_\nu^\mu)_{\nu\neq-}^{\mu\neq+}$ *is CPD with respect to the orthoprojector* $\boldsymbol{e}$ *, defined in (2.3) :*

$$\sum_y\boldsymbol{e}\boldsymbol{\zeta}^y=0\Rightarrow\sum_{x,z}\langle\boldsymbol{\zeta}^x|\boldsymbol{\lambda}(x^\star z)\,\boldsymbol{\zeta}^z\rangle\geq 0.$$

Proof. Similar to the dilation theorem in [4], see also [7], [8], [9] □

4. Pseudo-Poisson processes and their generators.

Let us consider the case $B = 1 \oplus \mathfrak{b}$ of the unital semigroup for a $\star$ -algebra $\mathfrak{b}$ with $\boldsymbol{\lambda}(1 \oplus b) = \boldsymbol{d} + \boldsymbol{\gamma}(b)$ given by a linear matrix -function

$$\boldsymbol{\gamma} = \begin{pmatrix} \gamma & \gamma_{\bullet} \\ \gamma^{\bullet} & \gamma_{\bullet}^{\bullet} \end{pmatrix} = \boldsymbol{\lambda} - \boldsymbol{d}, \quad \boldsymbol{d} = \begin{pmatrix} d & d_{\bullet} \\ d^{\bullet} & d_{\bullet}^{\bullet} \end{pmatrix} = \boldsymbol{\lambda}(1)$$

of $b \in \mathfrak{b}$ for $y = 1 \oplus b$. Following [4], the linear quantum stochastic process $\Lambda(t) : b \mapsto \boldsymbol{\gamma}(b)\,\boldsymbol{A}(t)$ with independent increments, generating together with $A(t, \mathbf{d}) = A_{\mu}^{\nu}(t)\, d_{\nu}^{\mu}$ the stochastic PD exponent

$$\phi_t(1 \oplus b) =: \exp\left[A(t, \mathbf{d}) + \Lambda(t, b)\right] : \quad b \in \mathfrak{b}$$

as the solution of the equation (2.1), will be called the pseudo-Poissonian[4] over the algebra $\mathfrak{b}$.

If B is a unit ball of an operator algebra $\mathcal{B}$, the linear form-generator can be extended to the whole algebra. The structure (3.3) of the linear form-generator for PD cocycles over an operator algebra $\mathcal{B}$ is a consequence of the cocycle equation (3.7), according to which $j(0)\, k(y) = 0$, where

$$k(y) = j(y)\, \varsigma - \varsigma, . \quad \varsigma = -k(0). \tag{4.1}$$

Denoting by $\varsigma^{\dagger}$ the linear functional $\xi^{\circ} \mapsto (\varsigma|\xi^{\circ})$ on $\mathcal{K}$ corresponding to the $\varsigma \in \mathcal{K}$, the condition (3.8) yields

$$l(y) = \frac{1}{2}\left(\varsigma^{\dagger} k(y) + k^{*}(y)\, \varsigma\right) = \varsigma^{\dagger} j(y)\, \varsigma - \varsigma^{\dagger}\varsigma. \tag{4.2}$$

Hence, in addition to $\lambda_n^m(y) = L_m^{\circ\dagger} j(y)\, L_n^{\circ}$ one can obtain the structure (3.3) with

$$\varphi(y) = \varsigma^{\dagger} j(y)\, \varsigma, \quad \varphi_n(y) = \varsigma^{\dagger} j(y)\, L_n^{\circ}, \quad \varphi^m(y) = L_m^{\circ\dagger} j(y)\, \varsigma, \tag{4.3}$$

and $\kappa = \varsigma^{\dagger}\varsigma - \delta$, $\kappa_n = \varsigma^{\dagger} L_n^{\circ} - L_n^{-}$. Thus, $\boldsymbol{\lambda}(y) = \boldsymbol{\varphi}(y) - \boldsymbol{\kappa}$, where $\boldsymbol{\varphi}$ is a completely positive nonlinear map of B into the space $\mathcal{M}\left(\mathbb{C} \oplus \ell_{\mathbb{N}}^2\right)$ of complex matrices $\boldsymbol{x} = (x_{\nu}^{\mu})$. Moreover, $\boldsymbol{\varphi}$ is uniquely defined as the birth-map by the condition $\boldsymbol{\varphi}(0) = 0$ with $\boldsymbol{\kappa} = -\boldsymbol{\lambda}(0) = (\kappa_{\nu}^{\mu})$, where $\kappa_{+}^{-} = \kappa, \kappa_n^{-} = \kappa_n, \kappa_{+}^{m} = \overline{\kappa}_m$, and $\kappa_n^m = -\lambda_{\nu}^{\mu}(0)$, constituting a negative-definite matrix $\kappa_{\bullet}^{\bullet} = [\kappa_n^m]$. Any germ-matrix $\boldsymbol{\lambda}$ whose components are decomposed into the sums of the components φ_{ν}^{μ} of a PD map $\boldsymbol{\varphi}$ and $\boldsymbol{\lambda}(0)$, are obviously CPD with respect to the orthoprojector $\boldsymbol{p}_0$ in (2.4). As follows from the dilation theorem, there exists a family $\varsigma_{-} = \varsigma = \varsigma_{+}, \quad \varsigma_n = L_n^{\circ} - j(0)\, L_n^{\circ}, \quad n \in \mathbb{N}$ of vectors $\varsigma_{\nu} \in \mathcal{K}$ with $j(0)\, \varsigma_{\nu} = 0$ such that $\varphi_{\nu}^{\mu}(y) = \varsigma_{\mu}^{\dagger} j(y)\, \varsigma_{\nu}$ for all $\mu \in \{-, 1, 2, ...\}$, $\nu \in \{+, 1, 2, ...\}$. Thus the equation (2.1) for a completely positive exponential cocycle with bounded stochastic derivatives has the following general form

$$\mathrm{d}\phi_t(y) + \left(\gamma - \varsigma^{\dagger} j(y)\, \varsigma\right) \phi_t(y)\, \mathrm{d}t = \sum_{m,n=1}^{\infty} \left(\varsigma_m^{\dagger} j(y)\, \varsigma_n - \gamma_n^m\right) \phi_t(y)\, \mathrm{d}A_m^n$$

$$+ \sum_{m=1}^{\infty} \left(\varsigma_m^{\dagger} j(y)\, \varsigma - \gamma_m^{\dagger}\right) \phi_t(y)\, \mathrm{d}A_m^{+} + \sum_{n=1}^{\infty} \left(\varsigma^{\dagger} j(y)\, \varsigma_n - \gamma_n\right) \phi_t(y)\, \mathrm{d}A_{-}^{n}, \tag{4.4}$$

where $\gamma_{\nu}^{\mu} = -\alpha_{\nu}^{\mu}(0)$. If $M_t = \phi_t(1)$ is a martingale, the normalization condition $\sum_{k=1}^{\infty} \varsigma^{k\dagger}\varsigma^{k} = \kappa$ ($\leq \kappa$ if submartingale).

In the particular case $\mathcal{K} = \mathbb{C} \oplus \mathfrak{h}$, $j(y) = 1 \oplus y$, where $\mathfrak{h}$ is a Hilbert space of a representation $\mathcal{B} \subseteq \mathcal{B}(\mathfrak{h})$ of the C*-algebra $\mathcal{B}$ in the operator algebra $\mathcal{B}(\mathfrak{h})$, this gives a quantum stochastic generalization of the Poissonian birth semigroups [1] with the affine generators $\alpha_{\nu}^{\mu}(y) = \varsigma_{\mu}^{\dagger} X \varsigma_{\nu} - \gamma_{\nu}^{\mu}$. In the more general case when the space $\mathcal{K}$ is embedded into the Hilbert sum of all tensor powers of the space $\mathfrak{h}$ such that $j(y) = \oplus_{k=0}^{\infty} y^{\otimes k}$, the birth function $\boldsymbol{\varphi}$ is described by the components

$$(4.5)\qquad \varphi_n^m(y) = \sum_{k=0}^{\infty} \varsigma_m^{k\dagger} y^{\otimes k} \varsigma_n^k, \qquad \varphi(y) = \sum_{k=1}^{\infty} \varsigma^{k\dagger} y^{\otimes k} \varsigma^k$$
$$\varphi^m(y) = \sum_{k=1}^{\infty} \varsigma_m^{k\dagger} y^{\otimes k} \varsigma^k, \qquad \varphi_n(y) = \sum_{k=1}^{\infty} \varsigma^{k\dagger} y^{\otimes k} \varsigma_n^k$$

with $\varsigma^k, \varsigma_n^k \in \mathfrak{h}^{\otimes k}$.

Note, if $\mathcal{B}$ is a W*-algebra and the germ map $\boldsymbol{\lambda}$ is w*-analytic, the completely positive function $\boldsymbol{\varphi}$ is also analytic, being defined by a w*-analytical representation $j = \oplus_{k=0}^{\infty} i^{\otimes k}$ in a full Fock space $\mathcal{K} = \oplus_{k=0}^{\infty} \mathcal{H}^{\otimes k}$, where i is a (linear) w*-representation of $\mathcal{B}$ on a Hilbert space $\mathcal{H}$. This gives the general form for the w*-analytical quantum stochastic quasi-Poisson birth process over the algebra $\mathcal{B}$.

The next theorem proves that these structural conditions which are necessary for complete positivity of the stochastic exponents, given by the equation (2.1), are also sufficient. In particular it proves the existence of the quantum birth cocycle ϕ for a given generating stochastic birth matrix-function $\boldsymbol{\varphi}$.

Theorem 3. *Let the structural maps $\boldsymbol{\lambda}$ of the quantum stochastic PD exponent ϕ over the unit ball of an operator algebra $\mathcal{B}$. Then they are bounded in the unit ball of $\mathcal{B}$,*

$$||\lambda|| < \infty, \qquad ||\lambda_{\bullet}|| = \left(\sum_{n=1}^{\infty} \|\lambda_n\|^2\right)^{\frac{1}{2}} = \|\lambda^{\bullet}\| < \infty, \qquad \|\lambda_{\bullet}^{\bullet}\| = \|\lambda_{\bullet}^{\bullet}(1)\| < \infty,$$

where $\|\lambda\| = \sup\{\|\lambda(y)\| : \|y\| < 1\}, \|\lambda_{\bullet}^{\bullet}(1)\| = \sup\{\langle\zeta^{\bullet}|\lambda_{\bullet}^{\bullet}(1)\zeta^{\bullet}\rangle \,|\|\zeta^{\bullet}\| < 1\}$, *and have the form (4.3) written as*

$$\boldsymbol{\lambda}(y) = \boldsymbol{\varphi}(y) - \boldsymbol{\kappa}$$

with $\varphi = \varphi_{+}^{-}, \quad \varphi^m = \varphi_{+}^{m}, \quad \varphi_n = \varphi_n^{-}$ *and* $\varphi_n^m = \lambda_n^m$ *, composing a bounded PD map*

$$(4.6)\qquad \boldsymbol{\varphi} = \begin{bmatrix} \varphi & \varphi_{\bullet} \\ \varphi^{\bullet} & \varphi_{\bullet}^{\bullet} \end{bmatrix}, \quad \text{and} \quad \boldsymbol{\kappa} = \begin{bmatrix} \kappa & \kappa_{\bullet} \\ \kappa_{\bullet}^{*} & 0 \end{bmatrix}$$

with arbitrary κ and $\kappa_{\bullet} = (\kappa_1, \kappa_{2,\dots})$. The equation (4.4) has the unique PD solution

$$(4.7)\quad \phi_t(y) = V_t^{\dagger} \exp\left[A_{\bullet}^{+}(t)\,\varphi^{\bullet}(y)\right] \varphi_{\bullet}^{\bullet}(y)^{A_{\bullet}^{\bullet}(t)} \exp\left[\varphi_{\bullet}(y)\,A_{-}^{\bullet}(t)\right] V_t \exp\left[t\varphi(y)\right],$$

where $V_t = \exp\left[-\kappa_{\bullet} A_{-}^{\bullet}(t) - \frac{1}{2}\kappa t I\right]$.

Proof. (Sketch) The PD solution to the quantum stochastic equation (4.4) can be obtained by the iteration of the equivalent quantum stochastic integral equation

$$\phi_t(y) = V_t^{\dagger} V_t + \int_0^t V_s^{\dagger} \phi_{t-s}^{s}(y)\, V_s \beta_{\nu}^{\mu}(y)\, \mathrm{d}A_{\mu}^{\nu}(s)$$

where $\beta_\nu^\mu(y) = \varphi_\nu^\mu(y) - \delta_\nu^\mu$ .Here V_t is the exponential vector cocycle $V_r^s V_s = V_{r+s}$, resolving the quantum stochastic differential equation

$$\mathrm{d}V_t + \kappa V_t \mathrm{d}t + \sum_{n=1}^{\infty} \kappa_n V_t \mathrm{d}A_-^n = 0$$

with the initial condition $V_0 = I$ in $\mathcal{D}$ and with $V_r^s = T_r^\dagger V_r T_s$, shifted by the time-shift co-isometry T_s in $\mathcal{D}$.

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