CHAOTIC STATES AND STOCHASTIC INTEGRATION IN QUANTUM SYSTEMS

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Abstract. Quantum chaotic states over a noncommutative monoid, a unitalization of a noncommutative Ito algebra parametrizing a quantum stochastic Levy process, are described in terms of their infinitely divisible generating functionals over the simple monoid-valued fields on an atomless 'space-time' set. A canonical decomposition of the logarithmic conditionally posive-definite generating functional is constructed in a pseudo-Euclidean space, given by a quadruple defining the monoid triangular operator representation and a cyclic zero pseudo-norm state in this space.

It is shown that the exponential representation in the corresponding pseudo-Fock space yields the infinitely-divisible generating functional with respect to the exponential state vector, and its compression to the Fock space defines the cyclic infinitly-divisible representation associated with the Fock vacuum state. The structure of states on an arbitrary Itô algebra is studied with two canonical examples of quantum Wiener and Poisson states.

A generalized quantum stochastic nonadapted multiple integral is explicitly defined in Fock scale, its continuity and quantum stochastic differentiability is proved. A unified non-adapted and functional quantum Itô formula is discovered and established both in weak and strong sense, and the multiplication formula on the exponential Itô algebra is found for the relatively bounded kernel-operators in Fock scale. The unitarity and projectivity properties of nonadapted quantum stochastic linear differential equations are studied, and their solution is constructed for the locally bounded nonadapted generators in terms of the chronological products in the underlying kernel algebra canonically represented by triangular operators in the pseudo-Fock space.

Contents

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Introduction. Non-commutative Itô algebra

Non-commutative stochastic analysis and calculus appeared in the eighties as a result of the mathematical justification of the notions of quantum white noise and the corresponding 'Langevin equations' discussed by physicists from the sixties onwards in connection with stochastic models of quantum optics and radio-physics [22], [24], [34]. The first rigorous results in quantum stochastic calculus are due to Hudson and Parthasarathy [34], who in 1983 described a quantum Itô formula for operator-valued integrals with respect to non-commutative canonical martingales of *creation* $A^{+}(t)$, *annihilation* $A_{-}(t)$, and *gage* (or vacuum quanta number) $N(t)$. Represented in the symmetric Fock space $\Gamma(\mathcal{K})$ over $\mathcal{K}=L^{2}(\mathbb{R}_{+})$ by noncommuting operators but commuting with their increments at each t, they determine three linear-independent self-adjoint combinations

$$M_1=A_-+A^+,\quad M_2=\mathrm{i}\left(A_--A^+\right),\quad M_3=N, \tag{0.1}$$

as the 'classical' martingales with respect to the vacuum state. Each $M_i\left(t\right)$ can be represented as a real-valued independent-increment classical martingale $m_i\left(t,\omega_i\right)$, however due to mutual noncommutativity $[M_i,M_k]\neq 0, i\neq k$ they cannot be *jointly represented* as a vector-valued stochastic process $m_\bullet\left(\omega,t\right)=\left(m_1,m_2,m_3\right)\left(\omega,t\right)$ in any Kolmogorovian probability space $(\Omega,\mathcal{F},\mathrm{P})$. They are *quantum martingales* with respect to the conditional expectations $\mathrm{E}_t:\mathcal{A}\to\mathcal{A}_t$ on an operator algebra $\mathcal{A}=\mathcal{A}\left(\Gamma\right)$ of multiple *quantum stochastic integrals* X with $\mathcal{A}_t=\mathcal{A}\left(\Gamma_t\right)$ corresponding to the natural filtration $\{\Gamma_t=\Gamma(\mathcal{K}_t):t\in\mathbb{R}_+\}$ of the Fock space defined by the subspaces $\mathcal{K}_t\subset\mathcal{K}$ of the functions with the support in $[0,t]$ and the unit state $1_\emptyset\in\cap_{t>0}\Gamma_t$ of the vacuum state $\mathrm{E}_0[X]=\langle 1_\emptyset\mid X1_\emptyset\rangle$. The triple $(\Gamma,\,\mathcal{A},\,\mathrm{E})$ is said to be a 'quantum probability space' [2], and in general it consists of a Hilbert space Γ, a unital algebra $\mathcal{A}$ of operators in Γ with involution, Hermitian conjugation $X\longmapsto X^*\in\mathcal{A}$, and the functional of mathematical expectation $\mathrm{E}:\mathcal{A}\to\mathbb{C}$ defined by the scalar product $\langle 1\mid x\rangle$ of a unit vector $1\in\Gamma$ and the vector $x=X1$. To any 'classical' probability space $(\Omega,\mathcal{F},P)$ there corresponds a canonical 'quantum' one consisting of the Hilbert space $\Gamma=L^2(\Omega)$ with the scalar product

$$\langle f\mid h\rangle=\int f\left(\omega\right)^{*}h\left(\omega\right)P\left(\mathrm{d}\omega\right),$$

the commutative algebra of bounded 'diagonal' operators $\left(Xf\right)\left(\omega\right)=x(\omega)f(\omega)$ given by multiplications by complex essentially bounded $\mathcal{F}$-measurable random variables $x:\Omega\to\mathbb{C}$, and the functional

$$\mathrm{E}(X)=\int x(\omega)P(\mathrm{d}\omega)=\langle 1\mid x\rangle, \tag{0.2}$$

defined by the probability vector $1(\omega) = 1$ for all $\omega \in \Omega$, see for example [32]. The converse is true only in the case of commutative C^*-algebra $\mathcal{A}$ when all operators have a joint spectrum Ω [20]. This proves considerably greater generality of the non-commutative probability theory, also covering the purely quantum case which corresponds to a simple, or irreducible algebra, the algebra $\mathcal{A} = \mathcal{L}(\Gamma)$ of all linear continuous operators in a Hilbert space Γ.

Using this analogy, Hudson and Parthasarathy introduced the notion of *adapted* operator-valued process as a family $\{X(t) : t \in \mathbb{R}_+\}$ of operators in $\Gamma(L^2(\mathbb{R}_+))$, each affiliated to the subalgebra $\mathcal{A}_t$ generated by the canonical operators $\{M_\bullet(s) : s \le t\}$. Due to the continual tensor-product structure $\Gamma_{t+\Delta} = \Gamma_t \otimes \Gamma^t_\Delta$ of Γ_t with $\Gamma^t_\Delta = \Gamma(\mathcal{K}^t_\Delta)$ for the subspaces $\mathcal{K}^t_\Delta$ of square-integrable functions with the support in $[t, t+\Delta t)$ the forward increments $\Delta M_i(t) = M_i(t+\Delta t) - M_i(t)$ turn out to commute with adapted $D_i(t)$, which allowed to introduce quantum stochastic integrals $X_t = \sum_i \int_0^t D_i(s)\,\mathrm{d}M_i(s)$ as the limits of integral Itô sums $\sum_{t\epsilon\tau} D_i(t)\,\Delta M_i(t)$, where $\tau = \{t_1 < \cdots < t_N\}$, $\Delta t_n = t_{n+1} - t_n \to 0$ as $N \to \infty$. Building on this approach, a quantum evolution was constructed in [26] as a solution of the linear stochastic differential equation $\mathrm{d}U_t = U_t L_j \mathrm{d}\Lambda^j_t$, $U_0 = I$, with constant bounded operator-valued coefficients and non-commutative increments $\mathrm{d}\Lambda^j_t = \mathrm{d}M_j(t)$, $j = 1, 2, 3$, and $\mathrm{d}\Lambda^0_t = \mathrm{d}t$ (Here and it what follows we employ Einstein summation convention $L_j \Lambda^j = \sum_{j\ge 0} L_j \Lambda^j$).

The unitarity condition $U^*_t = U^{-1}_t$ was studied using the *quantum Itô formula*

$$\begin{aligned} \mathrm{d}(X_t X^*_t) &= \mathrm{d}X_t X^*_t + X_t \mathrm{d}X^*_t + \mathrm{d}X_t \mathrm{d}X^*_t, \\ \mathrm{d}X_t \mathrm{d}X^*_t &= D_i c^{ik}_0 D^*_k \mathrm{d}t + \sum_{j\ge 1} D_i c^{ik}_j D^*_k \mathrm{d}M_j(t) = D_i c^{ik}_j D^*_k \mathrm{d}\Lambda^j_t, \end{aligned} \tag{0.3}$$

where $c^{ik}_j \in \mathbb{C}$ are the structural coefficients defining the product of quantum-stochastic differentials $\mathrm{d}X_t = D_j \mathrm{d}\Lambda^j_t$ and $\mathrm{d}X^*_t = D^*_j \mathrm{d}\Lambda^j_t$ corresponding to the Hudson-Parthasarathy (HP) multiplication table

$$\mathrm{d}N\mathrm{d}N = \mathrm{d}N, \quad \mathrm{d}N\mathrm{d}A^+ = \mathrm{d}A^+, \quad \mathrm{d}A_-\mathrm{d}N = \mathrm{d}A_-, \quad \mathrm{d}A_-\mathrm{d}A^+ = \mathrm{d}t$$

(other combinations are equal to zero). It follows from this table that $c^{i0}_j = 0 = c^{0k}_j$ for all $i, j, k = 0, 1, 2, 3$ corresponding to the completely degenerate adjoint representation of $\mathrm{d}\Lambda^0_t$, with $c^{\bullet 3}_0 = 0 = c^{3\bullet}_0$, $\left(c^{ik}_0\right)^{i,k=1,2} = \begin{pmatrix} 1 & -\mathrm{i} \\ +\mathrm{i} & 1 \end{pmatrix}$ and three Hermitian 3×3-matrices $c^{\bullet\bullet}_j = [c^{ik}_j]$,

$$c^{\bullet\bullet}_1 = \frac{1}{2}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & +\mathrm{i} \\ 1 & -\mathrm{i} & 0 \end{bmatrix}, \quad c^{\bullet\bullet}_2 = \frac{1}{2}\begin{bmatrix} 0 & 0 & -\mathrm{i} \\ 0 & 0 & 1 \\ +\mathrm{i} & 1 & 0 \end{bmatrix}, \quad c^{\bullet\bullet}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

indexed by $i, k = 1, 2, 3$, define the adjoint representations $c^{j\bullet}_\bullet, c^{\bullet j}_\bullet$ of the martingale differential algebra $\mathrm{d}\Lambda^j = \mathrm{d}M_j$, $j = 1, 2, 3$.

It can be directly verified that the three-dimensional subspace $\mathfrak{a}_\bullet$ of complex four-vectors $a_\bullet = (0, \alpha_\bullet)$, given by the rows $\alpha_\bullet = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{C}^3$, is an associative $*$-algebra with respect to the complex conjugation $\alpha^\star_\bullet = (\alpha^*_1, \alpha^*_2, \alpha^*_3)$ as an involution (not to be mixed up with Hermitian conjugation $\alpha^*_\bullet = \left[\alpha^*_j\right]$ defining the adjoint column to $\alpha_\bullet$) and the composition $\alpha_\bullet\alpha_\bullet^\star = \alpha_\bullet * \alpha_\bullet$ given by the Hermitian 3-vector-form

$$\alpha_\bullet * \alpha_\bullet := \left(\alpha_i c_1^{ik} \alpha_k^*,\ \ \alpha_i c_2^{ik} \alpha_k^*,\ \ \alpha_i c_3^{ik} \alpha_k^*\right) \equiv \alpha_i c_\cdot^{ik} \alpha_k.$$

Moreover, since four fundamental differentials $\mathrm{d}\Lambda^j$ form a linear basis of an associative algebra, the degenerate semi-positive scalar product

$$(\alpha_\bullet \mid \alpha_\bullet) := (\alpha_1 + \mathrm{i}\alpha_2)(\alpha_1 + \mathrm{i}\alpha_2)^* \equiv \alpha_i c_0^{ik} \alpha_k^*$$

satisfies the right $\star$-representation property

$$(\alpha_\bullet \beta_\bullet \mid \alpha_\bullet) = (\alpha_\bullet \mid \alpha_\bullet \beta_\bullet^\star)$$

for any $\beta_\bullet \in \mathfrak{a}_\bullet$. Thanks to this property one can combine the composition and inner product in $\mathfrak{a}_\bullet$ into a four-dimensional composition in the $\star$-algebra $\mathfrak{a} = \mathbb{C} \oplus \mathfrak{a}_\bullet \equiv \mathfrak{a}_\bullet + \mathbb{C}d_t$ of the pairs $a = (\alpha_0, \alpha_\bullet)$ with involution $a^\star = (\alpha_0^*, \alpha_\bullet^\star)$, the Hermitian sesquilinear composition

$$a \star a := \left(\alpha_i c_0^{ik} \alpha_k^*,\ \alpha_i c_\cdot^{ik} \alpha_k^*\right) \equiv a \cdot a^\star$$

and self-adjoint nilpotent element $d_t = (1, 0^\bullet) = d_t^\star$ representing the infinitesimal $\mathrm{d}t$. Here $a_\bullet = a - l(a)\, d_t$ is given by the linear functional $l(a) = \alpha_0$ for $a = (\alpha_0, \alpha_\bullet) \in \mathfrak{a}$, and

$$\langle a_\bullet, a_\bullet^\star \rangle := (\alpha_\bullet \mid \alpha_\bullet) \equiv \langle \alpha_\bullet, \alpha_\bullet \rangle$$

is bilinear form defining the associative multiplication

$$a \cdot a^\star := \left(\langle \alpha_\bullet, \alpha_\bullet^\star \rangle,\ \alpha_\bullet \alpha_\bullet^\star\right) \equiv \langle a_\bullet, a_\bullet^\star \rangle\, d_t + a_\bullet a_\bullet^\star \tag{0.4}$$

and the left semiscalar product $\langle a \mid a \rangle = \langle a_\bullet^\star, a_\bullet \rangle$ in $\mathfrak{a}$. We call this four-dimensional $\star$-algebra the Hudson-Parthasarathy quantum Itô algebra (HP-algebra) $\mathfrak{b}(\mathsf{k})$ of the "Hilbert" space $\mathsf{k} = \mathbb{C}$, or *achieved vacuum* Itô algebra, reflecting the degeneracy of the rank one form

$$\langle a \mid a \rangle = (\alpha_1 - \mathrm{i}\alpha_2)^* (\alpha_1 - \mathrm{i}\alpha_2) = l(a^\star \cdot a),$$

corresponding to the purity of the "vacuum state" l on $\mathfrak{b}(\mathsf{k})$. Note that the algebra $\mathfrak{a} = \mathfrak{b}(\mathbb{C})$ has no identity but "death" element d_t killing any element $a \in \mathfrak{a}$ in the sense $a \cdot d_t = 0 = d_t \cdot a$ and normalizing the linear functional l as $l(d_t) = 1$. The functional l is positive with respect to the multiplication in the usual sense $l(a \star a) \geq 0$,satisfying $\star$-property $l(a^\star) = l(a)^*$. One can easily see that $\mathbb{C}d_t$ is the ideal of $\mathfrak{a}$ since $d_t \cdot \mathfrak{a} = 0 = \mathfrak{a} \cdot d_t$ such that $\mathfrak{a}_\bullet = \{a \in \mathfrak{a} : l(a) = 0\}$ is identical with the factor-algebra $\mathfrak{a}/\mathbb{C}d_t$. Moreover, the two-sided ideal

$$\mathfrak{i} = \{b \in \mathfrak{a} : l(b) = l(a \cdot b) = l(b \cdot c) = l(a \cdot b \cdot c) = 0, \forall a, c \in \mathfrak{a}\},$$

which obviously does not contain d_t, is trivial in the Itô algebra $(\mathfrak{a}, l)$: $\mathfrak{i} = \{0\}$.

We take the above properties as the definition of an (abstract noncommutative) *quantum Itô algebra* $(\mathfrak{a}, l)$, and in this capacity we can consider any associative involutory algebra $\mathfrak{a} = \mathfrak{a}_\bullet + \mathbb{C}d_t$ with $l(a) = \alpha_0$ and the trivial ideal $\mathfrak{i} = \{0\}$. As for $\mathfrak{a}_\bullet$ one can take any $\star$-algebra with semi-positive scalar product $\langle a_\bullet \mid b_\bullet \rangle = \langle a_\bullet^\star, b_\bullet \rangle$ given by a bilinear form with the properties

$$\langle a_\bullet b_\bullet, c_\bullet \rangle = \langle a_\bullet, b_\bullet c_\bullet \rangle,\ \ \langle b_\bullet^\star, b_\bullet \rangle \geq 0 \quad \forall b_\bullet \in \mathfrak{a}_\bullet$$

and factorize it with respect to the ideal

$$\mathfrak{i} = \{b \in \mathfrak{a}_\bullet : \langle a_\bullet, b_\bullet \rangle = \langle b_\bullet, a_\bullet \rangle = \langle a_\bullet, b_\bullet c_\bullet \rangle = \langle a_\bullet b_\bullet, c_\bullet \rangle = 0, \forall a, c \in \mathfrak{a}_\bullet\}$$

if $\mathfrak{i} \neq \{0\}$. In this general case one can also write $\langle a_\bullet, b_\bullet \rangle = (a \cdot b)_0 = \langle a, b \rangle$ and implement the $\star$-composition notation $a \star a = a \cdot a^\star$ which should be distinguished from the associative $*$-composition

$$a * a := a \star a - l\,(a \star a)\,d_t = (0, \alpha_\bullet * \alpha_\bullet) \equiv a a^\star$$

with the values in $\mathfrak{a}_\bullet$, representing the composition in the factor-algebra $\mathfrak{a}/\mathbb{C}d_t$. Choosing a selfadjoint basis $\{e_j = e_j^\star : j = 0, 1, \ldots\}$ of $(\mathfrak{a}, l)$ in such a way that $l\,(a) = \alpha_0$ if $a = \sum \alpha_j e_j$, one can describe every finite-dimensional Itô algebra as above by the Hermitian structure coefficients

$$c_j^{ik} = \left(c_j^{ki}\right)^*, \quad c_i^{nj} c_j^{km} = c_j^{nk} c_i^{jm}, \;\; c_j^{0k} = 0 = c_j^{i0}, \tag{0.5}$$

defining a multiplication table $\mathrm{d}\Lambda_t^i \mathrm{d}\Lambda_t^k = c_j^{ik} \mathrm{d}\Lambda_t^j$ of basis quantum stochastic differentials with $\mathrm{d}\Lambda_t^0 = \mathrm{d}t$.

Note that for the Abelian Itô algebras all structure matrices $c_j^{\bullet\bullet}$ are real and symmetric with strictly positive-definite $c_0^{\bullet\bullet}$, as it is always so in the case of one-dimensional $\mathfrak{a}_\bullet$. For example, the standard Poisson calculus given by Itô multiplication rule

$$\mathrm{d}m_t \mathrm{d}m_t = \lambda \mathrm{d}t + \mathrm{d}m_t$$

for the compensated Poisson increments of the intensity λ is associated with one-dimensional algebra $\mathfrak{a}_\bullet \sim \mathbb{C}$ of

$$a_\bullet \sim \alpha, \; a_\bullet^\star \sim \alpha^*, \;\; a_\bullet \star a_\bullet \sim |\alpha|^2, \; \langle a_\bullet \mid a_\bullet \rangle = \lambda\,|\alpha|^2$$

containing the unit $1 \in \mathfrak{a}_\bullet$ such that $\mathrm{d}m_t = \mathrm{d}m_t^*$ can be identified with the real two-vector $e = (0, 1)$ in $\mathfrak{a} = \mathfrak{a}_\bullet + \mathbb{C}d_t$ and $(\mathrm{d}m_t)^2$ with $e \star e = e^2 = (\lambda, 1)$. The standard Wiener calculus

$$\mathrm{d}w_t \mathrm{d}w_t = \mathrm{d}t, \; \mathrm{d}w_t \mathrm{d}t = \mathrm{d}t \mathrm{d}t = \mathrm{d}t \mathrm{d}w_t = 0$$

is also associated with one-dimensional but nilpotent algebra $\mathfrak{a}_\bullet \sim \mathbb{C}, \; a_\bullet * a_\bullet \sim 0$ without unit such that $\mathrm{d}w_t = \mathrm{d}w_t^*$ is identified with the element $e = (0, 1) = e^\star$ of second order nilpotent algebra $\mathfrak{a} = \mathfrak{a}_\bullet + \mathbb{C}d_t$ with respect to the multiplication $e^2 = (1, 0) \equiv d_t$ defined by the semi-scalar product $\langle a \mid a \rangle = |\alpha|^2$ for $a = (\alpha_0, \alpha)$.

It is well known [31] that the Poisson calculus, as well as the Wiener one, can be realized as a sub-calculus of the quantum stochastic calculus in the Fock space with respect to the vacuum state $1_\emptyset$ putting, for example,

$$w_t = A_{\_}\,(t) + A^+(t), \quad m_t = \sqrt{\lambda} A_-\,(t) + \sqrt{\lambda} A^+\,(t) + N\,(t)\,.$$

A natural question arises as to whether we can realize in this way any (non-commutative) calculus corresponding to an (abstract quantum) Itô algebra $(\mathfrak{a}, l)$ as defined above. To be more precise, the question concerns a non-commutative calculus of stochastic integrals with respect to operator representations of the processes $\Lambda_t\,(a) = \alpha_j \Lambda_t^j$ with given expectations $\mathrm{E}\,[\Lambda_t\,(a)] = \alpha_0 t$, with independent increments $\mathrm{d}\Lambda_t\,(a) = \Lambda_{t+\mathrm{d}t}\,(a) - \Lambda_t\,(a)\,, a \in \mathfrak{a}$, and realizing the multiplication table $\mathrm{d}\Lambda_t^i \mathrm{d}\Lambda_t^k = \sum_{j \geq 0} c_j^{ik} \mathrm{d}\Lambda_t^j$:

$$\mathrm{d}\Lambda_t\,(a)\,\mathrm{d}\Lambda_t\,(a^\star) = \alpha_i c_j^{ik} \alpha_k^* \mathrm{d}\Lambda_t^j = \mathrm{d}\Lambda_t\,(\,a \star a)\,. \tag{0.6}$$

We shall give a positive answer to this question, reducing it to the construction of a canonical dilation of infinitely divisible generating functions

$$\phi_t\,(b) = \mathrm{E}\,[\pi_t\,(b)] = \exp\{tl\;(b)\}\,, \tag{0.7}$$

defined by vacuum expectation of adapted quantum stochastic 'exponential' operators $\pi_t(u+a) =: \exp[\Lambda_t(a)]:$ which represent an Itô $\star$-monoid $\mathfrak{b}$ in Fock space as a unitalization $b = u + a$ of Itô algebra $\mathfrak{a}$ with l trivially extended on the unit $u = u^\star$ by $l(u) = 0$. These exponential representations, satisfying the operator $\star$-multiplicativity condition

$$\pi_t(u+a)\,\pi_t(u+a)^* = \pi_t(u + a\bigstar a)$$

with respect to the new associative composition

$$a\bigstar a := a + a^\star + a\star a \equiv b\star b - u$$

in $\mathfrak{a}$, can be obtained as the solutions of the logarithmic quantum stochastic differential equations

$$\text{(0.8)} \qquad \mathrm{d}\pi_t(b) = \pi_t(b)\,\mathrm{d}\Lambda_t(b-u), \qquad \pi_0(b) = I$$

with $a = b - u \in \mathfrak{a}$. Note that one can always identify the Itô monoid $\mathfrak{b}$ with the algebra $\mathfrak{a}$ by taking $u = 0$ and defining the $\star$-monoidal operation as $b\star b = a\bigstar a$ such that $: \exp[\Lambda_t(a\bigstar a)] := \pi_t(a\bigstar a)$.

In Chapter I we define such functions as solutions $\phi_t(b) = \exp\{tl(b)\}$ of the equation

$$\mathrm{d}\phi_t(b) = \phi_t(b)\,l(b)\,\mathrm{d}t, \quad \phi_0(b) = 1$$

obtained by the averaging E of (0.8), taking into account the independence of the increments $\mathrm{d}\Lambda(t, a)$ and $\pi_t(b)$, and that $l(b) = l(a)$.

Application of the Itô formula

$$\begin{aligned} \mathrm{d}(\pi_t(b)\,\pi_t(b)^*) &= \mathrm{d}\pi_t(b)\,\mathrm{d}\pi_t(b)^* + \mathrm{d}\pi_t(b)\,\pi_t(b)^* + \pi_t(b)\,\mathrm{d}\pi_t(b)^* \\ &= \pi_t(b)\,\pi_t(b)^*\,\mathrm{d}\Lambda_t(a^\star + a\cdot a^\star + a) = \pi_t(b)\,\pi_t(b)^*\,\mathrm{d}\Lambda_t(a\bigstar a) \end{aligned}$$

gives the multiplication rule $\pi_t(b)^*\pi_t(b) = \pi_t(b^\star b)$. Hence we have positive definiteness $\sum_{a,c}\lambda_a\phi_t(a\bigstar c)\,\lambda_c^* \geq 0$ and normalization $\phi_t(0) = 1$ of ϕ_t defined on $\mathfrak{b} = \mathfrak{a}$ as the monoid for each t with respect to this new $\star$-semigroup composition $\bigstar$ and unit $u = 0$. This results from positivity $\mathrm{E}[X^*X] \geq 0$ and normalization $\mathrm{E}[I] = 1$ of the vacuum (and any) expectation on the operator algebra generated by linear combinations $X = \sum\lambda_b\pi_t(b)$. Any such function ϕ_t that is included into a continuous one-parameter semigroup $\{\phi_r : r \in \mathbb{R}_+\}$,

$$\phi_r(a)\,\phi_s(a) = \phi_{r+s}(a), \quad \phi_0(a) = 1$$

of generating functionals on Itô $\star$-algebra $\mathfrak{a}$ as the monoid $\mathfrak{b}$ is called infinitely divisible law [16].

In Chapter 2 we fulfil the Itô programme for quantum stochastic calculus in a dimension-free form, proving continuity of quantum stochastic integrals in Fock scales and constructing a noncommutative theory of multiple adapted and nonadapted quantum stochastic integrals which give solutions to linear quantum stochastic differential equations in the Wick form of time-ordered exponentials. We shall use the approach based upon explicit definition of these integrals in Fock representation, which allows to extend them to nonadapted operator-functions. We will also obtain a functional quantum Itô formula for a quantum stochastic "curve" X with adapted, or even nonadapted operator values $X_t \in \mathcal{A}$, having noncommuting quantum stochastic increments $\boldsymbol{D} = (D_j)$ with values in the tensor product $\mathbb{A} = \mathcal{A}\otimes\mathfrak{a}$ of an operator algebra $\mathcal{A}$ with the Itô algebra $\mathfrak{a}$. For a 'nice' function f the adapted Itô formula with respect to a filtration $(\mathcal{A}_t)_{t>0}$ generated by an initial algebra $\mathcal{A}_0$ and $(\Lambda_t^{\bullet})_{t>0}$ can be written for $D_j \in \mathcal{A}_t$ in the Pseudo-Poisson form [18] as

$$\mathrm{d}f(X_t) = (f(\mathbf{X}_t + \mathbf{D}_t) - f(\mathbf{X}_t))_j \, \mathrm{d}\Lambda_t^j. \tag{0.9}$$

Here $\mathbf{X}$ and $\mathbf{D}$ are canonical images $X \oplus \boldsymbol{O}$ and $O \oplus \boldsymbol{D}$ of $X_t \in \mathcal{A}_t$ and $\boldsymbol{D} \in \mathcal{A}_t \otimes \mathfrak{a} \equiv \mathbb{A}_t$ in the formal sums $\mathbf{X} + \mathbf{D} := X \oplus \boldsymbol{D}$ as the elements of the algebra $\mathbb{B}_t = \mathcal{A}_t \otimes \mathfrak{b} = \mathcal{A}_t \oplus \mathbb{A}_t$ equipped with the involution $(X \oplus \mathbf{D})^{\dagger} = X^* \oplus \mathbf{D}^{\star}$ and the product

$$(\mathbf{X} + \mathbf{D})(\mathbf{X} + \mathbf{D})^{\dagger} = XX^* \oplus (X\boldsymbol{D}^{\star} + \boldsymbol{D}X^* + \boldsymbol{D} \cdot \boldsymbol{D}^{\star}), \tag{0.10}$$

where $X\mathbf{D}^{\star} = \left(XD_j^*\right)$, $\mathbf{D}X^* = (D_j X^*)$ and $\boldsymbol{D} \cdot \boldsymbol{D}^{\star} = \left(D_i c_j^{ik} D_k^*\right)$. Since $f(\mathbf{X}) = f(X) \oplus \boldsymbol{O}$, the whole problem is reduced to computing the operator function $f(\mathbf{X} + \mathbf{D})$ using the product in $\mathbb{B}_t$. Thus, in the case $f(X) = X^m$, where

$$((\mathbf{X} + \mathbf{D})^m - \mathbf{X}^m)_j = D_j^{(m)}$$

with $D_j^{(0)} = 0$, and

$$D_j^{(n+1)} = XD_j^{(n)} + D_i c_j^{ik} D_k^{(n)}.$$

Note that in the nonstochastic case this new formula also gives an interesting difference form of the non-commutative chain rule $\mathrm{d}f(X_t) = B_t \mathrm{d}t$ for a smooth curve X_t in an initial algebra $\mathcal{A}_0$ with non-commuting derivative $D_t \in \mathcal{A}_0$. In this case the algebra $\mathfrak{a}_{\bullet}$ is zero-dimensional, $\mathfrak{a} = \mathbb{C}d_t$, and $\mathbb{A}_0 = \mathcal{A}_0 \otimes d_t$ is nilpotent algebra of first order, $A \cdot A^* = 0$, coinciding as the linear space with $\mathcal{A}_0$ such that

$$(\mathbf{X} + \mathbf{D})(\mathbf{X} + \mathbf{D})^{\star} = XX^* \oplus (XD^* + DX^*).$$

In particular, for any polynomial, $f(X) = X^m$ say, one immediately obtains

$$\mathrm{d}X_t^m = ((\mathbf{X}_t + \mathbf{D}_t)^m - \mathbf{X}_t^m)\,\mathrm{d}t = \sum_{n=1}^{m} X_t^{m-n} D_t X_t^{n-1} \mathrm{d}t$$

as a particular case of (0.9). Here $X = (X, 0)$, $D = (0, D)$ and we took into account that

$$(\mathbf{X} + \mathbf{D})^m = \mathbf{X}^m + \sum_{n=1}^{m} \mathbf{X}^{m-n} \mathbf{D} \mathbf{X}^{n-1} = \left(X^m, \sum_{n=1}^{m} X^{m-n} D X^{n-1}\right),$$

since $\mathbf{D}\mathbf{X}^n\mathbf{D} = \mathbf{0}$ for $\mathrm{d}\Lambda_t^0 \mathrm{d}\Lambda_t^0 = 0$ corresponding to $\mathrm{d}\Lambda^0 = \mathrm{d}t$.

In the nonadapted case the formula (0.9) also remains valid, with $\mathbf{X}_t = X_t \oplus \boldsymbol{\nabla} X_t$ given by quantum stochastic derivatives $\boldsymbol{\nabla}_t X_t = (\triangledown_{t,j} X_t) \in \mathbb{A} = \mathcal{A} \otimes \mathfrak{a}$, the noncommutative analog of Malliavin derivative with respect to the canonical integrators $\boldsymbol{\Lambda}_t = \left(\Lambda_t^j\right)$. As to author's knowledge, this general formula is not known even in the classical (commutative) case.

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Part 1. Infinitely divisible positive-definite functions and their representations

1. Introduction

In this paper we study two types of representations associated with a positive infinitely divisible state on an arbitrary $\star$-semigroup $\mathfrak{b}$ [6] with a unit $u \in \mathfrak{b}$. The first, 'differential' type, is connected with an indefinite metric space representation of conditionally positive functions $\mathfrak{b} \to \mathbb{C}$ in pseudo-Euclidean Minkowski space constructed in [14]. In the case when $\mathfrak{b}$ is a group, this representation was obtained by simple generalization [50] of the Gelfand-Naimark-Segal (GNS) construction from positive definite to conditionally positive definite functions on $\mathfrak{b}$. However our main interest will be the case when $\mathfrak{b}$ is obtained by a unitalization of a noncommutative Itô algebra $\mathfrak{a}$ as a parametrizing algebra for the quantum stochastic differentials of a quantum Levy process as operator-valued processes with independent increments in a quite general noncommutative sense.

In our construction the Hilbert space of the GNS representation is replaced by a pseudo-Hilbert (Minkowski) space which can be decomposed into a direct integral sum of a pre-Hilbert space and a one-dimensional complex space in accordance with the fact that the conditional positiveness (2.5) has co-dimension one. In the first section we show that this representation can be realized by block-triangular matrices of the form

$$\mathbf{B} = \begin{bmatrix} 1 & b^- & \beta \\ 0 & B & b_+ \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{B}^\dagger = \begin{bmatrix} 1 & b_+^* & \beta^* \\ 0 & B^* & b^{-*} \\ 0 & 0 & 1 \end{bmatrix} \tag{1.1}$$

with pseudo-Hermitian conjugation $(\boldsymbol{k}\mathbf{B}^\dagger \mid \boldsymbol{k}) = (\boldsymbol{k} \mid \boldsymbol{k}\mathbf{B})$ defined by the indefinite scalar product

$$(\boldsymbol{k} \mid \boldsymbol{k}') = k_-^* k_+' + (k_\circ \mid k_\circ') + k_+^* k_-', \tag{1.2}$$

on the rows $\boldsymbol{k} = (k_-, k_\circ, k_+)$, where $k_+ \in \mathbb{C} \ni k_-$ and $k_\circ$ is a vector-row from a complex Euclidean space $\mathcal{K}$. The algebra of the triangular matrices $\mathbf{A} = \mathbf{B} - \mathbf{I}$ realizes the non-matrix multiplication table

$$\begin{pmatrix} \alpha^* & a_+^* \\ a^{-*} & A^* \end{pmatrix} \cdot \begin{pmatrix} \alpha & a^- \\ a_+ & A \end{pmatrix} = \begin{pmatrix} a_+^* a_+ & a_+^* A \\ A^* a_+ & A^* A \end{pmatrix} \tag{1.3}$$

in terms of the 2×2 block-matrices (which are not matrices but tables)

$$\boldsymbol{A} = \begin{pmatrix} \alpha & a^- \\ a_+ & A \end{pmatrix}, \ \boldsymbol{A}^* = \begin{pmatrix} \alpha^* & a_+^* \\ a^{-*} & A^* \end{pmatrix},$$

defining the stochastic Itô differentials in the Hudson and Parthasarathy [26], [29] quantum calculus. Here $a^- = b^-$, $a_+ = b_+$, $A = B - I$, $\alpha = \beta$, with involution $\mathbf{A}^\dagger = \mathbf{B}^\dagger - \mathbf{I}$ defined in (1.1) by the usual Hermitian conjugation $\boldsymbol{A}^*$ of the tables $\boldsymbol{A}$ in terms of $A^* = B^* - I$ in $\mathcal{K}$, where I is the unit operator in $\mathcal{K}$.

This observation, which lays the foundation of a new formulation [7], [9] of quantum stochastic calculus, allows us to extend it to arbitrary algebras with infinitely divisible state ϕ. We mention two particular algebras of classical stochastic differentials in the case of one-dimensional $\mathcal{K} = \mathbb{C}$:

(1) the Wiener case: $A = 0$, $a^- = a_+^*$, $\alpha \in \mathbb{C}$,
(2) the Poisson case: $A \neq 0$, $a^- = a_+^* = 0 = \alpha$.

If we consider A as the coefficient $A_\circ^\circ$ at the standard Poisson differential $\mathrm{d}n = \mathrm{d}\Lambda_\circ^\circ$, $a^- = a_+^*$ as the coefficient $A_\circ^- = A_+^{\circ *}$ at the Wiener standard differential $\mathrm{d}w = \mathrm{d}\Lambda_-^\circ + \mathrm{d}\Lambda_\circ^+$, and α as the coefficient A_+^- at $\mathrm{d}t = \mathrm{d}\Lambda_-^+$, then in both cases we obtain the realization of the classical Itô formula for stochastic differential $\mathrm{d}x = \sum_{\mu,\nu} A_\nu^\mu \mathrm{d}\Lambda_\mu^\nu \equiv \langle \mathbf{A}, \mathrm{d}\Lambda \rangle$ in the form

$$\mathrm{d}(x^* x) = x^* \mathrm{d}x + \mathrm{d}x^* x + \mathrm{d}x^* \mathrm{d}x = \left\langle x^* \mathbf{A} + \mathbf{A}^\dagger x + \mathbf{A}^\dagger \mathbf{A}, \mathrm{d}\Lambda \right\rangle$$

of difference multiplication $\mathbf{Y}^\dagger\mathbf{Y} - x^* x\mathbf{I} = x^*\mathbf{A} + \mathbf{A}^\dagger x + \mathbf{A}^\dagger\mathbf{A}$ of the triangular matrices $\mathbf{Y} = x\mathbf{I} + \mathbf{A}$, $\mathbf{Y}^\dagger = x^*\mathbf{I} + \mathbf{A}^\dagger$, where $\mathbf{I}$ is the unit 3×3 matrix and $\mathbf{A}^\dagger\mathbf{A}$ is defined by the multiplication table (1.3).

In the second section we construct a second 'integral' type of representation of an infinitely divisible chaotic state on $\mathfrak{b}$ by means of exponential indefinite metric representation and we establish its relation with the calculus of Maassen-Meyer kernels [35], [38], [40], which define chaotic distribution of quantum random variables and processes.

The algebra of these kernels turns out to be isomorphic to the group algebra of the exponential representation of $\mathfrak{b}$ in a pseudo-Fock space, and its Fock projection defines an associated infinitely divisible representation of $\mathfrak{b}$ generating the corresponding quantum stochastic calculus in an appropriate Hilbert scale [15]. We note that this leads in a natural way to the Araki-Woods construction [4] associated with an infinitely divisible state in the case when $\mathfrak{b}$ is a group.

Finally, in the third section, we study the structure and consider examples of pseudo-Poisson chaotic states characterized by the linearity of conditionally positive functions $l(b) = \ln f(b)$ on a $\star$-algebra $\mathfrak{b}$. To this type belong the quantum Wiener states of Heisenberg commutation relations, as well as quantum Poisson states on noncommutative C^*-algebras $\mathfrak{b}$, studied in [10]. Unitary representations connected with infinite divisibility of states and their applications to the quantum probability theory were studied in [23], [27], [28], [43], [49] on groups and in [46] on bi-algebras.

2. Representations of conditionally positive functionals on $\star$-semigroups

Let $(X, \mathfrak{F}, \mu)$ be a measurable space X with a σ-algebra $\mathfrak{F}$ and a positive σ-finite atomless measure $\mu : \mathfrak{F} \ni \Delta \mapsto \mu_\Delta$, $\mu_{\mathrm{d}x} \equiv \mathrm{d}x := \mathrm{d}\mu(x)$, and let $\mathfrak{b}$ be a semigroup with involution

$$b \mapsto b^\star, \quad (b \cdot c)^\star = c^\star \cdot b^\star,$$

and with neutral element (unit) $u = u^\star$, $u \cdot b = b = b \cdot u$ for any $b \in \mathfrak{b}$. Typically $\mathfrak{b}$ will be the unitalization $u + \mathfrak{a}$ of a noncommutative Itô $\star$-algebra $\mathfrak{a}$, in which case

$$(u + a) \cdot (u + c) = u + a + c + a \cdot c \equiv u + a \bullet c$$

such that the momoidal product should be identified with $a \bullet c = a + c + a \cdot c$ if u is identified with zero, or simply write $b \cdot c = bc$ if $\mathfrak{a}$ is realized as a $\star$-subalgebra of a unital $\star$-algebra with $u = 1$. However in what follows one can take any group with $u = 1$ and $b^\star = b^{-1}$ as $\mathfrak{b}$, or any $\star$-submonoid of an operator algebra $\mathcal{B}$, a unit ball of a unital C*-algebra say, or even a submonoid of an idempotent algebra with trivial involution $b^\star = b$, e.g. a filter $\mathfrak{b}$ of a Boolean algebra $\mathfrak{B}$.

Denote $\mathfrak{m}$ the monoid of integrable step-maps $g : X \to \mathfrak{b}$, that is $\mathfrak{b}$-valued functions $x \mapsto g(x)$ having countable images $g(X) = \{g(x) : x \in X\} \subseteq \mathfrak{b}$, $|g(X)| < \infty$ and integrable co-images $\Delta(b) = \{x \in X : g(x) = b\} \in \mathfrak{F}$ in the sense $\mu_{\Delta(b)} < \infty$ for all $b \in \mathfrak{b}$ except $b = u$. We define on $\mathfrak{m}$ an inductive structure of a $\star$-monoid with pointwise defined operations $g^\star(x) = g(x)^\star$, $(g \cdot h)(x) = g(x) \cdot h(x)$ and unit $e(x) = u$ for all $x \in X$, considering $\mathfrak{m}$ as the union $\cup\mathfrak{m}_\Delta$ of subsemigroups $\mathfrak{m}_\Delta$ of functions $g : X \to \mathfrak{b}$ having integrable supports

$$\Delta = \mathrm{supp} g = \{x \in X : g(x) \neq u\}$$

in a $\Delta \in \mathfrak{F}$ with $\mu_\Delta < \infty$.

It is convenient to describe the $\star$-monoid $\mathfrak{b}$ by means of a single Hermitian operation $a \star c = a \cdot c^\star$ satisfying the relations

$$b \star u = b, \quad u \star (u \star b) = b \quad \forall b \in \mathfrak{b}$$

defining $u = u^\star$ as right unit for the composition $\star$, $b^\star$ as $u \star b$, and

$$u \star ((a \star c) \star b) = b \star ((u \star c) \star a)$$

corresponding to $(a \cdot c^\star)^\star = (a \star c)^\star = c \star a = c \cdot a^\star$ and associativity of the semigroup operation $a \cdot b$. This allows one to define both the product and involution in a $\star$-monoid $\mathfrak{m}$ by a single Hermitian binary operation $f \star h = g, g(x) = f(x) \star h(x)$ with left unit $e \in \mathfrak{m}$ which recovers the involution by $g^\star(x) = e \star g$ and the associative product by $g \cdot h = g \star (e \star h)$ for all g, $h \in \mathfrak{m}$.

Following [6] we say that a generating state functional over the monoid $\mathfrak{m}$, or briefly a state over $\mathfrak{m}$, is a mapping $\phi : \mathfrak{m} \to \mathbb{C}$ satisfying the condition $\phi(e) = 1$ and positive definiteness

$$\sum_{f,h \in \mathfrak{m}} \kappa_f \phi(f \star h) \kappa_h^* \geq 0, \quad \forall \kappa_g \in \mathbb{C} : |\mathrm{supp}\kappa| < \infty, \tag{2.1}$$

where $|\cdot|$ denotes the cardinality of the set $\mathrm{supp}\kappa = \{g \in \mathfrak{m} : \kappa_g \neq 0\}$.

Wewe introduce on $\mathfrak{m}$ a commutative and associative partial operation $f \sqcup h :=$ $f \cdot h$ for any functions $f, h \in \mathfrak{m}$ with disjoint supports $\mathrm{supp}\, f \cap \mathrm{supp}\, h = \emptyset$. Thus defined map $\mathfrak{m}_\Delta \times \mathfrak{m}_{\Delta'} \to \mathfrak{m}_\Delta \sqcup \mathfrak{m}_{\Delta'}$ for any measurable disjoint $\Delta, \Delta' \in \mathfrak{F}$ is obviously lifted to the tensor product $\mathbb{C}\mathfrak{m}_\Delta \otimes \mathbb{C}\mathfrak{m}_{\Delta'}$ of the enveloping semigroup algebras of the $\star$-monoids $\mathfrak{m}_\Delta$ and $\mathfrak{m}_{\Delta'}$. The operation $\sqcup$ is well defined even for an infinite countable family $\{g_n\} \in \mathfrak{m}$ with mutually disjoint supports $\Delta_n = \mathrm{supp}\, g_n$ by $\sqcup g_n(x) = g_m(x)$ for all $x \in \mathrm{supp}\, g_m$ and any m, otherwise $\sqcup g_n(x) = u$ if $x \notin \sum \Delta_n$. Any function $g \in \mathfrak{m}$ can be written as $\sqcup g_n$ in terms of the b_n-valued indicators g_n of its non-unit images $b_n \in g(X), b_n \neq u$ with respect to the partition $\mathrm{supp}\, g = \sum \Delta_n$ into the co-images $\Delta_n = \Delta(b_n)$.

We call a state ϕ over $\mathfrak{m}$ *chaotic* if

$$\phi\left(\bigsqcup_{n=1}^{\infty} g_n\right) = \prod_{n=1}^{\infty} \phi(g_n),$$

where $\prod_{n=1}^{\infty} \phi(g_n) = \lim_{N\to\infty} \prod_{n=1}^{N} \phi(g_n)$ for any functions $g_n \in \mathfrak{m}$ with pairwise disjoint supports: $\mathrm{supp}\, g_n \cap \mathrm{supp}\, g_m = \emptyset$ for all $n \neq m$. This condition is fulfilled for ϕ of the exponential form $\phi(g) = e^{\lambda(g)}$ with

$$\lambda(g) = \int l(x, g)\mathrm{d}x, \quad l(x, g) = l_x(g(x)), \tag{2.2}$$

which corresponds to absolute continuity (for all $\Delta \in \mathfrak{F}$ we have $\mu_\Delta = 0 \Rightarrow \lambda_\Delta(b) =$ 1) of the σ-additive measure $\lambda_\Delta(b) := \lambda(b_\Delta)$ for each $b \in \mathfrak{b}$, where $b_\Delta(x) = b$ for all $x \in \Delta$ and $b_\Delta(x) = u$ for $x \notin \Delta$ is the b-indicator of the subset $\Delta \subseteq X$ with $b \neq u$ as an elementary $\mathfrak{b}$-valued function $b_\Delta \in \mathfrak{m}$.

The function $\phi_\Delta : \mathfrak{b} \to \mathbb{C}$ given by

$$\phi_\Delta(b) = \exp\left\{\int_\Delta l_x(b)\mathrm{d}x\right\} = \phi(b_\Delta), \tag{2.3}$$

defines an *infinitely divisible state* over the monoid $\mathfrak{b}$ in the sense of the equality $\phi_\Delta(b) = \prod \phi_{\Delta_l}(b)$ in the limit of any integral sum sequence given by the decomposition $\Delta = \Sigma\Delta_i$, $\mu_{\Delta_i} \searrow 0$, where $\phi_{\Delta_i}(b) \to 1$ for any $b \in \mathfrak{b}$. If the Radon-Nikodym derivative $l_x(b) = \mathrm{d}\lambda(b)/\mathrm{d}x$ of the absolutely continuous measure $\mathrm{d}\lambda(b) = \lambda_{\mathrm{d}x}(b)$ does not dependent on x, functions $\phi_\Delta(b) = e^{l(b)\mu_\Delta} \equiv \phi^{\mu_\Delta}(b)$ are forming a continuous Abelian semigroup

$$\{\phi^t : t \in \mathbb{R}^+\}, \quad \phi^0(b) = 1, \quad [\phi^r \cdot \phi^s](b) = \phi^{r+s}(b)$$

with respect to the pointwise multiplication of $\phi^t = e^{tl(b)}$. Necessary and sufficient conditions for the function (2.2) corresponding to the infinitely divisible state (2.3) are given by the following theorem, where we assume that X admits a net of decompositions of the Vitali system in which $\mu_\Delta \searrow 0$, $x \in \Delta$, as $\Delta \searrow \{x\}$.

Theorem 1. *In our notation the following conditions are equivalent:*

(i) *The function $\phi_\Delta(b)$ defined for any set $\Delta \in \mathfrak{F}$ of finite measure $\mu_\Delta < \infty$ as $\phi(b_\Delta)$ by a functional $\phi : \mathfrak{m} \to \mathbb{C}$ on the b-indicator b_Δ is a generating function of an infinitely divisible state over $\mathfrak{b}$ such that for any $b \in \mathfrak{b}$ it is absolutely continuos multiplicative measure in the sense $\mu_\Delta = 0 \Rightarrow \phi_\Delta(b) = 1$ for all $\Delta \in \mathfrak{F}$, $b \in \mathfrak{b}$ with the limit*

$$l_x(b) = \lim_{\Delta \searrow \{x\}} \frac{1}{\mu_\Delta}(\phi_\Delta(b) - 1) \tag{2.4}$$

existing almost everywhere in the Lebesgue-Vitali sense [47]*.*

(ii) *$\phi(g) = \exp\{\lambda(g)\}$, where $\lambda(g) = \lambda_\Delta(b)$ for $g = b_\Delta$ is an absolutely continuous complex measure of $\Delta \in \mathfrak{F}$, and for any integrable set $\Delta \subseteq X$ the function $b \mapsto \lambda_\Delta(b)$ is conditionally positive definite*

$$\sum_{a,c\in\mathfrak{b}} \kappa_a \lambda_\Delta(a \star c)\kappa_c^* \geq 0, \, \forall \kappa : |\mathrm{supp}\kappa| < \infty, \, \sum_{b\in\mathfrak{b}} \kappa_b = 0, \tag{2.5}$$

where $\lambda_\Delta(u) = 0$ and $\lambda_\Delta(b^\star) = \lambda_\Delta(b)^*$ for any $b \in \mathfrak{b}$.

(iii) *There exists:*
1) an integral $\star$-functional $\lambda(g) := \int l(x, g)\,\mathrm{d}x$ with complex density $l : \mathfrak{m} \to L^1(X)$ such that $l(g)^* = l(g^\star)$ and whose values $l(x, g) = 0$ for all $g(x) = u$ and $l(x, b_\Delta) = l_x(b)$ with $x \in \Delta$ are independent of Δ;
2) a vector map $\mathrm{k} : g \mapsto \int^\oplus \mathrm{k}(x, g)\,\mathrm{d}x$ to the subspace $\mathrm{K} \subseteq \int^\oplus \mathrm{K}_x \mathrm{d}x$ of square integrable functions $\mathrm{k} : x \mapsto \mathrm{k}(x) \in \mathrm{K}_x$, $\|\mathrm{k}\|^2 = \int \|\mathrm{k}(x)\|_x^2 \mathrm{d}x < \infty$ with respect to scalar products $\langle \mathrm{k}_x \mid \mathrm{k}'_x\rangle \equiv \mathrm{k}_x^* \mathrm{k}'_x$ in the pre-Hilbert spaces K_x, with values $\mathrm{k}(x, b_\Delta) = \mathrm{k}_x(b) \in \mathrm{K}_x$ independent of $\Delta \ni x$ and $\mathrm{k}(x, b_\Delta) = 0$ if $x \notin \Delta$ such that $\mathrm{k}(x, g) = 0$ if $g(x) = u$; the map k, together with the adjoint functions $\mathrm{k}^\star(x, g) = \mathrm{k}(x, g^\star)^*$ as the linear functionals $\mathrm{k}^\star(g) = \int^\oplus \mathrm{k}^\star(x, g)\mathrm{d}x \in \mathrm{K}^*$, satisfies the condition

$$\mathrm{k}^\star(g)\mathrm{k}(h) = \lambda(g \cdot h) - \lambda(g) - \lambda(h), \quad \forall g, h \in \mathfrak{m}; \tag{2.6}$$

3) a unital $*$-representation $j : g \mapsto \int^\oplus j(x, g)\mathrm{d}x \equiv G$, $j(g)^* = j(g^\star)$

$$j(x, g)j(x, h) = j(x, g \cdot h), \quad j(x, g) = I_x \; \forall x : g(x) = u$$

of a $\star$-semiring $\mathfrak{m}$ in the $*$-algebra of decomposable operators $G : \mathrm{K} \ni \mathrm{k} \mapsto \int^\oplus j(x, g)\mathrm{k}(x)\mathrm{d}x$ with $j(x, b_\Delta) = j_x(b)$ independent of $\Delta \ni x$ and $j(x, b_\Delta) = I_x$ if $x \notin \Delta$, which satisfy the cocycle property

$$j(g)\mathrm{k}(h) = \mathrm{k}(g \cdot h) - \mathrm{k}(g), \quad \mathrm{k}^\star(g)j(h) = \mathrm{k}^\star(g \cdot h) - \mathrm{k}^\star(h), \quad \forall g, h \in \mathfrak{m} \tag{2.7}$$

and are continuous in K *with respect to the poly-norm*

$$\|\mathrm{k}\|^{f}=\left(\int\|j(x,f)\mathrm{k}(x)\|_{x}^{2}\mathrm{d}x\right)^{1/2},\quad f\in\mathfrak{m}. \tag{2.8}$$

(iv) *For almost all $x\in X$ there exists a pseudo-Hilbert space $\mathbb{K}_x$, a unital †-representation*

$$\mathbf{j}_x(b\cdot c)=\mathbf{j}_x(x,b)\mathbf{j}_x(x,c),\ \ \mathbf{j}_x(x,b^\star)=\mathbf{j}_x(x,b)^\dagger,\ \mathbf{j}_x(u)=\mathbf{I}_x$$

of $\mathfrak{b}$ in the algebra of linear operators $\mathcal{L}(\mathbb{K}_x)=\{\mathbf{L}:\mathbb{K}_x\to\mathbb{K}_x:\mathbf{L}^\dagger\mathbb{K}_x\subseteq\mathbb{K}_x\}$, where $\mathbf{L}^\dagger$ is pseudo-Hermitian conjugation $\left(\mathbf{k}\mid\mathbf{L}^\dagger\mathbf{k}\right)=(\mathbf{L}\mathbf{k}\mid\mathbf{k})$, $\mathbf{k}\in\mathbb{K}_x$, and a vector $\mathbf{e}_x\in\mathbb{K}_x$ of zero pseudo-norm $(\mathbf{e}_x\mid\mathbf{e}_x)=0$ such that the function

$$l_x(b)=(\mathbf{e}_x\mid\mathbf{j}_x(b^\star)\mathbf{e}_x)=(\mathbf{j}_x(b)\mathbf{e}_x\mid\mathbf{e}_x) \tag{2.9}$$

is integrable for each $b\in\mathfrak{b}$ on any $\Delta\subseteq X$ with $\mu_\Delta<\infty$ and $\int_\Delta l_x(b)\mathrm{d}x=\ln\phi_\Delta(b)$. Moreover, each $\mathbb{K}_x$ can be chosen in the complex Minkowski form $\mathbb{K}_x=\mathbb{C}\oplus\mathcal{K}_x^\circ\oplus\mathbb{C}\equiv\mathcal{K}_x^{\cdot}$ with $\mathbf{e}_x\in\mathbb{K}_x$ given as pseudo-adjoint $\mathbf{e}_x=\mathbf{e}_x^\dagger\equiv e^{\cdot}$ to the row $\boldsymbol{e}_x=(1,0,0)\equiv e_{.}$ of the dual space $\mathbb{K}_x^\dagger=\mathcal{K}_{.}$ of triples $k_{.}=(k_-,k_\circ,k_+)$ with the canonical pairing $\langle k_{.},h^{\cdot}\rangle_x=k_\iota h^\iota\equiv k_{.}h^{\cdot}$ and an antilinear embedding $\mathbf{k}\mapsto\mathbf{k}^\dagger$ of $\mathbf{k}\equiv k^{\cdot}\in\mathcal{K}_x^{\cdot}$ into $\mathcal{K}_{.}$ such that $k^\pm=k_\mp^*\in\mathbb{C}$, $k^\circ=k_\circ^*\in\mathcal{K}_x^\circ$, defining the Minkowski scalar product

$$(\mathbf{k}\mid\mathbf{k})_x:=k_-^*k_++\langle\mathrm{k}\mid\mathrm{k}\rangle_x+k_+^*k_-\equiv\left\langle\mathbf{k}^\dagger,\mathbf{k}\right\rangle_x \tag{2.10}$$

on $\mathcal{K}_x^{\cdot}$ in terms of the Euclidean scalar product $k_\circ k^\circ=\langle\mathrm{k}\mid\mathrm{k}\rangle_x$ for $k_\circ=\mathrm{k}^*$ and $k^\circ=\mathrm{k}\in\mathcal{K}_x^\circ$. The representation $\mathbf{j}_x$ is chosen then in the triangular form

$$j^{\cdot}_{\cdot}(x,b)=\begin{bmatrix}1&j_\circ^-(x,b)&j_+^-(x,b)\\0&j_\circ^\circ(x,b)&j_+^\circ(x,b)\\0&0&1\end{bmatrix},\quad j^{\cdot}_{\cdot}(x,b^\star)=\begin{bmatrix}1&j_+^\circ(x,b)^*&j_+^-(x,b)^*\\0&j_\circ^\circ(x,b)^*&j_\circ^-(x,b)^*\\0&0&1\end{bmatrix},$$

defining its dual action $\mathbf{j}_x(b^\star)^\dagger$ on $\mathcal{K}_{.}$ as right multiplication by this operator-matrix $j^{\cdot}_{\cdot}(x,b)$:

$$\mathbf{j}(b):(k_-,k_\circ,k_+)\mapsto(k_-,\ k_-j_\circ^-(b)+k_\circ j_\circ^\circ(b),\ k_-j_+^-(b)+k_\circ j_+^\circ(b)+k_+)\equiv k_{.}j^{\cdot}_{\cdot}(b).$$

The Hermitian conjugation $\mathbf{L}^\dagger=\mathbf{g}\mathbf{L}^*\mathbf{g}$ of the block-matrix operators $\mathbf{L}=[L_\nu^\mu]$ with respect to the indefinite form (2.10) is given by the metric tensor $\mathbf{g}=\left[\delta_{-\nu}^\mu\right]$ *corresponding to the inversion $-(-,\circ,+)=(+,\circ,-)$ of the ordered set $\{-<\circ<+\}$ of the indices $\mu,\nu=-,\circ,+$.*

Proof. We first establish the simple implications (iv) ⇒ (iii) ⇒ (ii) ⇒ (i), and then we prove (i) ⇒ (iv) constructing, similarly to the Gelfand-Naimark-Segal construction, a concrete pseudo-Euclidean representation of the logarithmic derivative of the generating functional ϕ_Δ of infinitely divisible state over $\mathfrak{b}$ with respect to λ_Δ.

(iv) ⇒ (iii). If $\mathbf{e}_x=(1,\mathrm{e},\varepsilon)_x^\dagger\equiv e_x^{\cdot}$ is a zero pseudo-norm vector-column with the components $e_x^-=\varepsilon_x^*\in\mathbb{C}$, $e_x^\circ=\mathrm{e}_x^*\in\mathcal{K}_x^\circ$ and $e_x^+=1$ such that $\|\mathrm{e}_x\|^2=2\operatorname{Re}\varepsilon_x$, defining (2.9) in the triangular matrix representation as $l_x(b)=e_{\cdot}^x j^{\cdot}_{\cdot}(x,b)e_x^{\cdot}$, where $e_-^x=1$, $e_\circ^x=\mathrm{e}_x,e_+^x=\varepsilon_x$, then we can take $\lambda(g)=\int l(x,g)\,\mathrm{d}x$, where $l(x,g)=l_x(g(x))$ has obviously properties

$$l(x,g^\star)=e_{\cdot}^x j^{\cdot}_{\cdot}(x,g)^\dagger e_x^{\cdot}=l(x,g)^*,$$

and $l(x, b_\Delta) = e^x_\cdot j^\cdot_\cdot(x, b)e^\cdot_x$ does not depend on Δ if $x \in \Delta$, otherwise

$$l\left(x, b_\Delta\right) = e^x_\cdot j^\cdot_\cdot(x, u)e^\cdot_x = e^x_\cdot e^\cdot_x = \left\|\mathrm{e}_x\right\|^2 - 2\operatorname{Re}\varepsilon_x = 0.$$

We denote by K_x the completion of the pre-Hilbert space $\mathcal{K}^\circ_x$ by the Cauchy 'kets' k with respect to the poly-norm

$$\|\mathrm{k}\|^\bullet_x = \{\|\mathrm{k}\|^a_x = \|j^\circ_\circ(x, a)\mathrm{k}\|, a \in \mathfrak{b}\}$$

as fundamental sequences $\{k^\circ_n\}$ in $\mathcal{K}^\circ_x$ which do not have limits in $\mathcal{K}^\circ_x$ simultaneously with respect to all seminorms $\|k^\circ\|^a_x$, $a \in \mathfrak{b}$. For each $g \in \mathfrak{m}$ let $\mathrm{k}\left(g\right)$ denote the vector function

$$\mathrm{k}\left(x, g\right) = (j^\circ_\circ(x, g) - 1)e^\circ_x + j^\circ_+(x, g) = j^\circ_\mu(x, g)e^\mu_x - e^\circ_x$$

with values $\mathrm{k}(x, g) \in \mathrm{K}_x$, with the adjoint 'bras' $\mathrm{k}^\star(x, g) = e^x_\mu j^\mu_\circ(x, g^\star) \in \mathrm{K}^*_x$, where $e^{x*}_{-\mu} = e^\mu_x$ and for short we use the notation $j^\mu_\nu\left(x, g\right) = j^\mu_\nu\left(x, g\left(x\right)\right)$. This function is square integrable since

$$\mathrm{k}\left(g\right)^* \mathrm{k}\left(g\right) = \lambda\left(g^\star \cdot g\right) - \lambda\left(g\right) - \lambda\left(g^\star\right) = \left\|\mathrm{k}\left(g\right)\right\|^2 < \infty$$

due to the condition (2.6) which is verified straightforward,

$$\begin{aligned}
\mathrm{k}^\star\left(g\right)\mathrm{k}\left(h\right) &= \int (e^x_\mu j^\mu_\circ(x, g) - e^x_\circ)(j^\circ_\nu(x, h)e^\nu_x - e^\circ_x)\mathrm{d}x \\
&= \int \{\|e^\circ_x\|^2_x + e^x_\mu\left[j^\mu_\lambda(g)j^\lambda_\nu(h) - j^\mu_-(g)j^-_\nu(h) - j^\mu_+(g)j^+_\nu(h)\right](x)e^\nu_x \\
&- e^x_\mu\left[j^\mu_\nu(g)e^\nu - j^\mu_-(g)e^- - j^\mu_+(g)\right](x) \\
&- \left[e_\mu j^\mu_\nu(h) - e_- j^-_\nu(h) - e_+ j^+_\nu(h)\right](x)e^\nu_x)\}\mathrm{d}x \\
&= \int [e^x_\mu j^\mu_\nu(x, g \cdot h)e^\nu_x - e^x_\mu j^\mu_\nu(x, h)e^\nu_x - e^x_\mu j^\mu_\nu(x, g)e^\nu_x]\mathrm{d}x \\
&= \lambda\left(g \cdot h\right) - \lambda\left(g\right) - \lambda\left(h\right)
\end{aligned}$$

for any $g, h \in \mathfrak{m}$, where $e^x_\mu j^\mu_\nu(x, g)e^\nu_x = l(x, g)$, $\int l(x, g)\mathrm{d}x < \infty$, and we have employed the condition $e^x_\mu e^\mu_x = 0$.

Let the subspace $\mathrm{K} \subseteq \prod \mathrm{K}_x$ be chosen as also the completion of the linear hull of square-integrable functions $\{\mathrm{k}\left(g\right) : g \in \mathfrak{m}\}$ with respect to all seminorms $\|\mathrm{k}\|^h = (\int \|j(x, h)\mathrm{k}(x)\|^2_x \mathrm{d}x)^{1/2}$, $h \in \mathfrak{m}$ given by operator-functions $j\left(x, h\right) = j^\circ_\circ(x, h)$. For any $g \in \mathfrak{m}$ we denote by $G = \int^\oplus j(x, g)\mathrm{d}x$ a linear decomposable operator in $\mathrm{K} = \int^\oplus \mathrm{K}_x \mathrm{d}x$ with G^* defined pointwise as

$$(G^*\mathrm{k})(x) = j^\circ_\circ(x, g(x)^\star)\,\mathrm{k}(x) = G(x)^*\mathrm{k}(x), \qquad \mathrm{k} \in \mathrm{K}.$$

This definition is correct since for almost all $x \in X$ and all $g, h \in \mathfrak{m}$ we have $j(g \cdot h) = j(g)j(h)$ pointwise, and any sequence of functions $\{\mathrm{k}_n\}$, $\mathrm{k}_n(x) \in \mathrm{K}_x$, fundamental with respect to all seminorms $\|\cdot\|^f$ is mapped by the operator $j(g)$ into a sequence $\{\mathrm{k}^g_n\}$, $\mathrm{k}^g_n(x) = j(x, g)\mathrm{k}_n(x) \in \mathrm{K}_x$ with the same fundamental property:

$$\|\mathrm{k}^g_m - \mathrm{k}^g_n\|^f = \|j(f)j(g)(\mathrm{k}_m - \mathrm{k}_n)\| = \|\mathrm{k}_m - \mathrm{k}_n\|^{f \cdot g} \longrightarrow 0.$$

This yields a decomposable non-degenerate representation $G\mathrm{k} = \int^\oplus G(x)\mathrm{k}(x)\mathrm{d}x$ of the $\star$-semiring $\mathfrak{m}$ in the poly-Hilbert space K:

$$e \mapsto I = j(e), \quad f \star h \mapsto FH^*, \ F = j(f), \ H = j(h).$$

This representation is closed in the sense of the completeness of K with respect to simultaneous convergence in all seminorms $\|\mathrm{k}\|^f = \|F\mathrm{k}\|$, $f \in \mathfrak{m}$ (which is equivalent to the convergence in the Hilbert norm $\|\mathrm{k}\|$ only in the case when the operator function $G(x) = j(x,g)$ is essentially bounded for every $g \in \mathfrak{m}$, in which case $\mathrm{K} = \int^{\oplus} \mathrm{K}_x \mathrm{d}x$ is called Hilbert integral). The map $\mathfrak{m} \ni g \mapsto \mathrm{k}\,(g)$ we have constructed, as well as $\mathrm{k}^\star$, is an additive cocycle in the sense (2.7) since the derivation property

$$\begin{aligned}\mathrm{k}\,(g \cdot h) = j^\circ_\mu(g \cdot h)e^\mu - e^\circ &= j^\circ_\mu(g)j^\mu_\nu(h)e^\nu - e^\circ \\ &= j^\circ_\circ(g)j^\circ_\nu(h)e^\nu + j^\circ_+(g) - e^\circ = j(g)\mathrm{k}\,(h) + \mathrm{k}\,(g)\end{aligned}$$

with respect to the representation $j(g)\mathrm{k}\,(h) = j^\circ_\circ(g)\mathrm{k}\,(h)$ of the monoid in K and the trivial representation $1(h) = 1$ of $\mathfrak{m}$ in $\mathbb{C}$.

(iii) $\Rightarrow$ (ii). It is obvious that the absolutely continuous measure $\lambda_\Delta(b) = \int_\Delta l(x,b)\mathrm{d}x$ defined by the functional $\lambda\,(g) = \int e^x_\mu j^\mu_\nu(x,g)e^x_\nu \mathrm{d}x$ satisfies the conditions $\lambda_\Delta(b^\star) = \lambda_\Delta(b)^*$ and $\lambda_\Delta(u) = 0$, since the functional $l(x,b)$ satisfies these conditions almost everywhere on X. The conditional positivity (2.5) follows from the positive definiteness $[\mathrm{k}\,(f)^*\,\mathrm{k}\,(h)] \geq 0$ of the scalar product $\mathrm{k}^*\mathrm{k}' = \langle \mathrm{k} \mid \mathrm{k}' \rangle$ which guarantees the conditional positivity of the form $\lambda\,(g)$:

$$\begin{aligned}\sum_{f,h\in\mathfrak{m}} \kappa_f \langle f \star h \rangle \kappa^*_h &= \sum_{f,h\in\mathfrak{m}} \kappa_f(\lambda\,(f \star h) + \lambda\,(f) + \lambda\,(h))\kappa^*_h \\ &= \sum_{f,h\in\mathfrak{m}} \kappa_f \lambda\,(f \star h)\,\kappa^*_h + \sum_{f\in\mathfrak{m}} \kappa_f \sum_{h\in\mathfrak{m}} \lambda\,(h)^*\,\kappa^*_h + \sum_{f\in\mathfrak{m}} \kappa_f \lambda\,(f) \sum_{h\in\mathfrak{m}} \kappa^*_h \\ &= \sum_{f,h\in\mathfrak{m}} \kappa_f \langle \mathrm{k}\,(f^\star)\,|\mathrm{k}\,(h^\star)\rangle \kappa_h \geq 0\end{aligned}$$

for any function $\kappa = \{\kappa_g\}$ with finite support and satisfying $\sum \kappa_g = 0$.

(ii) $\Rightarrow$ (i). If the function $\lambda_\Delta(b)$ is a (complex) absolutely continuous measure, then $\phi_\Delta(b) = \exp\{\lambda_\Delta(b)\}$ has the property $\phi_{\sqcup\Delta_l}(b) = \prod \phi_{\Delta_l}(b)$ of infinite divisibility. Moreover the limit (2.4) exists, and by virtue of $\phi_\Delta(b) \to 1$ as $\Delta \downarrow \{x\}$ it coincides with the Radon-Nikodým derivative $l_x(b) = \mathrm{d}\ln\phi(b)/\mathrm{d}x$ as the limit of the quotient $\lambda_\Delta(b)/\mu_\Delta$ over a net of subsets $\Delta \ni x$ of the system of Vitali decompositions of the measurable space X. For any integrable Δ the function $b \mapsto \phi_\Delta(b)$ is positive in the sense of (2.1). Indeed, for any complex function $b \mapsto \kappa_b$ with finite support we have due to (2.5)

$$\sum_{a,c\in\mathfrak{b}} \kappa_a(\lambda_\Delta(a \star c) - \lambda_\Delta(a) - \lambda_\Delta(c^\star))\kappa^*_c = \sum \kappa_a \langle \mathrm{k}^\star\,(a_\Delta)\,\mathrm{k}\,(c^\star_\Delta)\rangle \kappa^*_c \geq 0$$

since $\langle \mathrm{k}^\star\,(a_\Delta)\,\mathrm{k}\,(c^\star_\Delta)\rangle = \sum_{a,c\in\mathfrak{b}} \kappa^\circ_a \lambda_\Delta(a \star c)\kappa^{\circ *}_c$ with $\kappa^\circ_b = \kappa_b$, $b \neq u$, and $\kappa^\circ_\mu = \kappa_\mu - \sum_{b\in\mathfrak{b}} \kappa_b$ is a positive-definite kernel in a and c as $\sum_{b\in\mathfrak{b}} \kappa^\circ_b = 0$, and we have taken into account the fact that $\lambda_\Delta(u) = 0$. Since the exponent of any positive-definite kernel is a positive definite kernel, we have for any Δ

$$\sum_{a,c\in\mathfrak{b}} \kappa^*_a \exp\{\lambda_\Delta(a \star c)\}\kappa_c = \sum_{a,c\in\mathfrak{b}} \kappa^{a*}_\Delta \exp\{\langle \mathrm{k}^\star\,(a_\Delta)\,\mathrm{k}\,(c^\star_\Delta)\rangle\}\kappa^c_\Delta \geq 0,$$

where $\kappa^b_\Delta = \kappa_b \exp\{\lambda_\Delta(b)\}$ and we have taken into account (2.6) and $\lambda_\Delta(b^\star) = \lambda_\Delta(b)^*$.

(i) $\Rightarrow$ (iv). Since ϕ_Δ is an infinitely divisible state on $\mathfrak{b}$ and $\phi_\Delta(b) \to 1$ for all b as $\mu_\Delta \to 0$, the limit $l_x(b)$ is defined as the logarithmic derivative $\mu^{-1}_{\mathrm{d}x} \ln\phi_{\mathrm{d}x}(b)$

of the measure $\lambda_\Delta(b) = \ln \phi_\Delta(b)$ in the Radon-Nikodym sense. Consequently, the function $x \mapsto l_x(b)$ is integrable and almost everywhere satisfies the conditions $l_x(a \star c)^* = l_x(c \star a)$, $l_x(u) = 0$ and

$$\sum_{b \in \mathfrak{b}} \kappa_b = 0 \ \Rightarrow (\kappa' \mid \kappa)_x := \sum_{a,c \in \mathfrak{b}} \kappa_a l_x(a \star c) \kappa_c^* \geq 0$$

for all κ such that $|\operatorname{supp} \kappa| < \infty$, which can easily be verified directly for the difference derivative $l_\Delta(b) = (\phi_\Delta(b) - 1)/\mu_\Delta$ and next we can pass to the limit $\Delta \downarrow \{x\}$. In addition $\int_\Delta l_x(b) \mathrm{d}x = \ln \phi_\Delta(b)$ by absolute continuity.

We consider the space $\mathfrak{B}$ of complex functions $\kappa = (\kappa_b)_{b \in \mathfrak{b}}$ on $\mathfrak{b}$ with finite supports $\{b \in \mathfrak{b} : \kappa_b \neq 0\}$ as a unital $\star$-algebra with respect to the product $\kappa' \cdot \kappa$ defined as $\kappa' \star \kappa^\star$ by the Hermitian convolution

$$(\kappa' \star \kappa)_b = \sum_{a \star c = b} \kappa'_a \kappa_c^*, \qquad \delta_u \star \kappa = \kappa^\star \ , \qquad \kappa \star \delta_u = \kappa \ .$$

with right identity δ_u. Here $\delta_a = (\delta_{a,b})_{b \in \mathfrak{b}}$ is the Kronecker delta and it defines a $\star$-representation

$a \mapsto \delta_a$ of the monoid $\mathfrak{b}$ in $\mathfrak{B}$,

$$\delta_a \star \delta_c = \delta_{a \star c}, \quad \delta_u \star \delta_b = \delta_b, \quad \delta_b \star \delta_u = \delta_{b^\star},$$

with respect to the involution $\kappa^\star = (\kappa^*_{b^\star})_{b \in \mathfrak{b}}$. The linear subspace $\mathfrak{A} \subset \mathfrak{B}$ of distributions κ such that the sum $\kappa_- := \sum_{b \in \mathfrak{b}} \kappa_b$equals zero, is a $\star$-ideal since

$$\sum_{b \in \mathfrak{b}} (\kappa' \star \kappa)_b = \sum_{b \in \mathfrak{b}} \sum_{a \star c = b} \kappa'_a \kappa_c^* = \sum_{a \in \mathfrak{b}} \kappa'_a \sum_{c \in \mathfrak{b}} \kappa_c^* = 0,$$

Let us equip $\mathfrak{B}$ for every $x \in X$ with the Hermitian form $(\kappa' \mid \kappa)_x$ of the kernel $l_x(a \star c)$ which is positive on $\mathfrak{A}$ and can be written in terms of the kernel $\langle \delta_a, \delta_c^\star \rangle_x^\circ = l_x(a \star c) - l_x(a) - l_x(c^\star)$ as

$$(\kappa' \mid \kappa)_x = \kappa'_- \kappa_+^* + \langle \kappa', \kappa^\star \rangle_x^\circ + \kappa'_+ \kappa_-^*,$$

where $\kappa_+ := \sum_b \kappa_b l_x(b)$. We notice that the Hermitian form

$$\langle \kappa'^\star \mid \kappa^\star \rangle_x^\circ := \sum_{a.c \in \mathfrak{b}} \kappa'_a \langle \delta_a, \delta_c \rangle_x^\circ \kappa_c^\star \equiv \langle \kappa', \kappa^\star \rangle_x^\circ$$

is non-negative if $\kappa_- = 0$ or $\kappa'_- = 0$ as $\langle \kappa, \kappa^\star \rangle_x^\circ = \sum \kappa_a \langle \delta_a, \delta_c^\star \rangle_x^\circ \kappa_c^* \geq 0$, coinciding with $(\kappa' \mid \kappa)_x$. Since $(\kappa' \mid \kappa)_x = \sum_b (\kappa' \star \kappa)_b \, l_x(b)$, the form $(\kappa' \mid \kappa)_x$ has right associativity property

$$(\kappa' \cdot \kappa \mid \kappa)_x = (\kappa' \mid \kappa \star \kappa)_x = (\kappa' \mid \kappa \cdot \kappa^\star)_x,$$

for all $\kappa, \kappa' \in \mathfrak{B}$, and therefore its kernel $\mathfrak{R}_x = \{\kappa : (\kappa' \mid \kappa)_x = 0 \ \forall \kappa'\}$ is the right ideal

$$\mathfrak{R}_x = \{\kappa' \in \mathfrak{B} : (\kappa' \cdot \kappa \mid \kappa)_x = 0, \forall \kappa \in \mathfrak{B}\}$$

belonging to $\mathfrak{A}$. We factorize $\mathfrak{B}$ by this right putting $\kappa \approx 0$ if $\kappa \in \mathfrak{R}_x^\star := \{\kappa^\star : \kappa \in \mathfrak{R}_x\}$ and denoting the equivalence classes of the left factor-space $\mathcal{K}_x^{\cdot} = \mathfrak{B}/\mathfrak{R}_x^\star$ as the ket-vectors $|\kappa) = \{\kappa' : \kappa' - \kappa^\star \in \mathfrak{R}_x^\star\}$. The condition $\kappa \in \mathfrak{R}_x$ means in particular that $\kappa_x^- := (\delta_u \mid \kappa)_x = 0$, and therefore

$$(\kappa \mid \kappa)_x = \sum_{a,c \in \mathfrak{b}} \kappa_a \langle \delta_a, \delta_c^\star \rangle_x \kappa_c^* = \langle \kappa^\circ \mid \kappa^\circ \rangle_x = 0,$$

where $\kappa^{\circ}=(\kappa_{b}^{\circ})_{b\in\mathfrak{b}}$ denotes an element of $\mathfrak{A}$ obtained as $\kappa_{b}^{\circ}=\kappa_{b}^{\star}$ for all $b\neq u$ and $\kappa_{u}^{\circ}=\kappa_{u}^{\star}-\sum_{b\in\mathfrak{b}}\kappa_{b}^{\star}$ such that $\langle\kappa^{\circ}\mid\kappa^{\circ}\rangle_{x}=\langle\kappa,\kappa^{\star}\rangle_{x}^{\circ}$. Therefore it follows also that $\kappa^{+}:=\sum\kappa_{b}^{\star}$ is also zero for any $\kappa\in\mathfrak{R}_{x}$ since

$$0=(\kappa'\mid\kappa)_{x}=\kappa'_{-}\kappa_{+}^{*}+\langle\kappa',\kappa^{\star}\rangle_{x}^{\circ}+\kappa'_{+}\kappa_{-}^{*}=\kappa_{-}^{*}=\kappa^{+}$$

for any $\kappa'\in\mathfrak{B}$ with $\kappa'_{+}=1$ by virtue of $\kappa_{+}^{*}=\kappa^{-}=0$ and also due to the Schwartz inequality $(\kappa'\mid\kappa)=\langle\kappa',\kappa^{\star}\rangle^{\circ}=0$. This allows us to represent the left equivalence classes $|\kappa)_{x}$ by the columns $k^{\cdot}=[k^{\mu}]$ with $k^{\mp}=\kappa^{\mp}$ and $k^{\circ}=|\kappa^{\circ}\rangle$ in the Euclidean component $\mathcal{K}_{x}^{\circ}\subset\mathcal{K}_{x}^{\cdot}$ as the subspace of the left equivalence classes $|\kappa^{\circ}\rangle=|\kappa_{\circ})$ of the elements $\kappa_{\circ}=(\kappa_{b}-\delta_{u,b}\kappa_{-})_{b\in\mathfrak{b}}\in\mathfrak{A}$ such that $\kappa_{\circ}^{\star}=\kappa^{\circ}$. These columns are pseudo-adjoint to the rows $k.=(k_{-},k_{\circ},k_{+})$ as the right equivalence classes $(\kappa:=|\kappa)^{\dagger}\in\mathfrak{B}/\mathfrak{R}_{x}$ with $k_{\pm}=\kappa_{\pm}$ and $k_{\circ}=(\kappa_{\circ}$ defining the indefinite product (2.10) in terms of the canonical pairing

$$k.k^{\cdot}=k_{-}k^{-}+\langle k_{\circ},k^{\circ}\rangle+k_{+}k^{+}=\left(k^{\cdot}\mid k_{\cdot}^{\dagger}\right),$$

where $k^{\circ}=k_{\circ}^{*}\in\mathcal{K}_{x}^{\circ}$, $k^{\pm}=k_{\mp}^{*}\in\mathbb{C}$ with respect to the Euclidean scalar product $\langle k_{\circ},k^{\circ}\rangle=\langle k_{\circ}^{*}\mid k^{\circ}\rangle$ of the Euclidean space $\mathcal{K}_{x}^{\circ}=\{k^{\circ}=|\kappa^{\circ}\rangle:\kappa_{\circ}\in\mathfrak{A}\}$, and

$$\kappa_{+}^{*}=\sum_{b\in\mathfrak{b}}l_{x}(b^{\star})\kappa^{b}=\kappa_{x}^{-},\quad\kappa_{-}^{*}=\sum_{b\in\mathfrak{b}}\kappa_{b}^{*}=\kappa^{+}.$$

We notice that the representation $\delta.:\mathfrak{b}\ni b\mapsto\delta_{b}$ is Hermitian:

$$(\kappa\cdot\delta_{b}\mid\kappa)=\sum_{b\in\mathfrak{b}}l(b)(\kappa\cdot\delta_{b}\star\kappa)_{b}=(\kappa\mid\kappa\cdot\delta_{b^{\star}}),$$

and that it is well defined as right representation on $\mathfrak{B}/\mathfrak{R}_{x}$ (or left representation on $\mathfrak{B}/\mathfrak{R}_{x}^{\star}$) since $\kappa\cdot\delta_{b}\in\mathfrak{R}$ if $\kappa\in\mathfrak{R}_{x}$:

$$(\kappa\mid\kappa)=0,\ \forall\kappa\in\mathfrak{B}\Rightarrow(\kappa\cdot\delta_{b}\mid\kappa)=(\kappa\mid\kappa\star\delta_{b})=0,\ \forall\kappa\in\mathfrak{B}.$$

This allows us to define for each $b\in\mathfrak{b}$ an operator $(\kappa\ \mathbf{j}(b)=(\kappa\cdot\delta_{b}$ such that $\mathbf{j}(b^{\star})=\mathbf{j}(b)^{\dagger}$ with the componentwise action

$$\begin{aligned}&(\kappa\cdot\delta_{b})_{-}=\kappa_{-},\quad(\kappa\cdot\delta_{b})_{\circ}=\kappa_{-}(\delta_{b}-\delta_{u})+\kappa_{\circ}\cdot\delta_{b},\\&(\kappa\cdot\delta_{b})_{+}=\kappa_{-}l(b)+(\kappa_{\circ}\mid\delta_{b^{\star}}-\delta_{u})+\kappa_{+},\end{aligned}$$

given as the right multiplications $\boldsymbol{k}\mapsto\boldsymbol{k}\mathbf{B}$, $\boldsymbol{k}\mapsto\boldsymbol{k}\mathbf{B}^{\dagger}$ of the triangular matrices

$$\mathbf{B}=\begin{bmatrix}1&j_{\circ}^{-}(b)&j_{+}^{-}(b)\\0&j_{\circ}^{\circ}(b)&j_{+}^{\circ}(b)\\0&0&1\end{bmatrix}\equiv j_{\cdot}^{\cdot}(b)\,,\ j_{\cdot}^{\cdot}(b^{\star})=\begin{bmatrix}1&j_{+}^{\circ}(b)^{*}&j_{+}^{-}(b)^{*}\\0&j_{\circ}^{\circ}(b)^{*}&j_{\circ}^{-}(b)^{*}\\0&0&1\end{bmatrix}\equiv\mathbf{B}^{\dagger}$$

by the rows $\boldsymbol{k}=(k_{-},k_{\circ},k_{+})\in\mathcal{K}.$ (or as the left multiplications $\mathbf{Bk}$, $\mathbf{B}^{\dagger}\mathbf{k}$ by columns $\mathbf{k}\in\mathcal{K}_{x}^{\cdot}$). Here

$$\begin{aligned}j_{+}^{-}(x,b)&=&l_{x}(b),\quad(\kappa_{\circ}j_{\circ}^{\circ}(x,b)=(\kappa_{\circ}\cdot\delta_{b}=(\kappa_{\circ}j_{x}(b)\,,\\j_{+}^{\circ}(x,b^{\star})&=&\delta_{b}^{\star}\rangle_{x}=\mathrm{k}_{x}(b^{\star})=\mathrm{k}_{x}^{\star}(b)^{*}=\langle\delta_{b}^{\star}|^{*}=j_{\circ}^{-}(x,b)^{*},\end{aligned}$$

where $\delta_{b}^{\star}\rangle_{x}=|\delta_{b}-\delta_{u})$ and $B_{-\nu}^{\dagger\mu}=B_{-\mu}^{\nu*}$ is pseudo-Euclidean conjugation of the triangular matrix $\mathbf{B}=[B_{\nu}^{\mu}]$ corresponding to the map $\boldsymbol{k}\mapsto\boldsymbol{k}^{\dagger}$ into the adjoint columns $\mathbf{k} = [k^\mu]$ with the components $k^\mu = k^*_{-\mu}$ given by the pseudo-metric tensor $g^{\mu\nu} = \delta^\mu_{-\nu} = g_{\mu\nu}$:

$$\begin{bmatrix} b^-_- & b^-_\circ & b^-_+ \\ 0 & b^\circ_\circ & b^\circ_+ \\ 0 & 0 & b^+_+ \end{bmatrix}^\dagger = \begin{bmatrix} 0 & 0 & 1 \\ 0 & I & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} b^-_- & b^-_\circ & b^-_+ \\ 0 & b^\circ_\circ & b^\circ_+ \\ 0 & 0 & b^+_+ \end{bmatrix}^* \begin{bmatrix} 0 & 0 & 1 \\ 0 & I & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} b^{+*}_+ & b^{\circ *}_+ & b^{-*}_+ \\ 0 & b^{\circ *}_\circ & b^{-*}_\circ \\ 0 & 0 & b^{-*}_\circ \end{bmatrix}$$

Thus we can write the constructed canonical †-representation $\mathbf{j}(b) = [j^\mu_\nu(b)]$ of the monoid $\mathfrak{b}$ in the pseudo-Euclidean space $\mathbb{K}_x = \mathcal{K}^{\cdot}_x$ of columns $\mathbf{k} = [k^\mu]$ in terms of the usual matrix multiplication

$$\begin{bmatrix} 1 & \mathrm{k}^\star(b) & l(b) \\ 0 & j(b) & \mathrm{k}(b) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \mathrm{k}^\star(c) & l(c) \\ 0 & j(c) & \mathrm{k}(c) \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & \mathrm{k}^\star(c) + \mathrm{k}^\star(b) j(c), & l(c) + \mathrm{k}^\star(b)\mathrm{k}(c) + l(b) \\ 0 & j(b)j(c), & j(b)\mathrm{k}(c) + \mathrm{k}(b) \\ 0 & 0 & 1 \end{bmatrix}$$

This realizes a conditionally positive function $l(b)$ as the value of the vector form (2.9) on the column $\mathbf{e} = \left[\delta^\mu_+\right] = \boldsymbol{e}^\dagger$ as adjoint to row $\boldsymbol{e} = (1, 0, 0)$ of zero pseudonorm $\mathbf{e}^\dagger \mathbf{e} = e_\mu e^\mu = 0$ for each x as

$$\mathbf{e}^\dagger \mathbf{j}(b)\mathbf{e} = e_\mu j^\mu_\nu(b) e^\nu = j^-_+(b) = l(b).$$

The proof is complete. □

Remark 1. *Any indefinite-metric representation $(\mathcal{E}^{\cdot}, j^{\cdot}_{\cdot}, c^{\cdot})$ of a conditionally positive function l, written in the form $l(b) = e_\mu j^\mu_\nu(b) e^\nu$ with respect to a triangular $\star$-representation $j^{\cdot}_{\cdot} = [j^\mu_\nu]$ of $\mathfrak{b}$ in a pseudo-Hilbert space $\mathcal{E}^{\cdot} = \mathbb{C} \oplus \mathcal{E}^\circ \oplus \mathbb{C}$ with* (2.10) *and a zero-vector $e^{\cdot} = (e_-, e_\circ, e_+)^\dagger$ normalized as $e_- = 1$, $\|e_\circ\|^2 = -2\operatorname{Re} e_+$ can be reduced to the canonical form $(\mathbb{K}, \mathbf{j}, \mathbf{e})$ corresponding to*

$$j^-_+ = l, \;\; j^\circ_+ = k, \;\; j^-_{\cdot} = k^\star, \;\; j^\circ_\circ = j$$

with respect to the vector $\mathbf{e} = (1, 0, 0)^\dagger$ by a triangular pseudo-isometry $\mathbf{S} : \mathbb{K} \to \mathcal{E}^{\cdot}$. In particular, if $(\mathcal{E}^{\cdot}, j^{\cdot}_{\cdot}, e^{\cdot})$ is a minimal closed representation in the sense that the vector $e^{\cdot}$ is cyclic such that $\mathcal{E}^\circ$ is minimal poly-Hilbert space generated by the action on $e^{\cdot}$ of the linear hull of operators $j^\circ_{\cdot}(\mathfrak{b})$, then it is equivalent to the closed canonical representation on $\mathbb{K} = \mathbb{C} \oplus \mathrm{K} \oplus \mathbb{C}$ with the constructed minimal $\mathcal{K}^{\cdot} = \mathrm{K}$.

Indeed, taking an arbitrary isometry $U : \mathcal{K}^\circ \to \mathcal{E}^\circ$ of a minimal space $\mathcal{K}^\circ$ we can define the pseudo-isometry $\mathbf{S}$ in the form

$$\mathbf{S} = \begin{bmatrix} 1, & e_\circ U, & e^*_+ \\ 0, & -U, & e^*_\circ \\ 0, & 0, & 1 \end{bmatrix}, \quad \mathbf{S}^\dagger \mathbf{S} = \mathbf{I}, \;\; \mathbf{S}^\dagger = \begin{bmatrix} 1, & e_\circ, & e_+ \\ 0, & -U^*, & U^* e^*_\circ \\ 0, & 0, & 1 \end{bmatrix}, \tag{2.11}$$

converting the matrix $j^{\cdot}_{\cdot}(b)$ and the column $e^{\cdot} \in \mathcal{E}^{\cdot}$ into the canonical form

$$\mathbf{j}(b) = \begin{bmatrix} 1 & \mathrm{k}^\star(b) & l(b) \\ 0 & j(b) & \mathrm{k}(b) \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{S}^\dagger j^{\cdot}_{\cdot}(b)\mathbf{S}, \quad \mathbf{e} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \mathbf{S}^\dagger e^{\cdot},$$

since $e_{\cdot} S^{\cdot}_{\cdot} = S^-_{\cdot} + e_\circ S^\circ_{\cdot} + e_+ S^+_{\cdot} = (1, 0, 0)$ if $S^-_{\cdot} = (1, e_\circ U, e_+)$, $S^\circ_{\cdot} = (0, -U, e^*_\circ)$, $S^+_{\cdot} = (0, 0, 1)$ for $e_{\cdot} = (1, e_\circ, e_+)$ with $e_\circ e^*_\circ \equiv \langle e_\circ \mid e_\circ \rangle = e_+ + e^*_+$ corresponding to $l(u) = (e_{\cdot} \mid e_{\cdot}) = 0$. If the Euclidean space $\mathcal{E}^\circ$ is minimal containing $\{j^\circ_{\cdot}(b)e^{\cdot} : b \in \mathfrak{b}\}$

(or minimal closed with respect to the seminorms $\|k_\circ\|^c = \|k_\circ j_\circ^\circ(c)\|, c \in \mathfrak{b}$) then, defining the operator U by the isometricity condition

$$e.j^{\cdot}_{\cdot}(b)S^{\cdot}_{\circ} = (e_\circ - e.j^{\cdot}_{\circ}(b))U = \mathrm{k}(b)^*,$$
$$(e_\circ - e.j^{\cdot}_{\circ}(a) \mid e_\circ - e.j^{\cdot}_{\circ}(c)) = \mathrm{k}(a)^*\mathrm{k}(c),$$

we obtain a pseudo-unitary equivalence of the (closed) representation $(\mathcal{E}^{\cdot}, j^{\cdot}_{\cdot}, e^{\cdot})$ and the (closed) canonical representation $(\mathbb{K}, \mathbf{j}, \mathbf{e})$ constructed in the proof of the implication (i) $\Rightarrow$ (iv) of Theorem 1.

3. The Fock and pseudo-Fock representation of infinitely divisible states

We shall now describe an exponential indefinite-metric representation of the $\star$-monoid $\mathfrak{m}$ associated with the conditionally positive-definite functional $\lambda(g) = \int l_x(g(x))\mathrm{d}x$ and its connection with the generalized Araki-Woods construction [4] corresponding to the chaotic infinitely divisible state $\phi(g) = e^{\lambda(g)}$. Unlike the Fock representation of the Araki-Woods construction, the exponential representation, which we will construct in a pseudo-Fock space, has the property of decomposability in finite tensor representations, which can be used [15] to construct explicit solutions of quantum stochastic equations even in the case of non-adapted locally integrable generators.

We call pre-Fock space $\mathcal{F}^\circ$ over a pre-Hilbert space $\mathcal{K}^\circ$ the linear hull $\{\mathrm{f} = \Sigma\lambda_i \exp\{\mathrm{k}_i\} : \lambda_i \in \mathbb{C}, \mathrm{k}_i \in \mathcal{K}^\circ\}$ of *exponential* vectors $\exp\{\mathrm{k}\} := \oplus_{n=0}^\infty \frac{1}{n!}\mathrm{k}^{\otimes n}$ as direct weighted sums of finite tensor powers of vectors $\mathrm{k} \in \mathcal{K}^\circ$, with $\mathrm{k}^{\otimes 0} = 1$ and $\mathrm{k}^{\otimes 1} \equiv \mathrm{k}$ such that

$$\langle \exp\{\mathrm{k}\} \mid \exp\{\mathrm{k}\}\rangle = \sum_{n=0}^\infty \frac{1}{n!}\langle \mathrm{k} \mid \mathrm{k}'\rangle^n = e^{\langle \mathrm{k}|\mathrm{k}'\rangle}.$$

This positive-definite exponential kernel describes the scalar product

$$\langle \mathrm{f} \mid \mathrm{f}'\rangle = \Sigma_{ij}\lambda_i^* \langle \exp\{\mathrm{k}\} \mid \exp\{\mathrm{k}'\}\rangle \lambda_j$$

in $\mathcal{F}^\circ$, and the usual Fock space is defined as a completion of $\mathcal{F}^\circ$ with respect to the norm $\|\mathrm{f}\| = \langle \mathrm{f} \mid \mathrm{f}\rangle^{1/2}$. Below as such $\mathcal{K}^\circ$ we will take the poly-Hilbert space $\mathrm{K} = \int^\oplus \mathrm{K}_x \mathrm{d}x$ associated with the constructed canonical representation of a conditionally positive functional λ on the $\star$-monoid $\mathfrak{m}$ of simple functions $g : X \to \mathfrak{b}$, denoting by $\mathrm{K}_* \subseteq \mathcal{K}_\circ$ the dual subspace of those functionals $k_\circ \in \mathcal{K}_\circ$ which are represented as $k_\circ \mathrm{k} = \langle \mathrm{k} \mid \mathrm{k}\rangle \equiv \mathrm{k}^*\mathrm{k}$ on $\mathrm{k} \in \mathrm{K}$ (such $k_\circ = \mathrm{k}^*$ are continuous with respect to all seminorms of K).

Thanks to the fact that the measure $\mathrm{d}x$ is atomless, the space $\oplus_{n=0}^\infty \frac{1}{n!}\mathrm{K}^{(n)}$, with $\mathrm{K}^{(n)} \subseteq \mathrm{K}^{\otimes n}$ consisting of only *symmetric* tensor-functions $\mathrm{f}^{(n)} : X^n \to \mathrm{K}^{\otimes n}$, can be identified with the space $\Gamma(\mathrm{K}) = \oplus_{n=0}^\infty \Gamma_n(\mathrm{K})$ of square-integrable functions $\mathrm{f}(\omega)$ with *arbitrary* tensor values $\mathrm{f}(\omega) \in \mathrm{K}^\otimes(\omega)$ in *full* tensor products $\mathrm{K}^\otimes(\omega) = \otimes_{x\in\omega}\mathrm{K}_x$ of the component spaces K_x, with the indexing sets $\omega : |\omega| = n < \infty$ taken as *any* finite subsets $\omega \subseteq X$, $|\omega| = n < \infty$. The integrability of such tensor-valued functions $\mathrm{f}(\omega)$, defined on the space $\Omega = \sum_{n=0}^\infty \Omega_n$ of all finite subsets $\omega \subset X$ including empty subset $\omega = \emptyset$ with $\mathrm{K}^\otimes(\emptyset) = \mathbb{C}$ is understood with respect to the Lebesgue measure $\mathrm{d}\omega = \prod_{x\in\omega}\mathrm{d}x$ with $\mathrm{d}\emptyset = 1$, and the isometry of the components $\frac{1}{n!}\mathrm{K}^{(n)}$ and $\int_{\Omega_n}^\oplus \mathrm{K}^\otimes(\omega)\,\mathrm{d}\omega$ is given by the symmetric extension $f^{(n)}(x_1, \ldots, x_n) =$

$f(\{x_1,\dots,x_n\})$ defining $f^{(n)}$ almost everywhere on X^n with respect to $\mathrm{d}x_1\cdots\mathrm{d}x_n$ such that

$$\int_X\cdots\int_X \|\mathrm{f}^{(n)}(x_1,\dots,x_n)\|^2\mathrm{d}x_1\cdots\mathrm{d}x_n = n!\int_{\Omega_n}\|\mathrm{f}(\omega_n)\|^2\mathrm{d}\omega_n.$$

Denoting the series $\sum_{n=0}^{\infty}\int_{\Omega_n}\cdot\mathrm{d}\omega_n$ of integrals over the n-point subsets $\omega_n = \{x_1,\dots,x_n\}\subset X$ as the single integral on Ω with respect to the measure $\mathrm{d}\omega=\sum \mathrm{d}\omega_n$, the scalar product in $\Gamma(\mathrm{K}) = \int_\Omega^{\oplus}\mathrm{K}^{\otimes}(\omega)\,\mathrm{d}\omega$ can be written as the single Lebesgue integral on Ω,

$$\langle \mathrm{f}\mid\mathrm{f}\rangle := \sum_{n=0}^{\infty}\int_{\Omega_n}\langle\mathrm{f}(\omega_n)\mid\mathrm{f}(\omega_n)\rangle\,\mathrm{d}\omega_n \equiv \int_\Omega\langle\mathrm{f}(\omega)\mid\mathrm{f}(\omega)\rangle\,\mathrm{d}\omega$$

called the *multiple* Lebesgue integral on X. Obviously the symmetric extension from Ω onto $\sum_{n=0}^{\infty}\frac{1}{n!}X^n$ of the tensor-product functions $f(\omega)=\otimes_{x\in\omega}\mathrm{k}(x)\equiv \mathrm{k}^{\otimes}(\omega)$ defines almost everywhere the generating exponential functions $\exp\{\mathrm{k}\}$, and

$$\left\langle \mathrm{k}^{\otimes}\mid\mathrm{k}'^{\otimes}\right\rangle = e^{\langle\mathrm{k}|\mathrm{k}'\rangle},\qquad \langle\mathrm{k}\mid\mathrm{k}'\rangle=\int\langle\mathrm{k}(x)|\mathrm{k}'(x)\rangle_x\,\mathrm{d}x$$

since $\int_{\Omega_n}\|\mathrm{k}^{\otimes}(\omega)\|^2\mathrm{d}\omega_n = \frac{1}{n!}\left(\int_X\|\mathrm{k}(x)\|^2\,\mathrm{d}x\right)^n$ due to $\|\mathrm{k}^{\otimes}(\omega)\|^2=\prod_{x\in\omega}\|\mathrm{k}(x)\|_x^2$. In future we will refer to the Hilbert integral $\int_\Omega\mathrm{K}^{\otimes}(\omega)\,\mathrm{d}\omega$ as to the Fock space, denoting the exponential domain in it by $\Gamma(\mathrm{K})\ni\mathrm{k}^{\otimes}$.

We define decomposable operators $j(g)^{\otimes}=\oplus_{n=0}^{\infty}j(g)^{\otimes n}$ on $\Gamma(\mathrm{K})$ by a unital $*$-representation $j:\mathfrak{m}\to\mathcal{L}(\mathrm{K})$ on K associated with the conditionally positive function $\lambda(g)$ by means of the linear continuation $j(g)^{\otimes}\mathrm{f}=\Sigma\lambda_i j(g)^{\otimes}\mathrm{k}_i^{\otimes}$ of the tensor-product operators $j(g)^{\otimes}\mathrm{k}^{\otimes}=(j(g)\mathrm{k})^{\otimes}$. Obviously the correspondence $g\mapsto j(g)^{\otimes}$ is, like j itself, a unital $*$-representation

$$j^{\otimes}(g^{\star})=j^{\otimes}(g)^{*},\quad j^{\otimes}(f\cdot h)=j^{\otimes}(f)j^{\otimes}(h),\quad j^{\otimes}(e)=I^{\otimes}\quad\forall f,g,h\in\mathfrak{m}$$

on the pre-Hilbert space $\Gamma(\mathrm{K})$. In general the operators $j(g)^{\otimes}$ are unbounded and cannot be extended onto the complete Fock space over K (if only $\star$-monoid is not a group with $g^{\star}=g^{-1}$), however we can extend them to a closed $*$-representation $j^{\otimes}(g)=j(g)^{\otimes}$ on the completion F of the pre-Hilbert space $\Gamma(\mathrm{K})$ by fundamental sequences $\mathrm{f}_n\in\Gamma(\mathrm{K})$ converging in F with respect to all seminorms

$$\|\mathrm{f}\|^h=\left(\int\|j^{\otimes}(\omega,h)\mathrm{f}(\omega)\|^2\mathrm{d}\omega\right)^{1/2},\quad h\in\mathfrak{m}.$$

Note that the operators $j^{\otimes}(g)$ belong to the operator algebra $\mathcal{L}(\mathrm{F})$ of all linear continuous, together with their adjoints, operators $\mathrm{F}\to\mathrm{F}$ due to $\|j^{\otimes}(g)\mathrm{f}\|^h = \|\mathrm{f}\|^{g\cdot h}$, and in the case if they all are bounded, F is usual Fock space and $\mathcal{L}(\mathrm{F})=\mathcal{B}(\mathrm{F})$. All linear functionals $f_\circ\in\mathcal{F}_\circ$ of the form $f_\circ=f^{*}\in\mathrm{F}_{*}$ are also continuous on F since $f_\circ\mathrm{f}=\langle\mathrm{f}\mid\mathrm{f}\rangle$ converges for any sequence converging in all seminorms $\|\mathrm{f}\|^h$, $h\in\mathfrak{m}$.

Unfortunately, the representation $j^{\otimes}$ describes a dilation of an infinitely divisible state ϕ as a vector representation on F in the sense of the existence of an $\mathrm{f}\in\mathrm{F}$ such that $\phi(g)=\langle\mathrm{f}\mid j^{\otimes}(g)\mathrm{f}\rangle$ for all $g\in\mathfrak{m}$ only under special 'vector' choice $\lambda(g)=\langle\mathrm{k}\mid(j(g)-I)\mathrm{k}\rangle$ of the logarithmic function $\lambda(g)=\ln\phi(g)$. If such a vector $\mathrm{k}\in\mathrm{K}$ exists, then one can obviously take $\mathrm{f}=\exp\{-\langle\mathrm{k}\mid\mathrm{k}\rangle\}\mathrm{k}^{\otimes}$:

$$\left\langle\mathrm{f}\mid j^{\otimes}(g)\mathrm{f}\right\rangle=\exp\{-\langle\mathrm{k}\mid\mathrm{k}\rangle\}\left\langle\mathrm{k}^{\otimes}\mid j^{\otimes}(g)\mathrm{k}^{\otimes}\right\rangle=\exp\{\langle\mathrm{k}\mid(j(g)-I)\mathrm{k}\rangle\}.$$

Exploiting a similar exponential construction not for the pre-Hilbert K but for a pseudo-Hilbert extension $\mathbb{K}$ of the complex Euclidean space K, we can obtain a pseudo-Fock vector representation also for a general conditionally positive form $\lambda(g)$.

For we consider a vector-function space $\mathbb{K} = L^1(X) \oplus \mathrm{K} \oplus L^\infty(X)$ of the triples $\mathbf{k} = k^- \oplus k^\circ \oplus k^+$, where $k^\circ \in \mathrm{K}$ are square integrable vector-functions $k^\circ(x) \in \mathrm{K}_x$ from the poly-Hilbert space $\mathrm{K} = \{\|k^\circ\|^h < \infty : h \in \mathfrak{m}\}$, with $k^- \in L^1(X)$ and $k^+ \in L^\infty(X)$ taken as respectively absolutely integrable and essentially bounded complex functions:

$$\|k^-\|_1 = \int |k^-(x)|\mathrm{d}x < \infty, \quad \|k^+\|_\infty = \operatorname{ess\,sup} |k^+(x)| < \infty.$$

We equip this complex poly-Banach space with a pseudo-Hilbert scalar product

$$(\mathbf{k} \mid \mathbf{k}) = \left\langle k^- \mid k^+ \right\rangle + \langle k^\circ \mid k^\circ \rangle + \left\langle k^+ \mid k^- \right\rangle \equiv \langle k_\mu, k^\mu \rangle\,, \tag{3.1}$$

where $k^*_{-\nu}(x) = k^\mu(x)$ such that $\langle\, k_\mu, k^\mu \rangle = \int k_\mu(x)k^\mu(x)\mathrm{d}x$ is the integral product corresponding to (2.10) for the column-function $\mathbf{k} = k^\dagger_{\cdot} \equiv k^{\cdot}$ adjoint to $k_{\cdot}(x) = (k_-, k_\circ, k_+)(x)$ with the column $k(x) = [k^\mu(x)]$ such that $k^*_- = k^+,\ k^*_\circ = k^\circ, k^*_+ = k^-(x)$. Note that the products of the components k_μ and k^μ with the same $\mu = -, \circ, +$ are absolutely integrable for each μ, and thus all three integrals in 3.1 converge making $\mathbb{K}$ a generalized Krein space.

We define in $\mathbb{K}$ a closed decomposable $\star$-representation $(\mathbf{j}(g)k)(x) = \mathbf{j}(x,g)k(x)$ of $\mathfrak{b}$-valued functions $g(x)$ by triangular-operator functions $\mathbf{j}(x,g) = [j^\mu_\nu(x, g(x))]$ of the canonical form such that

$$\mathbf{j}(x, g^\star) = \begin{bmatrix} 1, & \mathrm{k}(x,g)^*, & l(x,g^\star) \\ 0, & j(x,g)^*, & \mathrm{k}(x,g^\star) \\ 0, & 0, & 1 \end{bmatrix} = \mathbf{j}(x,g)^\dagger, \tag{3.2}$$

where the functions $l(g) \in L^1(X)$, $\mathrm{k}(g) \in \mathrm{K}$, $j(g) : \mathrm{K} \to \mathrm{K}$ have been described in Theorem 1.

The operators $\mathbf{j}(g)$ are continuous on the whole of $\mathbb{K}$, together with the adjoint operators $\mathbf{j}(g)^\dagger$, with respect to the Hermitian form (3.1) by virtue of the inequalities

$$\|(\mathbf{j}(g)\mathbf{f})^-\|_1 \le \|f^-\|_1 + \|\mathrm{k}(g)\| \cdot \|f^\circ\| + \|l(g)\|_1 \|f^+\|_\infty < \infty,$$
$$\|(\mathbf{j}(g)\mathbf{k})^\circ\|^h \le \|f^\circ\|^{g\cdot h} + \|\mathrm{k}(g)\|^h \cdot \|e^+\|_\infty, \quad \|(\mathbf{j}(g)\mathbf{k})^+\|_\infty = \|f^+\|_\infty$$

for any $\mathbf{f} \in \mathbb{K}$, and satisfy conditions (2.6), (2.7) in the form

$$\mathbf{j}(g^\star) = \mathbf{j}(g)^\dagger, \ \ \mathbf{j}(f \cdot h) = \mathbf{j}(f)\mathbf{j}(h), \quad \mathbf{j}(e) = \mathbf{I}, \quad \forall f, g, h \in \mathfrak{m},$$

where $\mathbf{I} = [\delta^\mu_\nu]$ is the unit operator in $\mathbb{K}$.

We consider the space $\Gamma(\mathbb{K})$ generated by 'exponential' vectors $\mathbf{k}^\otimes = \oplus_{n=1}^\infty \mathbf{k}^{\otimes n}$ with a non-degenerate pseudo-Hilbert scalar product that extends to $\Gamma(\mathbb{K})$ the Hermitian form

$$(\mathbf{k}^\otimes \mid \mathbf{k}^\otimes) = \exp\left\{\int k_\mu(x)k^\mu(x)\mathrm{d}x\right\} = e^{(\mathbf{k}|\mathbf{e})} \tag{3.3}$$

Owing to the defining algebraic property

$$\Gamma(L^1(X) \oplus \mathrm{K} \oplus L^\infty(X)) = \Gamma(L^1(X)) \otimes \Gamma(\mathrm{K}) \otimes \Gamma(L^\infty(X))$$

of the exponential functor Γ, we shall write this scalar product as the triple integral over Ω,

$$(\mathbf{h} \mid \mathbf{h}) = \iiint h(\omega^-,\omega^\circ,\omega^+)\mathrm{h}(\omega^-,\omega^\circ,\omega^+)\mathrm{d}\omega^-\mathrm{d}\omega^\circ\mathrm{d}\omega^+ \equiv \langle \boldsymbol{h}, \mathbf{h}\rangle\,,$$

representing $\mathbf{h}\in\Gamma(\mathbb{K})$ by the ket-function $\mathrm{h}=[\mathrm{h}(\omega^-,\omega^\circ,\omega^+)]$, $\omega^\mu\in\Omega$, and $\boldsymbol{h}=h^\dagger$ by the pseudo-adjoint bra-function $h(\omega_-,\omega_\circ,\omega_+)=h(\omega_+,\omega_\circ,\omega_-)^*$ with values in $\mathrm{K}_*^\otimes(\omega_\circ)$ such that $h^\dagger(\omega^-,\omega^\circ,\omega^+)=h(\omega^+,\omega^\circ,\omega^-)\in\mathrm{K}^\otimes(\omega^\circ)$. The exponential correspondence $k_\cdot^\otimes\mapsto h(\omega.)$ for each $k.=(k_-,k_\circ,k_+)$ with $k_\circ\in\mathrm{K}_*$ is given by

$$k_\cdot^\otimes(\omega_-,\omega_\circ,\omega_+)=k_-^\otimes(\omega_-)k_\circ^\otimes(\omega_\circ)k_+^\otimes(\omega_+),\qquad k_\circ^\otimes(\omega)=\otimes_{x\in\omega}k_\circ(x),$$

with $k_\mp^\otimes(\omega)$ simply described as product functions $\prod_{x\in\omega}k_\mp(x)$ such that indeed

$$\left\langle k_\cdot^\otimes,k_\cdot^{\dagger\otimes}\right\rangle=\exp\left\{\left\langle k_-,k^-\right\rangle+\left\langle k_\circ,k^\circ\right\rangle+\left\langle k_+,k^+\right\rangle\right\}=e^{(\mathbf{k}|\mathbf{k})}.$$

The Banach space $\mathcal{F}.$ of such tensor-functions $h(\omega_-,\omega_\circ,\omega_+)$ with respect to the norm

$$\|\mathbf{h}\|=\int\mathrm{d}\omega^-\left(\int\mathrm{d}\omega^\circ\operatorname*{ess\,sup}_{\omega^+}\|\mathrm{h}(\omega^-,\omega^\circ,\omega^+)\|^2\right)^{1/2}<\infty,$$

equipped with the indefinite scalar product (3.3), will be called a *pseudo-Fock space*. It can be easily verified that $\mathcal{F}.$ contains the exponential vector $\boldsymbol{h}=k_\cdot^\otimes$ if and only if $\|k_-\|_\infty\leq 1$, in which case $\|k_\cdot^\otimes\|=\exp\left\{\|k_+\|_1+\|k_\circ\|_2\right\}$. The set $\mathcal{K}_-^1=\{k.\in\mathcal{K}.:\|k_-\|_\infty\leq 1\}$, where $\mathcal{K}_\cdot^\dagger=\mathbb{K}$, contains the canonical vector $e_\cdot^\otimes$ given by the constant vector-function $e.(x)=(1,0,0)$, and it is invariant under the action $k.\mapsto k.\mathbf{j}(g)$ of $\mathfrak{m}$ since $(k.\mathbf{j}(g))_-=k_-$ for any $g\in\mathfrak{m}$. Therefore the completion of the linear space $\Gamma\left(\mathcal{K}_-^1\right)$ with respect to the Banach poly-norm $\|\boldsymbol{h}\|^f=\{\|\boldsymbol{h}\mathbf{j}^\otimes(f)\|:f\in\mathfrak{m}\}$ is a dense subspace of $\mathcal{F}^\cdot$ which will be denoted by $\mathbb{F}_\star$, with $\mathbb{F}$ for $\mathbb{F}_\star^\dagger=\left\{\boldsymbol{h}^\dagger:\boldsymbol{h}\in\mathbb{F}_\star\right\}$. ($\mathbb{F}_\star$ coincides with $\mathcal{F}.$ if all $j(g)$ are bounded).

The canonical exponential vector is obviously state vector for the infinitely divisible state $\phi(g)$ which is represented as

$$\phi(g)=(\boldsymbol{e}^\otimes\mathbf{j}^\otimes(g)\mid\boldsymbol{e}^\otimes)=\exp\{(\boldsymbol{e}\mathbf{j}(g)\mid\boldsymbol{e})\}=\mathrm{e}^{\lambda(g)}.$$

What is more, as the next theorem shows, the representation $\mathbf{j}^\otimes$, compressed to the Fock subspace $\mathrm{F}\subset\mathbb{F}$ by means of a pseudo-conditional expectation

$$\epsilon[\mathbf{j}^\otimes(g)]:=J^\dagger\mathbf{j}^\otimes(g)J\equiv\pi(g),$$

remains multiplicative, with $J^\dagger\mathbf{e}^\otimes=1_\emptyset$ defined as the vacuum state the unital $*$-representation π associated with $\phi(g)=\exp\lambda(g)$. Here $\delta_\emptyset^\omega=1$ for $\omega=\emptyset$, $\delta_\emptyset^\omega=0$ for $\omega\neq\emptyset$, and

$$(J\mathrm{h})(\omega^-,\omega^\circ,\omega^+)=1_\emptyset(\omega^-)\mathrm{h}(\omega^\circ),\quad \mathrm{h}\in\mathrm{F},\tag{3.4}$$

is a pseudo-isometry $\mathrm{F}\to\mathbb{F}$, $(J\mathrm{h}\mid J\mathrm{h})=\langle\mathrm{h}\mid\mathrm{h}\rangle$ for all $\mathrm{h}\in\mathrm{F}$.

To obtain this result we note that any decomposable operator $\mathbf{K}=1\oplus\mathbf{G}\oplus\mathbf{G}^{\otimes 2}\oplus\ldots$ in $\Gamma(\mathbb{K})$, obtained by exponentiation $\mathbf{G}^\otimes$ of the triangular operator $\mathbf{G}=\mathbf{j}(g)$, can be written in the form

$$[\mathbf{K}\mathrm{h}](\omega^-,\omega^\circ,\omega^+)=\sum_{\bigsqcup_{\nu\geq\mu}\omega_\nu^\mu=\omega^\mu}^{\mu=-,\circ,+}K(\boldsymbol{\omega})\mathrm{h}(\omega_-^-,\omega_\circ^-\sqcup\omega_\circ^\circ,\omega_+^-\sqcup\omega_+^\circ\sqcup\omega_+^+),\tag{3.5}$$

where $\omega = \sqcup\omega_\nu$ denotes direct sum of pairwise disjoint ω_ν defining a decomposition $\omega. = (\omega_\nu)$ of ω. Here $K(\boldsymbol{\omega})$ is a function of the table $\boldsymbol{\omega} = (\omega_\nu^\mu)_{\nu=\circ,+}^{\mu=-,\circ}$ of four subsets $\omega_\nu^\mu \in \Omega$ with values in linear continuous operators

$$K\begin{pmatrix}\omega_+^- & \omega_\circ^- \\ \omega_+^\circ & \omega_\circ^\circ\end{pmatrix} : \mathrm{K}^\otimes(\omega_\circ^-) \otimes \mathrm{K}^\otimes(\omega_\circ^\circ) \to \mathrm{K}^\otimes(\omega_\circ^\circ) \otimes \mathrm{K}^\otimes(\omega_+^\circ),$$

$$K(\boldsymbol{\omega}) = l^\otimes(\omega_+^-, g)\mathrm{k}^\otimes(\omega_+^\circ, g)j^\otimes(\omega_\circ^\circ, g)\mathrm{k}^{\star\otimes}(\omega_\circ^-, g), \quad \mathrm{k}^\star(g) = \mathrm{k}(g^\star)^*, \tag{3.6}$$

where $\mathrm{K}^\otimes(\omega) = \bigotimes_{x\in\omega} \mathrm{K}_x$,

$$l^\otimes(\omega) = \prod_{x\in\omega} l(x), \;\; \mathrm{k}^\otimes(\omega) = \bigotimes_{x\in\omega} \mathrm{k}(x), \;\; \mathrm{k}^{\star\otimes}(\omega) = \bigotimes_{x\in\omega} \mathrm{k}^\star(x), \;\; j^\otimes(\omega) = \bigotimes_{x\in\omega} j(x).$$

Theorem 2. *Let $\mathbf{K} = \bigoplus_{n=0}^\infty \mathbf{K}^{(n)}$ be a decomposable operator in the pseudo-Fock space $\mathbb{F}$ defined in (3.5) by a linear combination of the kernels of the form (3.6). Then the operators $\epsilon(\mathbf{K}) = J^\dagger \mathbf{K} J$, defined by pseudo-projection $J^\dagger : \mathbb{F} \to \mathrm{F}$,*

$$(J^\dagger \mathbf{h})(\omega) = \int \mathrm{h}(\omega^-, \omega, \emptyset)\mathrm{d}\omega^-, \;\; \mathbf{h} \in \mathbb{F}, \tag{3.7}$$

as the adjoint to (3.4), can be extended to a continuous operator

$$[\epsilon(\mathbf{K})\mathrm{h}](\omega) = \sum_{\upsilon\subseteq\omega} \int K(\omega \setminus \upsilon, \upsilon, \vartheta_\circ)\mathrm{h}(\upsilon \sqcup \vartheta_\circ)\mathrm{d}\vartheta_\circ, \tag{3.8}$$

where $K(\vartheta^\circ, \upsilon, \vartheta_\circ) = \int K\begin{pmatrix}\vartheta & \vartheta_\circ \\ \vartheta^\circ & \upsilon\end{pmatrix}\mathrm{d}\vartheta$ defined on the completion F of the pre-Hilbert space $\Gamma(\mathrm{K})$ with respect to the family of the seminorms $\|\mathrm{h}\|^f = \|\pi(f)\mathrm{h}\|$, $f \in \mathfrak{m}$. The map $\epsilon : \mathbf{K} \mapsto \epsilon(\mathbf{K})$ defines a Fock $*$-representation $\epsilon(\lambda\mathbf{K}^\dagger + \lambda^*\mathbf{K}) = \lambda\epsilon(\mathbf{K})^* + \lambda^*\epsilon(\mathbf{K})$,

$$\epsilon(\mathbf{K}\mathbf{K}^\dagger) = \epsilon(\mathbf{K})\epsilon(\mathbf{K})^*, \; \epsilon(\mathbf{I}^\otimes) = I^\otimes$$

of a decomposable $\dagger$-algebra of operators $\mathbf{K}$ with respect to the involution $K^\star(\boldsymbol{\omega}) = K(\boldsymbol{\omega}')^*$, where $(\omega_\nu^\mu)' = (\omega_{-\mu}^{-\nu})$, and the associative product $[K \cdot M](\boldsymbol{\omega}) =$

$$= \sum_{\upsilon_\nu^\mu \subseteq \omega_\nu^\mu}^{\mu<\nu} \sum_{\sigma\cap\tau=\upsilon_+^-}^{\sigma\cup\tau=\omega_+^-} K\begin{pmatrix}\omega_+^- \setminus \tau, & \upsilon_\circ^- \sqcup \upsilon_+^- \\ \omega_+^\circ \setminus \upsilon_+^\circ, & \omega_\circ^\circ \sqcup \upsilon_+^\circ\end{pmatrix} M\begin{pmatrix}\omega_+^- \setminus \sigma, & \omega_\circ^- \setminus \upsilon_\circ^- \\ \upsilon_+^\circ \sqcup \upsilon_+^-, & \omega_\circ^\circ \sqcup \upsilon_\circ^-\end{pmatrix}. \tag{3.9}$$

It induces the involution $K^*(\vartheta^\circ, \upsilon, \vartheta_\circ) = K(\vartheta_\circ, \upsilon, \vartheta^\circ)^*$ and $[K \cdot M](\vartheta^\circ, \upsilon, \vartheta_\circ) =$

$$= \sum_{\upsilon_\circ\subset\vartheta_\circ} \sum_{\upsilon^\circ\subset\vartheta^\circ} \int K(\vartheta^\circ \setminus \upsilon^\circ, \upsilon \sqcup \upsilon^\circ, \upsilon_\circ \sqcup \vartheta)M(\vartheta \sqcup \upsilon^\circ, \upsilon \sqcup \upsilon_\circ, \vartheta_\circ \setminus \upsilon_\circ)\mathrm{d}\vartheta$$

for the kernels $K(\vartheta^\circ, \upsilon, \vartheta_\circ)$ and $M(\vartheta^\circ, \upsilon, \vartheta_\circ)$, and defines a factor algebra of the $\dagger$-algebra of operators $\mathbf{K}$ with respect to the zero $\dagger$-ideal $\{\mathbf{K} : \epsilon(\mathbf{K}\mathbf{K}^\dagger) = 0\}$. The compression $\pi = \epsilon \circ \mathbf{j}^\otimes$ of the $*$-representation ϵ to the operators $\mathbf{K}$ of the form (3.6) defined by the action (3.8):

$$[\pi(g)\mathrm{k}^\otimes](\omega) = \exp\left\{\int (l(x,g) + \mathrm{k}^\star(x,g)\mathrm{k}(x,g))\mathrm{d}x\right\}(\mathrm{k}(g) + j(g)\mathrm{k})^\otimes(\omega), \tag{3.10}$$

of the kernels $K(\omega^\circ, \upsilon, \omega_\circ) = \exp\{\lambda(g)\}\mathrm{k}^\otimes(\omega^\circ, g)j^\otimes(\omega, g)\mathrm{k}^{\star\otimes}(\omega_\circ, g)$ on $\mathrm{k}^\otimes(\omega) = \otimes_{x\in\omega}\mathrm{k}(x)$ yields the unital $*$-representation $\pi : \mathfrak{m} \to \mathcal{L}(\mathrm{K})$,

$$\pi(e) = I^\otimes, \quad \pi(g^\star) = \pi(g)^*, \quad \pi(f \cdot h) = \pi(f)\pi(h)$$

for all $f, g, h \in \mathfrak{m}$, and is associated with the infinitely divisible state $\phi : \mathfrak{m} \to \mathbb{C}$ in the sense that $\phi(g) = (1_\emptyset \mid \pi(g) 1_\emptyset)$, where $1_\emptyset = \mathrm{k}^\otimes$ for $\mathrm{k} = 0$.

Proof. The operator (3.4) is a pseudo-isometry:

$$(J\mathrm{h} \mid J\mathrm{h}) = \int (J\mathrm{h})(\omega^-, \omega^\circ, \omega^+)(J\mathrm{h})^*(\omega^+, \omega^\circ, \omega^-)\mathrm{d}\omega^-\mathrm{d}\omega^\circ\mathrm{d}\omega^+$$
$$= \int 1_\emptyset(\omega^-)\mathrm{h}(\omega^\circ)1_\emptyset(\omega^+)\mathrm{h}^*(\omega^\circ)\mathrm{d}\omega^-\mathrm{d}\omega^\circ\mathrm{d}\omega^+ = \int \mathrm{h}^*(\omega^\circ)\mathrm{h}(\omega^\circ)\mathrm{d}\omega^\circ = \langle \mathrm{h} \mid \mathrm{h} \rangle,$$

and consequently, the Hermitian adjoint operator (3.7) defined by the condition $\left\langle \mathrm{h} \mid J^\dagger \mathbf{h} \right\rangle = (\mathbf{h} \mid J\mathrm{h})$ for all $\mathrm{h} \in \mathrm{F}, \mathbf{h} \in \mathbb{F}$,

$$\left\langle \mathrm{h} \mid J^\dagger \mathbf{h} \right\rangle = \int \mathrm{h}^*(\omega)(J^\dagger \mathbf{h})(\omega)\mathrm{d}\omega = \iiint \mathrm{h}(\omega^-, \omega^\circ, \omega^+)\mathrm{h}^*(\omega^\circ)1_\emptyset(\omega^+)\mathrm{d}\omega^-\mathrm{d}\omega^\circ\mathrm{d}\omega^+,$$

is a pseudo-projection: $J^\dagger J\mathrm{h} = \mathrm{h}$ for all $\mathrm{h} \in \mathrm{F}$, where

$$(J\mathrm{h})(\omega_-, \omega_\circ, \omega_+) = 1_\emptyset(\omega_-)\mathrm{h}(\omega_\circ)1_\emptyset(\omega_+)$$

is the canonical embedding $\mathrm{F} \subset \mathbb{F}$. We now show that the action in $\mathbb{F}$ of the linear combinations of the operators $\mathbf{G}^\otimes$ with triangular $\mathbf{G} = [G^\mu_\nu]^{\mu=-,\circ,+}_{\nu=-,\circ,+}$, $G^\mu_\nu = 0$ for all $\mu < \nu$ with unit matrix entries $G^-_- = 1 = G^+_+$, can be written in the form (3.5).

For we have

$$(\mathbf{G}^\otimes \mathbf{k}^\otimes)(\omega^\cdot) = (\mathbf{G}\mathbf{k})^\otimes(\omega^\cdot) = \prod_\mu \left(\sum_\nu G^\mu_\nu k^\nu \right)^\otimes (\omega^\mu)$$
$$= \prod_\mu \sum_{\bigsqcup_\nu \omega^\mu_\nu = \omega^\mu} \prod_\nu (G^\mu_\nu)^\otimes(\omega^\mu_\nu) \prod_\nu (k^\nu)^\otimes(\omega^\mu_\nu),$$

where the sums over the decompositions $\omega^\mu = \omega^\mu_- \sqcup \omega^\mu_\circ \sqcup \omega^\mu_+$ in fact should be taken only over $\omega^\mu = \bigsqcup_{\nu \geq \mu} \omega^\mu_\nu$ since $G^\mu_\nu = 0$ for $\nu < \mu$. If $\omega = (\omega^-, \omega^\circ, \omega^+)$ do not intersect, then the same is true for $\omega^\cdot_\nu = (\omega^\cdot_-, \omega^\cdot_\circ, \omega^\cdot_+)$ since $\omega^\mu_\nu \subseteq \omega^\mu$. Consequently, $\prod_\mu \mathrm{h}(\omega^\mu_\cdot) = \mathrm{k}\left(\bigsqcup_\mu \omega^\mu_\cdot\right)$ for $\mathrm{h}(\omega_\cdot) = \prod_\nu (k^\nu)^\otimes(\omega_\nu)$, which yields

$$(\mathbf{G}^\otimes \mathbf{k}^\otimes(\omega^\cdot) = \sum_{\bigsqcup \omega^\cdot_\nu = \omega^\cdot} \prod_{\mu,\nu} (G^\mu_\nu)^\otimes(\omega^\mu_\nu)(\prod_\nu k^\nu)^\otimes(\bigsqcup_\mu \omega^\mu_\nu),$$

where $\bigsqcup_\mu \omega^\mu_\nu = \bigsqcup_{\mu \leq \nu} \omega^\mu_\nu$, since $(G^\mu_\nu)^\otimes(\omega) = \bigotimes_{x \in \omega} G^\mu_\nu(x)$ is equal to zero if $\omega = \omega^\mu_\nu \neq \emptyset$ for $\mu > \nu$. Thus we obtain (3.5) for exponential vectors $\mathbf{b} = \mathbf{k}^\otimes$ with the kernel $K(\omega) = \prod_{\mu \leq \nu} (G^\mu_\nu)^\otimes(\omega^\mu_\nu)$ of the form (3.6). Since this formula is linear with respect to the kernel $\mathbf{K}$, it is also valid for linear combinations $\mathbf{K} = \Sigma \lambda_i \mathbf{G}^\otimes_i$ at least on $\Gamma(\mathbf{K})$. We now define the operator $J^\dagger \mathbf{K} J$ in F, employing the formula

$$\int \sum_{\bigsqcup \omega_\mu = \omega} f(\omega_-, \omega_\circ, \omega_+)\mathrm{d}\omega = \iiint f(\omega_-, \omega_\circ, \omega_+)\mathrm{d}\omega_-\mathrm{d}\omega_\circ\mathrm{d}\omega_+.$$

Taking into account the forms (3.4), (3.7) of the operators J, $J^\dagger$ we obtain for $\mathrm{h} \in \Gamma(\mathrm{K})$ the formula

$$\begin{aligned}[J^\dagger \mathbf{K} J\mathrm{h}](\omega) &= \int (\mathbf{K}J\mathrm{h})(\omega^-, \omega, \emptyset)\mathrm{d}\omega^- \\ &= \iiint \sum_{\omega_\circ^\circ \sqcup \omega_+^\circ = \omega} K(\boldsymbol{\omega}) 1_\emptyset(\omega_-^-)\mathrm{h}(\omega_\circ^- \sqcup \omega_\circ^\circ)\mathrm{d}\omega_-^-\mathrm{d}\omega_\circ^-\mathrm{d}\omega_+^- \\ &= \sum_{\omega_\circ^\circ \sqcup \omega_+^\circ = \omega} \int K(\omega_+^\circ, \omega_\circ^\circ, \omega_\circ^-)\mathrm{h}(\omega_\circ^- \sqcup \omega_\circ^\circ)\mathrm{d}\omega_\circ^-,\end{aligned}$$

which can be written in the form (3.8) in the notation $\upsilon = \omega_\circ^\circ$ and $\omega_+^\circ = \omega\backslash\upsilon = \omega\cap\overline{\upsilon}$.

We shall now prove that the pseudo-conditional expectation $\mathbf{K} \to J^\dagger \mathbf{K} J$ is a $*$-representation on $\Gamma(\mathrm{K})$. To this end it is sufficient to show that this map is a homomorphism with respect to the binary operation (3.9), involution $\mathbf{K} \mapsto \mathbf{K}^\dagger$, and the unit $\mathbf{K} = \mathbf{I}^\otimes$ on the generating elements $\mathbf{G}^\otimes$ for which (3.8) yields (3.10) for $\mathrm{h} = \mathrm{k}^\otimes$: $[J^\dagger \mathbf{G}^\otimes J \mathrm{k}^\otimes](\omega) =$

$$\begin{aligned}&= \sum_{\omega_\circ^\circ \sqcup \omega_+^\circ = \omega} \iint \prod_{\mu<\nu} (G_\nu^\mu)^\otimes(\omega_\nu^\mu)\mathrm{k}^\otimes(\omega_\circ^- \sqcup \omega_\circ^\circ)\mathrm{d}\omega_\circ^-\mathrm{d}\omega_+^- \\ &= \sum_{\omega_\circ^\circ \sqcup \omega_+^\circ = \omega} (G_\circ^\circ \mathrm{k})^\otimes(\omega_\circ^\circ)(G_+^\circ)^\otimes(\omega_+^\circ) \int (G_\circ^- \mathrm{k})^\otimes(\omega_\circ^-)\mathrm{d}\omega_\circ^- \int (G_+^-)^\otimes(\omega_+^-)\mathrm{d}\omega_+^- \\ &= (G_\circ^\circ \mathrm{k} + G_+^\circ)^\otimes(\omega) \exp\{\int (G_\circ^- \mathrm{k} + G_+^-)(x)\mathrm{d}x\}.\end{aligned}$$

Using this formula we find that $[J^\dagger \mathbf{I}^\otimes J\mathrm{k}^\otimes](\omega) = \mathrm{k}^\otimes(\omega)$, that is, $J^\dagger \mathbf{I}^\otimes J = I^\otimes$, and

$$[J^\dagger \mathbf{G}^{\dagger\otimes} J\mathrm{k}^\otimes](\omega) = (G_\circ^\circ \mathrm{k} + G_\circ^{-*})^\otimes(\omega) \exp\left\{\int (G_+^{\circ *}\mathrm{k} + G_+^{-*})(x)\mathrm{d}x\right\},$$

that is, $J^\dagger \mathbf{K}^\dagger J = (J^\dagger \mathbf{K} J)$ for $\mathbf{K} = \Sigma\lambda_i G_i^\otimes$, $\mathbf{K}^\dagger = \Sigma\lambda_i \mathbf{G}_i^{\dagger\otimes}$, and

$$\begin{aligned}&[J^\dagger(\mathbf{FG})^\otimes J\mathrm{k}^\otimes](\omega) = (F_\nu^\circ G_\circ^\nu \mathrm{k} + F_\nu^\circ G_+^\nu)^\otimes(\omega) \exp\left\{\int (F_\nu^- G_\circ^\nu \mathrm{k} + F_\nu^- G_+^\nu)(x)\mathrm{d}x\right\} \\ &= (F_\circ^\circ G_\circ^\circ \mathrm{k} + F_\circ^\circ G_+^\circ + F_+^\circ)^\otimes(\omega) \exp\left\{\int (G_\circ^- \mathrm{k} + F_\circ^- G_\circ^\circ \mathrm{k} + G_+^- + F_\circ^- G_+^\circ + F_+^-)(x)\mathrm{d}x\right\} \\ &= (F_\circ^\circ \mathrm{k} + F_+^\circ)^\otimes(\omega) e^{\int (F_\circ^- \mathrm{k} + F_+^-)(x)\mathrm{d}x} (G_\circ^\circ \mathrm{k} + G_+^\circ)^\otimes(\omega) e^{\int (G_\circ^- \mathrm{k} + G_+^-)(x)\mathrm{d}x},\end{aligned}$$

where we have used the rule of multiplication

$$(\mathbf{FG})_\nu^\mu = \sum_\iota F_\iota^\mu G_\nu^\iota \equiv F_\iota^\mu G_\nu^\iota$$

of the triangular matrices $\mathbf{F} = [F_\nu^\mu]$, $\mathbf{G} = [G_\nu^\mu]$, $\mu, \nu \in \{-, \circ, +\}$, $F_\nu^\mu = 0 = G_\nu^\mu$ for $\mu > \nu$, with the entries $F_-^- = 1 = F_+^+$, $G_-^- = 1 = G_+^+$. Thus we have proved that $\epsilon(\mathbf{F}^\otimes \mathbf{G}^\otimes) = \epsilon(\mathbf{F}^\otimes)\epsilon(\mathbf{G}^\otimes)$, where $\epsilon(\mathbf{G}^\otimes)\mathrm{h} = J^\dagger \mathbf{G}^\otimes J\mathrm{h}$ for any $\mathrm{h} = \Sigma\lambda_i \mathrm{k}_i^\otimes \in \Gamma(\mathrm{K})$. We complete $\Gamma(\mathrm{K})$ by sequences $\mathrm{h}_n \in \Gamma(\mathrm{K})$ that are fundamental with respect to each of the seminorms $\|\mathrm{h}\|^f = \|\epsilon\,[\mathbf{j}^\otimes(f)]\,\mathrm{h}\|$, $f \in \mathfrak{m}$ (among others, also with respect to $\|\mathrm{h}\|e = \|\mathrm{h}\|$). Since $\pi(g) = \epsilon[\mathbf{j}^\otimes(g)]$ is a $*$-representation of $\mathfrak{m}$ on $\Gamma(\mathrm{K})$:

$$\pi(g \star f) = \epsilon\left[(\mathbf{j}(g)\mathbf{j}(f^\star))^\otimes\right] = \epsilon[\mathbf{j}^\otimes(g)]\epsilon[\mathbf{j}^\otimes(f)^\dagger] = \pi(g)\pi(f)^*,$$

any fundamental sequence remains fundamental after multiplying it by $\pi(g)$: $\|\pi(g)\mathrm{h}\|^{f}=\|\mathrm{h}\|^{g\cdot f}$. This allows us to extend the operators $\epsilon[\mathbf{j}^{\otimes}(g)]=J^{\dagger}\mathbf{j}^{\otimes}(g)J$ to continuous operators $\pi(g)$ on the completion F of $\Gamma(\mathrm{K})$ with respect to the convergence described above. The continuity implies that the algebraic relations in the decomposable †-algebra $\mathbb{B}=\mathbb{C}\mathbb{M}^{\otimes}$, where $\mathbb{M}=\int_{X}^{\oplus}\mathbb{M}_{x}\mathrm{d}x=\mathbf{j}(\mathfrak{m})$, become represented in the operator algebra $\mathcal{L}(\mathrm{F})$ of continuous operators $L,L^{*}:\mathrm{F}\rightarrow\mathrm{F}$ on the poly-Fock space F. Obviously, the linear hull $\Sigma\lambda_{i}\pi(g_{i})$ defines a $*$-subalgebra $\mathcal{B}$ of operators $\epsilon(\mathbf{K})\in\mathcal{L}(\mathrm{K})$ which is a homomorphic image of the †-algebra $\mathbb{B}$ genrated by linear combinations $\mathbf{K}=\Sigma\lambda_{i}\mathbf{j}^{\otimes}(g_{i})$ of decomposable operators $\mathbf{G}_{i}^{\otimes}=1\oplus\mathbf{G}_{i}\oplus\mathbf{G}_{i}^{\otimes2}\oplus\ldots$, where $\mathbf{G}_{i}=\mathbf{j}(g_{i})$. We recall that the elements of $\mathbb{B}$ as decomposable operators in the pseudo-Fock space $\mathbb{F}$ are represented by triangular kernels $\mathbf{K}\in\mathcal{L}(\mathbb{F})$ with $\mathbf{K}^{\dagger}\in\mathcal{L}(\mathbb{F})$ described in (3.5) by the kernels $K^{\star}(\boldsymbol{\omega})=K(\boldsymbol{\omega}')^{*}$, where the table $\boldsymbol{\omega}'$ of four subsets differs from $\boldsymbol{\omega}=(\omega_{\nu}^{\mu})\in\boldsymbol{\Omega}$ only by the interchange of $\omega_{\circ}^{-}$ and ω_{+}°, and the multiplication $\mathbf{K}^{\dagger}\mathbf{K}$ is defined, as in any semigroup algebra, by the operation $\mathbf{K}\mathbf{K}^{\dagger}=\Sigma\lambda_{i'}\lambda_{i}^{*}\mathbf{j}^{\otimes}(g_{i'}\star g_{i})$ in $\mathbb{M}$. Here $\mathbf{K}\mathbf{K}^{\dagger}$ is defined by the kernel (3.9) which can be verified straightforward for the generating kernels (3.6) by virtue of $\mathbf{j}^{\otimes}(f\cdot g)=\mathbf{j}^{\otimes}(f)\mathbf{j}^{\otimes}(g)$. Indeed,

$$l^{\otimes}(\omega_{+}^{-},f\cdot g)\mathrm{k}^{\otimes}\left(\omega_{+}^{\circ},f\cdot g)j^{\otimes}(\omega_{\circ}^{\circ},f\cdot g\right)\mathrm{k}^{\star\otimes}(\omega_{\circ}^{-},f\cdot g)$$

$$\begin{aligned}
&=[j(f)j(g)]_{\omega_{\circ}^{\circ}}^{\otimes}[j(f)\mathrm{k}(g)+\mathrm{k}\,(f)]_{\omega_{+}^{\circ}}^{\otimes}[l(f)+\mathrm{k}^{\star}(f)\mathrm{k}(g)+l(g)]_{\omega_{+}^{-}}^{\otimes}[\mathrm{k}^{*}(g)+\mathrm{k}^{\star}(f)j(g)]_{\omega_{\circ}^{-}}^{\otimes}\\
&=\sum_{\sigma_{+}^{-}\sqcup\tau_{+}^{-}\sqcup\upsilon_{+}^{-}=\omega_{+}^{-}}l(f)_{\sigma_{+}^{-}}^{\otimes}l(g)_{\tau_{+}^{-}}^{\otimes}[\mathrm{k}^{\star}(f)\mathrm{k}(g)]_{\upsilon_{+}^{-}}^{\otimes}j(f)_{\omega_{\circ}^{\circ}}^{\otimes}j(g)_{\omega_{\circ}^{\circ}}^{\otimes}\\
&\quad\otimes\sum_{\upsilon_{+}^{\circ}\sqcup\sigma_{+}^{\circ}=\omega_{+}^{\circ}}[j(f)\mathrm{k}(g)]_{\upsilon_{+}^{\circ}}^{\otimes}\otimes\mathrm{k}_{\sigma_{+}^{\circ}}^{\otimes}(f)\sum_{\upsilon_{\circ}^{-}\sqcup\tau_{\circ}^{-}=\omega_{\circ}^{-}}\mathrm{k}^{\star}(g)_{\tau_{\circ}^{-}}^{\otimes}\otimes[\mathrm{k}^{\star}(f)j(g)]_{\upsilon_{\circ}^{-}}^{\otimes}\\
&=\sum_{\upsilon_{\nu}^{\mu}\subseteq\omega_{\nu}^{\mu}}^{\mu<\nu}\sum_{\sigma_{+}^{-}\sqcup\tau_{+}^{-}=\omega_{+}^{-}\setminus\upsilon_{+}^{-}}l^{\otimes}(\sigma_{+}^{-},f)\mathrm{k}^{\otimes}(\omega_{+}^{\circ}\setminus\upsilon_{+}^{\circ},f)j^{\otimes}(\omega_{\circ}^{\circ}\sqcup\upsilon_{+}^{\circ},f)\mathrm{k}^{\star\otimes}(\upsilon_{\circ}^{-}\sqcup\upsilon_{+}^{-},f)\\
&\quad\times l^{\otimes}(\tau_{+}^{-},g)\mathrm{k}^{\otimes}(\upsilon_{+}^{\circ}\sqcup\upsilon_{+}^{-},g)j^{\otimes}(\omega_{\circ}^{\circ}\sqcup\upsilon_{\circ}^{-},g)\mathrm{k}^{\star\otimes}(\omega_{\circ}^{-}\setminus\upsilon_{\circ}^{-},g),
\end{aligned}$$

which can be written as (3.9) in terms of $\tau=\tau_{+}^{-}\sqcup\upsilon_{+}^{-}$ and $\sigma=\sigma_{+}^{-}\sqcup\upsilon_{+}^{-}$. Integrating (3.9) over $\omega_{+}^{-}\in\Omega$ we obtain the formula of multiplication of the kernels $K(\omega_{-}^{\circ},\omega_{\circ}^{\circ},\omega_{\circ}^{+})=\int K(\boldsymbol{\omega})\mathrm{d}\omega_{+}^{-}$: $\int[K\cdot K^{\star}](\boldsymbol{\omega})\mathrm{d}\omega_{+}^{-}=$

$$\begin{aligned}
&=\iiint\mathrm{d}\sigma\mathrm{d}\tau\mathrm{d}\upsilon_{+}^{-}\sum_{\upsilon_{\nu-}^{\mu}\omega_{\nu}^{\mu}}^{\mu<\nu}K\left(\ \sigma,\upsilon_{\circ}^{-}\sqcup\upsilon_{+}^{-\circ}\sqcup\upsilon_{+}^{\circ}\ \right)K^{\star}\left(\ \tau,\omega_{\circ}^{-}\setminus\upsilon_{\circ}^{-\circ}\sqcup\upsilon_{\circ}^{-}\ \right)\\
&=\sum_{\upsilon_{\circ}^{-}\subseteq\omega_{\circ}^{-}}\sum_{\upsilon_{+}^{\circ}\subseteq\omega_{+}^{\circ}}\int K(\omega_{+}^{\circ}\setminus\upsilon_{+}^{\circ},\omega_{\circ}^{\circ}\sqcup\upsilon_{n}^{\circ},\upsilon_{\circ}^{-}\sqcup\upsilon_{+}^{-})K^{*}(\upsilon_{+}^{-}\sqcup\upsilon_{+}^{\circ},\omega_{\circ}^{\circ}\sqcup\upsilon_{\circ}^{-},\omega_{\circ}^{-}\setminus\upsilon_{\circ}^{-})\mathrm{d}\upsilon_{+}^{-},
\end{aligned}$$

where $K^{*}(\omega_{-}^{\circ},\omega_{\circ}^{\circ},\omega_{\circ}^{+})=\int K^{\star}(\boldsymbol{\omega})\mathrm{d}\omega_{+}^{-}=K(\omega_{\circ}^{+},\omega_{\circ}^{\circ},\omega_{-}^{\circ})^{*}$. Thus we obtained a $*$-algebraic structure for three-argument kernels connected with the Maassen-Meyer kernels $M(\vartheta^{\circ},\vartheta,\vartheta_{\circ})$ [38], [40] by a one-to-one Möbius transformation

$$K(\vartheta^{\circ},\omega,\vartheta_{\circ})=\sum_{\vartheta\subseteq\omega}M(\vartheta^{\circ},\vartheta,\vartheta_{\circ})\otimes I^{\otimes}(\omega\setminus\vartheta).$$

We finally consider a $\star$-invariant subspace of the $\star$-algebra of kernels $K_0(\boldsymbol{\omega})$ defined by the condition $\int K_0(\boldsymbol{\omega})\mathrm{d}\omega_+^- = 0$. This is a zero ideal of the homomorphism $\{K(\boldsymbol{\omega})\} \mapsto \{K(\omega_+^\circ, \omega_\circ^\circ, \omega_\circ^-)\}$ for which we take a convention that it transforms the star conjugation $\star$ into $*$. Consequently, it is a two-sided ideal ($\dagger$-ideal in terms of $\mathbf{K}$, or $\star$-ideal in terms of the kernels K):

$$\int(\mathbf{K}\mathbf{K}_0)(\boldsymbol{\omega})\,\mathrm{d}\omega_+^- = 0 = \int(\mathbf{K}_0\mathbf{K})(\boldsymbol{\omega})\,\mathrm{d}\omega_+^-, \quad \forall \mathbf{K},$$

which is contained in the zero ideal of the representation ϵ : $\epsilon(\mathbf{K}_0) = 0$ if $K_0(\omega_+^\circ, \omega_\circ^\circ, \omega_\circ^-) = 0$. One can show that, owing to the fact that the measure $\mathrm{d}x$ on X is atomless, the zero ideal of the representation ϵ is exhausted in this way. This follows from the uniqueness of the stochastic representation (3.8), proved in terms of the Maassen-Meyer kernels in [38], [40]. Consequently, the integral $K(\omega_+^\circ, \omega_\circ^\circ, \omega_\circ^-) = \int K(\boldsymbol{\omega})\mathrm{d}\omega_+^-$ is a homomorphism of the factorization of the $\star$-algebra of kernels $K(\boldsymbol{\omega})$ also by the zero ideal of the representation ϵ. The proof is complete. □

Remark 2. *We introduce four types $G_\mu^\nu, \nu \neq -, \mu \neq +$, of elementary triangular decomposable operators in* K *described by matrices of the form*

$$\mathbf{G}_\circ^+(x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & I(x) & g_+^\circ(x) \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{G}_-^\circ(x) = \begin{bmatrix} 1 & g_\circ^-(x) & 0 \\ 0 & I(x) & 0 \\ 0 & 0 & 1 \end{bmatrix},$$
$$\mathbf{G}_-^+(x) = \begin{bmatrix} 1 & 0 & g_+^-(x) \\ 0 & I(x) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{G}_\circ^\circ(x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & G(x) & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and we write

$$G^N = \epsilon[(\mathbf{G}_\circ^\circ)^\otimes] \equiv G^\otimes,\ e^{\lambda(g)} = \epsilon[(\mathbf{G}_+^-)^\otimes], e^{\mathrm{k}^\star(g)A_-^\circ} = \epsilon[(\mathbf{G}_-^\circ)^\otimes], e^{A_\circ^+\mathrm{k}(g)} = \epsilon[(\mathbf{G}_\circ^+)^\otimes],$$

where ϵ is the map (3.8) for $\mathbf{K}(\boldsymbol{\omega}) = \otimes_{x\in\omega}\mathbf{G}_\mu^\nu(x)$.

Then the representation $\mathfrak{m} \ni g \mapsto \pi(g)$, associated with the infinitely divisible state $\phi(g) = e^{\lambda(g)}$ with respect to the vacuum vector $1_\emptyset \in$ K, can be written as a 'normally-ordered' product

$$\pi(g) = e^{\lambda(g)}e^{A_\circ^+\mathrm{k}(g)}G^N e^{\mathrm{k}(g)A_-^\circ}$$

for all $G \in \mathfrak{m}$, defined by the functions $G(x) = j(x,g)$, $g_+^-(x) = l(x,g)$, $g_\circ^-(x) = \mathrm{k}^\star(x,g)$, $g_+^\circ(x) = \mathrm{k}(x,g)$.

In fact, an arbitrary triangular operator $\mathbf{G}$ in K with the entries $G_-^- = 1 = G_+^+$ can be decomposed into a 'normally-ordered' product of elementary matrices:

$$\begin{bmatrix} 1 & g_\circ^- & g_+^- \\ 0 & G & g_+^\circ \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & g_+^- \\ 0 & I & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & I & g_+^\circ \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & g_\circ^- & 0 \\ 0 & I & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since the maps $\mathbf{G} \mapsto \mathbf{G}^\otimes$ and $\mathbf{K} \mapsto \epsilon(\mathbf{K})$ are multiplicative, we hence obtain for $\mathbf{K} = \mathbf{G}^\otimes$ the equality

$$\epsilon[\mathbf{G}^\otimes] = \epsilon[(\mathbf{G}_-^+)^\otimes]\epsilon[(\mathbf{G}_-^+)^\otimes]\epsilon[(\mathbf{G}_\circ^\circ)^\otimes]\epsilon[(\mathbf{G}_-^\circ)^\otimes],$$

which gives for $\mathbf{G} = \mathbf{j}(g)$ the corresponding representation for $\pi(g) = \epsilon[\mathbf{j}^\otimes(g)]$.

4. The pseudo-Poisson structure of chaotic sates on quantum Itô algebras

In this section we assume that the $\star$-semigroup $\mathfrak{b}$ is also an additive group with the same neutral element $0 \equiv u$, such that $\star$-monoid $\mathfrak{m}$ with respect to the multiplication, denoted now by bold dot, $f \bullet h$, has also the structure of an additive group with respect to the pointwise operations

$$(-g)(x) = -g(x),\ (f+h)(x) = f(x) + h(x),\ e(x) = 0.$$

In this terms $f \sqcup h$, whenever it is defined, can be written as $f + h$. We shall also assume that $(f+h)^\star = f^\star + h^\star$ and that the conditionally positive form (2.2) is an additive homomorphism $\mathfrak{m} \mapsto \mathbb{C}$ which will be denoted as $\lambda(g) = \langle g \rangle$:

$$\langle -g \rangle = -\langle g \rangle, \quad \langle f+h \rangle = \langle f \rangle + \langle h \rangle, \quad \langle 0 \rangle = 0.$$

Condition (2.5) of infinite divisibility of the state $\phi_\Delta(b) = e^{\lambda_\Delta(b)}$ for any integrable $\Delta \subseteq X$ can now be written in the form of positive definiteness

$$\sum_{a,c\in\mathfrak{b}} \kappa_a \left\langle (a \cdot c)_\Delta \right\rangle \kappa_c^\star \geq 0, \quad \forall \kappa_b \in \mathbb{C} : |\operatorname{supp}\kappa| < \infty \tag{4.1}$$

of the function $\lambda_\Delta(b) = \langle b_\Delta \rangle$, where $\kappa^\star = \kappa^*_{g^\star}$ and b_Δ is elementary function $b_\Delta(x) = b$ for $x \in \Delta$ and $b_\Delta(x) = 0$ otherwise, with respect to the new product $a \cdot c = a \bullet c - a - c$. This positive definiteness follows from the additivity of the form $\langle g \rangle$, which yields

$$\sum_{f,h\in\mathfrak{m}} \kappa_f \left\langle f \bullet h \right\rangle \kappa_h^\star = \sum_{f,h\in\mathfrak{m}} \kappa_f (\langle f \cdot h \rangle + \langle f \rangle + \langle h \rangle) \kappa_h^\star = \sum_{f,h\in\mathfrak{m}} \kappa_f \left\langle f \cdot h \right\rangle \kappa_h^\star$$

for any function $g \mapsto \kappa_g \in \mathbb{C}$ with $|\operatorname{supp}\kappa| < \infty$ and such that $\Sigma \kappa_g = 0$, where on the right-hand side we can arbitrarily change the value of κ_e since

$$0 \cdot b = 0 \bullet a - 0 - a = -0 = b \cdot 0$$

and therefore $\langle 0 \bullet g \rangle = 0 = \langle g \bullet 0 \rangle$.

We shall now assume that the additive $\star$-group $\mathfrak{b}$ has a ring structure with respect to the new product:

$$a \cdot (b + c) = a \cdot b + a \cdot c,\ a \cdot (b \cdot c) = (a \cdot b) \cdot c.$$

Note that the associativity of this product simply follows from its distributivity which is equivalent to the relation

$$a \cdot b + c \cdot b = b + (a + c) \cdot b.$$

This is particularly easy to see if the ring $\mathfrak{b}$ has identity 1 such that $1b = b = b1$, i.e.

$$1 \cdot b - b = 1 + b = b \cdot 1 - b,$$

by virtue of the relation

$$1 + a \cdot c = (1 + a) \cdot (1 + c), \quad \forall a, c \in \mathfrak{b}.$$

The function $\phi(g) = \mathrm{e}^{\langle g \rangle}$ corresponding to the additive and positive in the above sense form $\langle g \rangle$ is chaotic:

$$\phi(f \sqcup h) = e^{\langle f+h \rangle} = \phi(f)\phi(h), \quad \forall f, h \in \mathfrak{m} : fh = 0,$$

and will be called the *pseudo-Poisson state* over the $\star$-ring (or $\star$-algebra) $\mathfrak{m}$ with respect to the operations $+, \cdot$. In other words, a pseudo-Poisson state is described by an exponential generating functional $\phi(f+h)=\phi(f)\phi(h)$ on $\mathfrak{m}\ni f,h$ which is positive definite in the sense of (2.1) with respect to the operation $f\bigstar h=f+h^{\star}+f\star h$ and $e=0$, defined pointwisely by means of the ring $\mathfrak{m}$.

By this distributivity, the canonical maps

$$\mathrm{k}:\mathfrak{m}\ni g\mapsto \mathrm{k}\left(g\right)\equiv g\rangle \quad \text{and} \quad \mathrm{k}^{\star}:\mathfrak{m}\ni g\mapsto \mathrm{k}^{\star}\left(g\right)\equiv \langle g$$

defining the minimal decomposition (2.6) of the additive (linear) positive form $\langle g\rangle$ are additive (linear), and the $\star$-map $i:\mathfrak{m}\ni g\mapsto j(g)-I$ satisfying in accordance with (2.7) the conditions

$$\begin{aligned} i(g)h\rangle &= g\bullet h\rangle - g\rangle - h\rangle = g\cdot h\rangle, \quad \forall g,h\in\mathfrak{m},\\ \langle fi(g) &= \langle f\bullet g-\langle g-\langle f=\langle f\cdot g, \quad \forall g,f\in\mathfrak{m}, \end{aligned}$$

is also additive (linear):

$$i(f+h)=i(f)+i(h), \quad i(0)=0, \quad i(\lambda g)=\lambda i(g).$$

Moreover, the maps $i_x(b)=j_x(b)-I_x$ are $\star$-representations of the ring (algebra) $\mathfrak{b}$ in the operator $\star$-algebras $\mathcal{L}(\mathrm{K}_x)=\{L:\mathrm{K}_x\to\mathrm{K}_x, L^{*}\mathrm{K}_x\subseteq\mathrm{K}_x\}$ of the poly-Hilbert spaces $\mathrm{K}_x=\{\mathrm{k}:\|j_x(a)\mathrm{k}\|<\infty,\forall a\in\mathfrak{b}\}$:

$$\begin{aligned} i_x(a\cdot c) &= i_x(a\bullet c)-i_x(a)-i_x(c)=j_x(a\bullet c)-1-i_x(a)-i_x(c)\\ &= (i_x(a)+1)(i_x(c)+1)-1-i_x(a)-i_x(c)=i_x(a)i_x(c). \end{aligned}$$

Combining these relations and taking into account the fact that by additivity (linearity) of $l_x(b)$ in the integral (2.2) we have

$$l_x(a\cdot c)=l_x(a\bullet c)-l_x(a)-l_x(c)=\mathrm{k}_x^{\star}(a)\mathrm{k}_x(c)$$

almost everywhere on X, we obtain decomposable $\star$-representation $\boldsymbol{i}(x,b)=\boldsymbol{i}_x(g(x))$ with four-component

$$\text{(4.2)} \qquad \boldsymbol{i}_x(b)=\begin{pmatrix} l_x(b) & \mathrm{k}_x^{\star}(b)\\ \mathrm{k}_x(b) & i_x(b)\end{pmatrix}, \quad \boldsymbol{i}_x(b^{\star})=\begin{pmatrix} l_x(b) & \mathrm{k}_x^{\star}(b)\\ \mathrm{k}_x(b) & i_x(b)\end{pmatrix}^{\star}$$

of the $\star$-ring $\mathfrak{b}$ with the usual matrix Hermitian conjugation $\boldsymbol{i}_x(b^{\star})=\boldsymbol{i}_x(b)^{\star}$ and non-usual multiplication given by the Hudson-Parthasarathy table [26]

$$\text{(4.3)} \qquad \boldsymbol{i}_x(a\cdot c)=\begin{pmatrix} \mathrm{k}_x^{\star}(a)\mathrm{k}_x(c), & \mathrm{k}_x^{\star}(a)i_x(c)\\ i_x(a)\mathrm{k}_x(c), & i_x(a)i_x(c)\end{pmatrix}, \quad \forall a,c\in\mathfrak{b}.$$

It has a natural realization $\mathbf{i}(x,g)=\mathbf{j}(x,g)-\mathbf{I}_x$ given in the pseudo-Euclidean poly-Banach space $\mathbb{K}=L^1(X)\oplus\mathrm{K}\oplus L^{\infty}(X)$ by the canonical triangular representation $\mathbf{j}(x,g)=\mathbf{j}_x(g(x))$ of the $\star$-monoid $\mathfrak{m}$ with the usual matrix multiplication and non-usual pseudo-Hermitian conjugation (3.2):

$$\text{(4.4)} \qquad \mathbf{i}(x,g^{\star})=\begin{bmatrix} 0 & \mathrm{k}(x,g)^{*} & l(x,g)^{*}\\ 0 & i(x,g^{\star}) & \mathrm{k}(x,g^{\star})\\ 0 & 0 & 0\end{bmatrix}=\mathbf{i}(x,g)^{\dagger}.$$

All that has been said means that the factor in $\mathfrak{m}/\mathfrak{i}$, with zero $\star$-ideal $\mathfrak{i}=\{g\in\mathfrak{m}:\boldsymbol{i}(g)=0\}\equiv\boldsymbol{i}_x^{-1}(0)$, of step functions with values $g(x)\in\mathfrak{i}_x$, where $\boldsymbol{i}_x^{-1}(0)=\{b\in\mathfrak{b}:\boldsymbol{i}_x(b)=0\}$, can be described like $\boldsymbol{i}(\mathfrak{m})$ by four-component functions $\boldsymbol{g}=(g_{\nu}^{\mu})_{\nu=\circ,+}^{\mu=\circ,-}$, for example of the form $\boldsymbol{g}(x)=\boldsymbol{i}_x(g(x))$ with $g_{\circ}^{\circ}=i(g)$, $g_{+}^{\circ}=\mathrm{k}(g)$, $g_{\circ}^{-}=\mathrm{k}^{\star}(g)$, $g_{+}^{-}=l(g)$. These form a $\star$-ring with respect to the Hermitian conjugation

$(\boldsymbol{g}^\star(x))^\mu_\nu = g^{-\nu}_{-\mu}(x)^*$ and the table of componentwise multiplication (4.3). This allows us to represent additive integral Hermitian forms

$$\mu(g) = \int m(x,g)\mathrm{d}x, \quad m(x,g) = m_x(g(x)) \tag{4.5}$$

on the $\star$-ring $\mathfrak{m}$ by four-component functions

$$m(x) = \begin{pmatrix} \mu & m^+_\circ \\ m^\circ_- & m^\circ_\circ \end{pmatrix}(x), \quad \begin{array}{l} \mu(x,g) = \mu(x)g^-_+(x),\ m^+_\circ(x,g) = m^+_\circ(x)g^\circ_+(x), \\ m^\circ_-(x,g) = g^-_\circ(x)m^\circ_-(x),\ m^\circ_\circ(x,g) = \langle m^\circ_\circ(x), g^\circ_\circ(x)\rangle, \end{array}$$

in the form

$$m(x,g) = \langle m^\circ_\circ(x), i(x,g)\rangle + \mathrm{m}^*(x)\mathrm{k}(x,g) + \mathrm{k}^\star(x,g)\mathrm{m}(x) + \mu(x)l(x,g). \tag{4.6}$$

Here $\mu(x) \in \mathbb{R}$ (for almost all x), $m^\circ_-(x) : \mathcal{K}_x \to \mathbb{C}$ is a vector linear form $m^\circ_- = \mathrm{m}$ on the pre-Hilbert space $\mathcal{K}_x = \{\mathrm{k}^\star_x(b) : b \in \mathfrak{b}\} = \mathrm{K}^*_x$, adjoint to the form $m^+_\circ(x)$: $\mathrm{K}_x \ni \mathrm{k} \mapsto \mathrm{m}^*(x)\mathrm{k} \in \mathbb{C}$, and $m^\circ_\circ(x) : \mathcal{B}_x \ni B \mapsto \langle m^\circ_\circ(x), B\rangle \in \mathbb{C}$ is an operator linear form on the $*$-subalgebra $\mathcal{B}_x = \{i_x(b) : b \in \mathfrak{b}\}$ of operators $B, B^* : \mathrm{K}_x \to \mathrm{K}_x$. As the next theorem shows, we thus essentially exhaust all linear positive logarithmic forms $\mu : \mathfrak{m} \to \mathbb{C}$ of infinitely divisible states $\psi(g) = e^{\mu(g)}$ on $\star$-algebras $\mathfrak{m}$, absolutely continuous with respect to the Poisson state $\phi(g) = e^{\langle g\rangle}$ in the sense that $\mathfrak{i} \subseteq \mathfrak{i}^\mu$. Here $\mathfrak{i}^\mu$ is the $\star$-ideal of step functions $g : X \ni x \mapsto g(x) \in \mathfrak{i}^\mu_x$ with values in two-sided ideals

$$\mathfrak{i}^\mu_x = \{b \in \mathfrak{b} : m_x(b) = 0, m_x(ab) = 0,\ m_x(bc) = 0,\ m_x(abc) = 0, \quad \forall a, c \in \mathfrak{b}\}. \tag{4.7}$$

Theorem 3. *Suppose that* $\mathfrak{b}$ *is a* $\star$*-algebra over* $\mathbb{C}$ *and suppose that the linear positive form* (2.2) *on the* $\star$*-algebra* $\mathfrak{m}$ *satisfies the condition*

$$\forall g \in \mathfrak{m}\ \exists c < \infty : \langle h \cdot (g \star g) \cdot h^\star\rangle \le c\,\langle h \star h\rangle, \quad \forall h \in \mathfrak{m},$$

of boundedness $\|i(g)\| \le c$ *of the associated operator representation* $i(g) = j(g) - I$*. We equip* $\mathfrak{m}$ *with the inductive convergence* $g_n \to 0$ *if* $\|g_n\|^\Delta_p \to 0$ *for all* $p = 1, 2, \infty$ *and for some integrable* $\Delta \in \mathfrak{F}$*, where* $g_n \in \mathfrak{m}_\Delta$ *for all* n*,* $\|g\|^\Delta_\infty = \|i(g)\|$ *for* $\{x \in X : g(x) \neq 0\} \subseteq \Delta$*, and*

$$\|g\|^\Delta_2 = \left(\int_\Delta \|\mathrm{k}(x,g)\|^2\,\mathrm{d}x\right)^{1/2}, \quad \|g\|^\Delta_1 = \int_\Delta |l(x,g)|\,\mathrm{d}x$$

Then the following conditions are equivalent:

(i) *The functional* $\psi(g) = e^{\mu(g)}$*, continuous with respect to the inductive convergence on* $\mathfrak{m}$*, is a pseudo-Poisson state described by an absolutely continuous function* $\mu_\Delta(b) = \mu(b_\Delta)$ *in the sense that* $\mu_\Delta(b) = 0$ *for all* $b \in \mathfrak{b}$ *if* $\Delta \in \mathfrak{F}$ *and* $\mu_\Delta = \int_\Delta \mathrm{d}x = 0$.

(ii) *The functional* $\mu : \mathfrak{m} \to \mathbb{C}$ *has the integral form* (4.5)*, where* $m_x : \mathfrak{b} \to \mathbb{C}$ *is the linear function* (4.6) *defined almost everywhere on* X *by a positive numerical function* $\mu(x) \ge 0, \operatorname{ess\,sup}_{x\in\Delta}\mu(x) < \infty$ *for all* $\Delta \in \mathfrak{F}$ *with* $\mu_\Delta = \int_\Delta \mathrm{d}x < \infty$*, a vector-function* m *on* X *with values* $\mathrm{m}(x) \in \mathrm{K}_x$ *defined by the values*

$$\mathrm{m}^*(x) \in \mathrm{K}^*_x,\ \int_\Delta \|\mathrm{m}(x)\|^2_x\,\mathrm{d}x < \infty, \quad \forall \Delta \in \mathfrak{F} : \mu_\Delta = \int_\Delta \mathrm{d}x < \infty,$$

of continuous (for almost all $x \in X$) forms $\mathrm{m}^*(x)\mathrm{k} = \langle \mathrm{m}(x) \mid \mathrm{k}\rangle$ *on the Hilbert spaces* $\mathrm{K}_x = \mathcal{K}_x^*$*, and the function* $m_\circ^\circ$ *on* X *with values*

$$m_\circ^\circ(x) \in \mathcal{B}_x^*, \int_\Delta \sup_{0\le B\le 1_x} \langle m_\circ^\circ(x), B\rangle \,\mathrm{d}x < \infty, \forall \Delta \in \mathfrak{F} : \mu_\Delta = \int_\Delta \mathrm{d}x < \infty,$$

in positive forms on C^*-algebras $\mathcal{B}_x$ satisfying almost everywhere the inequality

$$\mu(x) \langle m_\circ^\circ(x), B^*B\rangle \ge \|B\mathrm{m}(x)\|^2, \quad \forall B \in \mathcal{B}_x. \tag{4.8}$$

(iii) *There is a triangular representation*

$$g \in \mathfrak{m} \mapsto \mathbf{g}(x) = \{g_\nu^\mu(x)\}, \quad g_-^\mu = 0 = g_\nu^+, \quad \forall \mu,\nu \in \{-,\circ,+\},$$

of the $\star$-algebra $\mathfrak{m}$ in the Banach space $\mathbb{K} = L^1(X) \oplus \mathcal{K} \oplus L^\infty(X)$ with indefinite metric (3.1) *defined by the scalar product* $\langle k^\circ \mid k^\circ\rangle = \int \|k^\circ(x)\|_x^2 \,\mathrm{d}x$ *of the Hilbert space* $\mathcal{K} = \int^\oplus \mathcal{K}_x \,\mathrm{d}x$*. This representation is locally pseudo-unitarily equivalent to the canonical representation* (4.4) *in the sense that* $\mathbf{g}(x) = \mathbf{S}^\dagger(x)\mathbf{i}(x,g)\mathbf{S}(x)$ *for decomposable operators* $\mathbf{S}(x)$ *in* $\mathbb{C} \oplus \mathcal{K}_x \oplus \mathbb{C}$ *of the form* (2.11)*, and is such that*

$$\mu(g) = \int (\mu(x) g_+^-(x) + \langle M(x), g_\circ^\circ(x)\rangle) \,\mathrm{d}x, \quad \forall g \in \mathfrak{m}, \tag{4.9}$$

where $\mu \ge 0$ is a locally bounded measurable function and $M \ge 0$ is a locally integrable function with positive values $M(x) \in \mathcal{B}_x^*$.

Proof. First of all we notice that if the decomposable operator-functions $i_x(b)$ are locally bounded, then the space K of the canonical representation $j(g) = I + i(g)$ of the $\star$-monoid $\mathfrak{m}$ of step functions $g : X \to \mathfrak{b}$, complete with respect to the family of seminorms (2.8), is a Hilbert space. This follows from the inequality

$$\|\mathrm{k}\|^h = \|j(h)\mathrm{k}\| < \|\mathrm{k}\| + \|i(h)\mathrm{k}\| \le (1 + \|h\|) \|\mathrm{k}\|,$$

where $\|f\| = \max_i \|b_i\|_{\Delta(i)} < \infty$ according to (4.6) for any step integrable function $f(x) = b_i, x \in \Delta(i)$, given by a finite partition $\Delta = \Sigma\Delta(i)$ of its support $\Delta = \{x \in X : f(x) \ne 0\}$.

We shall first prove the simple implications (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i), and next we shall construct the representation (4.8) of (iii) drawing on the conditions formulated in (i).

(iii) $\Rightarrow$ (ii). Suppose that $\mathbf{S}(x)$ is the triangular transformation of the form (2.11) given by an essentially measurable function $U(x) \in \mathcal{L}(\mathrm{K}_x)$ with unitary values, a function $e_\circ : X \ni x \mapsto e_\circ(x) \in \mathrm{K}_x^*$ given by the values $e_\circ^*(x) \in \mathrm{K}_x$ of a vector-function $e_\circ^*$ with $\int_\Delta \|e_\circ^*(x)\|_x^2 \,\mathrm{d}x < \infty$ for all Δ such that $\mu_\Delta = \int_\Delta \mathrm{d}x < \infty$, and a scalar locally integrable function e_+ such that $e_+(x) + e_+^*(x) = -\|e_\circ^*(x)\|_x^2$. Then $g_\circ^\circ(x) = U^*(x)i(x,g)U(x)$,

$$\begin{aligned} g_+^-(x) = e_\circ(x)U(x)\mathrm{k}(x,g) \qquad &+ e_\circ(x)U(x)i(x,g)U^*(x)e_\circ^*(x) \\ &+ \mathrm{k}^\star(x,g)U^*(x)e_\circ^*(x), \end{aligned}$$

and (4.9) takes the form (4.5) (4.6), where $\mathrm{m}^*(x) = e_\circ(x)U(x), \mathrm{m}(x) = U^*(x)e_\circ^*(x)$ is a locally square integrable function: $\int_\Delta \|\mathrm{m}(x)\|_x^2 \,\mathrm{d}x < \infty$, and

$$\langle m_\circ^\circ(x), B\rangle = \langle M(x), U^*(x)BU(x)\,\rangle + \mu(x)\mathrm{m}^*(x)B\mathrm{m}(x)$$

is a positive locally integrable function: $\int_\Delta \langle m_\circ^\circ(x), B_x\rangle \,\mathrm{d}x < \infty$, satisfying (4.8) by virtue of the positivity $\langle M(x), B^*B\rangle \geq 0$, $\mu(x) \geq 0$ for all $x \in X$.

(ii) $\Rightarrow$ (i). If μ is the integral (4.5) of the linear form (4.6) and (4.8) is fulfilled, then $\mu(g^\star g) \geq 0$ for all $g \in \mathfrak{m}$, since

$$\begin{aligned} 0 \leq m(g^\star g) &= \langle m_\circ^\circ, i(g^\star g)\rangle + 2\operatorname{Re}\mathrm{m}^*\mathrm{k}(g^\star g) + \mu l(g^\star g) \\ &= \langle m_\circ^\circ, i(g)^* i(g)\rangle + 2\operatorname{Re}\mathrm{m}^* i(g)^*\mathrm{k}(g) + \mu(x)\mathrm{k}(g)^*\mathrm{k}(g) \\ &= \langle m_\circ^\circ, i(g)^* i(g)\rangle - \tfrac{1}{\mu}\left\| i(g)\mathrm{m}\right\|^2 + \mu\|\mathrm{k}(g) + i(g)\mathrm{m}/\mu\|^2. \end{aligned}$$

By linearity of μ this is equivalent to positive definiteness

$$\sum_{a,c} \kappa_a^* \mu_\Delta(a^\star c)\kappa_c = \mu_\Delta(\sum_{a,c} \kappa_a^* a^\star c \kappa_c) = \mu_\Delta(b^\star b) \geq 0$$

of the form $\mu_\Delta(b) = \mu(b_\Delta)$. Hence it follows that $\phi_\mu(g) = e^{\mu(g)}$ is a pseudo-Poisson state given by an absolutely continuous complex measure $\mu_\Delta(b) = \int_\Delta m_x(b)\mathrm{d}x$ with density $m_x(b) = m(x, b_\Delta)$. It is continuous with respect to the inductive convergence with respect to the seminorms $\|g\|_p^\Delta, p = 1, 2, \infty$, because μ is locally bounded, m is locally L^2-integrable, and $m_\circ^\circ$ is locally L^1-integrable.

(i) $\Rightarrow$ (iii). If the function $\mu_\Delta(b) = \ln\psi(b)$ for a pseudo-Poisson state $\psi_\Delta(b) = \psi(b_\Delta)$ on $\mathfrak{b}$ is absolutely continuous with respect to $\Delta \in \mathfrak{F}$ for every $b \in \mathfrak{b}$, then it has the form (4.5), where the density $m_x : \mathfrak{b} \to \mathbb{C}$ is almost everywhere a linear positive functional. Since the kernel $\{g \in \mathfrak{m} : \|g\|_p = 0, p = 1, 2, \infty\}$ of the inductive convergence in $\mathfrak{m} = \cup\mathfrak{m}_\Delta$ coincides with the kernel $\mathfrak{i}$ of the canonical representation $\mathbf{i}(g) = \mathbf{j}(g) = \mathbf{I}$ in $\mathbb{K}$, which is equal, according to its construction, to step functions $g : x \mapsto g(x) \in \mathfrak{i}_x$, where

$$\mathfrak{i}_x = \{b \in \mathfrak{b} : l_x(b) = 0, l_x(ab) = 0,\ l_x(bc) = 0, l_x(abc) = 0, \quad \forall a, b, c \in \mathfrak{b}\},$$

the $\star$-ideal $\mathfrak{i}^\mu$ of functions $g \in \mathfrak{m}$ with values $g(x)$ in (4.7), corresponding to the form (4.5) continuous in the sense that $g_n \to 0 \Rightarrow \mu(g_n) \to 0$, necessarily contains $\mathfrak{i}$. This means that a linear functional $m_x(b)$ that vanishes on $\mathfrak{i}_x$ for almost all x can be written, by this continuity, in the form (4.6) of a linear Hermitian functional $m_x(b) = m(x, b_\Delta), x \in \Delta$, on the factor algebra $\mathfrak{b}/i_x^{-1}(0)$ isomorphic to the $\star$-subalgebra $i_x(\mathfrak{b})$ of quadruples (4.2) with the multiplication table (4.3). In addition, by the Hahn-Banach theorem and the duality between $L^p(\Delta)$ and $L^q(\Delta)$ for $1/p + 1/q = 1$ we can assume that μ is locally bounded, m is locally L^2-integrable, and $m_\circ^\circ$ is locally L^1-integrable on X.

For every $x \in X$ we define a triangular pseudo-unitary transform of $\mathbf{S}(x)$ into $\mathbb{K}_x = \mathbb{C} \oplus \mathrm{K}_x \oplus \mathbb{C}$ of the form (2.11), where $U = -I_x, e_\circ^*(x) = \mathrm{m}(x)$, and $e_+^*(x) = -\|\mathrm{m}(x)\|_x^2/2$. Denoting $g_\nu^\mu(x) = (\mathbf{S}^\dagger\mathbf{i}(x, g)\mathbf{S}(x))_\nu^\mu$, where $\mathbf{i}(x)$ is the triangular matrix representation (4.4) of the quadruple (4.2) for $b = g(x)$, we obtain

$$m(g) = \langle g_\circ^\circ, m_\circ^\circ\rangle - \mathrm{m}^* g_\circ^\circ \mathrm{m}/\mu + \mu g_+^-,$$

where we have taken into account the fact that $g_\circ^\circ(x) = i_x(g(x))$ and

$$\mu g_+^- = \mu l(g) + \mathrm{k}^\star(g)\mathrm{m} + \mathrm{m}^*\mathrm{k}(g) + \mathrm{m}^* i(g)\mathrm{m}/\mu.$$

In this representation the positivity condition $m(x, g^\star g) \geq 0$ takes the form

$$\langle g_\circ^{\circ *} g_\circ^\circ, M\rangle + \mu g_+^{\circ *} g_+^\circ \geq 0, \quad \forall g \in \mathfrak{m},$$

where $\langle B, M\rangle = \langle B, m_\circ^\circ\rangle - \mathrm{m}^* B\mathrm{m}/\mu$, $B \in \mathcal{B}_x$, and $g_+^\circ = \mathrm{k}_x(g) + i_x(g)\mathrm{m}$. The resulting inequality proves that $M(x)$ is positive for $g_+^\circ(x) = 0$ and $\mu(x) \geq 0$ if

$g_{\circ}^{\circ}(x) = 0$. This proves the existence of locally bounded measurable functions $\mu \geq 0$ and positive locally integrable functions M with values $M(x) \in \mathcal{B}_x^*$ defining the function $\mu(g)$ in the form (4.9). The proof is complete. □

Remark 3. *We consider an additive subgroup $\mathfrak{b} \subseteq \mathbb{C} \times H \times \mathcal{L}(K)$ of the triples $b = (\beta, \eta, B)$ with the involution $b^\star = (\beta^*, \eta^\#, B^*)$, where $\beta \mapsto \beta^* \in \mathbb{C}$ is the complex conjugation, $\eta \mapsto \eta^\# \in H$ is the involution $\eta^{\#\#} = \eta$ in a $\mathbb{C}$-linear subspace $H \subseteq K$ equipped with the Hermitian form $\langle \xi \mid \zeta \rangle = \xi^\# \cdot \zeta = \langle \xi \mid \zeta \rangle^*$ of a pseudo-Euclidean space K, and $B \mapsto B^* \in \mathcal{L}(K)$ is the Hermitian conjugation $\langle B^* \xi \mid \zeta \rangle = \langle \xi \mid B\zeta \rangle$ for all $\xi, \zeta \in K$ in the $*$-subalgebra $\mathcal{L} \subseteq \mathcal{L}(K)$ of operators $B : \eta \mapsto B\eta \in K$ leaving H invariant: $BH \subseteq H$ for all $B \in \mathcal{L}$.*

We define in $\mathfrak{b}$ the structure of a $\star$-algebra by putting

$$\lambda b = (\lambda\beta, \lambda\eta, \lambda B), a^\star c = (\xi^\# \cdot \zeta, \xi^\# C + A^* \zeta, A^* C)$$

for any $\lambda \in \mathbb{C}, b \in \mathfrak{b}, a = (\alpha, \xi, A), c = (\gamma, \zeta, C)$, where we use the notation $\xi^\# C = (C^* \xi)^\#$. It is easy to prove that this distributive algebra is associative, $(ab)c = a(bc)$, only in the case

$$(A\eta) \cdot \zeta = \xi \cdot (\eta C), (A\eta)C = A(\eta C), \quad \forall A, C \in \mathcal{L}, \xi, \eta, \zeta \in H,$$

which is possible only under the condition $(A\eta) \cdot \zeta = 0 = \xi \cdot (\eta C)$. This condition leads to $(A\eta)C = A(\eta C)$ if $\xi \cdot \zeta = (\xi^\# \mid \zeta)$ is a bilinear form on H non-degenerate in the sense that $\{\xi \cdot \eta = 0 = \eta \cdot \zeta : \forall \xi, \zeta \in H\} \Rightarrow \eta = 0$. A simple analysis of the positivity

$$l(b^\star b) = \lambda \langle \eta \mid \eta \rangle + \langle B\vartheta \mid \eta \rangle + \langle \eta \mid B\vartheta \rangle + \langle \Lambda, B^* B \rangle \geq 0$$

of the linear $\star$-form $l(b) = \lambda\beta + \vartheta_- \cdot \eta + \eta \cdot \vartheta_+ + \langle \Lambda, B \rangle$, where $\lambda = \lambda^*$, $\vartheta_+ = \vartheta = \vartheta_-^\#$, $\Lambda = \Lambda^*$, leads to the conditions $\langle \Lambda, B^* B \rangle \geq 0$ for all $B \in \mathcal{L}$ if $\lambda = 0$ and

$$\lambda \langle \eta \mid \eta \rangle \geq 0, \langle \Lambda, B^* B \rangle \geq \tfrac{1}{\lambda} \langle B\vartheta \mid B\vartheta \rangle, \quad \forall \eta \in H, B \in \mathcal{L}$$

if $\lambda \neq 0$. The latter is possible only if the form $\langle \eta \mid \eta \rangle = \eta^\# \cdot \eta$ is definite, that is, $\lambda > 0$ if $\langle \eta \mid \eta \rangle \geq 0$ for all $\eta \in H$ and $\lambda < 0$ if $\langle \eta \mid \eta \rangle \leq 0$ for all $\eta \in H$, which is a necessary condition for the existence of a pseudo-Poisson state on $\mathfrak{b} = \mathbb{C} \times H \times \mathcal{L}$.

Assuming without loss of generality that $\eta^\# \eta \geq 0$ for all η (otherwise we have to change the notation $b \mapsto (-\beta, \eta, B)$ and $\eta^\# \eta \mapsto -\eta^\# \eta$) we consider the following two cases, in which H is a Hilbert space with respect to the norm $\|\eta\| = (\eta^\#, \eta)^{1/2}$, where $(\xi, \zeta) = \frac{1}{2}(\xi \cdot \zeta + \zeta \cdot \xi)$.

Example 1 (Gaussian state)**.** *Let $\mathcal{L} = \{0\}$ and $\lambda = 1$, that is, $b = (\beta, \eta)$, and let $l(b) = \langle \eta, \theta \rangle + \beta$, where $\langle \eta, \theta \rangle = 2 \operatorname{Re}(\eta \mid \theta)$ for all $\eta = \eta^\#$. The algebra $\mathfrak{b} = \mathbb{C} \times H$ is now nilpotent: $ac = (\xi, \zeta, 0)$, $abc = (0, 0)$ for all $a, b, c \in \mathfrak{b}$, and commutative, $[a, c] = ac - ca = 0$, if the involution $\#$ is isometric on H in $K \supseteq H$:*

$$\left\langle \xi^\# \mid \zeta \right\rangle = \left\langle \zeta^\# \mid \xi \right\rangle, \quad \forall \xi, \zeta \in H.$$

The infinitely divisible functional $\phi_\Delta(b) = \exp\{[\beta + (\eta, \theta)] \mu_\Delta\}$ corresponding to the conditionally positive $\star$-form $\lambda_\Delta(b) = [\beta + (\eta, \theta)] \mu_\Delta$ with respect to the Hermitian operation

$$(\alpha, \xi) \star (\gamma, \zeta) = (\alpha^* + \langle \xi \mid \zeta \rangle + \gamma, \xi^\# + \zeta), \quad u = (0, 0),$$

defines a generating functional $\phi_\Delta(0, \eta) = 1$ of the factorial moments of a Gaussian chaotic state over H with mathematical expectation $\langle b_\Delta \rangle = (\eta, \theta) \mu_\Delta$ for $b = (0, \eta)$

and finite covariance $\langle b^\star_\Delta b_\Delta\rangle = \langle\eta \mid \eta\rangle\,\mu_\Delta \in \mathbb{R}_+$ *for every* $\Delta \in \mathfrak{F}$ *such that* $\mu_\Delta = \int_\Delta \mathrm{d}x < \infty$*. This covariance is symmetric only in the commutative* (*classical*) *case, and in the converse* (*quantum*) *case it satisfies the uncertainty relation*

$$\left\langle a^2_\Delta\right\rangle\left\langle c^2_\Delta\right\rangle \geq s(\xi,\zeta)^2\mu^2_\Delta, \quad \forall a = (\alpha,\xi), c = (\gamma,\zeta), \xi,\zeta \in \mathrm{Re}H$$

for the commutative Heisenberg relation $[a_\Delta, c_\Delta] = (\mathrm{i}s(\zeta,\xi)\mu_\Delta 0)$ *corresponding to the symplectic form* $s(\xi,\zeta) = 2\,\mathrm{Im}(\xi \mid \zeta)$ *on* $\mathrm{Re}H = \{\eta \in H : \eta^\# = \eta\}$*. The canonical representation* (4.4)*, defining a* $\star$*-representation* $\mathbf{j}(g) = \mathbf{I} + \mathbf{i}(g)$ *of the* $\star$*-monoid* $\mathfrak{m}$ *of step functions* $g : X \to \mathbb{C}\times H$*, and the corresponding representation* $\pi(g) = \epsilon[\mathbf{j}^\otimes(g)]$ *in the Fock space* K*, is described by the functions* $i_x(b) = 0$*,* $\mathrm{k}_x(b^\star) = \eta^\#$*,* $\mathrm{k}_x(b)^* = \eta^*$*,* $l_x(b) = \beta + (\eta,\theta)$*.*

Example 2 (Poisson state)**.** *Let* $H = \{0\}$ *and let* $\mathfrak{b} = \mathcal{L}$ *be the* $*$*-algebra of operators in* K *bounded by the identity* $I \in \mathfrak{b}$ *in the sense that*

$$\forall C = B^*B \;\; \exists c \in \mathbb{R}_+ : \langle\Lambda, A^*CA\rangle \leq c\,\langle\Lambda, A^*A\rangle\,, \quad \forall A \in \mathfrak{b},$$

where Λ *is a linear positive form defining* $l(b) = \langle\Lambda, B\rangle$*. Bearing in mind the Gelfand-Naimark-Segal construction, we may assume without loss of generality that this form is a vector one,* $\langle\Lambda, B\rangle = \langle eBe\rangle$*, represented in the Hilbert space* K *by an element* $e \in K, \|e\|^2 = \langle\Lambda, I\rangle$*. In the commutative case* $\mathfrak{b}$ *can be identified with a subalgebra of essentially bounded functions* $b : \omega \mapsto b(\omega) \in \mathbb{C}$ *on a measurable space* Ω *with finite positive measure* $\mathrm{d}\lambda$ *of the mass* $\lambda = \langle\Lambda, I\rangle$ *by putting* $(B\mathrm{k})(\omega) = b(\omega)\mathrm{k}(\omega)$ *on* $K = L^2_\lambda(\Omega)$*, and* $e(\omega) = 1$ *for all* $\omega \in \Omega$*, so that* $l(b) = \int b(\omega)\mathrm{d}\lambda$*. The infinitely divisible functional* $\phi_\Delta(b) = e^{\langle\Lambda,B\rangle\mu_\Delta}$*, corresponding to the conditionally positive* $\star$*-form* $\lambda_\Delta(b) = \langle\Lambda, B\rangle\,\mu_\Delta$ *with respect to the Hermitian operation* $A\cdot C = A^* + A^*C + C$ *with the neutral element* $U = 0$*, defines the generating functional of factorial moments of a Poisson chaotic state over* $\mathcal{L}$ *with mathematical expectation* $\langle b_\Delta\rangle = \langle\Lambda, B\rangle\,\mu_\Delta$ *and finite covariance* $\langle b^\star_\Delta b_\Delta\rangle = \langle\Lambda, B^*B\rangle\,\mu_\Delta \in \mathbb{R}_+$ *for each* $\Delta \in \mathfrak{F}$ *such that* $\mu_\Delta = \int_\Delta \mathrm{d}x < \infty$*. This covariance is symmetric not only in the commutative* (*classical*) *case* $[A, C] = AC - CA = 0$*, but also in the case when* $\Lambda \in \mathcal{L}^*$ *is central. The central form* $\langle\Lambda, B\rangle$*, described by the condition* $\langle\Lambda, [A, C]\rangle = 0$ *for all* $A, C \in \mathcal{L}$*, defines a* σ*-finite trace on the* $*$*-algebra* $\mathfrak{m}$ *of step functions* $G : X \ni x \mapsto G(x) \in \mathcal{L}$ *with the integral form* $\langle g\rangle = \int\langle\Lambda, G(x)\rangle\,\mathrm{d}x$ *or* $\langle g\rangle = \iint g(x,\omega)\mathrm{d}x\mathrm{d}\lambda$ *in the case of* $\mathfrak{b} \sim L^\infty_\gamma(\Omega)$*. Otherwise, the form* $\langle\Lambda, B\rangle$ *can also lead to the uncertainty relation*

$$\left\langle a^2_\Delta\right\rangle\left\langle c^2_\Delta\right\rangle \geq \left\langle\Lambda, \frac{1}{\mathrm{i}}[A, C]\right\rangle^2\mu^2_\Delta, \quad \forall A = A^*, C = C^*.$$

The canonical representation (4.3)*, defining the indefinite representation* $\mathbf{j}(g) = \mathbf{I} + \mathbf{i}(g)$ *of the* $\star$*-monoid* $\mathfrak{m}$ *and the corresponding representation* $\pi(g) = \epsilon[\mathbf{j}^\otimes(g)]$ *in the Fock space* K*, is described by the functions*

$$i_x(b) = B,\ \mathrm{k}_x(b^\star) = B^*e,\ \mathrm{k}^\star_x(b) = e^*B,\ l_x(b) = e^*Be,$$

where $e^*Be = \langle eBe\rangle = \langle\Lambda, B\rangle$*.*

Part 2. Non-commutative stochastic analysis and quantum evolution in scales

5. Introduction

Non-commutative generalization of the Itô stochastic calculus, developed in [1], [3], [21], [36], [41] and [44] gave an adequate mathematical tool for studying the behavior of open quantum dynamical systems singularly interacting with a boson quantum-stochastic field. Quantum stochastic calculus also made it possible to solve an old problem of describing such systems with continuous observation and constructing a quantum filtration theory which would explain a continuous spontaneous collapse under the action of such observation [8], [11] and [12]. This gave examples of stochastic non-unitary, non-stationary, and even non-adapted evolution equations in a Hilbert space whose solution requires a proper definition of chronologically ordered quantum stochastic semigroups and exponents of operators by extending the notion of the multiple stochastic integral to non-commuting objects.

Here we outline the solution to this important problem by developing a new quantum stochastic calculus in a natural scale of Fock spaces based on an explicit definition, introduced by us in [13], of the non-adapted quantum stochastic integral as a non-commutative generalization of the Skorokhod integral [48] represented in the Fock space. Using the indefinite $\star$-algebraic structure of the kernel calculus, which was obtained in the first chapter as a general property of a natural pseudo-Euclidean representation associated with infinitely divisible states, we establish the fundamental formula for the stochastic differential of a function of a certain number of non-commuting quantum processes, providing a non-commutative and non-adapted generalization of the Itô formula as the principal formula of the classical stochastic calculus. In the adapted case this formula coincides with the well-known Hudson-Parthasarathy formula [26] for the product of a pair of non-commuting quantum processes. In the commutative case it gives a non-adapted generalization of the Itô formula for classical stochastic processes which was recently obtained in a weak form by classical stochastic methods by Nualart [42] in the case of Wiener integrals. We also note that the classical stochastic calculus and the calculus of operators in the Fock scales was worked out by the group Hida, Kuo, Streit and Potthoff, see [25] and [45], and also by Berezanskii and Kondrat'ev [19].

Using the notion of a normal multiple quantum stochastic integral, which is formulated below, we construct explicit solutions of quantum stochastic evolution equations in the adapted as well as in the non-adapted case of operator-valued coefficients and we give a simple algebraic proof of the fact that this evolution is unitary if the generators of these equations are pseudounitary. In the adapted stationary case the quantum stochastic evolution was constructed by Hudson and Parthasarathy by means of the approximation by the Itô sums of quantum-stochastic generators. However, proving unitarity by this method turned out to be a difficult problem even in a simple case.

Within the framework of this approach Kholevo [30] constructed a solution of an adapted quantum-stochastic differential equation also for non-stationary generators by defining the chronological exponential as a quantum-stochastic multiplicative integral.

We note that our approach is close in spirit to the kernel calculus of Maassen-Lindsay-Meyer [36], [41], however the difference is that all the main objects are constructed not in terms of kernels but in terms of operators represented in the Fock space. In addition we employ a much more general notion of multiple stochastic integral, non-adapted in general, which reduces to the notion of the kernel representation of an operator only in the case of a scalar (non-random) operator function under the integral. The possibility of defining a non-adapted single integral in terms of the kernel calculus was shown by Lindsay [37], but the notion of the multiple quantum-stochastic integral has not been discussed in the literature even in the adapted case.

6. Non-adapted stochastic integrals and differentials in Fock scale

Let $(X, \mathfrak{F}, \mu)$ be an essentially ordered space, that is, a measurable space X with a σ-finite measure $\mu : \mathfrak{F} \ni \Delta \mapsto \mu_\Delta \geq 0$ and an ordering relation $x \leq x'$ with the property that any n-tuple $\boldsymbol{x} \in X^n$ is up to a permutation a chain $\varkappa = \{x_1 < \cdots < x_n\}$ modulo the product $\prod_{i=1}^n \mathrm{d}x_i$ of the measures $\mathrm{d}x := \mu_{\mathrm{d}x}$. In other words, we assume that the measurable ordering is almost total, or linear, that is, for any n the product measure of n-tuples $\boldsymbol{x} \in X^n$ with components $(x_1, \ldots, x_n)$ that are not comparable is zero. Hence, in particular, it follows that the measure μ on X is atomless. We may assume that this essentially total ordering on X is induced by a measurable map $t : X \to \mathbb{R}_+$ with respect to which μ is absolutely continuous in the sense of admitting the decomposition

$$\int_\Delta f(t(x))\mathrm{d}x = \int_0^\infty f(t)\mu_\Delta(t)\mathrm{d}t,$$

for any integrable set $\Delta \subseteq X$ and any essentially bounded function $f : \mathbb{R}_+ \to \mathbb{C}$, where $\Delta \mapsto \mu_\Delta(t)$ is a positive measure on X for each $t \in \mathbb{R}_+$ and $x_1 < \cdots < x_n$ means that $t(x_1) < \cdots < t(x_n)$. In any case we shall assume that we are given a map t such that the above condition holds and $t(x) \leq t(x')$ if $x \leq x'$, interpreting $t(x)$ as the time at the point $x \in X$. For example, $t(x) = t$ for $x = (\mathbf{x}, t)$ if $X = \mathbb{R}^d \times \mathbb{R}_+$ is the $(d+1)$-dimensional space-time with the casual ordering [5] and $\mathrm{d}x = \mathrm{d}\mathbf{x}\mathrm{d}t$, where $\mathrm{d}\mathbf{x}$ is the standard volume on d-dimensional space $\mathbb{R}^d \ni \mathbf{x}$.

We shall identify the finite chains $\varkappa$ with increasingly indexed n-tuples $\boldsymbol{x} = (x_1, \ldots, x_n)$ of $x_i \in X$, $x_1 < \cdots < x_n$, denoting by $\mathcal{X} = \sum_{n=0}^\infty \Gamma_n$ the set of all finite chains as the union of the sets $\Gamma_n = \{\boldsymbol{x} \in X^n : x_1 < \cdots < x_n\}$ with one-element $\Gamma_0 = \{\emptyset\}$ containing the empty chain as a subset of X: $\emptyset = X^0$. We introduce a measure 'element' $\mathrm{d}\varkappa = \prod_{x \in \varkappa} \mathrm{d}x$ on $\mathcal{X}$ induced by the direct sum $\sum_{n=0}^\infty \mu^n_{\Delta_n}, \Delta_n \in \mathfrak{F}^{\otimes n}$ of product measures $\mathrm{d}\boldsymbol{x} = \prod_{i=1}^n \mathrm{d}x_i$ on X^n with the unit mass $\mathrm{d}\varkappa = 1$ at the only atomic point $\varkappa = \emptyset$.

Let $\{\mathrm{K}_x : x \in X\}$ be a family of Hilbert spaces K_x, let $\mathcal{P}_0$ be an additive semigroup of positive essentially measurable locally bounded functions $p : X \to \mathbb{R}_+$ with zero $0 \in \mathcal{P}_0$, and let $\mathcal{P}_1 = \{1 + p_0 : p_0 \in \mathcal{P}_0\}$. For example, in the case $X = \mathbb{R}^d \times \mathbb{R}_+$ by $\mathcal{P}_1$ we mean the set of polynomials $p(x) = 1 + \sum_{k=0}^m c_k |x|^k$ with respect to the modulus $|\mathbf{x}| = (\Sigma x_i^2)^{1/2}$ of a vector $\mathbf{x} \in \mathbb{R}^d$ with positive coefficients $c_k \geq 0$. We denote by $\mathrm{K}(p)$ the Hilbert space of essentially measurable vector-functions $\mathrm{k} : x \mapsto \mathrm{k}(x) \in \mathrm{K}_x$ which are square integrable with the weight $p \in \mathcal{P}_1$:

$$\|\mathrm{k}\|(p) = \left(\int \|\mathrm{k}(x)\|_x^2 \, p(x)\mathrm{d}x \right)^{1/2} < \infty.$$

Since $p \geq 1$, any space $\mathrm{K}(p)$ can be embedded into the Hilbert space $\mathrm{K} = \mathrm{K}(1)$, and the intersection $\cap_{p \in \mathcal{P}_1} \mathrm{K}(p) \subseteq \mathrm{K}$ can be identified with the projective limit $\mathrm{K}_+ =$

$\lim_{p\to\infty} \mathrm{K}(p)$. This follows from the facts that the function $\|\mathrm{k}\|(p)$ is increasing: $p \le q \Rightarrow \|\mathrm{k}\|(p) \le \|\mathrm{k}\|(q)$, and so $\mathrm{K}(q) \subseteq \mathrm{K}(p)$, and the set $\mathcal{P}_1$ is directed in the sense that for any $p = 1 + r$ and $q = 1 + s$, $r, s \in \mathcal{P}_0$, there is a function in $\mathcal{P}_1$ majorizing p and q (we can take for example $p+q-1 = 1+r+s \in \mathcal{P}_1$). In the case of polynomials $p \in \mathcal{P}_1$ on $X = \mathbb{R}^d \times \mathbb{R}_+$ the decreasing family $\{\mathrm{K}(p)\}$, where $\mathrm{K}_x = \mathbb{C}$, is identical with the integer Sobolev scale of vector fields $\mathrm{k} : \mathbb{R}^d \to L^2(\mathbb{R}_+)$ with values $\mathrm{k}(x)(t) = \mathrm{k}(x,t)$ in the Hilbert space $L^2(\mathbb{R}_+)$ of square integrable functions on $\mathbb{R}_+$. If we replace $\mathbb{R}^d$ by $\mathbb{Z}^d$ and if we restrict ourselves to the positive part of the integer lattice $\mathbb{Z}^d$, then we obtain the Schwartz space in the form of vector fields $k \in \mathrm{K}_+$.

The space K_-, dual to K_+, of continuous functionals

$$\langle \mathrm{f} \mid \mathrm{k}\rangle = \int \langle \mathrm{f}(x) \mid \mathrm{k}(x)\rangle \,\mathrm{d}x, \quad \mathrm{k} \in \mathrm{K}_+,$$

is defined as the inductive limit $\mathrm{K}_- = \lim_{p\to 0} \mathrm{K}(p)$ in the scale $\{\mathrm{K}(p) : p \in \mathcal{P}_-\}$, where $\mathcal{P}_-$ is the set of functions $p : X \to (0,1]$ such that $1/p \in \mathcal{P}_1$. The space K_- of such generalized vector-functions $\mathrm{k} : X \ni x \mapsto \mathrm{k}(x) \in \mathrm{K}_x$ can be considered as the union $\cup_{p\in\mathcal{P}_-} \mathrm{K}(p)$ of the inductive family of Hilbert spaces $\mathrm{K}(p), p \in \mathcal{P}_-$, with the norms $\|\mathrm{k}\|(p)$, containing as the minimal the space $\mathrm{K} = \mathrm{K}(1)$. In the extended scale $\{\mathrm{K}(p) : p \in \mathcal{P}\}$, where $\mathcal{P} = \mathcal{P}_- \cup \mathcal{P}_1$, we obtain the Gel'fand chain $\mathrm{K}_+ \subseteq \mathrm{K}(p_+) \subseteq \mathrm{K} \subseteq \mathrm{K}(p^-) \subseteq \mathrm{K}_-$, where $p_+ \in \mathcal{P}_1$, $p_- \in \mathcal{P}_-$, and $\mathrm{K}_+ = \mathrm{K}_-^*$ coincides with the space of functionals on K_- continuous with respect to the inductive convergence.

We can similarly define a Gel'fand triple $(\mathrm{F}_+, \mathrm{F}, \mathrm{F}_-)$ for the Hilbert scale $\{\mathrm{F}(p) : p \in \mathcal{P}\}$ of Fock spaces $\mathrm{F}(p)$ over $\mathrm{K}(p)$ with $\mathrm{F}_+ = \cap_{p\in\mathcal{P}_1} \mathrm{F}(p)$, $\mathrm{F} = \mathrm{F}(1)$, $\mathrm{F}_- = \cup_{p\in\mathcal{P}_-} \mathrm{F}(p)$. We shall consider the Guichardet [23] representation of the symmetric Fock spaces $\mathrm{F}(p)$, regarding their elements $\mathrm{f} \in \mathrm{F}(p)$ as the functions $\mathrm{f} : \varkappa \mapsto \mathrm{f}(\varkappa) \in \mathrm{K}^{\otimes}(\varkappa)$ with sections in the Hilbert products $\mathrm{K}^{\otimes}(\varkappa) = \bigotimes_{x\in\varkappa} \mathrm{K}_x$, square integrable with the product weight $p(\varkappa) = \prod_{x\in\varkappa} p(x)$:

$$\|\mathrm{f}\|(p) = \left(\int \|\mathrm{f}(\varkappa)\|^2 p(\varkappa)\,\mathrm{d}\varkappa\right)^{1/2} < \infty.$$

The integral here is over all chains $\varkappa \in \mathcal{X}$ that define the pairing on F_- by

$$\langle \mathrm{f} \mid \mathrm{h}\rangle = \int \langle \mathrm{f}(\varkappa) \mid \mathrm{h}(\varkappa)\rangle \,\mathrm{d}\varkappa, \quad \mathrm{h} \in \mathrm{F}_+,$$

and in more detail we can write this in the form

$$\int \|\mathrm{f}(\varkappa)\|^2 p(\varkappa)\mathrm{d}\varkappa = \sum_{n=0}^{\infty} \int\limits_{0\le t_1<} \cdots \int\limits_{<t_n<\infty} \|\mathrm{f}(x_1,\ldots,x_n)\|^2 \prod_{i=1}^{n} p(x_i)\mathrm{d}x_i,$$

where the n-fold integrals are taken over simplex domains $\Gamma_n = \{\boldsymbol{x} \in X^n : t(x_1) < \cdots < t(x_n)\}$. In a similar way as is done in the case $X = \mathbb{R}_+$, $t(x) = x$, one can easily establish an isomorphism between the space $\mathrm{F}(p)$ and the symmetric (or antisymmetric) Fock space over $\mathrm{K}(p)$, the isomorphism defined by the isometry

$$\|\mathrm{f}\|(p) = \left(\sum_{n=0}^{\infty} \frac{1}{n!} \int \cdots \int \|\mathrm{f}(x_1,\ldots,x_n)\|^2 \prod_{i=1}^{n} p(x_i)\mathrm{d}x_i\right)^{1/2},$$

where the functions $\mathrm{f}(x_1,\ldots x_n)$ are extended to the whole of X^n in a symmetric (or antisymmetric) way.

Let $D = (D^\mu_\nu)^{\mu=-,\circ}_{\nu=\circ,+}$ be a quadruple of functions D^μ_ν on X with kernel values as continuous operators

$$
\begin{aligned}
&D^-_+(x) : \mathrm{F}_+ \to \mathrm{F}_-, \qquad D^\circ_\circ(x) : \mathrm{F}_+ \otimes \mathrm{K}_x \to \mathrm{F}_- \otimes \mathrm{K}_x, \\
&D^\circ_+(x) : \mathrm{F}_+ \to \mathrm{F}_- \otimes \mathrm{K}_x, \qquad D^-_\circ(x) : \mathrm{F}_+ \otimes \mathrm{K}_x \to \mathrm{F}_-,
\end{aligned} \tag{6.1}
$$

so that there is a $p \in \mathcal{P}_1$ that these operators are bounded from $\mathrm{F}(p) \supseteq \mathrm{F}_+$ to $\mathrm{F}(p)^* \subseteq \mathrm{F}_-$, where $\mathrm{F}(p)^* = \mathrm{F}(p^{-1})$. We assume that $D^-_+(x)$ is locally integrable in the sense that

$$
\exists\, p \in \mathcal{P}_1 : \|D^-_+\|^{(1)}_{p,t} = \int_{X^t} \|D^-_+(x)\|_p \mathrm{d}x < \infty, \quad \forall t < \infty,
$$

where $X^t = \{x \in X : t(x) < t\}$, and $\|D\|_p = \sup\{\|D\mathrm{h}\|(p^{-1})/\|\mathrm{h}\|(p)\}$ is the norm of the continuous operator $D : \mathrm{F}(p) \to \mathrm{F}(p)^*$ which defines a bounded Hermitian form $\langle \mathrm{f} \mid D\mathrm{h}\rangle$ on $\mathrm{F}(p)$. We also assume that $D^\circ_\circ(x)$ is locally bounded with respect to a strictly positive function s such that $1/s \in \mathcal{P}_0$ in the sense that

$$
\exists\, p \in \mathcal{P}_1 : \|D^\circ_\circ\|^{(\infty)}_{p,t}(s) = \operatorname*{ess\,sup}_{x \in X_t} \{s(x)\|D^\circ_\circ(x)\|_p\} < \infty, \quad \forall t < \infty,
$$

where $\|D\|_p$ is the norm of the operator $\mathrm{F}(p) \otimes \mathrm{K}_x \to \mathrm{F}(p)^* \otimes \mathrm{K}_x$. Finally, we assume that $D^\circ_+(x)$ and $D^-_\circ(x)$ are locally square integrable with strictly positive weight $r(x)$ such that $1/r \in \mathcal{P}_0$, in the sense that

$$
\exists\, p \in \mathcal{P}_1 : \|D^\circ_+\|^{(2)}_{p,t}(r) < \infty, \quad \|D^-_\circ\|^{(2)}_{p,t}(r) < \infty, \quad \forall t < \infty,
$$

where $\|D\|^{(2)}_{p,t}(r) = (\int_{X^t} \|D(x)\|^2_p r(x)\mathrm{d}x)^{1/2}$ and $\|D\|_p$ are the norms, respectively, of the operators

$$
D^\circ_+(x) : \mathrm{F}(p) \to \mathrm{F}(p)^* \otimes \mathrm{K}_x, \qquad D^-_\circ(x) : \mathrm{F}(p) \otimes \mathrm{K}_x \to \mathrm{F}(p)^*.
$$

Then for any $t \in \mathbb{R}_+$ we can define a *generalized quantum stochastic (QS) integral*

$$
i^t_0(\mathbf{D}) = \int_{X^t} \Lambda(\mathbf{D}, \mathrm{d}x), \qquad \Lambda(\mathbf{D}, \Delta) = \sum_{\mu,\nu} \Lambda^\mu_\nu(D^\mu_\nu, \Delta) \tag{6.2}
$$

introduced in [15] as the sum of four continuous operators $\Lambda^\nu_\mu(D^\mu_\nu) : \mathrm{F}_+ \to \mathrm{F}_-$ described as operator measures on $\mathfrak{F} \ni \Delta$ for $\Delta = X^t$ with values

$$
\begin{aligned}
[\Lambda^+_-(D^-_+, \Delta)\mathrm{h}](\varkappa) &= \int_\Delta [D^-_+(x)\mathrm{h}](\varkappa)\mathrm{d}x \quad \text{(preservation)}, \\
[\Lambda^+_\circ(D^\circ_+, \Delta)\mathrm{h}](\varkappa) &= \sum_{x \in \Delta \cap \varkappa} [D^\circ_+(x)\mathrm{h}](\varkappa \setminus x) \quad \text{(creation)}, \\
[\Lambda^\circ_-(D^-_\circ, \Delta)\mathrm{h}](\varkappa) &= \int_\Delta [D^-_\circ(x)\dot{\mathrm{h}}(x)](\varkappa)\mathrm{d}x \ \text{(annihilation)}, \\
[\Lambda^\circ_\circ(D^\circ_\circ, \Delta)h])\varkappa) &= \sum_{x \in \Delta \cap \varkappa} [D^\circ_\circ(x)\dot{\mathrm{h}}(x)](\varkappa \setminus x) \ \text{(exchange)}.
\end{aligned} \tag{6.3}
$$

Here $\mathrm{h} \in \mathrm{F}_+, \varkappa \setminus x = \{x' \in \varkappa : x' \neq x\}$ denotes the chain $\varkappa \in \mathcal{X}$ from which the point $x \in \varkappa$ has been eliminated, and $\dot{\mathrm{h}}(x) \in \mathrm{K}_x \otimes \mathrm{F}_+$ is the point derivative $\dot{\mathrm{h}}(x) = \nabla_x \mathrm{h}$ defined for each $\mathrm{h} \in \mathrm{F}_+$ almost everywhere (namely, for $\varkappa \in \mathcal{X}$: $x \notin \varkappa$) as the function $[\nabla_x \mathrm{h}](\varkappa) = \mathrm{h}(\varkappa \sqcup x) \equiv \dot{\mathrm{h}}(x, \varkappa)$, where the operation $\varkappa \sqcup x$ denotes the disjoint union $\omega = \varkappa \cup x$, $\varkappa \cap x = \emptyset$ of chains $\varkappa \in \mathcal{X}$ and $x \in X \backslash \varkappa$ with pairwise comparable elements. Note that the point derivative ∇_x is nothing but Malliavin derivative [39] densely defined in Fock-Guishardet representation as annihilation operator $\mathrm{F}_+ \to \mathrm{K}_x \otimes \mathrm{F}_+$ by $a(x)\mathrm{h}(\omega) = \dot{\mathrm{h}}(x, \omega \setminus x)$, $a(x)\,\mathrm{h}(\omega) = 0$ if $x \notin \omega$, and its right inverse operator $[\nabla_x^* \mathrm{f}](\omega) = \mathrm{f}(x, \omega \backslash x)$ with $[\nabla_x^* \mathrm{f}](\omega) = 0$ if $x \notin \omega$ defines in this representation the Skorochod nonadapted integral as the creation point integral $\sum_{x \in \varkappa} \mathrm{f}(x, \omega \setminus x)$ for any $\mathrm{f} \in \mathrm{K}_- \otimes \mathrm{F}_-$. The continuity of this derivative as the projective limit map $\mathrm{F}_+ \to \mathrm{K}_+ \otimes \mathrm{F}_+$ and the point integral as the adjoint map $\mathrm{K}_- \otimes \mathrm{F}_- \to \mathrm{F}_-$ will simply follow from the isometricity of the multiple point derivative and coisometricity of the adjoint multiple point integral introduced below as in [17].

We can consider the multipple annihilation operators $a(\vartheta) : \mathrm{h}(\omega) \mapsto \dot{\mathrm{h}}(\vartheta, \omega \setminus \vartheta)$ eliminating several points $\vartheta \subseteq \omega$ in a chain $\omega \in \mathcal{X}$ with $a(\vartheta)\mathrm{h}(\omega) = 0$ if $\vartheta \nsubseteq \omega$. They are described for each $\vartheta = \{x_1, \dots x_n\}$ in terms of the n-point derivatives

$$[\nabla_\vartheta^n \mathrm{h}](\varkappa) := \mathrm{h}(\varkappa \sqcup \vartheta) \equiv \dot{\mathrm{h}}(\vartheta, \varkappa), \ \vartheta \in \Gamma_n \tag{6.4}$$

defined almost everywhere ($\varkappa \cap \vartheta = \emptyset$) on $\varkappa \in \mathcal{X}$ as the n-th order point (or Malliavin) derivatives [39]. These annihilations as densely defined operators from F_+ into $\mathrm{K}^{\otimes}(\vartheta) \otimes \mathrm{F}_+$ are not continuous for each $\vartheta \in \Gamma_n$ (except $\vartheta = \emptyset$ corresponding to $n = 0$ for which $a(\emptyset) = I$), but they define projective-continuous linear maps into the functions $\vartheta \mapsto \dot{\mathrm{h}}(\vartheta)$ on $\Gamma_n \subset \mathcal{X}$ for each $n \in \mathbb{N}$ which are square-integrable with any $p_0 \in \mathcal{P}_0$ as partial isometric components of the *multiple point derivative* $\nabla^{\cdot} \mathrm{h} = \int_{\mathcal{X}}^{\otimes} \dot{\mathrm{h}}(\vartheta)\, p_0(\vartheta)\, \mathrm{d}\vartheta$ described as isometric map $\mathrm{F}(p_0 + p_1) \to \mathrm{F}(p_0) \otimes \mathrm{F}(p_1)$ in the following lemma.

Lemma 1. *The linear map $\nabla^{\cdot} : \mathrm{h} \mapsto [\nabla^{\cdot} \mathrm{h}]$ defined as $\nabla^{\cdot} \mathrm{h} = \oplus_{n=0}^{\infty} \nabla^n \mathrm{h}$ in (6.4) for all $\vartheta \in \mathcal{X}$ is an isometry of Fock scale $\{\mathrm{F}(p) : p \in \mathcal{P}\}$ into the scale $\{\mathrm{F}(p_0) \otimes \mathrm{F}(p_1) : p_0 \in \mathcal{P}_0, p_1 \in \mathcal{P}_1\}$ such that $\|\nabla^{\cdot} \mathrm{h}\|(p_0, p_1) = \|\mathrm{h}\|(p_0 + p_1)$. The adjoint coisometric operator $\langle \nabla_{\cdot}^* \mathrm{f} | \mathrm{h} \rangle = \langle \mathrm{f} | \nabla^{\cdot} \mathrm{h} \rangle$ is defined as the multiple point integral $\nabla_{\cdot}^* = \sum_{n=0}^{\infty} \nabla_n^*$, where*

$$[\nabla_n^* \mathrm{f}](\omega) = \sum_{\Gamma_n \ni \vartheta \subseteq \omega} \mathrm{f}(\vartheta, \omega \setminus \vartheta), \quad \omega \in \mathcal{X} \tag{6.5}$$

is n-th order point (or Skorochod) integral as a contraction from $\mathrm{F}(p_0^{-1}) \otimes \mathrm{F}\left(p_1^{-1}\right)$ into any $\mathrm{F}\left(p^{-1}\right)$ with $p \geq p_0 + p_1$.

Proof. We first of all establish the principal formula of the multiple integration

$$\int \sum_{\vartheta \subseteq \omega} f(\vartheta, \omega \setminus \vartheta) \mathrm{d}\omega = \iint f(\vartheta, \upsilon) \mathrm{d}\vartheta \mathrm{d}\upsilon, \quad \forall f \in L^1(\mathcal{X} \times \mathcal{X}), \tag{6.6}$$

which will allow us to define the adjoint operator $\nabla_{\cdot}^*$. Let $f(\vartheta, \upsilon) = g(\vartheta) h(\upsilon)$ be the product of integrable complex functions on $\mathcal{X}$ of the form $g(\vartheta) = \prod_{x \in \vartheta} g(x)$, $h(\upsilon) = \prod_{x \in \upsilon} h(x)$ for any ϑ, $\upsilon \in \mathcal{X}$. Employing the binomial formula

$$\sum_{\vartheta \subseteq \omega} g(\vartheta) h(\omega \setminus \vartheta) = \sum_{\vartheta \sqcup \upsilon = \omega} \prod_{x \in \vartheta} g(x) \prod_{x \in \upsilon} h(x) = \prod_{x \in \omega} (g(x) + h(x)),$$

and also the equality $\int f(\vartheta) \mathrm{d}\vartheta = \exp\{\int f(x) \mathrm{d}x\}$ for $f(\vartheta) = \prod_{x \in \vartheta} f(x)$, we obtain the formula

$$\int \sum_{\vartheta \subseteq \omega} g(\vartheta) h(\omega \setminus \vartheta) \mathrm{d}\omega = \exp\left\{ \int (g(x) + h(x)) \mathrm{d}x \right\} = \iint g(\vartheta) h(\upsilon) \mathrm{d}\vartheta \mathrm{d}\upsilon,$$

which proves (6.6) on a set of product-functions f dense in $L^1(\mathcal{X} \times \mathcal{X})$.

Applying this formula to the scalar product $\langle \mathrm{f}(\vartheta,\upsilon)|\mathrm{h}(\vartheta,\upsilon)\rangle \in L^1(\mathcal{X}\times\mathcal{X})$, we obtain

$$\int\sum_{\vartheta\subseteq\omega}\langle \mathrm{f}(\vartheta,\omega\setminus\vartheta)\mid \mathrm{h}(\omega)\rangle\,\mathrm{d}\omega=\iint\langle \mathrm{f}(\vartheta,\upsilon)\mid \mathrm{h}(\vartheta\sqcup\upsilon)\rangle\,\mathrm{d}\vartheta\mathrm{d}\upsilon,$$

that is, $\langle\nabla^*_{\cdot}\mathrm{f}\mid\mathrm{h}\rangle=\langle\mathrm{f}\mid\nabla^{\cdot}\mathrm{h}\rangle$, where $[\nabla^{\cdot}\mathrm{h}](\vartheta,\upsilon)=\mathrm{h}(\upsilon\sqcup\vartheta)\equiv\dot{\mathrm{h}}(\vartheta,\upsilon)$. Choosing arbitrary $\mathrm{f}\in\mathrm{F}(p_0^{-1})\otimes\mathrm{F}\left(p_1^{-1}\right)$, we find that the annihilation operators $a(\vartheta)\mathrm{h}=[\nabla_\vartheta\mathrm{h}]$ define the isometry $\nabla^{\cdot}:\mathrm{F}\left(p_0+p_1\right)\to\mathrm{F}\left(p_0\right)\otimes\mathrm{F}(p_1)$ with the operator $\nabla^*_{\cdot}$ defined as coisometry $\mathrm{F}(p_0^{-1})\otimes\mathrm{F}\left(p_1^{-1}\right)\to\mathrm{F}\left(p^{-1}\right)$ for $p=p_0+p_1$ with respect to the standard pairing of conjugate spaces $\mathrm{F}(p)$ and $\mathrm{F}\left(p^{-1}\right)$:

$$\begin{aligned}&\iint\|\dot{\mathrm{h}}(\vartheta,\upsilon\|^2p_0(\vartheta)p_1(\upsilon)\mathrm{d}\vartheta\mathrm{d}\upsilon\\ =\;&\int\sum_{\vartheta\subseteq\omega}\|\mathrm{h}(\omega)\|^2p_0(\vartheta)p_1(\omega\setminus\vartheta)\mathrm{d}\omega=\int\|\mathrm{h}(\omega)\|^2\sum_{\vartheta\sqcup\upsilon=\omega}p_0(\vartheta)p_1(\upsilon)\mathrm{d}\omega\\ =\;&\int\|\mathrm{h}(\omega)\|^2(p_0+p_1)(\omega)\mathrm{d}\omega.\end{aligned}$$

Hence it follows that $\nabla^{\cdot}$ is projective continuous from F_+ to $\mathrm{F}_0\otimes\mathrm{F}_+$, where $\mathrm{F}_0=\bigcap_{p\in\mathcal{P}_0}\mathrm{F}(p)$, and, in particular, so is the one-point derivative $\dot{\mathrm{f}}(x,\upsilon)=\mathrm{f}(x\sqcup\upsilon)$ from F_+ to $\mathrm{K}_+\otimes\mathrm{F}_+$ as a contracting map $\mathrm{F}\left(p_0+p_1\right)\to\mathrm{F}\left(p_0\right)\otimes\mathrm{F}(p_1)$ for all $p_0\in\mathcal{P}_0,p_1\in\mathcal{P}$. The lemma is proved. □

We are now ready to prove the inductive continuity of the integral (6.2) with respect to $\mathbf{D}=[D^\mu_\nu]$ by showing the inequality

$$\|(i_0^t(\mathbf{D})\mathrm{h})\|\left(q^{-1}\right)\le\|D\|^s_{p,t}(r)\|\mathrm{h}\|(q),\quad\forall q\le r^{-1}+p+s^{-1},$$

where $\|D\|^s_{p,t}(r)=\|D^-_+\|^{(1)}_{p,t}+\|D^\circ_+\|^{(2)}_{p,t}(r)+\|D^-_\circ\|^{(2)}_{p,t}(r)+\|D^\circ_\circ\|^{(\infty)}_{p,t}(s)$. We will establish this inequality as the single-integral case of the corresponding inequality for the *generalized multiple QS integral* [17]

$$[\iota_0^t(B)\mathrm{h}](\varkappa)\|=\sum_{\varkappa^\circ_\circ\sqcup\varkappa^\circ_+\subseteq\varkappa^t}\int_{\mathcal{X}^t}\int_{\mathcal{X}^t}[B(\boldsymbol{\vartheta})\dot{\mathrm{h}}(\vartheta^-_\circ\sqcup\vartheta^\circ_\circ)](\varkappa^\circ_-)\mathrm{d}\vartheta^-_+\mathrm{d}\vartheta^-_\circ\tag{6.7}$$

where $\varkappa^t=\varkappa\cap X^t,\mathcal{X}^t=\{\varkappa\in\mathcal{X}:\varkappa\subset X^t\}$ and the sum is taken over all decompositions $\varkappa=\varkappa^\circ_-\sqcup\vartheta^\circ_\circ\sqcup\vartheta^\circ_+$ such that $\vartheta^\circ_\circ\in\mathcal{X}^t$ and $\vartheta^\circ_+\in\mathcal{X}^t$. The multi-integrant $B\left(\boldsymbol{\vartheta}\right)$ is in general a kernel-valued function of the quadruple $\boldsymbol{\vartheta}=(\vartheta^\mu_\nu)^{\mu=-,\circ}_{\nu=\circ,+}$ of chains $\vartheta^\mu_\nu\in\mathcal{X}$, defined almost everywhere by its values in the operators

$$B\begin{pmatrix}\vartheta^-_+,\vartheta^-_\circ\\ \vartheta^\circ_+,\vartheta^\circ_\circ\end{pmatrix}:\mathrm{F}_+\otimes\mathrm{K}^\otimes(\vartheta^-_\circ)\otimes\mathrm{K}^\otimes(\vartheta^\circ_\circ)\to\mathrm{F}_-\otimes\mathrm{K}^\otimes(\vartheta^\circ_\circ)\otimes\mathrm{K}^\otimes(\vartheta^\circ_+).$$

We will assume that these operators are bounded from $\mathrm{F}(p)$ to $\mathrm{F}\left(p^{-1}\right)$ for some $p\in\mathcal{P}_1$ and that there exist strictly positive functions $r>0$, $r^{-1}\in\mathcal{P}_0$, and $s>0$, $s^{-1}\in\mathcal{P}_0$ such that

$$\|B\|^s_{p,t}(r)=\int_{\mathcal{X}^t}\|B^-_+(\vartheta)\|^s_{p,t}\mathrm{d}\vartheta<\infty,\quad\forall t<\infty,\tag{6.8}$$

where

$$\|B^-_+(\vartheta^-_+)\|^s_{p,t}(r)=\left(\int_{\mathcal{X}^t}\int_{\mathcal{X}^t}\operatorname*{ess\,sup}_{\vartheta^\circ_\circ\in\mathcal{X}^t}(s(\vartheta^\circ_\circ)\|B(\boldsymbol{\vartheta})\|_p)^2r(\vartheta^\circ_+\sqcup\vartheta^-_\circ)\mathrm{d}\vartheta^\circ_+\mathrm{d}\vartheta^-_\circ)^{1/2}\right),$$

and $s(\vartheta)=\prod_{x\in\vartheta}s(x)$, $r(\vartheta)=\prod_{x\in\vartheta}r(x)$. We mention that the single integral (6.2) corresponds to the case

$$B(\mathbf{x}_\nu^\mu)=D_\nu^\mu(x),\quad B(\boldsymbol{\vartheta})=0,\quad \forall\boldsymbol{\vartheta}:\sum_{\mu,\nu}|\vartheta_\nu^\mu|\neq 1,$$

where $\mathbf{x}_\nu^\mu$ denotes one of the atomic tables

$$\mathbf{x}_+^- = \begin{pmatrix} x, & \emptyset \\ \emptyset & \emptyset \end{pmatrix}, \mathbf{x}_+^\circ = \begin{pmatrix} \emptyset, & \emptyset \\ x & \emptyset \end{pmatrix}, \mathbf{x}_\circ^- = \begin{pmatrix} \emptyset, & x \\ \emptyset & \emptyset \end{pmatrix}, \mathbf{x}_\circ^\circ = \begin{pmatrix} \emptyset, & \emptyset \\ \emptyset & x \end{pmatrix}, \tag{6.9}$$

determined by $x\in X$. It follows from the next theorem that the function $B(\boldsymbol{\vartheta})$ in (6.7) can be defined up to equivalence, whose kernel is $B\approx 0\Leftrightarrow \|B\|_{p,t}^s(r)=0$ for all $t\in\mathbb{R}_+$ and for some p,r,s. In particular, B can be defined almost everywhere only for the tables $\vartheta=(\vartheta_\nu^\mu)$ that give disjoint decompositions $\varkappa=\cup_{\mu,\nu}\vartheta_\nu^\mu$ of the chains $\varkappa\in\mathcal{X}$, that is, are representable in the form $\boldsymbol{\vartheta}=\bigsqcup_{x\in\varkappa}\mathbf{x}$, where $\mathbf{x}$ is one of the atomic tables (6.9) with indices μ,ν for $x\in\vartheta_\nu^\mu$.

Theorem 4. *Suppose that $B(\boldsymbol{\vartheta})$ is a function locally integrable in the sense of* (6.8) *for some $p,r,s>0$. Then its integral* (6.7) *is a continuous operator $T_t=\iota_0^t(B)$ from F_+ to F_- satisfying the estimate*

$$\|T_t\|_q=\sup_{\mathrm{h}\in\mathrm{F}(q)}\left\{\|T_t\mathrm{h}\|\left(q^{-1}\right)/\|\mathrm{h}\|(q)\right\}\leq\|B\|_{p,t}^s(r) \tag{6.10}$$

for any $q\geq r^{-1}+p+s^{-1}$. The operator T_t^*, formally adjoint to T_t in F, is the integral

$$\iota_0^t(B)^*=\iota_0^t(B^\star),\quad B^\star\begin{pmatrix}\vartheta_+^-, & \vartheta_\circ^- \\ \vartheta_+^\circ, & \vartheta_\circ^\circ\end{pmatrix}=B\begin{pmatrix}\vartheta_+^-, & \vartheta_+^\circ \\ \vartheta_\circ^-, & \vartheta_\circ^\circ\end{pmatrix}^*, \tag{6.11}$$

which is continuous from F_+ to F_- with $\|B^\star\|_p^{s,t}(r)=\|B\|_p^{s,t}(r)$. Moreover, the operator-valued function $t\mapsto T_t$ has the quantum-stochastic differential $\mathrm{d}T_t=\mathrm{d}\iota_0^t(\mathbf{D})$ in the sense that

$$i_0^t(B)=B(\emptyset)+i_0^t(\mathbf{D}),\quad D_\nu^\mu(x)=i_0^{t(x)}(\dot{B}(\mathbf{x}_\nu^\mu)), \tag{6.12}$$

defined by the quantum-stochastic derivatives $\mathbf{D}=(D_\nu^\mu)$ with values (6.1) *acting from $\mathrm{F}(q)$ to $\mathrm{F}\left(q^{-1}\right)$ and bounded almost everywhere:*

$$\|D_+^-\|_{q,t}^{(1)}\leq\|B\|_{p,t}^s(r),\quad \|D\|_{p,t}^{(2)}(r)\leq\|B\|_{p,t}^s(r),\quad \|D_\circ^\circ\|_{q,t}^{(\infty)}(s)\leq\|B\|_{p,t}^s(r)$$

for $D=D_\circ^-$ and $D=D_+^\circ$, $q\geq r^{-1}+p+s^{-1}$. This differential is defined in the form of the multiple integrals (6.7) *with respect to $\boldsymbol{\vartheta}$ of pointwise derivatives $\dot{B}(\mathbf{x},\boldsymbol{\vartheta})=B(\boldsymbol{\vartheta}\sqcup\mathbf{x})$, where $\mathbf{x}$ is one of four atomic tables* (6.9) *at a fixed point $x\in X$.*

Proof. Using property (6.6) in the form

$$\int\sum_{\sqcup\vartheta_\nu^\circ=\vartheta}f(\vartheta_-^\circ,\vartheta_\circ^\circ,\vartheta_+^\circ)\mathrm{d}\vartheta=\iiint f(\vartheta_-^\circ,\vartheta_\circ^\circ,\vartheta_+^\circ)\prod_\nu\mathrm{d}\vartheta_\nu^\circ,$$

it is easy to find that from the definition (6.7) for $\mathrm{f},\mathrm{h}\in\mathrm{F}_+$ we have $\int\langle\mathrm{f}(\varkappa)\mid[T_t\mathrm{h}](\varkappa)\rangle\,\mathrm{d}\varkappa=$

$$\begin{aligned}
&=\int_{\mathcal{X}^t}\mathrm{d}\vartheta_+^-\int_{\mathcal{X}^t}\mathrm{d}\vartheta_+^\circ\int_{\mathcal{X}^t}\mathrm{d}\vartheta_\circ^-\int_{\mathcal{X}^t}\mathrm{d}\vartheta_\circ^\circ\left\langle\dot{\mathrm{f}}(\vartheta_\circ^\circ\sqcup\vartheta_+^\circ)\mid B(\vartheta)\dot{\mathrm{h}}(\vartheta_\circ^-\sqcup\vartheta_\circ^\circ)\right\rangle\\
&=\int_{\mathcal{X}^t}\mathrm{d}\vartheta_+^-\int_{\mathcal{X}^t}\mathrm{d}\vartheta_+^\circ\int_{\mathcal{X}^t}\mathrm{d}\vartheta_\circ^-\int_{\mathcal{X}^t}\mathrm{d}\vartheta_\circ^\circ\left\langle B(\boldsymbol{\vartheta})^*\dot{\mathrm{f}}(\vartheta_\circ^\circ\sqcup\vartheta_+^\circ)\mid\dot{\mathrm{h}}(\vartheta_\circ^-\sqcup\vartheta_\circ^\circ)\right\rangle\\
&=\int\langle[T_t^*\mathrm{f}](\varkappa)\mid\mathrm{h}(\varkappa)\rangle\,\mathrm{d}\varkappa,
\end{aligned}$$

that is, T_t^* acts as $\iota_0^t(B^\star)$ in (6.7) with $B^\star(\boldsymbol{\vartheta})=B(\boldsymbol{\vartheta}')^*$, where $(\vartheta_\nu^\mu)'=(\vartheta_{-\mu}^{-\nu})$ with respect to the inverstion $-:(-,\circ,+)\mapsto(+,\circ,-)$. More precisely, this yields $\|\iota_0^t(B)\|_q=\|\iota_0^t(B^\star)\|_q$, since $\|T\|_q=\|T^*\|_q$ by the definition (6.10) of q-norm and by

$$\sup\{|\langle\mathrm{f}\mid T\mathrm{h}\rangle|/\|\mathrm{f}\|(q)\|\mathrm{h}\|(q)\}=\sup\{|\langle T^*\mathrm{f}\mid\mathrm{h}\rangle|/\|\mathrm{f}\|(q)\|\mathrm{h}\|(q)\}.$$

We estimate the integral $\langle\mathrm{f}\mid T_t\mathrm{h}\rangle$ using the Schwartz inequality

$$\int\|\dot{\mathrm{f}}(\vartheta)\|(p)\|\dot{\mathrm{h}}(\vartheta)\|(p)s^{-1}(\vartheta)\mathrm{d}\vartheta\le\|\dot{\mathrm{f}}\|(s^{-1},p)\|\dot{\mathrm{h}}\|(s^{-1},p)$$

and the property (6.6) of the multiple integral according to which $\|\dot{\mathrm{f}}\|(s^{-1},p)=\|\mathrm{f}\|(p+s^{-1})$, $\|\dot{\mathrm{h}}\|(s^{-1},p)=\|\mathrm{h}\|(s^{-1}+p)$, $|(\mathrm{f}\mid T_t\mathrm{h})|\le$

$$\begin{aligned}
&\le\int_{\mathcal{X}^t}\mathrm{d}\vartheta_\circ^\circ\int_{\mathcal{X}^t}\int_{\mathcal{X}^t}\|\dot{\mathrm{f}}(\vartheta_\circ^\circ\sqcup\vartheta_+^\circ)\|(p)\left(\int_{\mathcal{X}^t}\|B(\vartheta)\|_p\mathrm{d}\vartheta_+^-\right)\|\dot{\mathrm{h}}(\vartheta_\circ^-\sqcup\vartheta_\circ^\circ)\|(p)\mathrm{d}\vartheta_\circ^-\mathrm{d}\vartheta_+^\circ\\
\le&\int_{\mathcal{X}^t}\mathrm{d}\vartheta\left(\int_{\mathcal{X}^t}\|\dot{\mathrm{f}}(\vartheta\sqcup\vartheta_+^\circ)\|^2(p)\frac{\mathrm{d}\vartheta_+^\circ}{r(\vartheta_+^\circ)}\int_{\mathcal{X}^t}\|\dot{\mathrm{h}}(\vartheta\sqcup\vartheta_\circ^-)\|^2(p)\frac{\mathrm{d}\vartheta_\circ^-}{r(\vartheta_\circ^-)}\right)^{\frac{1}{2}}\|B_\circ^\circ(\vartheta)\|_{p,t}(r)\\
\le&\int_{\mathcal{X}^t}\mathrm{d}\vartheta\|\dot{\mathrm{f}}(\vartheta)\|(r^{-1}+p)\|B_\circ^\circ(\vartheta)\|_{p,t}(r)\|\dot{\mathrm{h}}(\vartheta)\|(r^{-1}+p)\\
\le&\operatorname*{ess\,sup}_{\vartheta\in\mathcal{X}^t}\{s(\vartheta)\|B_\circ^\circ(\vartheta)\|_{p,t}(r)\}\|\mathrm{f}\|(r^{-1}+p+s^{-1})\|\mathrm{h}\|(r^{-1}+p+s^{-1}),
\end{aligned}$$

where $\|B_\circ^\circ(\vartheta_\circ^\circ)\|_{p,t}(r)=(\int_{\mathcal{X}^t}\int_{\mathcal{X}^t}(\int_{\mathcal{X}^t}\|B(\boldsymbol{\vartheta})\|_p\mathrm{d}\vartheta_+^-)^2r(\vartheta_\circ^-\sqcup\vartheta_+^\circ)\mathrm{d}\vartheta_\circ^-\mathrm{d}\vartheta_+^\circ)^{1/2}$. Thus $\|T_t\|_q\le\|B\|_{p,t}(r,s)$, where $q\ge r^{-1}+p+s^{-1}$ and

$$\|B\|_{p,t}(r,s):=\operatorname*{ess\,sup}_{\vartheta\in\mathcal{X}^t}\{s(\vartheta)\|B_\circ^\circ(\vartheta)\|_{p,t}(r)\}\le\|B\|_{p,t}^s(r).$$

Using the definition (6.7) and the property

$$\int_{\mathcal{X}^t}\mathrm{f}(\varkappa)\mathrm{d}\varkappa=\mathrm{f}(\emptyset)+\int_{X^t}\mathrm{d}x\int_{\mathcal{X}^{t(x)}}\dot{\mathrm{f}}(x,\varkappa)\mathrm{d}\varkappa,$$

where $\dot{\mathrm{f}}(x,\varkappa) = \mathrm{f}(\varkappa \sqcup x)$, it is easy to see that $[(T_t - T_0)\mathrm{h}](\varkappa) = [(\iota_0^t(B) - B(\emptyset))\mathrm{h}](\varkappa) =$

$$
\begin{aligned}
&= \int_{X^t} \mathrm{d}x \sum_{\vartheta_\circ^\circ \sqcup \vartheta_+^\circ \subseteq \varkappa}^{t(\vartheta_\nu^\circ)<t(x)} \{\int_{\mathcal{X}^{t(x)}} \mathrm{d}\vartheta_+^- [\int_{\mathcal{X}^{t(s)}} \mathrm{d}\vartheta_\circ^- (\dot{B}(\mathbf{x}_+^-, \boldsymbol{\vartheta})\dot{\mathrm{h}}(\vartheta_\circ^- \sqcup \vartheta_\circ^\circ) \\
&\quad + \dot{B}(\mathbf{x}_\circ^-, \boldsymbol{\vartheta})\dot{\mathrm{h}}(x \sqcup \vartheta_\circ^- \sqcup \vartheta_\circ^\circ))]\}(\varkappa \backslash \vartheta_\circ^\circ \backslash \vartheta_+^\circ) \\
&+ \sum_{x \in \mathcal{X}^t} \int_{\vartheta_\circ^\circ \sqcup \vartheta_+^\circ \subseteq \varkappa}^{t(\vartheta_\nu^\circ)<t(x)} \{\int_{\mathcal{X}^{t(x)}} \mathrm{d}\vartheta_+^- [\int_{\mathcal{X}^{t(x)}} \mathrm{d}\vartheta_\circ^- (\dot{B}(\mathbf{x}_+^\circ, \boldsymbol{\vartheta})\dot{\mathrm{h}}(\vartheta_\circ^- \sqcup \vartheta_\circ^\circ) \\
&\quad + \dot{B}(\mathbf{x}_\circ^\circ, \boldsymbol{\vartheta})\dot{\mathrm{h}}(x \sqcup \vartheta_\circ^- \sqcup \vartheta_\circ^\circ))]\}((\varkappa \backslash \vartheta_\circ^\circ \backslash \vartheta_+^\circ) \\
&= \int_{X^t} \mathrm{d}x [D_+^-(x)\mathrm{h} + D_\circ^-(x)\dot{\mathrm{h}}(x)](\varkappa) + \sum_{x \in \mathcal{X}^t} [D_+^\circ(x)\mathrm{h} + D_\circ^\circ(x)\dot{\mathrm{h}}(x)](\varkappa \setminus x).
\end{aligned}
$$

Consequently, $T_t - T_0 = \sum \Lambda_\mu^\nu(D_\nu^\mu, X^t)$, where $\Lambda_\nu^\mu(D, \Delta)$ are defined in (6.3) as operator-valued measures on X of operator-functions

$$
\begin{aligned}
[D_+^\mu(x)\mathrm{h}](\varkappa) &= \sum_{\vartheta_\circ^\circ \sqcup \vartheta_+^\circ \subseteq \varkappa}^{t(\vartheta_\nu^\circ)<t(x)} \int_{\mathcal{X}^{t(x)}} \mathrm{d}\vartheta_+^- \int_{\mathcal{X}^{t(x)}} \mathrm{d}\vartheta_\circ^- [\dot{B}(\mathbf{x}_+^\mu, \boldsymbol{\vartheta})\mathrm{h}(\vartheta_\circ^- \sqcup \vartheta_\circ^\circ)](\varkappa_-^\circ), \\
[D_\circ^\mu(x)\dot{\mathrm{h}}](\varkappa) &= \sum_{\vartheta_\circ^\circ \sqcup \vartheta_+^\circ \subseteq \varkappa}^{t(\vartheta_\nu^\circ)<t(x)} \int_{\mathcal{X}^{t(x)}} \mathrm{d}\vartheta_+^- \int_{\mathcal{X}^{t(x)}} \mathrm{d}\vartheta_\circ^- [\dot{B}(\mathbf{x}_\circ^\mu, \boldsymbol{\vartheta})\dot{\mathrm{h}}(\vartheta_\circ^- \sqcup \vartheta_\circ^\circ)](\varkappa_-^\circ),
\end{aligned}
$$

acting on $\mathrm{h} \in \mathrm{F}_+$ and $\dot{\mathrm{h}} \in \mathrm{K}_x \otimes \mathrm{F}_+$, where $\varkappa_-^\circ = \varkappa \cap \overline{(\vartheta_\circ^\circ \sqcup \vartheta_+^\circ)} = \varkappa \backslash \vartheta_\circ^\circ \backslash \vartheta_+^\circ$. This can be written in terms of (6.7) as $D_\nu^\mu(x) = \iota_0^t(\dot{B}(\mathbf{x}_\nu^\mu))$. Because of the inequality $\|T_t\|_q \le \|B\|_{p,t}^s(r)$ for all $q \ge r^{-1} + p + s^{-1}$ we obtain $\|D_+^-\|_{q,t}^{(1)} \le \|B\|_{p,t}^s(r)$, since $\|D_+^-(x)\|_q \le \|\dot{B}(\mathbf{x}_+^-)\|_{p,t(x)}^s(r)$:

$$
\begin{aligned}
&&\int_{X^t} \|D_+^-(x)\|_q \mathrm{d}x \le \int_{X^t} \|\dot{B}(\mathbf{x}_+^-)\|_{p,t(x)}^s(r)\mathrm{d}x \\
= \int_{X^t} \mathrm{d}x \int_{\mathcal{X}^{t(x)}} \|B_+^-(x \sqcup \vartheta)\|_{p,t(x)}^s(r)\mathrm{d}\vartheta &=& \int_{\mathcal{X}^t} \|B_+^-(\vartheta)\|_{p,t}^s(r)\mathrm{d}\vartheta - \|B_+^-(\emptyset)\|_{p,t}^s(r) \\
&=& \|B\|_{p,t}^s(r) - \|B_+^-(\emptyset)\|_{p,t}^s(r),
\end{aligned}
$$

where $B_+^-(\vartheta, \boldsymbol{\vartheta}) = B(\boldsymbol{\vartheta} \sqcup \boldsymbol{\vartheta}_+^-)\delta_\emptyset(\vartheta_+^-)$ for $\boldsymbol{\vartheta}_+^- = \begin{pmatrix} \vartheta, & \emptyset \\ \emptyset, & \emptyset \end{pmatrix}$, $\boldsymbol{\vartheta} = \begin{pmatrix} \vartheta_+^- & \vartheta_\circ^- \\ \vartheta_+^\circ & \vartheta_\circ^\circ \end{pmatrix}$.

In a similar way one can obtain

$$
\begin{aligned}
\|D_+^\circ\|_{q,t}^{(2)}(r) &\le \left(\int_{X^t} (\|\dot{B}(\mathbf{x}_+^\circ)\|_{p,t(x)}^s(r))^2 r(x)\mathrm{d}x \right)^{1/2} \\
&\le \int_{\mathcal{X}^t} \mathrm{d}\vartheta^- \left(\int_{\mathcal{X}^t} (\|B_+(\vartheta^-, \vartheta^\circ)\|_{p,t}^s(r))^2 r(\vartheta^\circ)\mathrm{d}\vartheta^\circ \right)^{1/2} \le \|B\|_{p,t}^s(r),
\end{aligned}
$$

where $B_+\left(\vartheta^-,\vartheta^\circ,\boldsymbol{\vartheta}\right)=B(\boldsymbol{\vartheta}\sqcup\boldsymbol{\vartheta}_+)\delta_\emptyset(\vartheta_+^-\sqcup\vartheta_+^\circ)$ for $\boldsymbol{\vartheta}_+=\begin{pmatrix}\vartheta^- & \emptyset\\ \vartheta^\circ & \emptyset\end{pmatrix}$,

$$\begin{aligned}\|D_\circ^-\|_{q,t}^{(2)}(r)&\le\Big(\int_{X^t}(\|\dot B(\mathbf{x}_\circ^-)\|_{p,t(x)}^s(r))^2r(x)\mathrm{d}x\Big)^{1/2}\\&\le\int_{\mathcal{X}^t}\mathrm{d}\vartheta_+\Big(\int_{\mathcal{X}^t}(\|B^-(\vartheta_+,\vartheta_\circ)\|_{p,t}^s(r))^2r(\vartheta_\circ)\mathrm{d}\vartheta_\circ\Big)^{1/2}\le\|B\|_{p,t}^s(r),\end{aligned}$$

where $B^-(\vartheta_+,\vartheta_\circ,\boldsymbol{\vartheta})=B(\boldsymbol{\vartheta}\sqcup\boldsymbol{\vartheta}^-)\delta_\emptyset(\vartheta_+^-\sqcup\vartheta_\circ^-)$, $\boldsymbol{\vartheta}^-=\begin{pmatrix}\vartheta_+ & \vartheta_\circ\\ \emptyset & \emptyset\end{pmatrix}$.

Finally, from $\|D_\circ^\circ(x)\|_q\le\|\dot B(\mathbf{x}_\circ^\circ\|_{p,t(x)}^s(r)$ we similarly obtain

$$\|D_\circ^\circ\|_{q,t}^{(\infty)}(s)\le\operatorname*{ess\,sup}_{x\in X^t}\{s(x)\|\dot B(\mathbf{x}_\circ^\circ)\|_{p,t(x)}^s(r)\}\le\|B\|_{p,t}^s(r)$$

if $q\ge r^{-1}+p+s^{-s}$, which concludes the proof. □

Remark 4. *The quantum-stochastic integral* (6.7) *constructed in* [17]*, as well as its single variations* (6.2) *introduced in* [13]*, are defined explicitly and do not require that the functions B and $\mathbf{D}$ under the integral be adapted. By virtue of the continuity we have proved above, they can be approximated in the inductive convergence by the sequence of integral sums $\iota_0^t(B_n)$, $i_0^t(\mathbf{D}_n)$ corresponding to step measurable operator functions B_n and $\mathbf{D}_n$ if the latter converge inductively to B and $\mathbf{D}$ in the poly-norm* (6.8)*.*

In fact, if there exist functions r, s with r^{-1}, $s^{-1}\in\mathcal{P}_0$ and $p\in\mathcal{P}_1$ such that $\|B_n-B\|_{p,t}^s(r)\to 0$, then there also exists a function $q\in\mathcal{P}_1$ such that $\|\iota_0^t(B_n-B)\|_q\to 0$, and we have $q\ge r^{-1}+p+s^{-1}$ by the inequality (6.10), which implies the inductive convergence $\iota_0^t(B_n)\to\iota_0^t(B)$ as a result of the linearity of ι_0^t. Suppose that $\mathbf{D}(x)$ is adapted in the sense that $D_\nu^\mu(x)(\mathrm{h}^{t(x)}\otimes\mathrm{h}_{[t(x)})=\mathrm{f}^{t(x)}\otimes\mathrm{h}_{[t(x)}$ or

$$[D_\nu^\mu(x)\mathrm{h}](\varkappa)=[D_\nu^\mu(x)\dot{\mathrm{h}}(\varkappa_{[t(x)})](\varkappa^{t(x)}),\quad\forall x\in X,$$

where $\dot{\mathrm{h}}(\varkappa_{[t},\varkappa^t)=\mathrm{h}(\varkappa^t\sqcup\varkappa_{[t})$ and $\varkappa^t\sqcup\varkappa_{[t}$ is the decomposition of the chain $\varkappa\in\mathcal{X}$ into $\varkappa^t=\{x\in\varkappa: t(x)<t\}$ and $\varkappa_{[t}=\{x\in\varkappa: t(x)\ge t\}$. In this case the above approximation in the class of adapted step functions leads to the definition of the quantum-stochastic integral $i_0^t(\mathbf{D})$ in the Itô sense given by Hudson and Parthasarathy for the case $X=\mathbb{R}_+, t(x)=x$ as the weak limit of integrals sums

$$i_0^t(\mathbf{D}_n)=\int_0^t\Lambda(\mathbf{D}_n,\mathrm{d}x)=\sum_{i=1}^nD_\nu^\mu(x_i)\Lambda_\mu^\nu(\Delta_i).$$

Here $\mathbf{D}_n(x)=\mathbf{D}(x_j)$ for $x\in[x_j,x_{j+1})$ is an adapted approximation corresponding to the decomposition $\mathbb{R}_+=\sum_{j=1}^n\Delta_i$ into the intervals $\Delta_j=[x_j,x_{j+1})$ given by the chain $x_0=0<x_1<\cdots<x_{n-1}<x_n=\infty$, and $D_\nu^\mu(x)\Lambda_\mu^\nu(\Delta)$ is the sum of the operators (4.3) with functions $D_\nu^\mu(x)$ constant on Δ which can therefore be pulled out in front of the integrals Λ_μ^ν. In particular, for $D_+^-=0=D_\circ^\circ$ and $D_\circ^-=\widehat{1}\otimes g=D_+^\circ$, where $\widehat{1}$ is the unit operator in F and $g(x)$ is a scalar locally square integrable function corresponding to the case $\mathrm{K}_x=\mathbb{C}$, we obtain the Itô definition of the Wiener integral

$$\mathrm{i}_0^t(g)=\int_0^tg(x)w(\mathrm{d}x),\quad\int_0^tg(x)\widehat{w}(\mathrm{d}x)=i_0^t(\mathbf{D})$$

with respect to the stochastic measure $w(\Delta)$, $\Delta \in \mathfrak{F}$ on $\mathbb{R}_+$, represented in F by the operators $\widehat{w}(\Delta) = \Lambda^+_\circ(\Delta) + \Lambda^\circ_-(\Delta)$. We also note that the multiple integral (6.7) in the scalar case $B(\boldsymbol{\vartheta}) = \widehat{1} \otimes b(\boldsymbol{\vartheta})$ defines the Fock representation of the generalized Maassen-Meyer kernels [21], [41] and in the case

$$b(\boldsymbol{\vartheta}) = \mathrm{f}(\vartheta^-_\circ \sqcup \vartheta^\circ_+)\delta_\emptyset(\vartheta^-_+)\delta_\emptyset(\vartheta^\circ_\circ), \quad \delta_\emptyset(\vartheta) = \begin{cases} 1, & \vartheta = \emptyset, \\ 0, & \vartheta \neq \emptyset \end{cases}$$

it leads to the multiple stochastic integrals $\iota^t_0(B) = \widehat{I}^t_0(\mathrm{f})$,

$$I^t_0(\mathrm{f}) = \sum_{n=0}^{\infty} \int \cdots \int_{0 \le t_1 < \cdots < t_n < t} \mathrm{f}(x_1, \ldots, x_n) w(\mathrm{d}x_1) \ldots w(\mathrm{d}x_n)$$

of the generalized functions $\mathrm{f} \in \bigcup_{r^{-1} \in \mathcal{P}_0} \mathrm{F}(r)$, that is, to the Hida distributions [23], [45] of the Wiener measure $w(\Delta)$ represented as $\widehat{w}(\Delta)$. Thus we have constructed a general non-commutative analogue of Hida distributions whose properties are described in the following corollary.

Corollary 1. *Suppose that the operator-function $B(\boldsymbol{\vartheta}) = \widehat{1} \otimes M(\boldsymbol{\vartheta})$ is defined by the kernel M such that $\|M\|^s_t(r) < \infty$,*

$$M \begin{pmatrix} \vartheta^-_+, & \vartheta^-_\circ \\ \vartheta^\circ_+, & \vartheta^\circ_\circ \end{pmatrix} : \mathrm{K}^\otimes(\vartheta^-_\circ \sqcup \vartheta^\circ_\circ) \to \mathrm{K}^\otimes(\vartheta^\circ_\circ \sqcup \vartheta^\circ_+),$$

where

$$\|M\|^s_t(r) = \int_{\mathcal{X}^t} \mathrm{d}\vartheta^-_+ \Big(\int_{\mathcal{X}^t} \mathrm{d}\vartheta^\circ_+ \int_{\mathcal{X}^t} \mathrm{d}\vartheta^-_\circ \operatorname{ess} \sup_{\vartheta^\circ_\circ \in \mathcal{X}^t} \{ s(\vartheta^\circ_\circ) \|M(\vartheta)\| \}^2 r(\vartheta^\circ_+ \sqcup \vartheta^-_\circ) \Big)^{1/2}$$

for all $t \in \mathbb{R}_+$ and for some $r(\vartheta) = \prod_{x \in \vartheta} r(x)$, $s(\vartheta) = \prod_{x \in \vartheta} s(x)$; $r^{-1}, s^{-1} \in \mathcal{P}_0$. Then the integral (6.7) defines an adapted family T_t, $t \in \mathbb{R}_+$, of q-bounded operators $T_t = \iota^t_0(\widehat{1} \otimes M)$, $\|T_t\|_q \le \|M\|^s_t(r)$ for $q \ge r^{-1} + 1 + s^{-1}$, with adapted p-bounded quantum-stochastic derivatives $D^\mu_\nu(x) = \iota^{t(x)}_0(\widehat{1} \otimes \dot{M}(\mathbf{x}^\mu_\nu))$.

7. Generalized Itô formula of unified quantum stochastic calculus

Let H be a Hilbert space. We consider a Hilbert scale $\mathrm{G}(p) = \mathrm{H} \otimes \mathrm{F}(p)$, $p \in \mathcal{P}$, of complete tensor products of H and the Fock spaces over $\mathrm{K}(p)$, and we put $\mathrm{G}_+ = \cap \mathrm{G}(p), \mathrm{G} = \mathrm{G}(1)$, and $\mathrm{G}_- = \cup \mathrm{G}(p)$ which constitute the corresponding Gel'fand triple $\mathrm{G}_+ \subseteq \mathrm{G} \subseteq \mathrm{G}_-$. We consider operators $T = \epsilon(K)$, not necessarily bounded, in the Hilbert space $\mathrm{G} = \mathrm{H} \otimes \mathrm{F}$ as the $*$-representation ϵ of operator-valued kernels

$$K \begin{pmatrix} \omega^-_+ & \omega^-_\circ \\ \omega^\circ_+ & \omega^\circ_\circ \end{pmatrix} : \mathrm{H} \otimes \mathrm{K}^\otimes(\omega^-_\circ \sqcup \omega^\circ_\circ) \to \mathrm{H} \otimes \mathrm{K}^\otimes(\omega^\circ_\circ \sqcup \omega^\circ_+), \tag{7.1}$$

satisfying the integrability condition $\|K\|_p(r) < \infty$ for some $r^{-1} \in \mathcal{P}_0$ and $p \in \mathcal{P}_1$, where

$$\|K\|_p(r) = \int \mathrm{d}\omega^-_+ \left(\iint \operatorname{ess\,sup}_{\omega^\circ_\circ} \{ \frac{\|K(\boldsymbol{\omega})\|}{p(\omega^\circ_\circ)} \}^2 r(\omega^\circ_+ \sqcup \omega^-_\circ) \mathrm{d}\omega^\circ_+ \mathrm{d}\omega^-_\circ \right)^{1/2}.$$

This representation ϵ is defined for $\mathrm{h} \in \mathrm{H} \otimes \mathrm{F}$ by

$$[\epsilon(K)\mathrm{h}](\varkappa) = \sum_{\omega^\circ_\circ \sqcup \omega^\circ_+ = \varkappa} \iint K \begin{pmatrix} \omega^-_+, & \omega^-_\circ \\ \omega^\circ_+, & \omega^\circ_\circ \end{pmatrix} \mathrm{h}(\omega^\circ_\circ \sqcup \omega^-_\circ) \mathrm{d}\omega^-_\circ \mathrm{d}\omega^-_+ \tag{7.2}$$

as the vacuum-adapted operator-valued multiple integral (6.7) with $t = \infty$ of the function $B(\boldsymbol{\vartheta}) = \widehat{\delta}_\emptyset \otimes K(\boldsymbol{\vartheta})$, where

$$[\widehat{\delta}_\emptyset \mathrm{f}](\varkappa) = \mathrm{f}(\emptyset)\delta_\emptyset(\varkappa), \quad \delta_\emptyset\,(\varkappa) = \{ \begin{matrix} 1 & \varkappa = \emptyset \\ 0 & \varkappa \neq \emptyset \end{matrix}$$

is the vacuum projection on F such that $[B(\boldsymbol{\vartheta})\dot{\mathrm{h}}(\vartheta^\circ_\circ \sqcup \vartheta^-_\circ)](\varkappa^\circ_-) = 0$ if $\varkappa^\circ_- = \varkappa \backslash \vartheta^\circ_\circ \backslash \vartheta^\circ_+ \neq \emptyset$. The operator $\epsilon(K)$ can be represented equivalently as the adapted (i.e. identity-adapted) integral (6.7) with $t = \infty$ of a scalar-valued integrant as the function $B(\boldsymbol{\vartheta}) = \widehat{1} \otimes M(\boldsymbol{\vartheta})$, where $\widehat{1}$ is the unit operator on F and $M(\boldsymbol{\vartheta})$ is the generalized Maassen-Meyer kernel-integrant. This follows from

$$[B(\boldsymbol{\vartheta})\dot{\mathrm{h}}(\vartheta^-_\circ \sqcup \vartheta^\circ_\circ)](\varkappa^\circ_-) = M(\boldsymbol{\vartheta})\mathrm{h}(\vartheta^-_\circ \sqcup \vartheta^\circ_\circ \sqcup \varkappa^\circ_-)$$

such that $\left[\iota^\infty_0(\widehat{1} \otimes M)\mathrm{h}\right](\varkappa) = [\epsilon(K)\mathrm{h}](\varkappa)$ for the kernel

$$K\begin{pmatrix} \omega^-_+, & \omega^-_\circ \\ \omega^\circ_+, & \omega^\circ_\circ \end{pmatrix} = \sum_{\vartheta \subseteq \omega^\circ_\circ} M\begin{pmatrix} \omega^-_+, & \omega^-_\circ \\ \omega^\circ_+, & \vartheta \end{pmatrix} \otimes I^\otimes(\omega^\circ_\circ \setminus \vartheta),$$

which is connected with M by a one-to-one relation

$$M\begin{pmatrix} \vartheta^-_+, & \vartheta^-_\circ \\ \vartheta^\circ_+, & \vartheta^\circ_\circ \end{pmatrix} = \sum_{\omega \subseteq \vartheta^\circ_\circ} K\begin{pmatrix} \vartheta^-_+, & \vartheta^-_\circ \\ \vartheta^\circ_+, & \omega \end{pmatrix} \otimes (-I)^\otimes(\vartheta^\circ_\circ \setminus \omega),$$

where $I^\otimes(v) = \bigotimes_{x \in v} I_x$ is the unit operator in $\mathrm{K}^\otimes(v)$.

According to Corollary 1, $\|T\|_q \leq \|M\|^s_\infty(r)$ for $q \geq r^{-1}+1+s^{-1}$. However, using the equivalent representation (7.2) in the form of the non-adapted integral (6.7) of $B(\boldsymbol{\vartheta}) = \widehat{\delta}_\emptyset \otimes K(\boldsymbol{\vartheta})$ and taking into account the fact that $\|\widehat{\delta}_\emptyset\|_p = 1$ for sufficiently small $p > 0$, we obtain as $p \to 0$ a more precise estimate $\|T\|_q \leq \|K\|_{s^{-1}}(r)$ for $q \geq r^{-1} + s^{-1} = \lim_{p_0 \searrow 0}(r^{-1} + p_0 + s^{-1})$. From this estimate the previous one follows, since

$$\| \sum_{\vartheta \subseteq \omega^\circ_\circ} M(\boldsymbol{\vartheta}) \otimes I^\otimes(\omega^\circ_\circ \setminus \vartheta)\| \leq \sum_{\vartheta \subseteq \omega^\circ_\circ} \|M(\boldsymbol{\vartheta})\| \leq (1 + s^{-1})(\omega^\circ_\circ)\|M\|^s_\infty,$$

where $\|M\|^s_\infty = \operatorname{ess\,sup}_{\vartheta \in \mathcal{X}}\{s(\vartheta)\|M(\vartheta)\|\}$,

$$s(\vartheta) = \prod_{x \in \vartheta} s(x), \quad (1 + s^{-1})(\omega^\circ_\circ) = \sum_{\vartheta \subseteq \omega^\circ_\circ} s^{-1}(\vartheta) = \prod_{x \in \omega^\circ_\circ}(1 + s^{-1}(x))$$

and consequently $\|K\|_p(r) \leq \|M\|^s_\infty(r)$ for $p \geq 1+1/s$. Hence in particular there follows the existence of the adjoint operator T^* bounded in norm $\|T^*\|_q \leq \|K^\star\|_p(r) = \|K\|_p(r)$ as the representation

$$\epsilon(K)^* = \epsilon(K^\star), \;\; K^\star\begin{pmatrix} \omega^-_+, & \omega^-_\circ \\ \omega^\circ_+, & \omega^\circ_\circ \end{pmatrix} = K\begin{pmatrix} \omega^-_+, & \omega^\circ_+ \\ \omega^-_\circ, & \omega^\circ_\circ \end{pmatrix}^* \tag{7.3}$$

of the $\star$-adjoint kernel $K^\star(\boldsymbol{\omega}) = K(\boldsymbol{\omega}')^*$.

In the next theorem we prove that the $\star$-map $\epsilon : K \mapsto \epsilon(K)$ is an operator representation of the $\star$-algebra of kernels $K(\boldsymbol{\omega})$ satisfying the boundedness condition

$$\|K\|_{\boldsymbol{\alpha}} = \operatorname*{ess\,sup}_{\omega = (\omega^\mu_\nu)}\{\|K(\boldsymbol{\omega})\| / \prod_{\mu \leq \nu} \alpha^\mu_\nu(\omega^\mu_\nu)\} < \infty \tag{7.4}$$

relative to the product of the quadruple $\boldsymbol{\alpha} = (\alpha^\mu_\nu)^{\mu=-,\circ}_{\nu=\circ,+}$ of positive essentially measurable product functions $\alpha^\mu_\nu(\omega) = \prod_{x \in \omega} \alpha^\mu_\nu(x)$, $\omega \in \mathcal{X}$. These are defined by an integrable function $\alpha_+^- : X \to \mathbb{R}_+$, by functions $\alpha_+^\circ, \alpha_\circ^- : X \to \mathbb{R}_+$, square integrable with a certain weight $r > 0$, $r^{-1} \in \mathcal{P}_0$, and by a function $\alpha_\circ^\circ : X \to \mathbb{R}_+$, essentially bounded by unity relative to some $p \in \mathcal{P}$:

$$\|\alpha_+^-\|^{(1)} < \infty,\ \|\alpha_+^\circ\|^{(2)}(r) < \infty,\ \|\alpha_\circ^-\|^{(2)}(r) < \infty,\ \|\alpha_\circ^\circ\|_p^{(\infty)} \le 1,$$
$$\|\alpha\|^{(1)} = \int |\alpha(x)| \mathrm{d}x,\ \|\alpha\|^{(2)}(r) = \Big(\int \alpha(x)^2 r(x) \mathrm{d}x \Big)^{1/2},$$
$$\|\alpha\|_p^{(\infty)} = \operatorname*{ess\,sup}_x \frac{|\alpha(x)|}{p(x)}. \tag{7.5}$$

The conditional boundedness (7.4) ensures the projective boundedness of K by the inequality $\|K\|_p(r) \le$

$$\le \int \mathrm{d}\omega_+^- \mathrm{d}(\iint \mathrm{d}\sup_{\omega_\circ^\circ} \{ \|K\|_\alpha \prod \alpha_\nu^\mu(\omega_\nu^\mu)/p(\omega_\circ^\circ)\}^2 r(\omega_+^\circ \sqcup \omega_\circ^-) \mathrm{d}\omega_+^\circ \mathrm{d}\omega_\circ^- \mathrm{d})^{1/2}, \tag{7.6}$$

$$\int \alpha_+^-(\omega)\mathrm{d}\omega \Big(\int \alpha_+^\circ(\omega)^2 r(\omega)\mathrm{d}\omega \int \alpha_\circ^-(\omega)^2 r(\omega)\mathrm{d}\omega \Big)^{1/2} \operatorname{ess\,sup} \frac{\alpha_\circ^\circ(\omega)}{p(\omega)} \|K\|_\alpha \tag{7.7}$$
$$\le \|K\|_\alpha \exp\left\{ \int (\alpha_+^-(x) + r(x)(\alpha_+^\circ(x)^2 + \alpha_\circ^-(x)^2)/2)\mathrm{d}x \right\},$$

where we have taken account of the fact that $\int \alpha(\omega)\mathrm{d}\omega = \exp \int \alpha(x)\mathrm{d}x$ for $\alpha(\omega) = \prod_{x\in\omega} \alpha(x)$ and

$$\operatorname*{ess\,sup}_\omega \{\alpha_\circ^\circ(\omega)/p(\omega)\} = \sup_n \operatorname{ess} \sup_{x \in X^n} \prod_{i=1}^n \{\alpha_\circ^\circ(x_i)/p(x_i)\} = 1 \text{ if } \alpha_\circ^\circ \le p.$$

Before we formulate the theorem we establish the following lemma.

Lemma 2. *Suppose that the multiple quantum-stochastic integral $T_t = \iota_0^t(B)$ is defined in (6.7) by a kernel operator-function $B(\boldsymbol{\vartheta}) = \epsilon(M(\boldsymbol{\vartheta}))$ with values in the operators of the form (7.2), where*

$$K \begin{pmatrix} \upsilon_+^-, & \upsilon_\circ^- \\ \upsilon_+^\circ, & \upsilon_\circ^\circ \end{pmatrix} = M \begin{pmatrix} \vartheta_+^-, & \vartheta_\circ^-, & \upsilon_+^-, & \upsilon_\circ^- \\ \vartheta_+^\circ, & \vartheta_\circ^\circ, & \upsilon_+^\circ, & \upsilon_\circ^\circ \end{pmatrix}, \vartheta_\nu^\mu \in \mathcal{X},$$

and $M(\boldsymbol{\vartheta}) : \boldsymbol{\upsilon} \mapsto M(\boldsymbol{\vartheta}, \boldsymbol{\upsilon})$ is a kernel-valued integrant

$$M(\boldsymbol{\vartheta}, \boldsymbol{\upsilon}) : \mathrm{H} \otimes \mathrm{K}^\otimes(\upsilon_\circ^- \sqcup \vartheta_\circ^-) \otimes \mathrm{K}^\otimes(\upsilon_\circ^\circ \sqcup \vartheta_\circ^\circ) \to \mathrm{H} \otimes \mathrm{K}^\otimes(\upsilon_\circ^\circ \sqcup \vartheta_\circ^\circ) \otimes \mathrm{K}^\otimes(\upsilon_+^\circ \sqcup \vartheta_+^\circ).$$

Then $T_t = \epsilon(K_t)$ for the kernel $K_t(\boldsymbol{\omega}) = \nu_0^t(\boldsymbol{\omega}, M)$ given by the multiple counting integral on the kernel-integrants M, that is, $\iota_0^t \circ \epsilon = \epsilon \circ \nu_0^t$, where

$$\nu_0^t(\boldsymbol{\omega}, M) = \sum_{\boldsymbol{\vartheta} \subseteq \boldsymbol{\omega}^t} M(\boldsymbol{\vartheta}, \boldsymbol{\omega} \setminus \boldsymbol{\vartheta}), \quad \boldsymbol{\omega}^t = (X^t \cap \omega_\nu^\mu)_{\nu=\circ,+}^{\mu=-,\circ} \tag{7.8}$$

(the sum is taken over all possible $\vartheta_\nu^\mu \subseteq X^t \cap \omega_\nu^\mu$, $\mu = -, \circ, \nu = \circ, +$). If $M(\boldsymbol{\vartheta})$ is relatively bounded in $\upsilon_\nu^\mu \in \mathcal{X}$ for each $\boldsymbol{\vartheta} = (\vartheta_\nu^\mu)$ such that

$$\|M(\boldsymbol{\vartheta})\|_{\boldsymbol{\gamma}} \le c \prod_{\mu,\nu} \beta_\nu^\mu(\vartheta_\nu^\mu), \quad \beta_\nu^\mu(\vartheta) = \prod_{x\in\vartheta} \beta_\nu^\mu(x)$$

for a pair of quadruples $\boldsymbol{\beta} = (\beta_\nu^\mu)$, $\beta_\nu^\mu \ge 0$ and $\boldsymbol{\gamma} = (\gamma_n^\mu)$, $\gamma_\nu^\mu \ge 0$ satisfying (7.5), then the kernel K is relatively bounded: $\|\nu_0^t(M)\|_{\boldsymbol{\alpha}} \le c$ if $\alpha_\nu^\mu(x) \ge \beta_\nu^\mu(x) + \gamma_\nu^\mu(x)$ for $t(x) < t$ and $\alpha_\nu^\mu(x) \ge \gamma_\nu^\mu(x)$ for all μ, ν when $t(x) \ge t$. In particular, the generalized *single integral* $i_0^t(\mathbf{D})$ *of the triangular operator-integrant* $\mathbf{D}(x) = \left[\delta_+^\mu \delta_\nu^- D_\nu^\mu(x)\right]$ *with* $D_\nu^\mu(x) = \epsilon(C_\nu^\mu(x))$ *is a representation* $i_0^t \circ \epsilon = \epsilon \circ n_0^t$ *of the single counting integral*

$$n_0^t(\boldsymbol{\omega}, \mathbf{C}) = \sum_{\mathbf{x}\in\boldsymbol{\omega}^t} C(\mathbf{x}, \boldsymbol{\omega}\setminus\mathbf{x}), \quad C(\mathbf{x}_\nu^\mu, \boldsymbol{\upsilon}) = C_\nu^\mu(x, \boldsymbol{\upsilon}),$$

of the triangular kernel-integrant $\mathbf{C}(x, \boldsymbol{\upsilon}) = \left[\delta_+^\mu \delta_\nu^- C_\nu^\mu(x, \boldsymbol{\upsilon})\right]$, *where the sum is taken over all possible* $x \in \omega_\nu^\mu \cap X^t$ *for* $\mu = -, \circ, \nu = \circ, +$, *and* $\mathbf{x} = \mathbf{x}_{\nu(x)}^{\mu(x)}$ *is one of the atomic matrices* (6.9) *with indices* $\mu(\mathbf{x}) = \mu$, $\nu(\mathbf{x}) = \nu$, *defined almost everywhere by the condition* $x \in \omega_\nu^\mu$. *Moreover, we have the estimate*

$$\|n_0^t(\mathbf{C})\|_p(r) \leq c\exp\left\{\int_{X^t}(\gamma_+^-(x) + \tfrac{1}{2}(\gamma_+^\circ(x)^2 + \gamma_\circ^-(x)^2)r(x))\mathrm{d}x\right\}.$$

given the kernel-valued quadruple-integrant $C_\nu^\mu(x, \boldsymbol{\gamma})$ *is relatively bounded for each* $x \in X$ *in* $\boldsymbol{\upsilon} = (\upsilon_\nu^\mu)$ *in the sense that there exist such* $\boldsymbol{\gamma} = (\gamma_\nu^\mu)$ *that*

$$c = \|C_+^-\|_{\boldsymbol{\gamma},t}^{(1)} + \|C_+^\circ\|_{\boldsymbol{\gamma},t}^{(2)}(r) + \|C_\circ^-\|_{\boldsymbol{\gamma},t}^{(2)}(r) + \|C_\circ^\circ\|_{\boldsymbol{\gamma},t}^{(\infty)}(1/p) < \infty,$$

$$\|C_+^-\|_{\boldsymbol{\gamma},t}^{(1)} = \int_{X^t}\|C_+^-(x)\|_{\boldsymbol{\gamma}}\mathrm{d}x,\ \|C\|_{\boldsymbol{\gamma},t}^{(2)} = \Big(\int_{X^t}\|C(x)\|_{\boldsymbol{\gamma}}^2 r(x)\mathrm{d}x\Big)^{1/2},$$

$$\|C_\circ^\circ\|_{\boldsymbol{\gamma},t}^{(\infty)}\left(\frac{1}{r}\right) = \sup_{x\in X^t}\left\{\frac{\|C_\circ^\circ(x)\|_{\boldsymbol{\gamma}}}{p(x)}\right\},$$

Proof. If $M(\boldsymbol{\vartheta}, \boldsymbol{\upsilon})$ is an operator-valued integrant-kernel that is bounded, $\|M\|_{\boldsymbol{\beta},\boldsymbol{\gamma}} \leq c$, relative to the pair $(\boldsymbol{\beta}, \boldsymbol{\gamma})$, then the relatively bounded operator $T_t = \epsilon(K_t)$ is well-defined for $K_t = \nu_0^t(M)$, since

$$\begin{aligned}\|K_t(\boldsymbol{\omega})\| &\leq c\sum_{\vartheta_+^-\subseteq\omega_+^-}^{t(\vartheta_+^-)<t}\ \sum_{\vartheta_+^\circ\subseteq\omega_+^\circ}^{t(\vartheta_+^\circ)<t}\ \sum_{\vartheta_\circ^-\subseteq\omega_\circ^-}^{t(\vartheta_\circ^-)<t}\ \sum_{\vartheta_\circ^\circ\subseteq\omega_\circ^\circ}^{t(\vartheta_\circ^\circ)<t}\|M(\boldsymbol{\vartheta}, \boldsymbol{\omega}\setminus\boldsymbol{\vartheta})\| \\ &\leq c\prod_{\nu=\circ,+}^{\mu=-,\circ}\ \sum_{\vartheta_\nu^\mu\subseteq\omega_\nu^\mu}^{t(\vartheta_\nu^\mu)<t}\beta_\nu^\mu(\vartheta_\nu^\mu)\gamma_\nu^\mu(\omega_\nu^\mu\setminus\vartheta_\nu^\mu) = c\prod_{\nu=\circ,+}^{\mu=-,\circ}\alpha_\nu^\mu(\omega_\nu^\mu),\end{aligned}$$

where $\alpha_\nu^\mu(\omega) = \prod_{x\in\omega}^{t(x)<t}[\beta_\nu^\mu(x) + \gamma_\nu^\mu(x)] \cdot \prod_{x\in\omega}^{t(x)\geq t}\gamma_\nu^\mu(x)$ for $\beta_\nu^\mu(\vartheta) = \prod_{x\in\vartheta}\beta_\nu^\mu(x)$ and $\gamma_\nu^\mu(\upsilon) = \prod_{x\in\upsilon}\gamma_\nu^\mu(x)$. Applying the representation (7.2) to $K_t(\boldsymbol{\omega}) = \nu_0^t(\boldsymbol{\omega}, M)$ it is easy to obtain the representation of the operator $\epsilon(K_t)$ in the form of the generalized multiple integral (6.7) of $B(\boldsymbol{\vartheta}) = \epsilon(M(\boldsymbol{\vartheta}))$. Indeed, $[T_t\mathrm{h}](\varkappa) =$

$$\begin{aligned}&= \sum_{\omega_\circ^\circ\sqcup\omega_+^\circ=\varkappa}\iint\sum_{\boldsymbol{\vartheta}\subseteq\boldsymbol{\omega}^t}M(\boldsymbol{\vartheta}, \boldsymbol{\omega}\setminus\boldsymbol{\vartheta})\mathrm{h}(\omega_\circ^\circ\sqcup\omega_\circ^-)\mathrm{d}\omega_\circ^-\mathrm{d}\omega_+^- \\ &= \sum_{\vartheta_\circ^\circ\sqcup\vartheta_+^\circ\subseteq\varkappa^t}\int_{\mathcal{X}^t}\mathrm{d}\vartheta_\circ^-\int_{\mathcal{X}^t}\mathrm{d}\vartheta_+^-\sum_{\upsilon_\circ^\circ\sqcup\upsilon_+^\circ=\varkappa_-^\circ}\iint M(\boldsymbol{\vartheta}, \boldsymbol{\upsilon})\dot{\mathrm{h}}(\vartheta_\circ^\circ\sqcup\vartheta_\circ^-, \upsilon_\circ^\circ\sqcup\upsilon_\circ^-)\mathrm{d}\upsilon_\circ^-\mathrm{d}\upsilon_+^-,\end{aligned}$$

where $\varkappa_-^\circ = \varkappa\setminus(\vartheta_\circ^\circ\sqcup\vartheta_+^\circ)$, $\dot{\mathrm{h}}(\vartheta, \upsilon) = \mathrm{h}(\upsilon\sqcup\vartheta)$. Consequently, $T_t = \iota_0^t(B)$, where

$$[B(\boldsymbol{\vartheta})\dot{\mathrm{h}}(\vartheta_\circ^\circ\sqcup\vartheta_\circ^-)](\varkappa) = \sum_{\upsilon_\circ^\circ\sqcup\upsilon_+^\circ=\varkappa}\iint M(\boldsymbol{\vartheta}, \boldsymbol{\upsilon})\dot{\mathrm{h}}(\vartheta_\circ^\circ\sqcup\vartheta_\circ^-,\ \upsilon_\circ^\circ\sqcup\upsilon_\circ^-)\mathrm{d}\upsilon_\circ^-\mathrm{d}\upsilon_+^-,$$

that is, we have proved that $\epsilon\circ\nu_0^t=\iota_0^t\circ\epsilon$. In particular, if $M(\boldsymbol{\vartheta},\boldsymbol{\upsilon})=0$ for $\sum|\vartheta_\nu^\mu|\neq 1$, then, obviously

$$\nu_0^t(\boldsymbol{\omega},M)=n_0^t(\boldsymbol{\omega},\mathbf{C}),\quad \iota_0^t(B)=i_0^t(\mathbf{D}),$$

where $C_\nu^\mu(x,\boldsymbol{\upsilon})=M(\mathbf{x}_\nu^\mu,\boldsymbol{\upsilon})\equiv C\left(\mathbf{x}_\nu^\mu,\boldsymbol{\upsilon}\right)$ and $B(\boldsymbol{\vartheta})=0$ for $\sum|\vartheta_\nu^\mu|\neq 1$, $D_\nu^\mu(x)=B(\mathbf{x}_\nu^\mu)$. This yields the representation $\epsilon\circ n_0^t=i_0^t\circ\epsilon$ for the single generalized non-adapted integral (6.2) for $\Delta=X^t$ in the form of the sum

$$\sum_{\mu,\nu}\Lambda_\nu^\mu(\epsilon(C_\nu^\mu),\Delta)=\epsilon\Big(\sum_{\mu,\nu}(N_\mu^\nu(C_\nu^\mu,\Delta)\Big),\quad N_\mu^\nu(\boldsymbol{\omega},C,\Delta)=\sum_{x\in\omega_\nu^\mu\cap\Delta}C(x,\boldsymbol{\omega}\setminus\mathbf{x}_\nu^\mu)$$

of representations of four kernel measures $N_\nu^\mu(\boldsymbol{\omega},C_\nu^\mu,\Delta)$ for that define kernel representations $\epsilon\circ N(\Delta)=\Lambda(\Delta)\circ\epsilon$ of the canonical measures (4.3) with $D_\nu^\mu(x)=\epsilon(C_\nu^\mu(x))$. □

In the following theorem, which generalizes Itô formula to noncommutative and nonadapted quantum stochastic processes $T_t=\epsilon\left(K_t\right)$ given by an operator-valued kernel $K_t\left(\boldsymbol{\omega}\right)$, we use the following triangular-matrix notation

$$\mathbf{T}\left(x\right)=\left[T\left(\mathbf{x}_\nu^\mu\right)\right],\ \ T\left(\mathbf{x}\right)=\nabla_{\mathbf{x}}T_t|_{t=t(x)}$$

for the *quantum stochastic germs* $\nabla_{\mathbf{x}}T=\epsilon\left(\dot{K}\left(\mathbf{x}\right)\right)$ given by the point derivatives of the kernel $\dot{K}\left(\mathbf{x},\boldsymbol{\upsilon}\right)=K\left(\boldsymbol{\upsilon}\sqcup\mathbf{x}\right)$, with $T_\nu^\mu\left(x\right)=T\left(\mathbf{x}_\nu^\mu\right)$ equal zero for $\mu=+$ or $\nu=-$ and $T_-^-\left(x\right)=T_{t(x)}=T_+^+\left(x\right)$. We notice that if $K_t\left(\boldsymbol{\omega}\right)=K_0\left(\boldsymbol{\omega}\right)+n_0^t\left(\mathbf{C}\left(\boldsymbol{\omega}\right)\right)$, corresponding to the single-integral representation $T_t-T_0=i_0^t\left(\mathbf{D}\right)$ with $\mathbf{D}\left(x\right)=\epsilon\left(\mathbf{C}\left(x\right)\right)$, then $\dot{K}_t(\mathbf{x},\boldsymbol{\upsilon})=K_t(\boldsymbol{\upsilon}\sqcup\mathbf{x})$ is given by

$$\dot{K}_t(\mathbf{x},\boldsymbol{\upsilon})=\dot{K}_{t\wedge t(x)}\left(\mathbf{x},\boldsymbol{\upsilon}\right)+\sum_{\mathbf{z}\in\boldsymbol{\upsilon}}^{t(x)\leq t(z)<t}C\left(\mathbf{z},\boldsymbol{\upsilon}\backslash\mathbf{z}\sqcup\mathbf{x}\right).$$

This proves that $\dot{K}_t(\mathbf{x},\boldsymbol{\upsilon})$ does not depend on $t\in\left(t\left(x\right),t^+(x)\right]$, where $t^+\left(x\right)=\min\left\{t\left(x'\right)>t\left(x\right):x'\in\sqcup\upsilon_\nu^\mu\right\}$, and therefore the right limit

$$\dot{K}_{t(x)]}(\mathbf{x},\boldsymbol{\upsilon}):=\lim_{t\searrow t(x)}\dot{K}_t(\mathbf{x},\boldsymbol{\upsilon})=\dot{K}_{t(x)}\left(\mathbf{x},\boldsymbol{\upsilon}\right)+C\left(\mathbf{x},\boldsymbol{\upsilon}\right)$$

trivially exists for each $\mathbf{x}\in\{\mathbf{x}_\nu^\mu\}$ and $\boldsymbol{\upsilon}$ with $\dot{K}_{t(x)]}(\mathbf{x}_-^-,\boldsymbol{\upsilon})=K_{t(x)}(\boldsymbol{\upsilon})=\dot{K}_{t(x)]}(\mathbf{x}_-^-,\boldsymbol{\upsilon})$ for $\dot{K}_t(\mathbf{x}_-^-,\boldsymbol{\upsilon})=K_t(\boldsymbol{\upsilon})=\dot{K}_t(\mathbf{x}_+^+,\boldsymbol{\upsilon})$ due to the independency of $K\left(\boldsymbol{\omega}\right)$ on ω_-^- and ω_+^+. We may assume that the germs $\nabla_{\mathbf{x}}T_t=\epsilon\left(\dot{K}_t\left(\mathbf{x}\right)\right)$ also converge from the right to $G\left(\mathbf{x}\right)=T\left(\mathbf{x}\right)+D\left(\mathbf{x}\right)$ with $D\left(\mathbf{x}\right)=\epsilon\left(C\left(\mathbf{x}\right)\right)$ at $t\searrow t\left(x\right)$ for $x\in X$ corresponding to each atomic table $\mathbf{x}$ in (6.9) as they have limits $\epsilon\left(\dot{K}_{t(x)}\left(\mathbf{x}_-^-\right)\right)=T_{t(x)}=\epsilon\left(\dot{K}_{t(x)}\left(\mathbf{x}_+^+\right)\right)$ for $\mathbf{x}\in\left\{\mathbf{x}_-^-,\mathbf{x}_+^+\right\}$. As it is proved in the following theorem, these germ-limits $G\left(\mathbf{x}\right)$ are given by the matrix elements $D\left(\mathbf{x}_\nu^\mu\right)$ of the QS-derivatives $\mathbf{D}=\left[D_\nu^\mu\left(\mathbf{x}\right)\right]$ at least in the case $K_t=\nu_0^t(M)$ (7.8) corresponding to the multiple integral representation $T_t=\iota_0^t(B)$ (see (6.7)) with $B(\boldsymbol{\vartheta})=\epsilon(M(\boldsymbol{\vartheta}))$.

Theorem 5. *If kernel $K(\boldsymbol{\omega})$ is relatively bounded, then the same is true for the kernel $K^\star(\boldsymbol{\omega}):\|K^\star\|_{\boldsymbol{\gamma}}=\|K\|_{\boldsymbol{\gamma}'}$, where $\begin{pmatrix}\gamma_+^- & \gamma_\circ^-\\ \gamma_+^\circ & \gamma_\circ^\circ\end{pmatrix}'=\begin{pmatrix}\gamma_+^- & \gamma_+^\circ\\ \gamma_\circ^- & \gamma_\circ^\circ\end{pmatrix}$, and the operator $T^*=\epsilon(K^\star)$, as well as the operator $T=\epsilon(K)$, is q-bounded by the estimate (7.6) for $q\geq p+1/r$. For any such kernels $K(\boldsymbol{\vartheta})$ and $K^\star(\boldsymbol{\vartheta})$, bounded relative to* *the quadruples* $\boldsymbol{\alpha} = (\alpha^\mu_\nu)$ *and* $\boldsymbol{\gamma} = (\gamma^\mu_\nu)$ *of functions* $\alpha^\mu_\nu(x)$, $\gamma^\mu_\nu(x)$ *satisfying* (7.5), *the operator*

$$\epsilon(K)\epsilon(K)^* = \epsilon(K \cdot K^\star), \quad \epsilon(I^\otimes) = I$$

is well-defined as a $*$*-representation of kernel product* (2.9) *of* Chapter I *with the estimate* $\|K \cdot K^\star\|_{\boldsymbol{\beta}} \leq \|K\|_{\boldsymbol{\alpha}}\|K^\star\|_{\boldsymbol{\gamma}}$ *if* $\beta^\mu_\nu \geq (\boldsymbol{\alpha}\cdot\boldsymbol{\gamma})^\mu_\nu$, *where* $(\boldsymbol{\alpha}\cdot\boldsymbol{\gamma})^\mu_\nu(x) = \sum \alpha^\mu_\lambda(x)\gamma^\lambda_\nu(x)$ *is defined by the product of triangular matrices*

$$\begin{bmatrix} 1 & \alpha^-_\circ & \alpha^-_+ \\ 0 & \alpha^\circ_\circ & \alpha^\circ_+ \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \gamma^-_\circ & \gamma^-_+ \\ 0 & \gamma^\circ_\circ & \gamma^\circ_+ \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \alpha^-_\circ\gamma^\circ_\circ + \gamma^-_\circ, & \gamma^-_+ + \alpha^-_\circ\gamma^\circ_+ + \alpha^-_+ \\ 0, & \alpha^\circ_\circ\gamma^\circ_\circ, & \alpha^\circ_\circ\gamma^\circ_+ + \gamma^\circ_+ \\ 0, & 0, & 1 \end{bmatrix}.$$

Let $T_t = \epsilon(K_t)$ *with* $\dot{K}_t(\mathbf{x}, \boldsymbol{\upsilon})$ *defining the right limit* $\nabla_{\mathbf{x}} T_t|_{t=t(x)]} = \epsilon(\dot{K}_{t(x)]}(\mathbf{x}))$ *of* $\nabla_{\mathbf{x}} T_t$ *at* $t \searrow t(x)$. *Let* $\mathbf{T}(x) = [T^\mu_\nu(x)]$ *and* $\mathbf{G}(x) = [G^\mu_\nu(x)]$ *denote the triangular matrices of germs* $T(\mathbf{x}) = \nabla_{\mathbf{x}} T_t|_{t=t(x)}$ *and* $G(\mathbf{x}) = \nabla_{\mathbf{x}} T_t|_{t=t(x)]}$ *as operator-valued matrix elements*

$$T^\mu_\nu(x) = \epsilon(\dot{K}_{t(x)}(\mathbf{x}^\mu_\nu)), \quad G^\mu_\nu(x) = \epsilon(\dot{K}_{t(x)]}(\mathbf{x}^\mu_\nu)) \tag{7.9}$$

corresponding to point-derivatives $\dot{K}_t(\mathbf{x}^\mu_\nu)$ *at* $t = t(x)$ *and their right limits at* $t = t(x)]$ *respectively. Then the operator-functions* $D^\mu_\nu(x) = G^\mu_\nu(x) - T^\mu_\nu(x)$ *are quantum-stochastic derivatives of the function* $t \mapsto T_t$ *which define the QS differential* $\mathrm{d}T_t = \mathrm{d}i^t_0(\mathbf{D})$ *in the difference form so that* $T_t - T_0 = i^t_0(\mathbf{G} - \mathbf{T})$. *Moreover,* $T^*_t - T^*_0 = i^t_0(\mathbf{G}^\dagger - \mathbf{T}^\dagger)$, *and we have the generalized non-adapted Itô formula*

$$T_t T^*_t - T_0 T^*_0 = i^t_0(\mathbf{T}\mathbf{D}^\dagger + \mathbf{D}\mathbf{T}^\dagger + \mathbf{D}\mathbf{D}^\dagger) = i^t_0(\mathbf{G}\mathbf{G}^\dagger - \mathbf{T}\mathbf{T}^\dagger), \tag{7.10}$$

where $\mathbf{D} \mapsto \mathbf{D}^\dagger$ *is the pseudo-Euclidean conjugation* $[D^\mu_\nu(x)]^\dagger = [D^{-\nu}_{-\mu}(x)]^*$ *of the triangular operators*

$$\mathbf{T} = \begin{bmatrix} T & T^-_\circ & T^-_+ \\ 0 & T^\circ_\circ & T^\circ_+ \\ 0 & 0 & T \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 0 & D^-_\circ & D^\circ_+ \\ 0 & D^\circ_\circ & D^\circ_+ \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} T & G^-_\circ & G^-_+ \\ 0 & G^\circ_\circ & G^\circ_+ \\ 0 & 0 & T \end{bmatrix}$$

with the standard block-matrix multiplication $(\mathbf{TG})^\mu_\nu = \Sigma T^\mu_\lambda G^\lambda_\nu$.

Proof. The adjoint operators $\epsilon(K)$ and $\epsilon(K^\star)$, which define the $*$-representation (7.2) with respect to the kernels K bounded in the sense of (7.4) and (7.5), are q-bounded for $q \geq p + 1/r$ by the estimate $\|\epsilon(K)\|_q \leq \|K\|_p(r)$ and inequality (7.6), which leads to the exponential estimate

$$\|\epsilon(K)\|_q \leq \|K\|_{\boldsymbol{\alpha}} \exp\{\|\alpha^-_+\|^{(1)} + \tfrac{1}{2}(\|\alpha^\circ_+\|^{(2)}(r)^2 + \|\alpha^-_\circ\|^{(2)}(r)^2)\}.$$

The formula for the kernel multiplication $K^\star \cdot K$, which corresponds to the operator product $\epsilon(K^\star)\epsilon(K)$, has already been found for scalar $\mathrm{H} = \mathbb{C}$ in the case of linear combinations of exponential kernels

$$\mathbf{f}^\otimes(\boldsymbol{\vartheta}) = f^-_+(\vartheta^-_+) f^\circ_+(\vartheta^\circ_+) \otimes f^\circ_\circ(\vartheta^\circ_\circ) \otimes f^-_\circ(\vartheta^-_\circ),$$

where $f_\nu^\mu(\vartheta) = \bigotimes_{x\in\vartheta} f(x)(f_+^-(\vartheta) = \prod_{x\in\vartheta} f_+^-(x))$. We shall now verify this formula for operator-valued kernels $K(\omega)$ and $K^\star(\omega)$, noticing that their product is $\boldsymbol{\beta}$-bounded for $\boldsymbol{\beta} = \boldsymbol{\alpha}\cdot\boldsymbol{\gamma}$, since $\|K^\star \cdot K\|(\omega) \leq$

$$\begin{aligned}
&\leq \sum \left\| K \begin{pmatrix} \omega_+^- \setminus \sigma_+^-, & \upsilon_\circ^- \sqcup \upsilon_+^- \\ \omega_+^\circ \setminus \upsilon_+^\circ, & \omega_\circ^\circ \sqcup \upsilon_+^\circ \end{pmatrix} \right\| \cdot \left\| K^\star \begin{pmatrix} \omega_+^- \setminus \tau_+^-, & \omega_\circ^- \setminus \upsilon_\circ^- \\ \upsilon_+^- \sqcup \upsilon_+^\circ, & \omega_\circ^\circ \sqcup \upsilon_\circ^- \end{pmatrix} \right\| \\
&\leq \|K\|_{\boldsymbol{\alpha}} \|K^\star\|_{\boldsymbol{\gamma}} \sum \boldsymbol{\alpha}^\otimes \begin{pmatrix} \omega_+^- \setminus \sigma_+^-, & \upsilon_\circ^- \sqcup \upsilon_+^- \\ \omega_+^\circ \setminus \upsilon_+^\circ, & \omega_\circ^\circ \sqcup \upsilon_+^\circ \end{pmatrix} \boldsymbol{\gamma}^\otimes \begin{pmatrix} \omega_+^- \setminus \tau_+^-, & \omega_\circ^- \setminus \upsilon_\circ^- \\ \upsilon_+^- \sqcup \upsilon_+^\circ, & \omega_\circ^\circ \sqcup \upsilon_\circ^- \end{pmatrix} \\
&= \|K^\star\|_{\boldsymbol{\gamma}} \|K\|_{\boldsymbol{\alpha}} (\boldsymbol{\alpha}\cdot\boldsymbol{\gamma})^\otimes(\omega); (\boldsymbol{\alpha}\cdot\boldsymbol{\gamma})_\nu^\mu = \sum_{\mu\leq\lambda\leq\nu} \alpha_\lambda^\mu \gamma_\nu^\lambda,
\end{aligned}$$

where we have employed the multiplication formula $\boldsymbol{\alpha}^\otimes \cdot \boldsymbol{\gamma}^\otimes = (\boldsymbol{\alpha}\cdot\boldsymbol{\gamma})^\otimes$ for scalar exponential kernels

$$\boldsymbol{\beta}^\otimes(\boldsymbol{\omega}) = \prod \beta_\nu^\mu(\omega_\nu^\mu);\ \beta_\nu^\mu(\omega) = \prod_{x\in\omega} \beta_\nu^\mu(x) : (\boldsymbol{\alpha}\cdot\boldsymbol{\gamma})_\nu^\mu(x) = \sum \gamma_\lambda^\mu(x)\alpha_\nu^\lambda(x).$$

Using the main formula (6.6) of the scalar integration we write the scalar square of the action (7.2) in the form $\|\epsilon(K)\mathrm{h}\|^2 =$

$$\begin{aligned}
&= \int \left\| \sum_{\omega_\circ^\circ \sqcup \omega_+^\circ = \varkappa} \iint K^\star(\boldsymbol{\omega}) \mathrm{h}(\omega_\circ^\circ \sqcup \omega_\circ^\circ) \mathrm{d}\omega_+^- \mathrm{d}\omega_\circ^- \right\|^2 \mathrm{d}\varkappa \\
&= \iint \iiint \int \sum_{\sigma_\circ^\circ \sqcup \sigma_+^\circ = \varkappa} \sum_{\tau_\circ^\circ \sqcup \tau_+^\circ = \varkappa} \langle K^\star(\boldsymbol{\sigma}) \mathrm{h}(\sigma_\circ^- \sqcup \sigma_\circ^\circ) \mid \\
&\quad K^\star(\boldsymbol{\tau}) \mathrm{h}(\tau_\circ^- \sqcup \tau_\circ^\circ) \rangle \mathrm{d}\varkappa \mathrm{d}\sigma_+^- \mathrm{d}\sigma_\circ^- \mathrm{d}\tau_+^- \mathrm{d}\tau_\circ^- \\
&= \iint \iint \iiint \langle K^\star \begin{pmatrix} \sigma_+^-, & \sigma_\circ^- \\ \upsilon_\circ^- \sqcup \upsilon_+^-, & \upsilon_\circ^\circ \sqcup \upsilon_+^\circ \end{pmatrix} \mathrm{h}(\sigma_\circ^- \sqcup \upsilon_\circ^\circ \sqcup \upsilon_+^\circ) \mid \\
&\quad K^\star \begin{pmatrix} \tau_+^-, & \tau_\circ^- \\ \upsilon_+^\circ \sqcup \upsilon_+^-, & \upsilon_\circ^\circ \sqcup \upsilon_\circ^- \end{pmatrix} \mathrm{h}(\tau_\circ^- \sqcup \upsilon_\circ^\circ \sqcup \upsilon_\circ^-) \rangle \mathrm{d}\boldsymbol{\upsilon} \mathrm{d}\sigma_+^- \mathrm{d}\sigma_\circ^-\, \mathrm{d}\tau_+^- \mathrm{d}\tau_\circ^- \\
&= \iint \iint \iiint \langle \mathrm{h}(\sigma_\circ^- \sqcup \upsilon_\circ^\circ \sqcup \upsilon_+^\circ) \mid K \begin{pmatrix} \sigma_+^-, & \upsilon_\circ^- \sqcup \upsilon_+^- \\ \sigma_\circ^-, & \upsilon_\circ^\circ \sqcup \upsilon_+^\circ \end{pmatrix} \cdot \\
&\quad K^\star \begin{pmatrix} \tau_+^-, & \tau_\circ^- \\ \upsilon_+^\circ \sqcup \upsilon_+^-, & \upsilon_\circ^\circ \sqcup \upsilon_\circ^- \end{pmatrix} \mathrm{h}(\tau_\circ^- \sqcup \upsilon_\circ^\circ \sqcup \upsilon_\circ^-) \rangle \mathrm{d}\boldsymbol{\upsilon} \mathrm{d}\sigma_+^- \mathrm{d}\sigma_\circ^-\, \mathrm{d}\tau_+^- \mathrm{d}\tau_\circ^- \\
&= \int (\mathrm{h}(\varkappa) \mid \sum_{\omega_\circ^\circ \sqcup \omega_+^\circ = \varkappa} \iint (K \cdot K^\star)(\boldsymbol{\omega}) \mathrm{h}(\omega_\circ^- \sqcup \omega_\circ^\circ) \mathrm{d}\omega_+^- \mathrm{d}\omega_\circ^-) \mathrm{d}\varkappa,
\end{aligned}$$

where $\upsilon_\circ^\circ = \sigma_\circ^\circ \cap \tau_\circ^\circ$, $\upsilon_+^\circ = \sigma_\circ^\circ \cap \tau_+^\circ$, $\upsilon_\circ^- = \tau_\circ^\circ \cap \sigma_+^\circ$, $\upsilon_+^- = \sigma_+^\circ \cap \tau_+^\circ$, and the integral over $\mathrm{d}\varkappa$ of the double sum $\sum_{\sigma_\circ^\circ \sqcup \sigma_+^\circ = \varkappa} \sum_{\tau_\circ^\circ \sqcup \tau_+^\circ = \varkappa} = \sum_{\upsilon_\circ^\circ \sqcup \upsilon_+^\circ \sqcup \upsilon_\circ^- \sqcup \upsilon_+^- = \varkappa}$ is replaced by the quadruple integral over $\mathrm{d}\boldsymbol{\upsilon} = \mathrm{d}\upsilon_\circ^\circ \mathrm{d}\upsilon_+^\circ \mathrm{d}\upsilon_\circ^- \mathrm{d}\upsilon_+^-$. Since $\mathrm{h} \in \mathrm{H} \otimes \mathrm{F}(q)$ is arbitrary, this proves the kernel multiplication formula (2.9) of Chapter I for K and $K^\star$, which extends to any relatively bounded kernels K and M because of the polarization formula for the Hermitian function $K \cdot K^\star$.

We shall now consider the stochastic differential $\mathrm{d}T_t$ of the multiple integral $T_t = \iota_0^t(B)$ of the operator function $B(\boldsymbol{\vartheta}) = \epsilon(M(\boldsymbol{\vartheta}))$ defined by the quantum-stochastic derivatives

$$D_\nu^\mu(x) = \iota_0^{t(x)}(\dot{B}(\mathbf{x}_\nu^\mu)) = \epsilon(C_\nu^\mu(x)),$$

representing the differences of the kernels

$$C^{\mu}_{\nu}(x,\boldsymbol{\upsilon}) = \nu_0^{t(x)}(\boldsymbol{\upsilon},\dot{M}(\mathbf{x}^{\mu}_{\nu})) = \dot{K}_{t(x)]}(\mathbf{x}^{\mu}_{\nu},\boldsymbol{\upsilon}) - \dot{K}_{t(x)}(\mathbf{x}^{\mu}_{\nu},\boldsymbol{\upsilon}).$$

Here $\nu_0^t(\boldsymbol{\upsilon},\dot{M}(\mathbf{x})) = \sum_{\boldsymbol{\vartheta}\subseteq\boldsymbol{\upsilon}^t} M(\boldsymbol{\vartheta}\sqcup\mathbf{x},\boldsymbol{\upsilon}\setminus\boldsymbol{\vartheta})$, $\mathbf{x}$ is one of the atomic matrices (6.9), and

$$\begin{aligned}
\dot{K}_{t(x)}(\mathbf{x},\boldsymbol{\upsilon}) &= \sum_{\boldsymbol{\vartheta}\subseteq\upsilon^{t(x)}} M(\boldsymbol{\vartheta},(\boldsymbol{\upsilon}\sqcup\mathbf{x})\setminus\boldsymbol{\vartheta}) = K_{t(x)}(\boldsymbol{\upsilon}\sqcup\mathbf{x}),\\
\dot{K}_{t(x)]}(\mathbf{x},\boldsymbol{\upsilon}) &= \sum_{\boldsymbol{\vartheta}\subseteq\upsilon^{t(x)}\sqcup\mathbf{x}} M(\boldsymbol{\vartheta},(\boldsymbol{\upsilon}\sqcup\mathbf{x})\setminus\boldsymbol{\vartheta})\\
&= K_{t(x)}(\boldsymbol{\upsilon}\sqcup\mathbf{x}) + \sum_{\boldsymbol{\vartheta}\subseteq\boldsymbol{\upsilon}^{t(x)}} M(\boldsymbol{\vartheta}\sqcup\mathbf{x},\boldsymbol{\upsilon}\setminus\boldsymbol{\vartheta})\\
&= \dot{K}_{t(x)}(\mathbf{x},\boldsymbol{\upsilon}) + \nu_0^{t(x)}(\boldsymbol{\upsilon},\dot{M}(\mathbf{x})).
\end{aligned}$$

We note that $K_{t]}(\boldsymbol{\omega}) = \sum_{\boldsymbol{\vartheta}\subseteq\boldsymbol{\omega}^{t]}} M(\boldsymbol{\vartheta},\boldsymbol{\omega}\setminus\boldsymbol{\vartheta}) = K_{t_+}(\boldsymbol{\omega})$, where $t_+ = \min\{t(\mathbf{x}) > t : \mathbf{x}\in\boldsymbol{\omega}\}$, $\boldsymbol{\omega}^{t]} = \{\mathbf{x}\in\boldsymbol{\omega} : t(x)\le t\}$, so that $\dot{K}_{t(x)]}(\mathbf{x},\boldsymbol{\upsilon}) = \dot{K}_t(\mathbf{x},\boldsymbol{\upsilon})$ for any $t\in(t(\mathbf{x}),t_+(\mathbf{x})]$. Thus the derivatives $D^{\mu}_{\nu}(x), x\in X^t$, defining the increment $T_t - T_0 = i_0^t(\mathbf{D})$, can be written in the form of the differences

$$D^{\mu}_{\nu}(x) = \epsilon[\dot{K}_{t(x)]}(\mathbf{x}^{\mu}_{\nu})] - \epsilon[\dot{K}_{t(x)}(\mathbf{x}^{\mu}_{\nu})]$$

of the operators (7.9). If we consider $\dot{K}_t(\mathbf{x})$ as one of the four entries $\dot{K}_t(\mathbf{x}^{\mu}_{\nu}) = K_t(x)^{\mu}_{\nu}$ in the triangular operator kernel $\mathbf{K}_t(x)$ with $K_t(x)^-_- = K_{t(x)} = K_t(x)^+_+$, we define the triangular functions

$$\mathbf{T}(x) = \epsilon(\mathbf{K}_{t(x)}(x)),\qquad \mathbf{G}(x) = \epsilon(\mathbf{K}_{t(x)]}(x)).$$

This allows us to obtain the quantum non-adapted Itô formula in the form

$$T_tT_t^* - T_0T_0^* = i_0^t(\mathbf{T}\mathbf{D}^{\dagger} + \mathbf{D}\mathbf{T}^{\dagger} + \mathbf{D}\mathbf{D}^{\dagger}),$$

where $\mathbf{D}(x) = \mathbf{G}(x) - \mathbf{T}(x)$. This is a consequence of the fact that the map (7.2) is a $\star$-homomorphism, $T_tT_t^* = \epsilon(K\cdot K^{\star})$, and the formula (3.9) of Chapter I for the product of the operator kernels K_t and $K_t^{\star}$, which can be written in the form

$$(K_t\cdot K_t^{\star})(\boldsymbol{\omega}\sqcup\mathbf{x}^{\mu}_{\nu}) = \sum_{\lambda=\mu}^{\nu}[K_t(x)^{\mu}_{\lambda}\cdot K_t^{\star}(x)^{\lambda}_{\nu}](\boldsymbol{\omega}) = [\mathbf{K}_t(x)\mathbf{K}_t^{\dagger}(x)]^{\mu}_{\nu}(\boldsymbol{\omega}),$$

where the right-hand side is computed as an entry in the product of triangular matrices $\mathbf{K}(x) = [K^{\mu}_{\nu}(x)]$ which defines the multiplication of the entries as operator-valued kernels $K_t(x,\boldsymbol{\omega})^-_- = K_t(\boldsymbol{\omega}) = K_t(x,\boldsymbol{\omega})^+_+$, $\dot{K}(\mathbf{x},\boldsymbol{\omega}) = K(\boldsymbol{\omega}\sqcup\mathbf{x})$. For from (3.9) of Chapter I we obtain

$$\begin{aligned}
[K\cdot K^{\star}](\boldsymbol{\omega}\sqcup\mathbf{x}^{\circ}_{\circ}) &= [\dot{K}(\mathbf{x}^{\circ}_{\circ})\cdot\dot{K}^{\star}(\mathbf{x}^{\circ}_{\circ})](\boldsymbol{\omega}),\\
[K\cdot K^{\star}](\boldsymbol{\omega}\sqcup\mathbf{x}^{\circ}_{+}) &= [K\cdot\dot{K}^{\star}(\mathbf{x}^{-}_{\circ}) + \dot{K}(\mathbf{x}^{-}_{\circ})\dot{K}^{\star}(\mathbf{x}^{\circ}_{\circ})](\boldsymbol{\omega}),\\
[K\cdot K^{\star}](\boldsymbol{\omega}\sqcup\mathbf{x}^{-}_{\circ}) &= [\dot{K}(\mathbf{x}^{\circ}_{\circ})\dot{K}^{\star}(\mathbf{x}^{\circ}_{+}) + \dot{K}(\mathbf{x}^{\circ}_{+})\cdot K^{\star}](\boldsymbol{\omega}),\\
[K\cdot K^{\star}](\boldsymbol{\omega}\sqcup\mathbf{x}^{-}_{+}) &= [K\cdot\dot{K}^{\star}(\mathbf{x}^{-}_{+}) + \dot{K}(\mathbf{x}^{-}_{\circ})\cdot\dot{K}^{\star}(\mathbf{x}^{\circ}_{+}) + \dot{K}(\mathbf{x}^{-}_{+})\cdot K^{\star}](\boldsymbol{\omega}),
\end{aligned}$$

which are the matrix elements of

$$[K\cdot K^{\star}](\boldsymbol{\omega}\sqcup\mathbf{x}) = \left[\dot{K}(x)^{\mu}_{\lambda}\cdot\dot{K}_t^{\star}(x)^{\lambda}_{\nu}\right](\boldsymbol{\omega}) = \left(\mathbf{K}\cdot\mathbf{K}^{\dagger}\right)(x,\boldsymbol{\omega}).$$

This allows us to write $\epsilon[(\dot{K}_t \cdot \dot{K}_t^\star)(x_\nu^\mu)] = \sum_{\lambda=\mu}^{\nu} \epsilon(\dot{K}_t(x)_\lambda^\mu \dot{K}_t^\star(x)_\nu^\lambda)$ in the form of the triangular operator

$$\epsilon(\mathbf{K}_t(x)\mathbf{K}_t^\dagger(x)) = \epsilon(\mathbf{K}_t(x))\epsilon(\mathbf{K}_t(x))^*$$

which is the product of the triangular matrices $\mathbf{T}_t(x)$ and $\mathbf{T}_t^\dagger(x)$ with operator product of the entries. We put $t = t(x)$ and $t = t_+(x)$ in the formula, and we obtain

$$\epsilon[(\mathbf{K}_{t(x)]} \cdot \mathbf{K}_{t(x)]}^\dagger)(x) - (\mathbf{K}_{t(x)} \cdot \mathbf{K}_{t(x)}^\dagger)(x)] = \mathbf{G}(x)\mathbf{G}^\dagger(x) - \mathbf{T}(x)\mathbf{T}^\dagger(x),$$

which allows us to write the stochastic derivative of the quantum non-adapted process $T_t T_t^*$ in the form

$$\mathrm{d}(T_t T_t^*) = \mathrm{d}i_0^t(\mathbf{G}\mathbf{G}^\dagger - \mathbf{T}\mathbf{T}^\dagger),$$

corresponding to (7.10). The theorem has been proved. □

Remark 5. *Using the non-adapted table of stochastic multiplication*

$$\begin{aligned}\mathbf{G}^\dagger\mathbf{G} - \mathbf{T}^\dagger\mathbf{T} &= \mathbf{D}^\dagger\mathbf{T} + \mathbf{T}^\dagger\mathbf{D} + \mathbf{D}^\dagger\mathbf{D} \\ &= \begin{bmatrix} 0, & T^* D_\circ^-, & T^* D_+^- + D_+^{-*} T \\ 0, & 0, & D_\circ^{-*} T \\ 0, & 0, & 0 \end{bmatrix} + \begin{bmatrix} 0, & D_+^{\circ *} D_\circ^\circ, & D_+^{\circ *} D_+^\circ \\ 0, & D_\circ^{\circ *} D_\circ^\circ, & D_\circ^{\circ *} D_+^\circ \\ 0, & 0, & 0 \end{bmatrix} \\ &\quad + \begin{bmatrix} 0, & D_+^{\circ *} T_\circ^\circ + T_+^{\circ *} D_\circ^\circ, & D_+^{\circ *} T_+^\circ + T_+^{\circ *} D_+^\circ \\ 0, & D_\circ^{\circ *} T_\circ^\circ + T_\circ^{\circ *} D_\circ^\circ, & D_\circ^{\circ *} T_+^\circ + T_\circ^{\circ *} D_+^\circ \\ 0, & 0, & 0 \end{bmatrix}\end{aligned}$$

we can write (7.10) *in a weak form*

$$\begin{aligned}\|T_t \mathrm{h}\|^2 - \|T_0 \mathrm{h}\|^2 &= \int_{X^t} 2\mathrm{Re}\left\langle T_{t(x)}\mathrm{h} \mid D_+^-(x)\mathrm{h} + D_\circ^-(x)\dot{\mathrm{h}}(x)\right\rangle \mathrm{d}x \\ &+ \int_{X^t}\left[\left\|D_+^\circ(x)\dot{\mathrm{h}} + D_\circ^\circ(x)\dot{\mathrm{h}}(x)\right\|^2 + 2\mathrm{Re}\left\langle \nabla_x T_{t(x)}\mathrm{h} \mid D_+^\circ(x)\dot{\mathrm{h}} + D_\circ^\circ(x)\dot{\mathrm{h}}(x)\right\rangle\right]\mathrm{d}x,\end{aligned} \tag{7.11}$$

where $\nabla_x T_{t(x)}\mathrm{h} = T_+^\circ(x)\mathrm{h} + T_\circ^\circ(x)\dot{\mathrm{h}}(x)$. *This formula is valid for any non-adapted single integral* $T_t = T_0 + i_0^t(\mathbf{D})$ *with square integrable values* $T_t\mathrm{h}$ *for all* $\mathrm{h} \in \mathrm{G}_+$ *if we understand by* ∇_x *the Fock space representation of the Malliavin derivative* $[\nabla_x T_{t(x)}\mathrm{h}](\varkappa) = [T_{t(x)}\mathrm{h}](\varkappa \sqcup x)$ *at the point* $x \in X$.

Indeed, taking into account that

$$\left\langle \mathrm{f} \mid i_0^t(\mathbf{D})\mathrm{h}\right\rangle = \int_{X^t}[\left\langle \mathrm{f} \mid D_+^-(x)\mathrm{h} + D_\circ^-\dot{\mathrm{h}}(x)\right\rangle + \left\langle \dot{\mathrm{f}}(x) \mid D_+^\circ(x)\mathrm{h} + D_\circ^\circ(x)\dot{\mathrm{h}}(x)\right\rangle]\mathrm{d}x,$$

we readily obtain the weak form of the non-adapted Itô formula if we substitute $\mathbf{D}^\dagger\mathbf{T} + \mathbf{D}^\dagger\mathbf{D} + \mathbf{T}^\dagger\mathbf{D}$ in place of $\mathbf{D}$. This formula can also be obtained by a direct computation

$$\left\|i_0^t(\mathbf{D})\mathrm{h}\right\|^2 + 2\mathrm{Re}\left\langle i_0^t(\mathbf{D})\mathrm{h} \mid T_0\mathrm{h}\right\rangle = \|T_t\mathrm{h}\|^2 - \|T_0\mathrm{h}\|^2$$

without assuming that the family T_t is defined by the kernels (7.8) which represent it in the form of the multiple stochastic integral (6.7) of $B = \epsilon(M)$. For we compute the square of the norm of the full single integral

$$\begin{aligned}[i_0^t(\mathbf{D})\mathrm{h}](\varkappa) &= \int_{X^t}[D_+^-(x)\mathrm{h} + D_\circ^-(x)\dot{\mathrm{h}}(x)](\varkappa)\mathrm{d}x \\ &+ \sum_{x\in\varkappa^t}[D_+^\circ(x)\mathrm{h} + D_\circ^\circ(x)\dot{\mathrm{h}}(x)](\varkappa\setminus x),\end{aligned}$$

and we obtain $\|i_0^t(\mathbf{D}\mathrm{h}\|^2 = \|\int\|^2 + 2\,\mathrm{Re}\,\langle\sum \mid \int\rangle + \|\sum\|^2$, where

$$\begin{aligned}\left\|\int\right\|^2 &= \int_{X^t}\int_{X^t}\left\langle D_+^-(z)\mathrm{h} + D_+^-(z)\dot{\mathrm{h}}(z) \mid D_+^-(x)\mathrm{h} + D_\circ^-(x)\dot{\mathrm{h}}(x)\right\rangle \mathrm{d}x\,\mathrm{d}z \\ &= \int_{X^t} 2\mathrm{Re}\left\langle\int_{X^{t(x)}}\left[D_+^-(z)\mathrm{h} + D_\circ^-(z)\dot{\mathrm{h}}(z)\mathrm{d}z\right] \mid D_+^-(x)\mathrm{h} + D_\circ^-(x)\dot{\mathrm{h}}(x)\right\rangle \mathrm{d}x,\end{aligned}$$

the cross-term can be written as $2\,\mathrm{Re}\,\langle\sum \mid \int\rangle =$

$$\begin{aligned}&2\,\mathrm{Re}\int\left\langle\sum_{z\in\varkappa^t}[D_+^\circ(z)\mathrm{h} + D_\circ^\circ(z)\dot{\mathrm{h}}(z)](\varkappa\setminus z) \mid \int_{X^t}[D_+^-(x)\mathrm{h} + D_\circ^-(x)\dot{\mathrm{h}}(x)](\varkappa)\mathrm{d}x\right\rangle \mathrm{d}\varkappa \\ &= \int_{X^t} 2\,\mathrm{Re}\int\left\langle\sum_{z\in\varkappa^{t(x)}}[D_+^\circ(z)\mathrm{h} + D_\circ^\circ(z)\dot{\mathrm{h}}(z)](\varkappa) \mid D_+^-(x)\mathrm{h} + D_\circ^-(x)\dot{\mathrm{h}}(x)\right\rangle \mathrm{d}\varkappa\,\mathrm{d}x \\ &+ \int_{X^t} 2\,\mathrm{Re}\left\langle\nabla_x\int_{X^{t(x)}}[D_+^-(z)\mathrm{h} + D_\circ^-(z)\dot{\mathrm{h}}(z)]\mathrm{d}s \mid D_+^\circ(x)\mathrm{h} + D_\circ^\circ(x)\dot{\mathrm{h}}(x)\right\rangle \mathrm{d}x\end{aligned}$$

and $\|\sum\|^2 - \int\sum_{x\in\varkappa^t}\left\|[D_+^\circ(x)\mathrm{h} + D_\circ^\circ(x)\dot{\mathrm{h}}(x)](\varkappa\setminus x)\right\|^2 \mathrm{d}\varkappa =$

$$\begin{aligned}&= \left\|\sum\right\|^2 - \int\sum_{x\in\varkappa^t}\left\|[D_+^\circ(x)\mathrm{h} + D_\circ^\circ(x)\dot{\mathrm{h}}(x)](\varkappa\setminus x)\right\|^2 \mathrm{d}\varkappa \\ &= \int\sum_{x,z\in\varkappa^t}^{x\neq z}\left\langle[D_+^\circ(z)\mathrm{h} + D_\circ^\circ(z)\dot{\mathrm{h}}(z)](\varkappa\setminus z) \mid [D_+^\circ(x)\mathrm{h} + D_\circ^\circ(x)\dot{\mathrm{h}}(x)](\varkappa\setminus x)\right\rangle \mathrm{d}\varkappa \\ &= \int_{X^t} 2\,\mathrm{Re}\int\left\langle\nabla_x\sum_{z\in\varkappa^{t(x)}}\mathrm{f}\,(z,\varkappa\setminus z) \mid \mathrm{f}\,(x,\varkappa)\right\rangle \mathrm{d}\varkappa\,\mathrm{d}x,\end{aligned}$$

where $\mathrm{f}\,(x,\varkappa) = [D_+^\circ(x)\mathrm{h} + D_\circ^\circ(x)\dot{\mathrm{h}}(x)]\,(\varkappa)$. Here we have used (6.6) in the form

$$\int\sum_{x\in\varkappa^t}\langle\mathrm{f}(x,\varkappa) \mid \mathrm{h}(x,\varkappa\setminus x)\rangle\,\mathrm{d}\varkappa = \int_{X^t}\int\langle\mathrm{f}(x,\varkappa\sqcup x) \mid \mathrm{h}(x,\varkappa)\rangle\,\mathrm{d}\varkappa\,\mathrm{d}x,$$

which gives the Itô term of the Hudson-Parthasarathy formula for the adapted integrals of the form

$$\int\sum_{x\in\varkappa^t}\left\|[D_+^\circ(x)\mathrm{h} + D_\circ^\circ(x)\dot{\mathrm{h}}(x)](\varkappa\setminus x)\right\|^2 \mathrm{d}\varkappa = \int_{X^t}\left\|D_+^\circ(x)\mathrm{h} + D_\circ^\circ(x)\dot{\mathrm{h}}(x)\right\|^2 \mathrm{d}x,$$

and $[\nabla_x \mathrm{f}(z)](\varkappa) = \mathrm{f}(z, \varkappa \sqcup x)$ is the annihilation operator at $x \in X$. Adding up all three integrals, we obtain

$$\left\| i_0^t(\mathbf{D})\mathrm{h} \right\|^2 = \int_{X^t} 2\,\mathrm{Re}\left\langle (i_0^{t(x)}(\mathbf{D})\mathrm{h} \mid D_+^-(x)\mathrm{h} + D_\circ^-(x)\dot{\mathrm{h}}(x) \right\rangle \mathrm{d}x$$

$$+ \int_{X^t} \left\| D_+^\circ(x)\mathrm{h}(x) + D_\circ^\circ(x)\dot{\mathrm{h}}(x) \right\|^2 \mathrm{d}x$$

$$+ \int_{X^t} 2\ \mathrm{Re}\left\langle \nabla_x i_0^{t(x)}(\mathbf{D})\mathrm{h} \mid D_+^\circ(x)\mathrm{h}(x) + D_\circ^\circ(x)\dot{\mathrm{h}}(x) \right\rangle^2 \mathrm{d}x,$$

which leads to the weak form (7.11) of the non-adapted generalization of the quantum Itô formula for $T_t = T_0 + i_0^t(\mathbf{D})$.

If $T_t = \epsilon(K_t)$ is the representation (7.2) of the kernel (7.6), then obviously

$$[\epsilon(K_t)\mathrm{h}](\varkappa \sqcup x) = [\epsilon(\dot{K}_t(x_+^\circ))\mathrm{h} + \epsilon(\dot{K}(x_\circ^\circ))\dot{\mathrm{h}}(x)](\varkappa),$$

and therefore $\nabla_x T_{t(x)}\mathrm{h} = T_+^\circ(x)\mathrm{h} + T_\circ^\circ \dot{\mathrm{h}}(x)$.

In particular, in the scalar case $\mathrm{K}_x = \mathbb{C}$ for $D_+^- = 0 = D_\circ^\circ$, $D_\circ^-(x) = D(x) = D_+^\circ(x)$ and $T_\circ^\circ(x) = T_{t(x)}$, $T_\circ^-(x) = T_+^\circ(x) \equiv \partial\,(x)\,T$ we obtain

$$\|T_t\mathrm{h}\|^2 - \|T_0\mathrm{h}\|^2 = \int_{X^t} 2\,\mathrm{Re}\left\langle T_{t(x)}\mathrm{h} \mid \mathrm{d}T_{t(x)}\mathrm{h}\right\rangle$$

$$+ \int_{X^t} [\|D(x)\mathrm{h}\|^2 + 2\ \mathrm{Re}\left\langle \partial_x T\mathrm{h} \mid D(x)\mathrm{h}\right\rangle]\mathrm{d}x,$$

where $\partial_x T\mathrm{h} = \nabla_x T_{t(x)}\mathrm{h} - T_{t(x)}\dot{\mathrm{h}}(x) \equiv [\nabla_x, T_{t(x)}]\mathrm{h}$. This gives the Itô formula for the normally-ordered non-adapted integral

$$T_t - T_0 = \int_{X^t} (\Lambda_\circ^+(\mathrm{d}x)D(x) + D(x)\Lambda_-^\circ(\mathrm{d}x)) = \int_{X^t} \mathrm{d}T_{t(x)}$$

with respect to the Wiener stochastic measure $w(\Delta)$, $\Delta \in \mathfrak{F}$, which is represented in F by commuting operators $\widehat{w}(\Delta) = \Lambda_\circ^+(\Delta) + \Lambda_-^\circ(\Delta)$. Consider a particular case when the operators T_0, $D(x)$, and consequently T_t are anticipating functions $T_0(w)$, $D(x, w)$, and $T_t(w)$ of w, that is, $T_0 = T_0(\widehat{w})$, $D(x) = D(x, \widehat{w})$, and $T_t = T_t(\widehat{w})$. Then the operators $T(x) = [\nabla_x, T_{t(x)}] = \epsilon(\dot{K}_{t(x)}(x))$ are defined by the Malliavin derivative $\partial_x T_t(w)|_{t=t(x)}$ as the Wiener representation of the pointwise derivative $\dot{K}_{t(x)}(x, \varkappa) = K_{t(x)}(x \cup \varkappa)$ of operator-valued kernels in the multiple stochastic integral $T_t(w) = \int K_t(\varkappa)w(\mathrm{d}\varkappa) = I(K_t)$. In this particular case (7.11) was recently obtained by Nualart in [42].

We note that in the adapted case we always have $T_\circ^\circ(x) = T_{t(x)} \otimes I(x)$ and $T_\nu^\mu(x) = 0$ for $\mu \neq \nu$ except, possibly, $T_+^-(x) = \epsilon(K_+^-(x))$. Hence we readily obtain the following result.

Corollary 2. *The quantum stochastic process $T_t = \epsilon(K_t)$ is adapted if and only if the kernel process K_t is adapted in the sense that*

$$K_t(\sigma, \upsilon, \tau) = \int K_t \begin{pmatrix} \omega, & \tau \\ \sigma, & \upsilon \end{pmatrix} \mathrm{d}\omega = \delta_\emptyset(\sigma_{[t})I^\otimes(\upsilon_{[t})\delta_\emptyset(\tau_{[t}) \otimes K_t(\sigma^t, \upsilon^\tau, \tau^t),$$

where $\delta_\emptyset(\varkappa) = 1$ if $\varkappa = \emptyset$, $\delta_\emptyset(\varkappa) = 0$ if $\varkappa \neq \emptyset$, $I^\otimes(\varkappa) = \bigotimes_{x\in\varkappa} I(x)$, $\varkappa^t = \varkappa \cap X^t$, $\varkappa_{[t} = \{x \in \varkappa : t(x) \geq t\}$. The quantum-stochastic Itô formula (7.10) *for such* *processes can be written in the strong form*

$$\begin{aligned} T_t^* T_t - T_0^* T_0 &= \int_{X^t} (T_{t(x)}^* \mathrm{d}T(x) + \mathrm{d}T^*(x) T_{t(x)} + \mathrm{d}T^*(x)\mathrm{d}T(x)) \\ &= i_0^t (\mathbf{G}^\dagger \mathbf{G} - T^* T \otimes 1), \end{aligned}$$

where

$$\mathrm{d}T(x) = \Lambda(\mathbf{D}, \mathrm{d}x),\ \mathrm{d}T^*(x) = \Lambda(\mathbf{D}^\dagger, \mathrm{d}x),$$
$$\mathrm{d}T^*(x)\mathrm{d}T(x) = \Lambda(\mathbf{D}^\dagger \mathbf{D}, \mathrm{d}x), \quad 1(x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & I(x) & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and in the weak form as (7.11)*, where* $\nabla_x T_{t(x)} \mathrm{h} = [T_{t(x)} \otimes I(x)]\dot{\mathrm{h}}(x)$,

8. Non-stationary quantum evolutions and chronological products

We have proved the continuity of the $*$-representation ϵ of an inductive $\star$-algebra $\mathfrak{B}$ of relatively bounded operator-valued kernels $K(\boldsymbol{\omega})$ in the operator $*$-algebra $\mathfrak{B}(\mathrm{G}_+)$ of the inductive limit $\mathrm{G}_+ = \cap_{p\in\mathcal{P}_1} \mathrm{G}(p)$, and this property allows us to construct the quantum-stochastic functional calculus. Namely, if $K = f(Q_1, \ldots, Q_m)$ is an analytic function of the kernels $Q_i \in \mathfrak{B}$ as the limit of polynomials K_n with fixed ordering of non-commuting $Q_1 \ldots, Q_n$, the limit taken in the sense of $\|K_n - K\|_{\boldsymbol{\alpha}} \to 0$ for (p,q)-admissible quadruple $\boldsymbol{\alpha} = (\alpha_\nu^\mu)$ of positive functions $\alpha_\nu^\mu(x) > 0$, then $T = \epsilon(K)$ is an ordered function $f(Z_1, \ldots, Z_m)$ of operators $Z_i = \epsilon(Q_i)$ as the limit $\|T_n - T\|_q \to 0$ for $q \geq p + 1/r$ of the corresponding polynomials $T_n = \epsilon(K_n)$. The function $T^* = f^\star(Z_1^*, \ldots, Z_m^*)$ with transposed order of action of the operators $Z_i^* = \epsilon(Q_i^\star)$ is also defined as a q-bounded operator $T^* = \epsilon(K^\star)$ in the scale $\{\mathrm{G}(p)\}$ for $K^\star = f^\star(Q_1^\star, \ldots, Q_m^\star)$.

The differential form of this unified QS calculus is given by the non-commutative and non-adapted generalization of the function Itô formula

$$\mathrm{d}Z_t = \mathrm{d}i_0^t(\mathbf{A}) \quad \Rightarrow \quad \mathrm{d}f(Z_t) = \mathrm{d}i_0^t(f(\mathbf{Z} + \mathbf{A}) - f(\mathbf{Z})), \tag{8.1}$$

defined for any analytic function $T_t = f(Z_t)$ of an operator-valued quantum stochastic curve $Z_t = \epsilon(Q_t)$ as the generalized QS-differential of $\epsilon(K_t)$ for $K_t = f(Q_t)$ as soon as this function is well-defined also on the germs $\mathbf{Y}(x) = Z(x) + \mathbf{A}(x)$, $Z(x) = [Z(\mathbf{x}_\nu^\mu)]$ of Z_t as the triangular matrix-functions with the elements $Y(\mathbf{x}) = \boldsymbol{\nabla}_{\mathbf{x}} Z_{t(x)]}$, $Z(\mathbf{x}) = \nabla_{\mathbf{x}} Z_{t(x)}$ for $\mathbf{x} \in \{\mathbf{x}_\nu^\mu\}$. Here

$$T_\nu^\mu(x) = f(\mathbf{Z})_\nu^\mu(x), \quad G_\nu^\mu(x) = f(\mathbf{Z} + \mathbf{A})_\nu^\mu(x),$$

where $f(\mathbf{Z})(x) = f(\mathbf{Z}(x))$ is a triangular matrix which as an analytic function of the triangular matrix

$$\mathbf{Z}(x) = \left[\epsilon\left(\dot{K}(\mathbf{x}_\nu^\mu)\right)\right] = \epsilon(\mathbf{K}(x)), \ \ \dot{K}(\mathbf{x}_\nu^\mu, \boldsymbol{\upsilon}) = K(\boldsymbol{\upsilon} \sqcup \mathbf{x}_\nu^\mu)$$

with the elements representing $\dot{Q}_{t(x)}(\mathbf{x})$ and $\dot{Q}_{t(x)]}(\mathbf{x})$, respectively, as

$$\begin{aligned} \mathbf{Z}(x) &= \left[\epsilon(\dot{Q}_{t(x)}(\mathbf{x}_\nu^\mu))\right], \ \ \mathbf{Y}(\mathbf{x}) = [Z_\nu^\mu(x) + A_\nu^\mu(x)], \\ A_\nu^\mu(x) &= \epsilon\left[\dot{Q}_{t(x)]}(\mathbf{x}_\nu^\mu) - \dot{Q}_{t(x)}(\mathbf{x}_\nu^\mu)\right]. \end{aligned}$$

For an ordered function $T_t = f(Z_{1t}, \dots, Z_{mt})$ this can be written in terms of Z_{it}, with the differential $\mathrm{d}Z_{it} = \mathrm{d}i_0^t(A_i)$, and $\mathbf{Y}_i = \mathbf{Z}_i + \mathbf{A}_i$ as

$$\mathrm{d}T_t = \mathrm{d}i_0^t(f(\mathbf{Y}_1, \dots, \mathbf{Y}_m) - f(\mathbf{Z}_1, \dots, \mathbf{Z}_m)).$$

In particular, if all triangular operator-matrices $\{\mathbf{Y}_i, \mathbf{Z}_i\}$ commute, then we can obtain the exponential function $T_t = \exp\{Z_t\}$ for $Z_t = \sum_{i=1}^m Z_{it}$ as a solution of the following quantum-stochastic non-adapted differential equation:

$$\mathrm{d}T_t = \mathrm{d}i_0^t[\mathbf{T}(\mathbf{S} - \widehat{\mathbf{1}})],\ T_0 = I, \tag{8.2}$$

where $\mathbf{S} = \exp\big\{\sum_{i=1}^m \mathbf{A}_i\big\}$, $\mathbf{T} = \exp\big\{\sum_{i=1}^m \mathbf{Z}_i\big\}$ corresponding to

$$\mathbf{TS} = \exp\Big\{\sum_{i=1}^m \mathbf{Y}_i\Big\}.$$

We shall now deal with the problem of solving a general quantum-stochastic equation

$$T_t = T_0^t + i_0^t(\mathbf{T}\mathbf{A}^t) \tag{8.3}$$

of type (8.2) corresponding to the integral equation $T_t = I + i_0^t(\mathbf{TA})$ with $T_0^t = I$ and $\mathbf{A}^t(x) = \mathbf{S}(x) - \hat{1}(x)$ independent of t. In general T_0^t is given as a nonadapted function of $t \in \mathbb{R}_+$ with values in continuous operators $\mathrm{G}_+ \to \mathrm{G}_-$, and $\mathbf{A}^t(x) = [A^t(x)^\mu_\nu]$ is a triangular matrix-function of $x \in X$, where $A^t(x)^\mu_\nu = 0$ for $\mu = +$ or $\nu = -$ and the non-zero values are continuous operators

$$A^-_+(x) : \mathrm{G}_+ \to \mathrm{G}_-, \quad A^\circ_\circ(x) : \mathrm{G}_+ \otimes \mathrm{K}_x \to \mathrm{G}_- \otimes \mathrm{K}_x,$$
$$A^\circ_+(x) : \mathrm{G}_+ \to \mathrm{G}_- \otimes \mathrm{K}_x, \quad A^-_\circ(x) : \mathrm{G}_+ \otimes \mathrm{K}_x \to \mathrm{G}_-,$$

for example $T_0^t = T_0 U_0^t$, $\mathbf{A}^t(x) = \mathbf{A}(x)(U^t_{t(x)} \otimes \mathbf{I}(x))$, where $\{U_s^t : t > s \in \mathbb{R}_+\}$ is a given two-parameter family of evolution operators on G_+. First of all we prove the following lemma.

Lemma 3. *Suppose that the operator-functions*

$$T_0^t = \epsilon(K_0^t), \quad A^t(x)^\mu_\nu = \epsilon(L^t(\mathbf{x}^\mu_\nu))^{\mu=-,\circ}_{\nu=\circ,+}$$

are the representations (7.2) *of the kernel functions* $K_0^t(\omega)$, $L^t(\mathbf{x}^\mu_\nu, \boldsymbol{\upsilon})$, *where* $\boldsymbol{\omega} = (\omega^\mu_\nu), \omega^\mu_\nu \in \mathcal{X}, \boldsymbol{\upsilon} = (\upsilon^\mu_\nu), \upsilon^\mu_\nu \in \mathcal{X}$, *and* $\mathbf{x}^\mu_\nu$ *are the atomic tables* (6.9). *Then the integral equation* (8.3) *is the operator representation* $T_t = \epsilon(K_t)$ *of a triangular system of recurrence equations*

$$K_t(\boldsymbol{\omega}) = K_0^t(\boldsymbol{\omega}) + \sum_{x \in \boldsymbol{\omega}^t} [K_{t(x)} \cdot L_x^t](\boldsymbol{\omega}), \tag{8.4}$$

where the kernel-operators $L_x^t(\boldsymbol{\omega})$ *are defined almost everywhere (for pairwise disjoint* $(\omega^\mu_\nu)^{\mu=-,\circ}_{\nu=+,\circ}$*) as*

$$L_x^t(\boldsymbol{\upsilon} \sqcup \mathbf{x}^\mu_\nu) := L^t\,(\mathbf{x}^\mu_\nu, \boldsymbol{\upsilon}) \equiv L^t\,(x, \boldsymbol{\upsilon})^\mu_\nu$$

by the matrix elements of $\mathbf{L}^t(x, \boldsymbol{\upsilon})$, *with* $L_x^t(\boldsymbol{\omega}) = 0$ *if* $x \notin \sqcup\omega^\mu_\nu$, *and* $K_{t(x)} \cdot L_x^t$ *is the kernel product. The solution of* (8.4) *is uniquely defined almost everywhere (if* $t(x) \neq t(x')$ *for all* $x \neq x' \in \sqcup\omega^\mu_\nu$*) as the sum*

$$K_t(\boldsymbol{\omega}) = \sum_{\boldsymbol{\vartheta} \subseteq \boldsymbol{\omega}^t} M_t(\boldsymbol{\vartheta}, \boldsymbol{\omega} \setminus \boldsymbol{\vartheta}) = \nu_0^t(\boldsymbol{\omega}, M_t)$$

of chronological kernel products

$$M_t(\boldsymbol{\vartheta},\boldsymbol{\upsilon}) = [K_0^{t(x_1)} \cdot L_{x_1}^{t(x_2)} \cdot \ldots \cdot L_{x_{m-1}}^{t(x_m)} \cdot L_{x_m}^t](\boldsymbol{\vartheta} \sqcup \boldsymbol{\upsilon}) \tag{8.5}$$

over all decompositions $\vartheta = \mathbf{x}_1 \sqcup \ldots \sqcup \mathbf{x}_m$ *of the tables* $\vartheta = (\vartheta_\nu^\mu)$ *into atomic tables* $\mathbf{x}_i$*, each of the form* (6.9)*, with the correspondence* $x_i \in \vartheta_\nu^\mu \Leftrightarrow \mathbf{x}_i = \mathbf{x}_{i\nu}^\mu$*. It gives the unique solution* (8.3) *in the form of the generalized multiple integral*

$$T_t = \iota_0^t(B_t), \quad B_t(\boldsymbol{\vartheta}) = \epsilon(M_t(\boldsymbol{\vartheta})),$$

if Fock representation $B_t(\boldsymbol{\vartheta})$ *of the products* (8.5) *satisfies the condition* $\|B_t\|_p^s(r) < \infty$ *for some admissible* $p \in \mathcal{P}_1$ *and* r^{-1}, $s^{-1} \in \mathcal{P}_0$.

Proof. We substitute $T_0^t = \epsilon(K_0^t)$, $\mathbf{A}^t(x) = \epsilon(\mathbf{L}^t(x))$, and $T_t = \epsilon(K_t)$ in (8.3) and we take into account the fact that

$$\mathbf{T}(x)\mathbf{A}^t(x) = \epsilon(\mathbf{K}_{t(x)}(x) \cdot \mathbf{L}^t(x)),$$

where $\mathbf{K}_t(x) = [K_t(x)_\nu^\mu]$ is a triangular matrix, $K_t(x)_\nu^\mu = 0$ if $\mu > \nu$, with the non-zero kernel entries

$$K_t(x,\boldsymbol{\upsilon})_-^- = K_t(\boldsymbol{\upsilon}) = K(x,\boldsymbol{\upsilon})_+^+, \;\; K_t(x,\boldsymbol{\upsilon})_\nu^\mu = K_t(\boldsymbol{\upsilon} \sqcup \mathbf{x}_\nu^\mu),$$

and $\mathbf{L}^t(x) = [L^t(x)_\nu^\mu]$, where $L^t(x)_\nu^\mu = 0$ if $\mu > \nu$ and the entries

$$L^t(x,\boldsymbol{\upsilon})_-^- = 0 = L^t(x,\boldsymbol{\upsilon})_+^+, \quad x \notin \sqcup \upsilon_\nu^\mu, \;\; L^t(x)_\nu^\mu = \dot{L}_x^t(\mathbf{x}_\nu^\mu)$$

are defined by the kernels $L_x^t(\boldsymbol{\omega}) = L^t(\mathbf{x}, \boldsymbol{\omega} \setminus \mathbf{x})$, with $L_x^t(\boldsymbol{\omega}) = 0$ if $x \notin \omega_\nu^\mu$ for all $\mu \neq +, \nu \neq -$, in the same way as the entries $K_t(x,\boldsymbol{\upsilon})_\nu^\mu$ are defined by the kernels $K_t(\boldsymbol{\omega})$. As a result we found that (8.3) is satisfied if

$$K_t(\boldsymbol{\omega}) = K_0^t(\boldsymbol{\omega}) + \sum_{\mathbf{x} \in \boldsymbol{\omega}^t} [\mathbf{K}_{t(x)}(x)\mathbf{L}^t(x)]_{\nu(\mathbf{x})}^{\mu(\mathbf{x})}(\boldsymbol{\omega} \setminus \mathbf{x})$$
$$= K_0^t(\boldsymbol{\omega}) + \sum_{\mu<+}^{\nu>-} \sum_{x \in \omega_\nu^\mu}^{t(x)<t} [K_{t(x)} \cdot L_x^t](\mathbf{x}_\nu^\mu \sqcup \boldsymbol{\omega} \setminus \mathbf{x}_\nu^\mu),$$

which corresponds to (8.4). The solution of this equation for any table $\boldsymbol{\omega} = (\omega_\nu^\mu)_{\nu=\circ,+}^{\mu=-,\circ}$ with chronologically ordered entries is represented as the sum (7.6) of the chronological products (8.5) of the operator-valued kernels $M_t(\emptyset,\boldsymbol{\omega}) = K_0^t(\boldsymbol{\omega})$ and $L_x^t(\boldsymbol{\omega})$, since

$$\begin{aligned} K_t(\boldsymbol{\omega}) &= \sum_{\boldsymbol{\vartheta} \subseteq \boldsymbol{\omega}^t} M_t(\boldsymbol{\vartheta}, \boldsymbol{\omega} \setminus \boldsymbol{\vartheta}) = M_t(\emptyset,\boldsymbol{\omega}) + \sum_{|\boldsymbol{\vartheta}|\geq 1}^{\boldsymbol{\vartheta} \subseteq \boldsymbol{\omega}^t} M_t(\boldsymbol{\vartheta}, \boldsymbol{\omega} \setminus \boldsymbol{\vartheta}) \\ &= M_t(\emptyset,\boldsymbol{\omega}) + \sum_{\mathbf{x} \in \boldsymbol{\omega}^t} \sum_{\boldsymbol{\vartheta} \in \boldsymbol{\omega}^{t(x)}} M_t(\boldsymbol{\vartheta} \sqcup \mathbf{x}, \boldsymbol{\omega} \setminus (\boldsymbol{\vartheta} \sqcup \mathbf{x})) \\ &= K_0^t(\boldsymbol{\omega}) + \sum_{\mathbf{x} \in \boldsymbol{\omega}^t} \sum_{\boldsymbol{\vartheta} \subseteq \boldsymbol{\omega}^{t(x)}} [M_{t(x)} \cdot L_x^t](\boldsymbol{\omega}) = K_0^t(\boldsymbol{\omega}) + \sum_{\mathbf{x} \in \boldsymbol{\omega}^t} [K_{t(x)} \cdot L_x^t](\boldsymbol{\omega}), \end{aligned}$$

where we have used the representation (8.5) in the recurrent form

$$M_t(\boldsymbol{\vartheta} \sqcup \mathbf{x}, \boldsymbol{\upsilon}) = [M_{t(x)}(\mathbf{x}) \cdot L^t(\mathbf{x})]_{\nu(\mathbf{x})}^{\mu(\mathbf{x})}(\boldsymbol{\vartheta} \sqcup \boldsymbol{\upsilon}) = [M_{t(x)} \cdot L_x^t](\mathbf{x} \sqcup \boldsymbol{\vartheta} \sqcup \boldsymbol{\upsilon}).$$

This defines the representation of the solution $T_t = \epsilon(K_t)$ in the form of the non-adapted quantum-stochastic integral (6.7) of $B_t = \epsilon(M_t)$, since by Lemma 2, $\epsilon \circ \nu_0^t = \iota_0^t \circ \epsilon$ if the integrability condition (6.8) is fulfilled. □

Theorem 6. *Suppose that* $U_s^t = \epsilon(V_s^t)$ *is the representation on* G *of the evolution family* $\{V_s^t : t \geq s \in \mathbb{R}_+\}$ *of relatively bounded operator-valued kernels*

$$V_s^t \begin{pmatrix} \omega_+^-, & \omega_\circ^- \\ \omega_+^\circ, & \omega_\circ^\circ \end{pmatrix} : \mathrm{H} \otimes \mathrm{K}^\otimes(\omega_\circ^-) \otimes \mathrm{K}^\otimes(\omega_\circ^\circ) \to \mathrm{H} \otimes \mathrm{K}^\otimes(\omega_\circ^\circ) \otimes \mathrm{K}^\otimes(\omega_+^\circ),$$

satisfying the condition $V_r^s \cdot V_s^t = V_r^t$ *for all* $r < s < t$*, where the representation is considered with respect to the kernel product* (7.10) *of* Chapter I *with the unit* $V_t^t(\boldsymbol{\omega}) = I \otimes \mathbf{I}^\otimes(\boldsymbol{\omega})$*. Suppose that*

$$K_0^t(\boldsymbol{\omega}) = [K_0^s \cdot V_s^t](\boldsymbol{\omega}), \quad L_x^t(\boldsymbol{\omega}) = [L_x^s \cdot V_s^t](\boldsymbol{\omega}), \quad \forall t > s$$

are the kernel products defining the representation (8.4) *of equation* (8.3)*. Then the kernels chronological product*

$$K_t(\boldsymbol{\omega}) = [K_0^{t(x_1)} \cdot F_{x_1}^{t(x_2)} \cdot \ldots \cdot F_{x_{n-1}}^{t(x_n)} \cdot F_{t(x_n)}^t](\boldsymbol{\omega}) \tag{8.6}$$

for $F_x^t(\boldsymbol{\omega}) = L_x^t(\boldsymbol{\omega}) + V_{t(x)}^t(\boldsymbol{\omega})$*,* $\boldsymbol{\omega}^t = \mathbf{x}_1 \sqcup \ldots \sqcup \mathbf{x}_n$*, is a unique solution of the system* (8.4) *for almost all* $\boldsymbol{\omega} = (\omega_\nu^\mu)$ (*if* $t(x) \neq t(x')$ *for all* $x \neq x' \in \sqcup\omega_\nu^\mu$)*. This yields the representation of the solution of* (8.3) *in the form* $T_t = \epsilon(K_t)$ *defined on* G *for each* t *as a relatively bounded operator if the product* (8.6) *satisfies the condition* $\|K_t\|_{\boldsymbol{\alpha}} < \infty$ *with respect to the norm* (7.4) *for the quadruple* $\boldsymbol{\alpha} = (\alpha_\nu^\mu)$ *of functions admissible in the sense of* (7.5) *and equal to zero for* $t(x) > t$*. The operators* T_t *are isometric, that is,* $T_t^* T_t = \hat{I}$ (*unitary:* $T_t^* = T_t^{-1}$) *if and only if the operators* T_0 *and* $U_s^t, t \geq s \geq 0$*, are isometric* (*unitary*)*. Consequently, for all* t *we have* $T_0^t = T_0 U_0^t$ *and the triangular operator-matrices* $\mathbf{S}(x) = [S_\nu^\mu(x)]$ *that define the generators of the equation* (8.3) *in the form* $\mathbf{A}^t(x) = (\mathbf{S}(x) - \hat{\mathbf{1}})(U_{t(x)}^t \otimes \mathbf{1}(x))$ *are pseudo-isometric, that is,* $\mathbf{S}^\dagger(x)\mathbf{S}(x) = \hat{I} \otimes \mathbf{1}(x)$ (*pseudo-unitary:* $\mathbf{S}^\dagger(x) = \mathbf{S}(x)^{-1}$)*, and such that*

$$\begin{aligned} &S_\circ^\circ(x)^* S_\circ^\circ(x) = \hat{I} \otimes I(x), \ \ S_+^-(x)^* + S_+^\circ(x)^* S_+^\circ(x) + S_+^-(x) = 0, \\ &S_\circ^-(x)^* + S_\circ^\circ(x)^* S_+^\circ(x) = 0, \qquad S_+^\circ(x)^* S_\circ^\circ(x) + S_\circ^-(x) = 0 \end{aligned} \tag{8.7}$$

(*and* $S_\circ^\circ(x)$ *are unitary, that is,* $S_\circ^\circ(x)^* = S_\circ^\circ(x)^{-1}$) *for almost all* $x \in X^t$*.*

Proof. Suppose that $\boldsymbol{\upsilon} = \boldsymbol{\upsilon}_0 \sqcup \boldsymbol{\upsilon}_1 \sqcup \ldots \sqcup \boldsymbol{\upsilon}_m$ is a decomposition of the table $\boldsymbol{\upsilon} = (\upsilon_\nu^\mu) = \boldsymbol{\omega} \backslash \vartheta$ into the subtables $\boldsymbol{\upsilon}_i = \mathbf{x}_i^1 \sqcup \ldots \sqcup \mathbf{x}_i^{n_i}$ determined by the points $x_i \in X^t$ of the atomic tables $\mathbf{x}_i$ in the chronological decomposition $\vartheta = \mathbf{x}_1 \sqcup \ldots \sqcup \mathbf{x}_m$, so that $t(x_i) < t(x_i^1) < \cdots < t(x_i^{n_i}) < t(x_{i+1}), t(x_0) = 0$. Then

$$\begin{aligned} K_0^{t(x_1)} &= K_0^{t(x_0^1)} \cdot V_{t(x_0^1)}^{t(x_0^2)} \ldots V_{t(x_0^{n_i})}^{t(x_1)}, L_{x_i}^{t(x_{i+1})} = L_{x_i}^{t(x_i^1)} \cdot V_{t(x_i^1)}^{t(x_i^2)} \ldots V_{t(x_i^n i)}^{t(x_{i+1})}, \\ K_t(\boldsymbol{\omega}) &= \sum_{\boldsymbol{\vartheta} \subseteq \boldsymbol{\omega}^t} [K_0^{t(x_1)} \cdot L_{x_1}^{t(x_2)} \ldots L_{m-1}^{t(x_m)} \cdot L_{x_m}^t](\boldsymbol{\vartheta} \sqcup \boldsymbol{\upsilon}) \\ &= [K_0^{t(z_1)} \cdot (V_{t(z_1)}^{t(z_1)} + L_{z_1}^{t(z_2)}) \ldots (V_{t(z_n)}^t + L_{z_n}^t)](\boldsymbol{\omega}), \end{aligned}$$

where the points $z_1, \ldots, z_n \in X^t$, $t(z_1) < \ldots < t(z_n)$ define the decomposition $\boldsymbol{\omega} = \sqcup \mathbf{z}_i$ into atomic tables (6.9). Thus the chronological products (8.6) of the kernels $F_z^t = L_z^t + V_{t(z)}^t$ defines a unique solution of the system (8.4), which is a pseudo-isometric (pseudo-unitary) kernel if and only if the same is true for each factor $K_0^{t(z_1)}, F_{z_1}^{t(z_2)}, \ldots, F_{z_n}^t$. If, in addition, the kernel $K_t(\boldsymbol{\omega})$ is locally bounded for each t relative to the quadruple $\boldsymbol{\alpha} = (\alpha^\mu_\nu)$ of positive functions $\alpha^\mu_\nu(x)$ locally integrable in the sense

$$\int_{X^t} \alpha^-_+(x)\mathrm{d}x < \infty, \ \int_{X^t} (\alpha^\circ_+(x)^2 + \alpha^-_\circ(x)^2) r(x)\mathrm{d}x < \infty, \ \operatorname*{ess\,sup}_{x\in X^t} \frac{\alpha^\circ_\circ(x)}{p(x)} < \infty,$$

then, in accordance with Theorem 5, the representation (7.2) defines the map $\epsilon : K_t \to T_t$ as $\star$-homomorphism in the $*$-algebra of q-bounded, $q \geq p + 1/r$, operators on G_+ satisfying the exponential estimate (7.6). Moreover, T_t is an isometry (a unitary operator) if the kernel K_t is pseudo-symmetric, that is, $K^\star_t \cdot K_t = I \otimes 1^\otimes$ (pseudo-unitary, that is, $K^\star_t = K^{-1}_t$), with respect to the kernel product (7.10) of Chapter I and the pseudo-involution $K_t \mapsto K^\star_t$. For any chronologically ordered collection $\boldsymbol{\omega} = (\omega^\mu_\nu)$ this is guaranteed by the corresponding properties of the kernels $K_0, V^t_s, s \leq t$ and F_z (for almost all $z \in X^t$) by virtue of the representation of (8.6) in the form of a finite product of the kernels $K_0 = K^0_0, V^{t(x_1)}_0$, and $F_z = F^{t(z)}_z, V^t_{t(z)}, z \in \omega, t > t(z)$. Hence the kernel-matrices $\mathbf{F}(x) = [F^\mu_\nu(x)]$, with the entries

$$F^\mu_\nu(x) = 0, \mu > \nu, \ \ F^-_-(x) = I = F^+_+(x), \ \ F^\mu_\nu(x) = \dot{F}_x(\mathbf{x}^\mu_\nu),$$

are pseudo-isometric (pseudo-unitary). This implies that the operators $T_0 = \epsilon(K_0)$ and $U^t_s = \epsilon(V^t_s)$ are isometric (unitary) and the triangular matrix $\mathbf{S}(x) = [\epsilon(F^\mu_\nu(x))]$ (where $S^\mu_\nu(x) = 0$ if $\mu > \nu$ $S^-_-(x) = I = S^+_+(x)$ and $S^\mu_\nu(x) = \epsilon(\dot{F}(x^\mu_\nu))$ if $\mu \neq +, \nu \neq -$), defining the generator $A(x) \equiv A^{t(x)}(x)$ as $\mathbf{S}(x) - I \otimes \mathbf{1}(x)$, is pseudo-isometric (pseudo-unitary).

By virtue of the uniqueness of the representation $T_0 = \epsilon(K_0), U^t_s = \epsilon(V^t_s)$, and $\mathbf{S}(x) = \epsilon(\mathbf{F}(x))$ up to the $\star$-ideal described in Section 2 of Chapter I, the resulting conditions are necessary and sufficient for the solution $T_t = \epsilon(K_t)$ of the non-adapted quantum-stochastic equation (8.3), uniquely (up to the ideal mentioned above) determined by the pseudo-isometric (pseudo-unitary) kernels (8.6), to be isometric (unitary). Writing the condition $\mathbf{S}^\dagger\mathbf{S} = \hat{I} \otimes \mathbf{1}$ in terms of the matrix entries $S^\mu_\nu(x), S^{\dagger\mu}_{-\nu} = S^{\nu *}_{-\mu}$, we obtain the system (8.7):

$$[\mathbf{S}^\dagger\mathbf{S}](x) = \begin{bmatrix} 1, & S^\circ_+(x)^*, & S^-_+(x)^* \\ 0, & S^\circ_\circ(x)^*, & S^-_\circ(x)^* \\ 0, & 0, & 1 \end{bmatrix} \begin{bmatrix} 1, & S^-_\circ(x), & S^-_+(x) \\ 0, & S^\circ_\circ(x), & S^\circ_+(x) \\ 0, & 0, & 1 \end{bmatrix} = I \otimes \begin{bmatrix} 1, & 0, & 0 \\ 0, & I(x), & 0 \\ 0, & 0, & 1 \end{bmatrix}.$$

Thus Theorem 6 is proved. □

Remark 6. *Suppose that the evolution family $\{U^t_s\}$ is a solution of the non-stochastic non-adapted equation*

$$U^t_s = \hat{I} + \int_{s\leq t(x)<t} U^{t(x)}_s S^-_+(x)\mathrm{d}x, \quad S < t, \tag{8.8}$$

and in the dissipative case $S^-_+(x) + S^-_+(x)^* < 0$ this solution is defined as an adapted family of contractions $U^t_s : \mathrm{G} \to \mathrm{G}, \|U^t_s\| \leq \mathbf{1}$. Then the solution of the differential equation (8.2) *can be written in the form of a purely stochastic quantum multiple integral $T_t = \iota^t_0(B^t)$ satisfying* (8.3) *with $T^t_0 = U^t_0$ and the generators $\mathbf{A}^t(x) = \mathbf{A}(x)(U^t_{t(x)} \otimes \mathbf{1}(x))$, where*

$$A^-_+(x) = 0, A^\circ_+(x) = S^\circ_+(x), A^-_\circ(x) = S^-_\circ(x), A^\circ_\circ(x) = S^\circ_\circ(x) - \hat{I} \otimes I(x).$$

In the case when the operator function $S^-_+(x)$ *is locally absolutely integrable in the sense that* $\int_{X^t} \|S^-_+(x)\| \mathrm{d}x < \infty$ *for all* t, *and if we have*

$$U^t_s = \sum_{n=0}^{\infty} \int\cdots\int_{s<t(x_1)<\cdots<t(x_n)<t} S^-_+(x_1)\dots S^-_+(x_n)\prod_{i=1}^{n}\mathrm{d}x_i = \int_{\mathcal{X}^t_s} S^-_+(\vartheta)\mathrm{d}\vartheta,$$

where $\mathcal{X}^t_s = \{\vartheta \in \mathcal{X} : \vartheta \le [s,t)\}, S^-_+(x_1,\dots,x_n) = S^-_+(x_1)\dots S^-_+(x_n)$, *then this representation can be directly obtained by the integration with respect to* $\omega^-_+ \in \mathcal{X}$ *of the kernel* $K_t(\boldsymbol{\omega}) = [F_{z_1}\dots F_{z_n}](\boldsymbol{\omega})$, *defined for* $\boldsymbol{\omega}^t = \mathbf{z}_1 \sqcup \dots \sqcup \mathbf{z}_n$ *as the chronological product of the kernels* $F_x(\boldsymbol{\omega}) = F^\mu_\nu(x, \boldsymbol{\omega}\backslash\mathbf{x}^\mu_\nu)$ *for all* $\boldsymbol{\omega} = (\omega^\mu_\nu)$ *if* $x \in \omega^\mu_\nu$ *and* $F_x(\boldsymbol{\omega}) = I \otimes \mathbf{1}^\otimes(\boldsymbol{\omega})$ *if* $x \notin \sqcup\omega^\mu_\nu$, *which correspond to the representation* $S^\mu_\nu(x) = \epsilon(F^\mu_\nu(x))$.

For we write the solution of the equation $T_t = \hat{I} + i^t_0(\mathbf{T}(\mathbf{S} - \hat{\mathbf{I}}))$ in the form $T_t = \epsilon(K_t)$, where K_t is the kernel (8.6) with $K^t_0 = I^\otimes$ and $F^t_x = F_x$ independent of t. We denote by $\{z_1,\dots,z_n\}$ the subchain of the chain $\{x_1,\dots x_m\}$ of the decomposition $\boldsymbol{\omega}^t = \mathbf{x}_1 \sqcup \dots \sqcup \mathbf{x}_m$ that corresponds to the elements $z_i \notin \omega^-_+$, and we write the integral of $K_t(\boldsymbol{\omega})$ with respect to $\omega^-_+ \in \mathcal{X}$ in the form of a multiple integral in $\vartheta_i \in \mathcal{X}^{t(z_{i+1})}_{t(z_i)}, i = 0,1,\dots,n$, where $t(z_0) = 0, t(z_{n+1}) = t$, and $z_i \in \mathcal{X}, i = 1,\dots,n$. Then, in accordance with (7.10) of Chapter I we obtain the kernel chronological product

$$K_t(\omega^\circ, \upsilon, \omega_\circ) = [V^{t(z_1)}_0 \cdot F_{z_1} \cdot V^{t(z_2)}_{t(z_1)} \cdots F_{z_n} \cdot V^t_{t(z_n)}](\omega^\circ, \upsilon, \omega_\circ).$$

Here in the square brackets we have the product of integral kernels $F_x(\omega^\circ, \upsilon, \omega_\circ) = \int F_x \begin{pmatrix} \omega & \omega_\circ \\ \omega^\circ & \upsilon \end{pmatrix} \mathrm{d}\omega$ and

$$V^t_s(\omega^\circ, \upsilon, \omega_\circ) = \sum_{n=0}^{\infty} \int\cdots\int_{s\le t(x_1)<\cdots<t(x_n)<t} [F^-_+(x_1)\dots F^-_+(x_n)](\omega^\circ, \upsilon, \omega_\circ)\prod_{i=1}^{n}\mathrm{d}x_i,$$

where $[F^-_+(x)](\omega^\circ, \upsilon, \omega_\circ) = F^-_+ \begin{pmatrix} x, & \omega_\circ \\ \omega^\circ, & \upsilon \end{pmatrix}, x \in X$. On the other hand, we can obtain the same result if we integrate the kernel product

$$K_t(\boldsymbol{\omega}) = [V^{t(z_1)}_0 \cdot F_{z_1} \cdot V^{t(z_2)}_{t(z_1)} \cdots F_{z_n} V^t_{t(z_n)}](\boldsymbol{\omega})$$

with respect to $\omega^-_+ \in \mathcal{X}$, where the kernels $V^t_s(\boldsymbol{\omega}) = [F_{x_1}\dots F_{x_n}](\boldsymbol{\omega})$ (for $X^t_s \cap \omega^-_+ = x_1 \sqcup \dots \sqcup x_n$) define the representation $U^t_s = \epsilon(V^t_s)$ of the solution of (8.8) for $S^-_+(x) = \epsilon(F^-_+(x))$. Putting $F_x(\boldsymbol{\omega}) = I \otimes 1^\otimes(\boldsymbol{\omega})$ for $x \in \omega^-_+ \cap X^t$ and taking into account the consistency condition $V^s_r \cdot V^t_s = V^t_r$, we find the solution of (8.2) as the solution of (8.3) with the generators $A^t(x)^\mu_\nu = \epsilon(L^t(\mathbf{x}^\mu_\nu))$, where

$$L^t(\mathbf{x}^\mu_\nu, \boldsymbol{\upsilon}) = [(F_x - \mathbf{I}^\otimes) \cdot V^t_{t(x)}](\boldsymbol{\upsilon} \sqcup \mathbf{x}^\mu_\nu) = 0 \text{ for } (\mu,\nu) = (-,+).$$

This solution can be written in the form of the quantum-stochastic multiple non-adapted integral (6.7) of $B_t(\boldsymbol{\vartheta}) = \epsilon(M_t(\boldsymbol{\vartheta}))$, where $M_t(\boldsymbol{\vartheta}, \boldsymbol{\upsilon})$ is defined in (8.5) by the kernels $K^t_0 = V^t_0$ and $L^t_x = (F_x - \mathbf{I}^\otimes) \cdot V^t_{t(x)}$. The operator-function $B_t(\boldsymbol{\vartheta})$ is equal to zero if $\vartheta^-_+ \ne \emptyset$, since the product (8.5) is zero for $x_i \in \vartheta^-_+$. From this we can readily obtain the following corollary.

Corollary 3. *Suppose that* $S^\mu_\nu(x) = F^\mu_\nu(x) \otimes \hat{\mathbf{1}}$, *where* $F^-_+(x)$ *are closed dissipative operators such that there exists a consistent family* $\{V^t_s\}$ *of contractions in* H *which allows us to write the solution of* (8.8) *in the form* $U_s^t = V_s^t \otimes \hat{\mathbf{1}}$. *(It is sufficient, for example, to require that* $F_+^-(x)$ *be locally absolutely integrable, that is,* $\int_{X_t} \|F_+^-(x)\| \mathrm{d}x < \infty$ *for all* t.)

Suppose that the operator-functions

$$F_+^\circ(x) : \mathrm{H} \to \mathrm{H} \otimes \mathrm{K}_x, \quad F_\circ^-(x) : \mathrm{H} \otimes \mathrm{K}_x \to \mathrm{H}$$

are locally square integrable in the sense that

$$\|F\|_t^{(2)}(r) = \Big(\int_{X^t} \|F(x)\|^2 r(x) \mathrm{d}x \Big)^{1/2} < \infty$$

and $\|F_\circ^\circ\|_{t,p}^{(\infty)} = \operatorname{ess\,sup}_{x \in X^t} \{\|F_\circ^\circ(x)\|/p(x)\} \leq 1$ *for some* $r^{-1} \in \mathcal{P}_0$ *and* $p \in \mathcal{P}_1$. *Then the solution* $T_t = \iota_0^t(B)$, $B(\boldsymbol{\vartheta}) = M(\boldsymbol{\vartheta}) \otimes \hat{\mathbf{1}}$ *of the quantum-stochastic equation* (8.2) *is uniquely determined for each* $t \geq 0$ *as a relatively bounded operator* $T_t = \epsilon(K_t)$ *representing by means of* (7.9) *the adapted chronological products*

$$K_t(\omega^\circ, \upsilon, \omega_\circ) = V_0^{t(x_1)} \odot F(\mathbf{x}_1) \odot V_{t(x_1)}^{t(x_2)} \odot \ldots \odot F(\mathbf{x}_n) \odot V_{t(x_n)}^t. \tag{8.9}$$

Here $\{x_1, \ldots, x_n\} = (\omega^\circ \sqcup \upsilon \sqcup \omega_\circ) \cap X^t$ *is the chronologically ordered chain* $0 \leq t(x_1) < \cdots < t(x_n) < t$, $\mathbf{x} = \mathbf{x}_+^\circ$ *if* $x \in \omega^\circ$, $\mathbf{x} = \mathbf{x}_\circ^\circ$ *if* $x \in \upsilon$, $\mathbf{x} = \mathbf{x}_\circ^-$ *if* $x \in \omega_\circ$ *are atomic tables* (6.9), $F(\mathbf{x}_\nu^\mu) = F_\nu^\mu(x)$ *is one of the three functions* $F_+^\circ, F_\circ^\circ, F_\circ^-$, *and* $\odot$ *denotes the semi-tensor product defined recurrently by*

$$K(\boldsymbol{\upsilon}) \odot F(\boldsymbol{\vartheta}) = (K(\boldsymbol{\upsilon}) \otimes I^\otimes(\vartheta_\circ^\circ \sqcup \vartheta_+^\circ))(F(\boldsymbol{\vartheta}) \otimes I^\otimes(\upsilon_\circ^- \sqcup \upsilon_\circ^\circ))$$

where $\upsilon_\circ^- = \omega_\circ, \upsilon_\circ^\circ = \upsilon, \upsilon_+^\circ = \omega^\circ, \boldsymbol{\vartheta} = \mathbf{x}_\circ^-, \mathbf{x}_\circ^\circ, \mathbf{x}_+^\circ$, *and* $F(\mathbf{x}_+^-) = V_{t(x)}^t$.

Moreover, the family T_t *is adapted, it can be written as the purely quantum-stochastic integral* (6.7) *of the Maassen-Meyer kernels*

$$M_t(\omega^\circ, \upsilon, \omega_\circ) = V_\circ^{t(x_1)} \odot L(x_1) \odot V_{t(x_1)}^{t(x_2)} \odot \cdots \odot L(x_n) \odot V_{t(x_n)}^t,$$

where $\omega^\circ \sqcup \upsilon \sqcup \omega_\circ = \{x_1, \ldots, x_n\}$, $L(\mathbf{x}_\nu^\mu) = F(\mathbf{x}_\nu^\mu) - I \otimes \delta_\nu^\mu \equiv L_\nu^\mu(x)$, *and the following estimate holds:*

$$\|T_t\|_p(r) \leq \exp\big\{\tfrac{1}{2} \int_{X^t} (\|L_\circ^-(x)\|^2 + \|L_+^\circ(x)\|^2) r(x) \mathrm{d}x\big\}. \tag{8.10}$$

In fact since $\|V_s^t\| \leq 1$, the kernels (8.9) are bounded:

$$\|K_t(\omega^\circ, \upsilon, \omega_\circ) \leq \|F_+^\circ(\omega^\circ)\|_t \|F_\circ^\circ(\upsilon)\| \, \|F_\circ^-(\omega_\circ)\|_t,$$

relative to $\|F(\omega)\|_t = \prod_{x \in \omega^t} \|F(x)\|$. To obtain (8.10) we use (7.6), where we put $\alpha_+^\circ(x) = \|L_+^\circ(x)\|$ and $\alpha_\circ^-(x) = \|L_\circ^-(x)\|$ for $x \in X^t$, $\alpha_\circ^\circ(x) = 0 = \alpha_\circ^-(x)$ for $x \in X^t$, $\alpha_+^\circ(x) = 0 = \alpha_\circ^-(x)$ for $t(x) \geq t$, $\alpha_\circ^\circ(x) = \|F_\circ^\circ(x)\|$ for $x \in X^t$, $\alpha_\circ^\circ(x) = 1$ for $t(x) \geq t$, and $\alpha_+^-(x) = 0$ for all $x \in X$, and now the estimate (8.10) corresponds to $\|T_t\|_{\boldsymbol{\alpha}} = 1$.

Example 3. *We construct the solution of* (8.2) *corresponding to the pseudo-unitary operators* $\mathbf{S}(x) = \mathbf{F}(x) \otimes \hat{\mathbf{1}}$ *with the triangular operators* $\mathbf{F}(x) = e^{\mathrm{i}\mathbf{H}(x)}$, *where* $\mathbf{H}^\dagger(x) = \mathbf{H}(x)$ *are pseudo-selfadjoint operators with the entries* $H_\nu^\mu = 0$ *for* $\mu = +$ *or* $\nu = -$, $H_\circ^-(x)^* = H_+^\circ(x)$, *and* $H_\circ^\circ(x)^* = H_\circ^\circ(x)$. *We assume that the local absolute integrability condition* $\|F_+^-\|_t^{(1)} = \int_{X^t} \|F_+^-(x)\| \mathrm{d}x < \infty$ *is satisfied, which leads, since* $\mathbf{F}$ *is pseudo-unitary, to*

$$\|F_+^\circ\|_t^{(2)} = \Big(\int_{X^t} \|F_+^\circ(x)\|^2 \mathrm{d}x \Big)^{1/2} < \infty, \quad \|F_\circ^-\|_t^{(2)} = \Big(\int_{X^t} \|F_\circ^-(x)\|^2 \mathrm{d}x \Big)^{1/2} < \infty$$

and $\|F^\circ_\circ\|^{(\infty)}_t = \operatorname{ess\,sup}_{x\in X^t}\|F^\circ_\circ(x)\| = 1$. *We can now define the operators* $T_t = \epsilon(K_t)$ *as the representations of the chronologically ordered products* $K_t(\boldsymbol{\omega}) = F(\mathbf{x}_1)\odot\cdots\odot F(\mathbf{x}_n)$ *for* $\sqcup_{i=1}^n \mathbf{x}_i = \boldsymbol{\omega}^t$, *where* $F(\mathbf{x}^\mu_\nu) = F^\mu_\nu(x)$ *are entries in the exponential matrix* $\exp\{i\mathbf{H}(x)\}$. *We compute these entries by induction finding the powers* $\mathbf{H}^0 = \mathbf{I}$, $\mathbf{H}^1 = \mathbf{H}$,

$$\mathbf{H}^2 = \begin{bmatrix} 0, & H^-_\circ H^\circ_\circ, & H^-_\circ H^\circ_+ \\ 0, & H^\circ_\circ H^\circ_\circ, & H^\circ_\circ H^\circ_+ \\ 0, & 0, & 0 \end{bmatrix}, \mathbf{H}^{n+2} = \begin{bmatrix} 0, & H^-_\circ H^{\circ n-1}_\circ, & H^-_\circ H^{\circ n}_\circ H^\circ_+ \\ 0, & H^{\circ n+2}_\circ, & H^{\circ n-1}_\circ H^\circ_+ \\ 0, & 0, & 0 \end{bmatrix}.$$

As a result we obtain $\mathbf{F} = \sum_{n=0}^\infty (\mathrm{i}\mathbf{H}^n/n!$ *as the triangular matrix with*

$$F^\mu_\nu = 0, \mu > \nu, \qquad F^-_- = I = F^+_+,$$
$$F^\circ_\circ = e^{\mathrm{i}H^\circ_\circ}, \quad F^-_+ = H^-_\circ[(e^{\mathrm{i}H^\circ_\circ} - I^\circ_\circ - \mathrm{i}H^\circ_\circ)/H^\circ_\circ H^\circ_\circ]H^\circ_+ + \mathrm{i}H^-_+,$$
$$F^\circ_+ = [(e^{\mathrm{i}H^\circ_\circ} - I^\circ_\circ)/H^\circ_\circ]H^\circ_+, \qquad F^-_\circ = H^-_\circ[(e^{\mathrm{i}H^\circ_\circ} - I^\circ_\circ)/H^\circ_\circ].$$

Substituting the adjoint operators $H^-_\circ, H^\circ_+$ *in the form*

$$H^-_\circ = F^* H^\circ_\circ - \mathrm{i}E^*,\ H^\circ_+ = H^\circ_\circ F + \mathrm{i}E,$$

where the operators $E(x)$ *are uniquely determined by the conditions* $H^\circ_\circ(x)E(x) = 0$, *we can obtain the following canonical decomposition for the operators*

$$L^\mu_\nu(x) = F^\mu_\nu(x) - I\otimes\delta^\mu_\nu I(x)$$

of the unitary quantum-stochastic evolution T_t*:*

$$\begin{pmatrix} L^-_+ & L^-_\circ \\ L^\circ_+ & L^\circ_\circ \end{pmatrix} = \begin{pmatrix} F^* L^\circ_\circ F, & F^* L^\circ_\circ \\ L^\circ_\circ F, & L^\circ_\circ \end{pmatrix} + \begin{pmatrix} \frac12 E^*E, & E^* \\ -E, & 0 \end{pmatrix} + \begin{pmatrix} \mathrm{i}H, & 0 \\ 0, & 0 \end{pmatrix},$$

where $H = H^-_+ - F^* H^\circ_\circ F$, $L^\circ_\circ = \exp\{\mathrm{i}H^\circ_\circ\} - I^\circ_\circ$. *Each of these three tables* $\mathbf{L}_i$, $i = 1, 2, 3$, *corresponds to a pseudo-unitary matrix* $\mathbf{F}_i = \mathbf{I} + \mathbf{L}_i$, *these matrices commute, and we have* $\prod_{i=1}^3 \mathbf{F}_i = \mathbf{I} + \sum_{i=1}^3 \mathbf{L}_i = \mathbf{F}$ *by the orthogonality of* $\mathbf{L}_i$. *The first matrix can be diagonalized by means of the pseudo-unitary transform* $\mathbf{F}_0^\dagger \mathbf{F}_1 \mathbf{F}_0$ *so that*

$$\mathbf{F}_0 = \begin{bmatrix} 1, & F^*, & -K \\ 0, & I, & -F \\ 0, & 0, & 1 \end{bmatrix}, \quad \mathbf{F}_0^\dagger \mathbf{L}_1 \mathbf{F}_0 = \begin{bmatrix} 0, & 0, & 0 \\ 0, & L^\circ_\circ, & 0 \\ 0, & 0, & 0 \end{bmatrix},$$

where $K = F^*F/2$. *This defines the decomposition of the quantum stochastic evolution into three types:*

(1) *Poissonian quantum unitary evolution, which is given by the diagonal matrix* $\mathbf{F}$ *corresponding to* $H^\mu_\nu = 0$ *except* $\mu, \nu = 0$*:*

$$T_t = \epsilon(K_t) = F^\triangleright_{[0,t)}, \quad F^\triangleright_{[0,t)} =: \exp\{\mathrm{i}\int_{X^t} H^\circ_\circ(x)\Lambda^\circ_\circ(\mathrm{d}x)\}:$$

where $[F^\triangleright_{[0,t)}\mathrm{h}](\varkappa) = F^\circ_\circ(x_1)\odot\cdots\odot F^\circ_\circ(x_n)\mathrm{h}(\varkappa)$ *for the chain* $\varkappa^t = \{x_1, \dots, x_n\}$, $t(x_1) < \cdots < t(x_n)$*;*

(2) *Brownian quantum unitary evolution corresponding to* $H^\circ_\circ = 0 = H^-_+$ *and* $\mathrm{i}H^{\circ *}_+ = E = \mathrm{i}H^-_\circ$*;*

(3) *Lebesgue quantum unitary evolution corresponding to* $H^\mu_\nu = 0$ *for all* $(\mu, \nu) \neq (-, +)$*:*

$$T_t = \epsilon(K_t) = \int_{\mathcal{X}^t} \mathrm{i}^{|\varkappa|}\left(\overrightarrow{\prod_{x\in\varkappa}} H^-_+(x)\right)\mathrm{d}\varkappa = \overrightarrow{\exp}\Big\{\mathrm{i}\int_{X^t} H^-_+(x)\mathrm{d}x\Big\}\otimes\hat{\mathbf{1}},$$

where $\overrightarrow{\prod_{x\in\varkappa}} H_{+}^{-}(x) = H_{+}^{-}(x_1)\dots H_{+}^{-}(x_n)$ *for* $\varkappa = \{x_1 < \dots < x_n\}$.

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