INFINITE DIMENSIONAL ITÔ ALGEBRAS OF QUANTUM WHITE NOISE

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Abstract. A simple axiomatic characterization of the general (infinite dimensional, noncommutative) Itô algebra is given and a pseudo-Euclidean fundamental representation for such algebra is described. The notion of Itô B*-algebra, generalizing the C*-algebra is defined to include the Banach infinite dimensional Itô algebras of quantum Brownian and quantum Lévy motion, and the B*-algebras of vacuum and thermal quantum noise are characterized. It is proved that every Itô algebra is canonically decomposed into the orthogonal sum of quantum Brownian (Wiener) algebra and quantum Lévy (Poisson) algebra. In particular, every quantum thermal noise is the orthogonal sum of a quantum Wiener noise and a quantum Poisson noise as it is stated by the Lévy-Khinchin theorem in the classical case.

1. Introduction

The classical differential calculus for the infinitesimal increments $\mathrm{d}x = x(t+\mathrm{d}t) - x(t)$ became generally accepted only after Newton gave a very simple algebraic rule $(\mathrm{d}t)^2 = 0$ for the formal computations of first order differentials for smooth trajectories $t \mapsto x(t)$ in a phase space. The linear space of the differentials $\mathrm{d}x = \alpha \mathrm{d}t$ for a (complex) trajectory became treated at each $x = x(t) \in \mathbb{C}$ as a one-dimensional algebra $\mathfrak{a} = \mathbb{C}\,đ$ of the elements $a = \alpha đ$ with involution $a^\star = \bar{\alpha} đ$ given by the complex conjugation $\alpha \mapsto \bar{\alpha}$ of the derivative $\alpha = \mathrm{d}x/\mathrm{d}t \in \mathbb{C}$ and the nilpotent multiplication $a \cdot a^\star = 0$ corresponding to the multiplication table $đ \cdot đ^\star = 0$ for the basic nilpotent element $đ = đ^\star$, the abstract notation of $\mathrm{d}t$. Note that the nilpotent $\star$-algebra $\mathfrak{a}$ of abstract infinitesimals $\alpha đ$ has no realization in complex numbers, as well as no operator representation $\alpha Đ$ with a Hermitian nilpotent $Đ = Đ^\dagger$ in a Euclidean (complex pre-Hilbert) space, but it can be represented by the algebra of complex nilpotent 2×2 matrices $\hat{a} = \alpha \hat{đ}$, where $\hat{đ} = \frac{1}{2}(\hat{\sigma}_3 + i\hat{\sigma}_1) = \hat{đ}^\dagger$ with respect to the standard Minkowski metric $(\mathrm{x}|\mathrm{x}) = |\zeta|^2 - |\eta|^2$ for $\mathrm{x} = \zeta \mathrm{e}_+ + \eta \mathrm{e}_-$ in $\mathbb{C}^2$. The complex pseudo-Hermitian nilpotent matrix $\hat{đ}$, $\hat{đ}^2 = 0$, representing the multiplication $đ^2 = đ \cdot đ = 0$, has the canonical triangular form

$$(1.1) \qquad Đ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad Đ^* = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

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$$\mathrm{Đ}^{\dagger}=\begin{bmatrix}0&1\\1&0\end{bmatrix}\mathrm{Đ}^{*}\begin{bmatrix}0&1\\1&0\end{bmatrix}=\mathrm{Đ}$$

in the basis $\mathrm{h}_{\pm}=(\mathrm{e}_{+}\pm\mathrm{e}_{-})/\sqrt{2}$ in which $(\mathbf{x}\mathrm{Đ}|\mathbf{x})=\overline{(\mathbf{x}\mathrm{Đ}|\mathbf{x})}$ for all $\mathbf{x}=(\xi_{-},\xi_{+})$ with respect to the pseudo-Euclidean scalar product $(\mathbf{x}|\mathbf{x})=\xi_{-}\xi^{-}+\xi_{+}\xi^{+}$, where $\xi^{\pm}=(\zeta\pm\eta)/\sqrt{2}=\bar{\xi}_{\mp}\in\mathbb{C}$.

The Newton's formal computations can be generalized to non-smooth paths to include the calculus of first order forward differentials $\mathrm{d}y\simeq(\mathrm{d}t)^{1/2}$ of continuous diffusions $y(t)\in\mathbb{R}$ which have no derivative at any t, and the forward differentials $\mathrm{d}n\in\{0,1\}$ of left continuous counting trajectories $n(t)\in\mathbb{Z}_{+}$ which have zero derivative for almost all t (except the points of discontinuity when $\mathrm{d}n=1$). The first is usually done by adding the rules

$$(\mathrm{d}w)^{2}=\mathrm{d}t,\quad \mathrm{d}w\mathrm{d}t=0=\mathrm{d}t\mathrm{d}w \tag{1.2}$$

in formal computations of continuous trajectories having the first order forward differentials $\mathrm{d}x=\alpha\mathrm{d}t+\beta\mathrm{d}w$ with the diffusive part given by the increments of standard Brownian paths $w(t)$. The second can be done by adding the rules

$$(\mathrm{d}m)^{2}=\mathrm{d}m+\mathrm{d}t,\quad \mathrm{d}m\mathrm{d}t=0=\mathrm{d}t\mathrm{d}m \tag{1.3}$$

in formal computations of left continuous and smooth for almost all t trajectories having the forward differentials $\mathrm{d}x=\alpha\mathrm{d}t+\gamma\mathrm{d}m$ with jumping part $\mathrm{d}z\in\{\gamma,-\gamma\mathrm{d}t\}$ given by the increments of standard Lévy paths $m(t)=n(t)-t$. These rules, well known since the beginning of this century, were formalized by Itô [1] into the form of a stochastic calculus: the first one is now known as the multiplication rule for the forward differential of the standard Wiener process $w(t)$, and the second one is the multiplication rule for the forward differential of the standard Poisson process $n(t)$, compensated by its mean value t.

The linear span of $\mathrm{d}t$ and $\mathrm{d}w$ forms a two-dimensional differential Itô algebra $\mathfrak{b}=\mathbb{C}đ+\mathbb{C}d_{w}$ for the complex Brownian motions $x(t)=\int\alpha\mathrm{d}t+\int\eta\mathrm{d}w$, where $d_{w}=d_{w}^{\star}$ is a nilpotent of second order element, representing the real increment $\mathrm{d}w$, with multiplication table $d_{w}^{2}=đ$, $d_{w}\cdot đ=0=đ\cdot d_{w}$, while the linear span of $\mathrm{d}t$ and $\mathrm{d}m$ forms a two-dimensional differential Itô algebra $\mathfrak{c}=\mathbb{C}đ+\mathbb{C}d_{m}$ for the complex Lévy motions $x=\int\alpha\mathrm{d}t+\int\zeta\mathrm{d}m$, where $d_{m}=d_{m}^{\star}$ is a basic element, representing the real increment $\mathrm{d}m$, with multiplication table $d_{m}^{2}=d_{m}+đ$, $d_{m}\cdot đ=0=đ\cdot d_{m}$. As in the case of the Newton algebra, the Itô $\star$-algebras $\mathfrak{b}$ and $\mathfrak{c}$ have no Euclidean operator realization, but they can be represented by the algebras of triangular matrices $\mathrm{B}=\alpha\mathrm{Đ}+\eta\mathrm{D}_{w}$, $\mathrm{C}=\alpha\mathrm{Đ}+\zeta\mathrm{D}_{m}$ with pseudo-Hermitian basis elements

$$\mathrm{Đ}=\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}=\mathrm{Đ}^{\dagger},\quad \mathrm{D}_{w}=\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}=\mathrm{D}_{w}^{\dagger}, \tag{1.4}$$

$$\mathrm{D}_{m}=\begin{bmatrix}0&1&0\\0&1&1\\0&0&0\end{bmatrix}=\mathrm{D}_{m}^{\dagger},$$

where $(\mathbf{x}\mathrm{B}^{\dagger}|\mathbf{x})=\overline{(\mathbf{x}\mathrm{B}|\mathbf{x})}$ for all $\mathbf{x}=(\xi_{-},\xi_{\circ},\xi_{+})\in\mathbb{C}^{3}$ in the complex three-dimensional Minkowski space with respect to the indefinite metric $(\mathbf{x}|\mathbf{x})=\xi_{-}\xi^{-}+\xi_{\circ}\xi^{\circ}+\xi_{+}\xi^{+}$, where $\xi^{\mu}=\bar{\xi}_{-\mu}$ with $-(-,\circ,+)=(+,\circ,-)$.

Note that according to the Lévy-Khinchin theorem, every stochastic process $x(t)$ with independent increments can be canonically decomposed into a smooth, Wiener and Poisson parts as in the mixed case of one-dimensional complex motion $x(t) = \int \alpha \mathrm{d}t + \int \eta \mathrm{d}w + \int \zeta \mathrm{d}m$ given by the orthogonal and thus commutative increments $\mathrm{d}w\mathrm{d}m = 0 = \mathrm{d}m\mathrm{d}w$. In fact such generalized commutative differential calculus applies not only to the stochastic integration with respect to the processes with independent increments; these formal algebraic rules, or their multidimensional versions, can be used for formal computations of forward differentials for any classical trajectories decomposed into the smooth, diffusive and jumping parts.

Two natural questions arrise: are there other then these two commutative differential algebras which could be useful, in particular, for formal computations of the noncommutative differentials in quantum theory, and if there are, is it possible to characterize them by simple axioms and to give a generalized version of the Lévy-Khinchin decomposition theorem? The first question has been already positively answered since the well known differential realization of the simplest non-commutative table $d_w d_w^\star = \rho_+ đ$, $d_w^\star d_w = \rho_- đ$ for $\rho_+ > \rho_- \geq 0$ was given in the mid of 60-th in terms of the annihilators $\hat{w}(t)$ and creators $\hat{w}^\dagger(t)$ of a quantum Brownian thermal noise [2]. This paper gives a systematic answer on the second question, the first part of which has been in principle positively resolved in our papers [3, 4].

Although the orthogonality condition $d_w \cdot d_m = 0 = d_w \cdot d_m$ for the classical independent increments $\mathrm{d}w$ and $\mathrm{d}m$ can be realized only in a higher, at least four, dimensional Minkowski space, it is interesting to make sense of the non-commutative $\star$-algebra, generated by three dimensional non-orthogonal matrix representations (1.4) of these differentials with $d_w \cdot d_m \neq d_w \cdot d_m$:

$$\begin{aligned} \mathrm{D}_w \mathrm{D}_m &= (\mathrm{D}_m \mathrm{D}_w)^\dagger = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &\neq \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = (\mathrm{D}_w \mathrm{D}_m)^\dagger = \mathrm{D}_m \mathrm{D}_w . \end{aligned}$$

This is the four-dimensional $\star$-algebra $\mathfrak{a} = \mathbb{C}Đ + \mathbb{C}\mathrm{E}_- + \mathbb{C}\mathrm{E}^+ + \mathbb{C}\mathrm{E}$ of triangular matrices $\mathrm{A} = \alpha + z^- \mathrm{E}_- + z_+ \mathrm{E}^+ + z\mathrm{E}$, where $\mathrm{E}^+ = \mathrm{G}\mathrm{E}_-^* \mathrm{G} = \mathrm{E}_-^\dagger$, $\mathrm{E} = \mathrm{E}^\dagger$ with respect to the Minkowski metric tensor G in the canonical basis,

$$\mathrm{G} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \mathrm{E}_- = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \mathrm{E}^+ = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \mathrm{E} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

given by the algebraic combinations

$$\mathrm{E}_- = \mathrm{D}_w \mathrm{D}_m - , \mathrm{D}^+ = \mathrm{D}_m \mathrm{D}_w - , \mathrm{E} = \mathrm{D}_m - \mathrm{D}_w$$

of three matrices (1.4). It realizes the multiplication table

$$e_- \cdot e^+ = đ, \quad e_- \cdot e = e_-, \quad e \cdot e^+ = e^+, \quad e^2 = e$$

with the products for all other pairs being zero, unifying the commutative tables (1.2), (1.3). It is well known in the quantum stochastic calculus as the HP (Hudson-Parthasarathy) table [5]

$$\mathrm{d}\Lambda_- \mathrm{d}\Lambda^+ = I\mathrm{d}t, \quad \mathrm{d}\Lambda_- \mathrm{d}\Lambda = \mathrm{d}\Lambda_-, \quad \mathrm{d}\Lambda \mathrm{d}\Lambda^+ = \mathrm{d}\Lambda^+, \quad (\mathrm{d}\Lambda)^2 = \mathrm{d}\Lambda,$$

with zero products for all other pairs, for the multiplication of the canonical number $\mathrm{d}\Lambda$, creation $\mathrm{d}\Lambda^{+}$, annihilation $\mathrm{d}\Lambda_{-}$, and preservation $\mathrm{d}\Lambda_{+}^{-}=I\mathrm{d}t$ differentials in Fock space over the Hilbert space $L^{2}\left(\mathbb{R}_{+}\right)$ of square-integrable complex functions $f\left(t\right), t\in\mathbb{R}_{+}$.

Note that any two-dimensional Itô $\star$-algebra $\mathfrak{a}$ is commutative as $\text{đ}a=0=a\text{đ}$ for any other element $a\neq\text{đ}$ of the basis $\{a,\text{đ}\}$ in $\mathfrak{a}$. Moreover, each such algebra is either of the Wiener or of the Poisson type, as it is either second order nilpotent, or contains a unital one-dimensional subalgebra, as the cases of the subalgebras $\mathfrak{b},\mathfrak{c}$. Other two-dimensional sub-algebras containing đ, are generated by either Wiener $d_{w}=\bar{\xi}e_{-}+\xi e^{+}$ or Poisson $d_{m}=e+\lambda d_{w}$ element with the special case $d_{m}=e$, corresponding to the only non-faithful Itô algebra of the Poisson process with zero intensity $\lambda^{2}=0$. However there is only one three dimensional $\star$-subalgebra of the four-dimensional HP algebra with đ, namely the noncommutative subalgebra of vacuum Brownian motion, generated by the creation e^{+} and annihilation e_{-} differentials. Thus our results on the classification of noncommutative Itô $\star$-algebras will be nontrivial only in the higher dimensions of $\mathfrak{a}$.

The well known Lévy-Khinchin classification of the classical noise can be reformulated in purely algebraic terms as the decomposability of any commutative Itô algebra into Wiener (Brownian) and Poisson (Lévy) orthogonal components. In the general case we shall show that every Itô $\star$-algebra is also decomposable into a quantum Brownian, and a quantum Lévy orthogonal components.

Thus classical stochastic calculus developed by Itô, and its quantum stochastic analog, given by Hudson and Parthasarathy in [5], has been unified in a $\star$-algebraic approach to the operator integration in Fock space [3], in which the classical and quantum calculi become represented as two extreme commutative and noncommutative cases of a generalized Itô calculus.

In the next section we remind the definition of the general Itô algebra and show that every such algebra can be embedded as a $\star$-subalgebra into in general infinite dimensional Hudson-Parthasarathy algebra as it was first proved in [4].

2. Representations of Itô $\star$-algebras

The generalized Itô algebra was defined in [3] as a linear span of the differentials

$$\mathrm{d}\Lambda\left(t,a\right)=\Lambda\left(t+\mathrm{d}t,a\right)-\Lambda\left(t,a\right),\quad a\in\mathfrak{a}$$

for a family $\{\Lambda\left(a\right):a\in\mathfrak{a}\}$ of operator-valued integrators $\Lambda\left(t,a\right)$ on a pre-Hilbert space, satisfying for each $t\in\mathbb{R}_{+}$ the $\star$-semigroup conditions

$$\Lambda\left(t,a^{\star}\right)=\Lambda\left(t,a\right)^{\dagger},\quad \mathrm{d}\Lambda\left(t,a\cdot b\right)=\mathrm{d}\Lambda\left(t,a\right)\mathrm{d}\Lambda\left(t,b\right),\quad \Lambda\left(t,\text{đ}\right)=tI, \tag{2.1}$$

with mean values $\langle\mathrm{d}\Lambda\left(t,a\right)\rangle=l\left(a\right)\mathrm{d}t$ in a given vector state $\langle\cdot\rangle$, absolutely continuous with respect to $\mathrm{d}t$. Here $\Lambda\left(t,a\right)^{\dagger}$ means the Hermitian conjugation of the (unbounded) operator $\Lambda\left(t,a\right)$, which is defined on the pre-Hilbert space for each $t\in\mathbb{R}_{+}$ as the operator $\Lambda\left(t,a^{\star}\right)$,

$$\begin{aligned}&\mathrm{d}\Lambda\left(t,a\right)\mathrm{d}\Lambda\left(t,b\right)\\ =\;&\mathrm{d}\left(\Lambda\left(t,a\right)\Lambda\left(t,b\right)\right)-\mathrm{d}\Lambda\left(t,a\right)\Lambda\left(t,b\right)-\Lambda\left(t,a\right)\mathrm{d}\Lambda\left(t,b\right),\end{aligned}$$

and $\mathrm{d}t$ is embedded into the family of the operator-valued differentials as $\mathrm{d}\Lambda\left(t,\text{đ}\right)$ with the help of a special element $\text{đ}=\text{đ}^{\star}$ of the parametrizing $\star$-semigroup $\mathfrak{a}$.

Assuming that the parametrization is exact such that $\mathrm{d}\Lambda(t,a)=0\Rightarrow a=0$, where $0=a\text{đ}$ for any $a\in\mathfrak{a}$, we can always identify $\mathfrak{a}$ with the linear span,

$$\sum\lambda_i\mathrm{d}\Lambda(t,a_i)=\mathrm{d}\Lambda\left(t,\sum\lambda_i a_i\right),\quad\forall\lambda_i\in\mathbb{C},a_i\in\mathfrak{a},$$

and consider it as a complex associative $\star$-algebra, having the death $\text{đ}\in\mathfrak{a}$, a $\star$-invariant annihilator $\mathfrak{a}\cdot\text{đ}=\{0\}$ corresponding to $\mathrm{d}\Lambda(t,\mathfrak{a})\,\mathrm{d}t=\{0\}$. The derivative l of the differential expectations $a\mapsto l(a)\,\mathrm{d}t$ with respect to the Lebesgue measure $\mathrm{d}t$, called the Itô algebra state, is a linear positive $\star$-functional

$$l:\mathfrak{a}\to\mathbb{C},\quad l(a\cdot a^\star)\geq 0,\quad l(a^\star)=\overline{l(a)},\quad\forall a\in\mathfrak{a},$$

normalized as $l(\text{đ})=1$ correspondingly to the determinism $\langle I\mathrm{d}t\rangle=\mathrm{d}t$ of $\mathrm{d}\Lambda(t,\text{đ})$. We shall identify the Itô algebra $(\mathrm{d}\Lambda(\mathfrak{a}),l\mathrm{d}t)$ and the parametrizing algebra $(\mathfrak{a},l)$ and assume that it is faithful in the sense that the $\star$-ideal

$$\mathfrak{i}=\{b\in\mathfrak{a}:l(b)=l(b\cdot c)=l(a\cdot b)=l(a\cdot b\cdot c)=0\quad\forall a,c\in\mathfrak{a}\}\tag{2.2}$$

is trivial, $\mathfrak{i}=\{0\}$, otherwise $\mathfrak{a}$ should be factorized with respect to this ideal. Note that the associativity of the algebra $\mathfrak{a}$ as well as the possibility of its noncommutativity is inherited from the associativity and noncommutativity of the operator product $\Delta\Lambda(t,a)\,\Delta\Lambda(t,b)$ on the pre-Hilbert space.

Now we can study the representations of the Itô algebra $(\mathfrak{a},l)$. Because any Itô algebra contains the Newton nilpotent subalgebra $(\mathbb{C}\text{đ},l)$, it has no identity and cannnot be realized by operators in a Euclidean space even if it is finite-dimensional $\star$-algebra. Thus we have to consider the operator representations of $\mathfrak{a}$ in a pseudo-Euclidean space, and we shall find such representations in the simplest one, in a complex Minkowski space.

Let $\mathbb{H}$ be a complex pseudo-Euclidean space with respect to a separating indefinite metric $(\mathrm{x}|\mathrm{x})$, and $\mathrm{h}\in\mathbb{H}$ be a non-zero vector. We denote by $\mathcal{B}(\mathbb{H})$ the $\dagger$-algebra of all operators $\mathrm{A}:\mathbb{H}\to\mathbb{H}$ with $\mathrm{A}^\dagger\mathbb{H}\subseteq\mathbb{H}$, where $\mathrm{A}^\dagger$ is defined as the kernel of the Hermitian adjoint sesquilinear form $\left(\mathrm{x}|\mathrm{A}^\dagger\mathrm{x}\right)=\overline{(\mathrm{x}|\mathrm{A}\mathrm{x})}$. A linear map $\mathrm{i}:\mathfrak{a}\to\mathcal{B}(\mathbb{H})$ is a representation of the Itô $\star$-algebra $(\mathfrak{a},l)$ on $(\mathbb{H},\mathrm{h})$ if

$$\mathrm{i}(a^\star)=\mathrm{i}(a)^\dagger,\quad\mathrm{i}(a\cdot b)=\mathrm{i}(a)\,\mathrm{i}(b),\quad(\mathrm{h}|\mathrm{i}(a)\,\mathrm{h})=l(a)\quad\forall a,b\in\mathfrak{a}.\tag{2.3}$$

We can always assume that $(\mathrm{h}|\mathrm{h})=0$, otherwise h should be replaced by the vector $\mathrm{h}_+=\mathrm{h}-\frac{1}{2}(\mathrm{h}|\mathrm{h})\,\mathrm{h}_-$, where $\mathrm{h}_-=\mathrm{i}(\text{đ})\,\mathrm{h}$, inducing the same state

$$(\mathrm{h}_+|\mathrm{i}(a)\,\mathrm{h}_+)\tag{2.4}$$

$$=l(a)-\frac{1}{2}(\mathrm{h}|\mathrm{h})\left(\mathrm{h}|\mathrm{i}\left(\text{đ}a+a\text{đ}-\frac{1}{2}(\mathrm{h}|\mathrm{h})\,\text{đ}a\text{đ}\right)\mathrm{h}\right)\tag{2.5}$$

$$=l(a).\tag{2.6}$$

Proposition 2.1. *Every operator representation $(\mathbb{H},\mathrm{i},\mathrm{h})$ of an Itô algebra $(\mathfrak{a},l)$ is equivalent to the triangular-matrix representation $\mathbf{i}=[i^\mu_\nu]^{\mu=-,\bullet,+}_{\nu=-,\bullet,+}$ with $i^\mu_\nu(a)=0$ if $\mu=+$ or $\nu=-$ and $i^-_+(a)=l(a)$ for all $a\in\mathfrak{a}$. Here $a^\mu_\nu=i^\mu_\nu(a)$ are linear operators $\mathbb{H}_\nu\to\mathbb{H}_\mu$ on a pseudo-Hilbert (pre-Hilbert if minimal) space $\mathbb{H}_\bullet$ and on $\mathbb{H}_+=\mathbb{C}=\mathbb{H}_-$, having the adjoints $a^{\mu\dagger}_\nu:\mathbb{H}_\mu\to\mathbb{H}_\nu$, which define the pseudo-Euclidean involution $\mathbf{a}\mapsto\mathbf{a}^\dagger$ by $a^{\star\mu}_{-\nu}=a^{\nu\dagger}_{-\mu}$, where $-(-,\bullet,+)=(+,\bullet,-)$. Moreover, if the representation is minimal, then $\mathbb{H}_\bullet$ is a pre-Hilbert space and $i^\mu_\nu(\text{đ})=\delta^\mu_-\delta^+_\nu$.*

Proof. In the matrix notation $i^{\mu}_{\nu}(a) = \mathrm{h}^{\mu}\mathrm{i}(a)\,\mathrm{h}_{\nu}$, where $\mathrm{h}^{-} = \mathrm{h}^{\dagger}_{+}$, $\mathrm{h}^{+} = \mathrm{h}^{\dagger}_{-}$ are defined by $\mathrm{h}^{\dagger}\mathrm{x} = (\mathrm{h}|\mathrm{x})$ for all $\mathrm{h},\mathrm{x} \in \mathbb{H}$, (2.4) can be written as $i^{-}_{+}(a) = l(a)$, and $i^{+}_{+}(a) = 0 = i^{-}_{-}(a)$ and $i^{+}_{-}(a) = 0$ as

$$\mathrm{i}(a)\,\mathrm{h}_{-} = \mathrm{i}(a\not{d})\,\mathrm{h} = 0 = \mathrm{h}^{\dagger}\mathrm{i}(\not{d}a) = \mathrm{h}^{+}\mathrm{i}(a) \quad \forall a \in \mathfrak{a}.$$

Moreover, due to the pseudo-orthogonality

$$(\mathrm{x}|\mathrm{x}) = \xi_{-}\xi^{-} + (\mathrm{x}_{\bullet}|\mathrm{x}_{\bullet}) + \xi_{+}\xi^{+} \equiv (\mathbf{x}|\mathbf{x}),$$

of the decomposition $\mathrm{x} = \xi^{-}\mathrm{h}_{-} + \mathrm{x}_{\bullet} + \xi^{+}\mathrm{h}_{+}$, where $\xi^{-} = \mathrm{h}^{-}\mathrm{x} = \bar{\xi}_{+}$, $\xi^{+} = \mathrm{h}^{+}\mathrm{x} = \bar{\xi}_{-}$, $\mathbf{x} = \left(\xi_{-}, \mathrm{x}^{\dagger}_{\bullet}, \xi_{+}\right)$, the representation of the Itô $\star$-algebra $(\mathfrak{a}, l)$ is defined by the homomorphism $\mathbf{i} : a \mapsto [i^{\mu}_{\nu}(a)]$ into the space of triangular block-matrices $\mathbf{a} = [a^{\mu}_{\nu}]^{\mu=-,\bullet,+}_{\nu=-,\bullet,+}$ with $a^{\mu}_{\nu} = 0$ if $\mu = +$ or $\nu = -$.

If the representation is minimal in the sense that $\mathbb{H}=\mathrm{i}(\mathfrak{a})\,\mathrm{h}$, and h has zero length, it is pseudo-unitary equivalent to the triangular representation on the complex Minkowski space $\mathbb{C}\oplus\mathbb{H}_{\bullet}\oplus\mathbb{C}$, as it can be easily seen in the basis $\mathrm{h}_{+} = \mathrm{h}$, $\mathrm{h}_{-} = \mathrm{i}(\not{d})\,\mathrm{h}$. Indeed, the pseudo-orthogonal to the zero length vectors $\mathrm{h}_{-}, \mathrm{h}_{+}$ space $\mathbb{H}_{\bullet}$ in this case is the pre-Hilbert space $\mathbb{H}_{\bullet} = \{\mathrm{i}(a)\,\mathrm{h} : l(a) = 0\}$ as

$$\xi^{+} = \mathrm{h}^{+}\mathrm{i}(a)\,\mathrm{h} = 0, \quad \xi^{-} = \mathrm{h}^{-}\mathrm{i}(a)\,\mathrm{h} = l(a) \quad \forall a \in \mathfrak{a},$$

and $(\mathrm{x}_{\bullet}|\mathrm{x}_{\bullet}) = l(a^{\star} \cdot a) \geq 0$ for all $\mathrm{x}_{\bullet} = \mathrm{i}(a)\,\mathrm{h} - \xi^{-}\mathrm{h}_{-} = \mathrm{i}(a - l(a)\,\not{d})\,\mathrm{h}$. Moreover, in the minimal representation $i^{-}_{\bullet}(\not{d}) = 0 = i^{\bullet}_{+}(\not{d})$ and $i^{\bullet}_{\bullet}(\not{d}) = 0$ as

$$\mathrm{i}(\not{d})\,\mathrm{x}_{\bullet} = \mathrm{i}(\not{d} \cdot a)\,\mathrm{h} = 0 = \mathrm{i}(a^{\star} \cdot \not{d})\,\mathrm{h} = \mathrm{x}^{\dagger}_{\bullet}\mathrm{i}(\not{d}) \quad \forall \mathrm{x}_{\bullet} \in \mathbb{H}_{\bullet}.$$

Thus, the only nonzero matrix element of $\mathrm{i}(\not{d})$ is $i^{-}_{+}(\not{d}) = 1$. □

Note that the constructed equivalent matrix representation is also defined as the right representation $\mathbf{x} \mapsto \mathbf{x}\mathbf{a}$ on all raw-vectors $\mathbf{x} = \left(\xi_{-}, \xi, \xi_{+}\right)$, $\xi \in \mathbb{H}^{\dagger}_{\bullet}$, into the dual space $\mathbb{H}^{*} = \mathbb{C} \times \mathbb{H}^{*}_{\bullet} \times \mathbb{C} \supseteq \mathbb{H}^{\dagger}$ with the invariant $\mathbb{H}^{\dagger} = \left\{\mathrm{x}^{\dagger} : \mathrm{x} \in \mathbb{H}\right\}$ such that $a^{-}_{+} = (\mathrm{h}^{-}\mathbf{a}|\mathrm{h}^{-}) = l(a)$, where $\mathrm{h}^{-} = (1, 0, 0)$. Moreover, in the Minkowski $\mathbb{C} \oplus \mathbb{H}_{\bullet} \oplus \mathbb{C}$ space it can be extended by a continuity onto $\mathbb{C} \oplus \mathcal{H} \oplus \mathbb{C}$, where $\mathcal{H}$ is the closure of the pre-Hilbert space $\mathbb{H}_{\bullet}$ with respect to $\|\mathrm{x}_{\bullet}\|$ and all the seminorms $\|i^{\bullet}_{\bullet}(a)\,\mathrm{x}_{\bullet}\|$, $a \in \mathfrak{a}$ simultaneously. We shall call such representation closed if $\mathbb{H}$ is the minimal closed Minkowski space $\mathbb{C} \oplus \mathcal{H} \oplus \mathbb{C}$, i.e. if $\mathcal{H}$ is the closure of the minimal pre-Hilbert space $\mathbb{H}_{\bullet} = i^{\bullet}_{+}(\mathfrak{a})$.

Theorem 2.2. *Every Itô $\star$-algebra $(\mathfrak{a}, l)$ can be canonically realized as the triangular matrix subalgebra in a complex Minkowski space. Moreover, every minimal closed representation is equivalent to the canonical one.*

Proof. Now we shall construct a faithful canonical operator representation for any Itô algebra $(\mathfrak{a}, l)$. The functional l defines for each $a \in \mathfrak{a}$ the canonical quadruple

$$a^{\bullet}_{\bullet} = i(a), \quad a^{\bullet}_{+} = k(a), \quad a^{-}_{\bullet} = k^{\dagger}(a), \quad a^{-}_{+} = l(a), \tag{2.7}$$

where $i(a) = i(a^{\star})^{\dagger}$ is the GNS representation $k(a \cdot b) = i(a)\,k(b)$ of $\mathfrak{a}$ on the pre-Hilbert space $\mathbb{H}_{\bullet} = \{k(b) : b \in \mathfrak{a}\}$ of the Kolmogorov decomposition $l(a \cdot b) = k^{\dagger}(a)\,k(b)$, and $k^{\dagger}(a) = k(a^{\star})^{\dagger}$. Because the operators $i(a)$ are continuous on $\mathbb{H}_{\bullet}$ w.r.t. the topology induced by $\|k(b)\|$ and all the seminorms $\|k(a \cdot b)\|$, $a \in \mathfrak{a}$ on $\mathbb{H}_{\bullet}$, can define the representation i on the clousure $\mathcal{H} = \overline{k(\mathfrak{a})}$ of $\mathbb{H}_{\bullet}$ w.r.t. these seminorms. The obtained quadrupole representation $\boldsymbol{i} : a \mapsto \boldsymbol{a} = (a^{\mu}_{\nu})^{\mu=-,\bullet}_{\nu=+,\bullet}$ of $\mathfrak{a}$

is multiplicative, $\boldsymbol{i}(a\cdot b)=(a^{\mu}_{\bullet}b^{\bullet}_{\nu})^{\mu=-,\bullet}_{\nu=+,\bullet}$ with respect to the product given by the convolution of the components a_{ν} and b^{μ} over the common index values $\mu=\bullet=\nu$:

$$\begin{aligned} i(a)\,i(b) &= i(a\cdot b)\,, \quad k^{\dagger}(a)\,i(b)=k^{\dagger}(a\cdot b)\\ i(a)\,k(b) &= k(a\cdot b)\,, \quad k^{\dagger}(a)\,k(b)=l(a\cdot b)\,. \end{aligned}$$

It is faithful because of the triviality of the ideal (2.2). Now we can use the convenience $a^{\mu}_{-}=0=a^{+}_{\nu}$ of the tensor notations (2.7), extending the quadruples $\boldsymbol{a}=\boldsymbol{i}(a)$ to the triangular matrices $\mathbf{a}=[a^{\mu}_{\nu}]^{\mu=-,\bullet,+}_{\nu=-,\bullet,+}$, in which (2.8) is simply given by $\mathbf{i}(a\cdot b)=\mathbf{ab}$ in terms of the usual product of the matrices $\mathbf{a}=\mathbf{i}(a)$ and $\mathbf{b}=\mathbf{i}(b)$. However the involution $a\mapsto a^{\star}$, which is given by the Hermitian conjugation $\boldsymbol{i}(a^{\star})=\left(a^{-\nu\dagger}_{-\mu}\right)^{\mu=-,\bullet}_{\nu=+,\bullet}$ of the quadruples $\boldsymbol{a}$, where $-(-)=+$, $-\bullet=\bullet$, $-(+)=-$, is represented by the adjoint matrix $\mathbf{a}^{\dagger}=\mathrm{G}\mathbf{a}^{*}\mathrm{G}$ w.r.t. the pseudo-Euclidean (complex Minkowski) metric tensor $\mathrm{G}=\left[\delta^{\mu}_{-\nu}\right]^{\mu=-,\bullet,+}_{\nu=-,\bullet,+}$. Thus, we have constructed the canonical representation

$$\mathbf{i}(a)=\begin{bmatrix} 0 & k^{\dagger}(a) & l(a)\\ 0 & i(a) & k(a)\\ 0 & 0 & 0\end{bmatrix},\quad \mathbf{i}(a^{\star})=\mathrm{G}\mathbf{i}(a)^{*}\mathrm{G},\quad \mathrm{G}=\begin{bmatrix} 0 & 0 & 1\\ 0 & I & 0\\ 1 & 0 & 0\end{bmatrix}$$

in the Minkowski space $\mathbb{C}\oplus\mathcal{H}\oplus\mathbb{C}$ with $\mathcal{H}=\overline{k(\mathfrak{a})}$ and $\mathrm{h}^{-}=(1,0,0)$.

The second part of the Theorem follows from the fact that every minimal representation space is the Minkowski one. All Minkowski spaces of the minimal closed representations are pseudo-unitary equivalent because all minimal closed pre-Hilbert spaces containing $i^{\bullet}_{+}(\mathfrak{a})$ are unitary equivalent. □

Definition 2.1. *Let $\mathcal{H}$ be a pre-Hilbert space, and $\mathfrak{b}(\mathcal{H})$ be the associated $\star$-algebra of all quadrupoles $A=(a^{\mu}_{\nu})^{\mu=-,\bullet}_{\nu=+,\bullet}$, where a^{μ}_{ν} are linear operators $\mathbb{H}_{\nu}\to\mathbb{H}_{\mu}$ with $\mathbb{H}_{\bullet}=\mathcal{H}$, $\mathbb{H}_{+}=\mathbb{C}=\mathbb{H}_{-}$, having the adjoints $a^{\mu\dagger}_{\nu}:\mathbb{H}_{\mu}\to\mathbb{H}_{\nu}$, with the product and the involution*

$$A\cdot B=(a^{\mu}_{\bullet}b^{\bullet}_{\nu})^{\mu=-,\bullet}_{\nu=+,\bullet}\,,\quad A^{\star}=\left(a^{-\nu\dagger}_{-\mu}\right)^{\mu=-,\bullet}_{\nu=+,\bullet}. \tag{2.8}$$

It is an Itô algebra with respect to $l(\boldsymbol{a})=a^{-}_{+}$ and the death $Đ=\left(\delta^{\mu}_{-}\delta^{+}_{\nu}\right)^{\mu=-,\bullet}_{\nu=+,\bullet}=Đ^{\star}$, $A\cdot Đ=0$, $\forall A\in\mathfrak{b}(\mathcal{H})$, called the HP (Hudson-Parthasarathy) algebra associated with the space $\mathcal{H}$. The fundamental representation of an Itô algebra $(\mathfrak{a},l)$ is given by the constructed homomorphism $\boldsymbol{i}:\mathfrak{a}\to\mathfrak{b}(\mathcal{H})$

$$\boldsymbol{i}(a)=\begin{pmatrix} l(a) & k^{\dagger}(a)\\ k(a) & i(a)\end{pmatrix},\quad \boldsymbol{i}(a^{\star})=\begin{pmatrix} l(a^{\star}) & k(a)^{\dagger}\\ k(a^{\star}) & i(a)^{\dagger}\end{pmatrix} \tag{2.9}$$

$\boldsymbol{i}(a\cdot b)=\boldsymbol{i}(a)\cdot\boldsymbol{i}(b)$, into the HP algebra, associated with the space $\mathcal{H}$ of its canonical representation.

Note that because the Itô algebra is assumed faithful in the sense of the triviality of the ideal (2.2), the fundamental representation, and so the canonical representation (2.9), is also faithful. It proves in this case that $\Lambda(t,a)=a^{\mu}_{\nu}\Lambda^{\nu}_{\mu}(t)$, where

$$a^{\mu}_{\nu}\Lambda^{\nu}_{\mu}(t)=a^{\bullet}_{\bullet}\Lambda^{\bullet}_{\bullet}(t)+a^{\bullet}_{+}\Lambda^{+}_{\bullet}(t)+a^{-}_{\bullet}\Lambda^{\bullet}_{-}(t)+a^{-}_{+}\Lambda^{+}_{-}(t)\,, \tag{2.10}$$

is the canonical decomposition of Λ into the exchange $\Lambda^{\bullet}_{\bullet}$, creation $\Lambda^{+}_{\bullet}$, annihilation $\Lambda^{\bullet}_{-}$ and preservation (time) $\Lambda^{+}_{-}=t\mathrm{I}$ operator-valued processes of the HP quantum stochastic calculus, having the mean values $\left\langle \Lambda_{\mu}^{\nu}(t)\right\rangle = t\delta_{+}^{\nu}\delta_{\mu}^{-}$. This was already noted in [3, 4] that any (classical or quantum) stochastic noise described by a process $t \in \mathbb{R}_{+} \mapsto \Lambda(t,a), a \in \mathfrak{a}$ with independent increments $\mathrm{d}\Lambda(t,a) = \Lambda(t+\mathrm{d}t,a) - \Lambda(t,a)$ forming an Itô †-algebra, can be represented in the Fock space $\mathfrak{F}$ over the space of $\mathcal{H}$-valued square-integrable functions on $\mathbb{R}_{+}$, with the vacuum vector state.

3. Two basic Itô B*-algebras

Here we consider two extreme cases of Banach Itô algebras as closed sub-algebras of the vaccum HP-algebra $\mathfrak{b}(\mathcal{H})$ associated with a Hilbert space $\mathcal{H}$. The first case correspondes to a pure state l on $\mathfrak{a}$ as it is in the case of a quantum noise of zero temperature, and the second case corresponds to a completely mixed l as in the case of a quantum noise of a finite temperature.

3.1. **Vacuum noise B*-algebra.** Let $\mathcal{H}$ be a Hilbert space of ket-vectors ζ with scalar product $(\zeta|\zeta) \equiv \zeta^{\dagger}\zeta$ and $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ be a C*-algebra, represented on $\mathcal{H}$ by the operators $\mathcal{A} \ni A : \zeta \mapsto A\zeta$ with $\left(A^{\dagger}\zeta|\xi\right) = (\zeta|A\xi)$, $\xi \in \mathcal{H}$. We denote by $\mathcal{H}^{\dagger}$ the dual Hilbert space of bra-vectors $\eta = \zeta^{\dagger}$, $\zeta \in \mathcal{H}$ with the scalar product $\left(\eta|\xi^{\dagger}\right) = \eta\xi = \left(\eta^{\dagger}|\xi\right)$ given by inverting anti-linear isomorphism $\mathcal{H}^{\dagger} \ni \eta \mapsto \eta^{\dagger} \in \mathcal{H}$, and the dual representation of $\mathcal{A}$ as the right representation $A' : \eta \mapsto \eta A$, $\eta \in \mathcal{H}^{\dagger}$, given by $(\eta A)\zeta = \eta(A\zeta)$ such that $\left(\eta A^{\dagger}|\eta\right) = (\eta|\eta A)$ on $\mathcal{H}^{\dagger}$. Then the direct sum $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}^{\dagger}$ of $\xi = \zeta \oplus \eta$ becomes a two-sided $\mathcal{A}$-module

$$A(\zeta \oplus \eta) = A\zeta, \quad (\zeta \oplus \eta)A = \eta A, \quad \forall \zeta \in \mathcal{H}, \eta \in \mathcal{H}^{\dagger}, \tag{3.1}$$

with the flip-involution $\xi^{\star} = \eta^{\dagger} \oplus \zeta^{\dagger}$ and two scalar products

$$\left\langle \zeta \oplus \eta' | \zeta' \oplus \eta\right\rangle_{+} = \left(\zeta|\zeta'\right), \quad \left\langle \zeta \oplus \eta' | \zeta' \oplus \eta\right\rangle^{-} = \left(\eta'|\eta\right). \tag{3.2}$$

The space $\mathfrak{a} = \mathbb{C} \oplus \mathcal{K} \oplus \mathcal{A}$ of triples $a = (\alpha, \xi, A)$ becomes an Itô $\star$-algebra with respect to the non-commutative product

$$a^{\star} \cdot a = \left(\langle \xi|\xi\rangle_{+}, \xi^{\star}A + A^{\dagger}\xi, A^{\dagger}A\right), \quad a \cdot a^{\star} = \left(\langle \xi|\xi\rangle^{-}, A\xi^{\star} + \xi A^{\dagger}, AA^{\dagger}\right), \tag{3.3}$$

where $(\alpha, \xi, A)^{\star} = \left(\bar{\alpha}, \xi^{\star}, A^{\dagger}\right)$, with death $\text{đ} = (1,0,0)$ and $l(\alpha,\xi,A) = \alpha$. Obviously $a^{\star} \cdot a \neq a \cdot a^{\star}$ if $\|\xi\|_{+} = \|\zeta\| \neq \|\eta\| = \|\xi\|^{-}$ even if the operator algebra $\mathcal{A}$ is commutative, $A^{\dagger}A = AA^{\dagger}$. It is separated by four semi-norms

$$\|a\| = \|A\|, \ \|a\|_{+} = \|\zeta\|, \ \|a\|^{-} = \|\eta\|, \ \|a\|_{+}^{-} = |\alpha|, \tag{3.4}$$

and is jointly complete as $a = (\alpha, \zeta \oplus \eta, A) \in \mathfrak{a}$ have independent components from the Banach spaces $\mathbb{C}$, $\mathcal{H}$, $\mathcal{H}^{\dagger}$and $\mathcal{A}$.

We shall call such Banach Itô algebra the vacuum algebra as $l(a^{\star} \cdot a) = 0$ for any $a \in \mathfrak{a}$ with $\xi \in \mathcal{H}^{\dagger}$ (it is Hudson-Parthasarathy algebra $\mathfrak{a} = \mathfrak{b}(\mathcal{H})$ if $\mathcal{A} = \mathcal{B}(\mathcal{H})$). Every closed Itô subalgebra $\mathfrak{a} \subseteq \mathfrak{b}(\mathcal{H})$ of the HP algebra $\mathfrak{b}(\mathcal{H})$ equipped with four norms (3.4) on a Hilbert space $\mathcal{H}$ is called the operator Itô B*-algebra.

If the algebra $\mathcal{A}$ is completely degenerated on $\mathcal{H}$, $\mathcal{A} = \{0\}$, the Itô algebra $\mathfrak{a}$ is nilpotent of second order, and contains only the two-dimensional subalgebras of Wiener type $\mathfrak{b} = \mathbb{C} \oplus \mathbb{C} \oplus \{0\}$ generated by an $a = (\alpha, \zeta \oplus \eta, 0)$ with $\|\zeta\| = \|\eta\|$. Every closed Itô subalgebra $\mathfrak{b} \subseteq \mathfrak{a}$ of the HP B*-algebra $\mathfrak{a} = \mathfrak{b}(\mathcal{H})$ is called the B*-Itô algebra of a vacuum Brownian motion if it is defined by a $\star$-invariant direct sum $\mathcal{G} = \mathcal{G}_{+} \oplus \mathcal{G}^{-} \subseteq \mathcal{K}$ given by a Hilbert subspace $\mathcal{G}_{+} \subseteq \mathcal{H}$, $\mathcal{G}^{-} = \mathcal{G}_{+}^{\dagger}$ and $\mathcal{A} = \{0\}$.

In the case $I \in \mathcal{A}$ the algebra $\mathcal{A}$ is not degenerated and contains also the vacuum Poisson subalgebra $\mathbb{C} \oplus \{0\} \oplus \mathbb{C}I$ of the total quantum number on $\mathcal{H}$, and other Poisson two-dimensional subalgebras, generated by $a = (\alpha, \zeta \oplus \eta, I)$ with $\eta = e^{i\theta}\zeta^{\dagger}$. We shall call a closed Itô subalgebra $\mathfrak{c} \subseteq \mathfrak{a}$ of the HP B*-algebra $\mathfrak{a} = \mathfrak{b}(\mathcal{H})$ the B*-algebra of a vacuum Lévy motion if it is given by a direct sum $\mathcal{E} = \mathcal{E}_{+} \oplus \mathcal{E}^{-} \subseteq \mathcal{K}$ with $\mathcal{E}_{-} = \mathcal{E}_{+}^{\dagger}$ and a $\dagger$-subalgebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ nondegenerated on the subspace $\mathcal{E}_{+} \subseteq \mathcal{H}$.

Theorem 3.1. *Every vacuum B*-algebra can be decomposed into an orthogonal sum* $\mathfrak{a} = \mathfrak{b} + \mathfrak{c}$, $\mathfrak{b} \cdot \mathfrak{c} = \{0\}$ *of the Brownian vacuum B*-algebra* $\mathfrak{b}$ *and the Lévy vacuum B*-algebra* $\mathfrak{c}$. *This decomposition is unique on the zero mean kernel* $\mathfrak{x} = \{x \in \mathfrak{a} : l(x) = 0\}$.

Proof. This decomposition is uniquely defined for all $a = (\alpha, \xi, A)$ by $a = \alpha đ + y + z$, with $y = (0, \eta, 0)$, $z = (0, \zeta, A)$, $\eta = P\xi \oplus \xi P \in \mathcal{G}$, $\zeta = \xi - \eta \in \mathcal{E}$, where $P = P^{\dagger}$ is the maximal projector in $\mathcal{H}$, for which $\mathcal{A}P = \{0\}$, $\mathcal{G}_{+} = P\mathcal{H}$, and $\mathcal{E}_{+} = \mathcal{G}_{+}^{\perp}$. □

3.2. **Thermal noise B*-algebra.** Let $\mathcal{D}$ be a left Tomita $\star$-algebra [6] with respect to a Hilbert norm $\|\xi\|_{+} = 0 \Rightarrow \xi = 0$, and thus a right pre-Hilbert $\star$-algebra with respect to $\|\xi\|^{-} = \|\xi^{\star}\|_{+}$. This means that $\mathcal{D}$ is a complex pre-Hilbert space with continuous left (right) multiplications $C : \zeta \mapsto \xi\zeta$ $(C' : \eta \mapsto \eta\xi)$ w.r.t. $\|\cdot\|_{+}$ (w.r.t. $\|\cdot\|^{-}$) of the elements $\zeta, \eta \in \mathcal{D}$ respectively, defined by an associative product in $\mathcal{D}$, and the involution $\mathcal{D} \ni \xi \mapsto \xi^{\star} \in \mathcal{D}$ such that

$$\langle \eta\zeta^{\star}|\xi\rangle^{-} = \langle \eta|\xi\zeta\rangle^{-}, \quad \langle \eta^{\star}\zeta|\xi\rangle_{+} = \langle \zeta|\eta\xi\rangle_{+} \quad \forall \xi, \zeta, \eta \in \mathcal{D}, \tag{3.5}$$

$$\langle \eta|\xi^{\star}\rangle^{-} = \left\langle \xi|\eta^{\sharp}\right\rangle^{-}, \quad \langle \zeta|\xi^{\star}\rangle_{+} = \left\langle \xi|\zeta^{\flat}\right\rangle_{+} \quad \forall \eta \in \mathcal{D}^{-}, \zeta \in \mathcal{D}_{+}. \tag{3.6}$$

Here $\langle \eta|\xi^{\star}\rangle^{-} = \langle \eta^{\star}|\xi\rangle_{+}$ is the right scalar product, $\mathcal{D}_{+} = \mathcal{D}_{+}^{\flat}$ is a dense domain for the left adjoint involution $\zeta \mapsto \zeta^{\flat}$, $\zeta^{\flat\flat} = \zeta$, and $\mathcal{D}^{-} = \mathcal{D}_{+}^{\star}$ is the invariant domain for the right adjoint involution $\eta \mapsto \eta^{\sharp}$, $\left(\eta^{\sharp}\eta\right)^{\sharp} = \eta^{\sharp}\eta$ such that $\zeta^{\flat\star} = \zeta^{\star\sharp}$, $\eta^{\sharp\star} = \eta^{\star\flat}$.

Since the adjoint operators $C^{\dagger}\zeta = \xi^{\star}\zeta$, $\eta C^{\dagger} = \eta\xi^{\star}$ are also given by the multiplications, they are bounded:

$$\|\xi\| = \sup\left\{\|\xi\zeta\|_{+} : \|\zeta\|_{+} \le 1\right\} = \sup\left\{\|\eta\xi\|^{-} : \|\eta\|^{-} \le 1\right\} < \infty. \tag{3.7}$$

Note that we do not require the sub-space $\mathcal{D}\mathcal{D} \subseteq \mathcal{D}$ of all products $\{\eta\zeta : \eta, \zeta \in \mathcal{D}\}$ to be dense in $\mathcal{D}$ w.r.t. any of two Hilbert norms on $\mathcal{D}$, but it is always dense w.r.t. the operator semi-norm (3.7) on $\mathcal{D}$. Hence the operator $\dagger$-algebra $\mathcal{C} = \{C : \mathcal{D} \ni \zeta \mapsto \xi\zeta | \xi \in \mathcal{D}\}$ w.r.t. the left scalar product, which is also represented on the $\mathcal{D} \ni \eta$ equipped with $\langle\cdot|\cdot\rangle^{-}$ by the right multiplications $\eta C = \eta\xi$, $\xi \in \mathcal{D}$, can be degenerated on $\mathcal{D}$.

Thus the direct sum $\mathfrak{a} = \mathbb{C} \oplus \mathcal{D}$ of pairs $a = (\alpha, \xi)$ becomes an Itô $\star$-algebra with the product

$$a^{\star} \cdot a = \left(\langle \xi|\xi\rangle_{+}, \xi^{\star}\xi\right), \quad a \cdot a^{\star} = \left(\langle \xi|\xi\rangle^{-}, \xi\xi^{\star}\right), \tag{3.8}$$

where $(\alpha, \xi)^{\star} = (\bar{\alpha}, \xi^{\star})$, with death $đ = (1, 0)$ and $l(\alpha, \xi) = \alpha$. Obviously $a^{\star}a \neq aa^{\star}$ if the involution $a \mapsto a^{\star}$ is not isometric w.r.t. any of two Hilbert norms even if the algebra $\mathcal{D}$ is commutative. It is a Banach algebra separated by the semi-norms

$$\|a\| = \|\xi\|, \|a\|^{-} = \|\xi\|^{-}, \|a\|_{+} = \|\xi\|_{+}, \|a\|_{+}^{-} = |\alpha|. \tag{3.9}$$

if its normed $\star$-algebra $\mathcal{D}$ is complete jointly w.r.t. to the first three norms.

We shall call such complete Itô algebra the thermal B*-algebra as $l(a^{\star}a) = \|\xi\|_{+}^{2} \neq 0$ for any $a \in \mathfrak{a}$ with $\xi \neq 0$. If $\zeta\eta = 0$ for all $\zeta, \eta \in \mathcal{D}$, it is the Itô B*-algebra of thermal Brownian motion. A thermal B*-subalgebra $\mathfrak{b} \subseteq \mathfrak{a}$ with such trivial product is given by any involutive pre-Hilbert $\star$-invariant two-normed subspace $\mathcal{G} \subseteq \mathcal{D}$ which is closed w.r.t. the Hilbert sum $\langle\eta|\zeta\rangle^{-} + \langle\zeta|\eta\rangle_{+}$. We shall call such Brownian algebra $\mathfrak{b} = \mathbb{C} \oplus \mathcal{G}$ the quantum (if $\|\cdot\|_{+} \neq \|\cdot\|^{-}$) Wiener B*-algebra associated with the space $\mathcal{G}$.

In the opposite case, if $\mathcal{DD} = \{\zeta\eta : \zeta, \eta \in \mathcal{D}\}$ is dense in $\mathcal{D}$, it has nondegenerated operator representation $\mathcal{C}$ on $\mathcal{D}$. Any closed involutive sub-algebra $\mathcal{E} \subseteq \mathcal{D}$ which is non-degenerated on $\mathcal{E}$ defines an Itô B*-algebra $\mathfrak{c} = \mathbb{C} \oplus \mathcal{E}$ of thermal Lévy motion. We shall call such Itô algebra the quantum (if $\mathcal{E}$ is non-commutative) Poisson B*-algebra.

Theorem 3.2. *Every thermal Itô B*-algebra is an orthogonal sum* $\mathfrak{a} = \mathfrak{b} + \mathfrak{c}$, $\mathfrak{bc} = \{0\}$ *of the Wiener B*-algebra* $\mathfrak{b} = \mathfrak{b}^{\star}$ *and the Poisson B*-algebra* $\mathfrak{c} = \mathfrak{c}^{\star}$. *This decomposition is unique on the zero mean kernel* $\mathfrak{x} = l^{-1}(0)$.

Proof. The orthogonal decomposition $a = \alpha đ + b + c$ for all $a = (\alpha, \xi) \in \mathfrak{a}$, uniquely given by the decomposition $\xi = \eta + \zeta$, where η is the orthogonal projection of ξ onto the orthogonal complement $\mathcal{G}$ of $\mathcal{DD}$ w.r.t. any of two scalar products in $\mathcal{D}$, and $\zeta = \xi - \eta$.

Indeed, if $\xi \in \mathcal{D}$ is left orthogonal to $\mathcal{DD}$, then it is also right orthogonal to $\mathcal{DD}$ and vice versa:

$$\begin{aligned} \langle\eta\zeta^{\star}|\xi\rangle^{-} &= \langle\zeta\eta^{\star}|\xi^{\star}\rangle_{+} = \left\langle\xi|\eta^{\sharp\star}\zeta^{\flat}\right\rangle_{+} = 0, \quad \forall\eta \in \mathcal{D}^{-}, \zeta \in \mathcal{D}_{+}, \\ \langle\eta^{\star}\zeta|\xi\rangle_{+} &= \langle\zeta^{\star}\eta|\xi^{\star}\rangle^{-} = \left\langle\xi|\eta^{\sharp}\zeta^{\flat\star}\right\rangle^{-} = 0, \quad \forall\eta \in \mathcal{D}^{-}, \zeta \in \mathcal{D}_{+}, \end{aligned}$$

and $\xi^{\sharp}, \xi^{\flat}$ are also orthogonal to $\mathcal{DD}$:

$$\left\langle\eta\zeta^{\star}|\xi^{\sharp}\right\rangle^{-} = \langle\xi|\zeta\eta^{\star}\rangle^{-} = 0, \quad \left\langle\eta^{\star}\zeta|\xi^{\flat}\right\rangle_{+} = \langle\xi|\zeta^{\star}\eta\rangle_{+} = 0$$

From these and (3.5) equations it follows that $\eta\xi = 0 = \xi\zeta$ for all $\zeta, \eta \in \mathcal{D}$ if ξ is (right or left) orthogonal to $\mathcal{DD}$, and so $\|\xi\| = 0$ for such ξ and vice versa. Thus the maximal orthogonal subspace to $\mathcal{DD}$ is the $\star$-invariant space $\mathcal{G} = \{\xi \in \mathcal{D} : \|\xi\| = 0\}$, which is complete w.r.t. the two norms $\|\cdot\|^{-}, \|\cdot\|_{+}$, i.e. is a closed subspace of the pre-Hilbert space $\mathcal{D}$ with the $\star$-invariant scalar product

$$\langle\zeta^{\star}|\eta\rangle = \langle\zeta^{\star}|\eta\rangle_{+} + \langle\eta|\zeta^{\star}\rangle^{-} = \langle\zeta|\eta^{\star}\rangle^{-} + \langle\eta^{\star}|\zeta\rangle_{+} = \langle\eta^{\star}|\zeta\rangle .$$

Denoting by P the right orthogonal projector in $\mathcal{D}$ onto the orthogonal complement to $\mathcal{DD}$, we obtain $P\xi = \eta \in \mathcal{G}$, $P\xi^{\star} = \eta^{\star}$ as

$$\langle\eta^{\star}|P\xi\rangle_{+} = \left\langle P\xi|\eta^{\sharp}\right\rangle^{-} = \left\langle\xi|\eta^{\sharp}\right\rangle^{-} = \langle\eta|\xi^{\star}\rangle^{-} = \langle\eta|P\xi^{\star}\rangle^{-}, \ \eta \in \mathcal{G}^{-}, \xi \in \mathcal{D}$$

and thus $P\xi$ is also the left projection, and so the symmetric projection $\langle\eta|P\xi\rangle = \langle\eta|\xi\rangle$ of $\xi \in \mathcal{D}$ onto $\mathcal{G} \ni \eta$. So $\zeta = \xi - \eta \in \mathcal{D}$ is in the left (right) closure $\mathcal{E} \subseteq \mathcal{D}$ of $\mathcal{DD}$. □

4. Decomposition of Itô B*-algebras

Now we shall study the general uniformly bounded infinite dimensional Ito $\star$-algeblas, unifying the considered two basic cases.

Let $\mathfrak{a}$ be an associative infinite-dimensional complex algebra with involution $b^\star = a \in \mathfrak{a}, \forall b = a^\star$ which is defined by the properties

$$(b \cdot b^\star)^\star = b \cdot b^\star, \quad \left(\sum \lambda_i b_i\right)^\star = \sum \bar{\lambda}_i b_i^\star, \quad \forall b_i \in \mathfrak{a}, \lambda_i \in \mathbb{C}.$$

We shall suppose that this algebra is a normed space with respect to four semi-norms $\|\cdot\|_\nu^\mu$, indexed as

$$\|\cdot\|_\bullet^\bullet \equiv \|\cdot\|, \quad \|\cdot\|_+^\bullet \equiv \|\cdot\|_+, \quad \|\cdot\|_\bullet^- \equiv \|\cdot\|^-, \quad \|\cdot\|_+^- \tag{4.1}$$

by $\mu = -, \bullet$, $\nu = +, \bullet$, satisfying the following conditions

$$\|b^\star\| = \|b\|, \quad \|b^\star\|_+ = \|b\|^-, \quad \|b^\star\|_+^- = \|b\|_+^-, \tag{4.2}$$

$$\left(\|a \cdot c\|_\nu^\mu \leq \|a\|_\bullet^\mu \, \|c\|_\nu^\bullet, \right)_{\nu=+,\bullet}^{\mu=-,\bullet} \quad \forall\, a, b, c \in \mathfrak{a}. \tag{4.3}$$

Thus the semi-norms (4.1) separate $\mathfrak{a}$ in the sense

$$\|a\| = \|a\|_+ = \|a\|^- = \|a\|_+^- = 0 \quad \Rightarrow \quad a = 0,$$

and the product $(a, c) \mapsto ac$ with involution $\star$ is uniformly continuous in the induced topology due to (4.3).

If $\mathfrak{a}$ is a $\star$-algebra equipped with a linear positive $\star$-functional l such that

$$l\,(b) = l\,(a \cdot b) = l\,(b \cdot c) = l\,(a \cdot b \cdot c) = 0, \forall a, c \in \mathfrak{a} \quad \Rightarrow \quad b = 0,$$

and it is bounded with respect to l in the sense

$$\|b\| = \sup\left\{\|a \cdot b \cdot c\|_+^- / \|a\|^- \, \|c\|_+ : a, c \in \mathfrak{a}\right\} < \infty \quad \forall b \in \mathfrak{a}, \tag{4.4}$$

where $\|a\|_+^- = |l\,(a)|$, $\|a\|^- = l\,(a \cdot a^\star)^{1/2}$, $\|a\|_+ = l\,(a^\star \cdot a)^{1/2}$, then it is four-normed in the above sense. The defined by l semi-norms $\|\cdot\|_\nu^\mu$ are obviously separating, satisfy the inequalities (4.3), and they also satisfy the $\star$-equalities of the following definition

Definition 4.1. *An associative four-normed $\star$-algebra $\mathfrak{a}$ is called B^*-algebra if it is complete in the uniform topology, induced by the semi-norms $(\|a\|_\nu^\mu)_{\nu=+,\bullet}^{\mu=-,\bullet}$, satisfying the following equalities*

$$\|a \cdot a^\star\| = \|a\| \, \|a^\star\|, \quad \|a \cdot a^\star\|_+^- = \|a\|^- \, \|a^\star\|_+ \quad \forall a \in \mathfrak{a}.. \tag{4.5}$$

The Itô B^*-algebra is a B^*-algebra with self-adjoint annihilator $đ = đ^\star$, $a \cdot đ = 0 = đ \cdot a$, $\forall a \in \mathfrak{a}$ called the death of $\mathfrak{a}$, and the semi-norms (4.1) given by a linear positive $\star$-functional $l\,(a^\star) = \overline{l\,(a)}$, $l\,(a \cdot a^\star) \geq 0, \forall a \in \mathfrak{a}$ normalized as $l\,(đ) = 1$.

Obviously, any C*-algebra can be considered as a B*-algebra in the above sense with three trivial semi-norms $\|a\|_+^- = \|a\|^- = \|a\|_+ = 0, \forall a \in \mathfrak{a}$. Moreover, as it follows from the inequalities (4.3) for $c = 1$, every unital B*-algebra is a C*-algebra, the three nontrivial semi-norms on which might be given by a state l, normalized as $l\,(1) = \|1\|_+^-$. However, if a B*-algebra $\mathfrak{a}$ contains only approximative identity $e_i \nearrow 1$, and $\|e_i\|_+^- \longrightarrow \infty$, it is a proper dense sub-algebra of its C*-algebraic completion w.r.t. the norm $\|\cdot\|$.

Note that the use of the term B*-algebra, the obsolete name for the C*-algebras, in a more general sense is not contradictive, and will never make a confusion in the context of Itô algebras, as there is no Itô algebra which is simultaneously a C*-algebra. Every C*-algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ with a ciclic vector $\eta \in \mathcal{H}$ can be embedded into a faithful Itô B*-algebra $\mathfrak{a} = \mathbb{C}d + \mathfrak{x}$, where the subspace $\mathfrak{x} \subseteq \mathfrak{a}$ identified with the factor-algebra $\mathfrak{a}/\mathbb{C}d$, is the operator C*-algebra $\mathcal{A}$. This can be done by $a = (\eta|A\eta)\, d + x$ with $x = A$, $x^\star = A^\dagger$ such that $\mathfrak{x} = \mathcal{A}$, $l(x) = 0$, $\|x\|_+ = \|A\eta\|$, $\|x\|^- = \|A^\dagger\eta\|$. However in the general case the zero mean algebra $\mathfrak{x} = \{a \in \mathfrak{a} : l(a) = 0\}$ equipped with the factor-product

$$aa^\star = a \cdot a^\star - l(a \cdot a^\star)\, d = x \cdot x^\star - l(x \cdot x^\star)\, d = xx^\star, \tag{4.6}$$

where $x = a - l(a)\, d$, is not a C*-algebra for an arbitrary Itô B*-algebra $\mathfrak{a}$, although it is complete with respect to the C*-semi-norm $\|x\| = \|a\|$, jointly with two Hilbert semi-norms $\|x\|_+ = \|a\|_+$, $\|x\|^- = \|a\|^-$.

Thus in order to classify the Itô B*-algebras we should study the structure of the zero-mean algebras $\mathfrak{x} = \{x = a - l(a) : a \in \mathfrak{a}\}$ with the normes, given by the semi-scalar products $\langle x^\star|x\rangle_+ = \langle x|x^\star\rangle^-$, defining the Itô algebras $\mathfrak{a} = \{\alpha d + x\}$ with $l(a) = \alpha$ by

$$a \cdot a^\star = \langle x|x\rangle^-\, d + xx^\star = x \cdot x^\star, \quad a^\star \cdot a = \langle x|x\rangle_+\, d + x^\star x = x^\star \cdot x. \tag{4.7}$$

As was proved in the first section, every Itô B*-algebra is algebraically and isometrically isomorphic to a closed $\star$-subalgebra $\mathfrak{a} \ni D$ of the simple vacuum B*-algebra $\mathfrak{b}(\mathcal{H})$ associated with the GNS Hilbert space $\mathcal{H}$ by (**??**) so that $\boldsymbol{x} = \boldsymbol{i}(x)$ for each $x = a - l(a)\, d$ is given by the tripple

$$\hat{x} = i(x), \quad x\rangle_+ = k(x), \quad \langle x = k^\dagger(x),$$

with $\|x\| = \|i(x)\|$, $\|x\|_+ = \|k(x)\|$, $\|x\|^- = \|k^\dagger(x)\|$. Thus the factor-algebra $\mathfrak{x} = \mathfrak{a}/\mathbb{C}d$ can be identified with a closed $\star$-subalgebra of the B*-algebra $\mathcal{H} \oplus \mathcal{A} \oplus \mathcal{H}^\dagger$ equipped with the product

$$x^\star x = \hat{x}^\dagger x\rangle_+ \oplus \hat{x}^\dagger\hat{x} \oplus \langle x|\hat{x}, \quad xx^\star = \hat{x}|x\rangle^- \oplus \hat{x}\hat{x}^\dagger \oplus \langle x\hat{x}^\dagger, \tag{4.8}$$

for $x = x\rangle_+ \oplus \hat{x} \oplus \langle x \in \mathcal{H} \oplus \mathcal{A} \oplus \mathcal{H}^\dagger \ni |x\rangle^- \oplus \hat{x}^\dagger \oplus \langle x| = x^\star$, where

$$|x\rangle^- = \langle x^\dagger = x^\star\rangle_+, \quad \langle x| = x\rangle_+^\dagger = \langle x^\star.$$

Let P be the maximal orthoprojector in $\mathcal{H}$ for which $i(a)\, P = 0$ for all $a \in \mathfrak{a}$ such that $E = I - P$ is the support for the operator algebra $\mathcal{A} = i(\mathfrak{a})$. We shall prove that the $\star$-projections

$$\pi(x) = Px\rangle_+ \oplus \langle xP, \quad \varepsilon(x) = Ex\rangle_+ \oplus \hat{x} \oplus \langle xE \tag{4.9}$$

define the homomorphisms of $\mathfrak{x} \subseteq \mathcal{H} \oplus \mathcal{A} \oplus \mathcal{H}^\dagger$ respectively onto the $\star$-ideal $\mathfrak{y} = \{y \in \mathfrak{x} : \mathfrak{a}y = 0 = y\mathfrak{a}\}$ and the maximal subalgebra $\mathfrak{z} \subseteq \mathfrak{x}$ having the dense part $\mathfrak{a}\mathfrak{a} = \{ac : a, c \in \mathfrak{a}\}$ w.r.t. any of two Hilbert seminorms. An Ito B*-algebra $\mathfrak{a}$ with the trivial factor-product $\mathfrak{a}\mathfrak{a} = \{0\}$ and thus $\|\cdot\| = 0$ is called the (general) Brownian algebra. In the opposite case, when $\mathfrak{a}\mathfrak{a}$ is dense in $\mathfrak{x}$ w.r.t. any of the Hilbert semi-norms $\|\cdot\|^-$, $\|\cdot\|_+$, it is called the (general) Lévy algebra.

Theorem 4.1. *Let $\mathfrak{a}$ be a B*-Itô algebra. Then it is an orthogonal sum $\mathfrak{b} + \mathfrak{c}$, $\mathfrak{b} \cdot \mathfrak{c} = \{0\}$ of a quantum Brownian B*-algebra $\mathfrak{b}$ and a quantum Lévy B*-algebra $\mathfrak{c}$. This decomposition is unique up to the ideal $\mathfrak{b} \cap \mathfrak{c} = \mathbb{C}d$.*

Proof. We want to find the orthogonal decomposition

$$x = y + z, \quad \langle y|z\rangle_+ = 0 = \langle y|z\rangle^- , \quad y \in \mathfrak{y}, z \in \mathfrak{z}$$

for all $x = a - l(a)$, $a \in \mathfrak{a}$, and to prove that it is unique. First note that the condition $\mathfrak{a}y = \{0\} = y\mathfrak{a}$ for the elements $y \in \mathfrak{y}$ is equivalent to $\|y\| = 0$ with the right and left orthogonality of $y \in \mathfrak{x}$ to $\mathfrak{aa} \subseteq \mathfrak{x}$:

$$\langle y|a^\star c\rangle_+ = 0 = \|y\| \Leftrightarrow \langle ay|c\rangle_+ = 0 = \langle ya|c\rangle_+ \quad \forall a, c \in \mathfrak{a}$$

$$\langle y|ac^\star\rangle^- = 0 = \|y\| \Leftrightarrow \langle yc|a\rangle^- = 0 = \langle cy|a\rangle^- \quad \forall a, c \in \mathfrak{a}$$

So the $\star$-ideal $\mathfrak{y} \subseteq \mathfrak{x}$ with all trivial products $xy = 0 = yx$ is defined as the closed $\star$-subspace $\left\{y \in \mathfrak{x}_0 : \langle y|\mathfrak{aa}\rangle_+ = 0 = \langle y|\mathfrak{aa}\rangle^-\right\}$ of the Hilbert space $\{x \in \mathfrak{x} : \|x\| = 0\}$ with the $\star$-invariant scalar product

$$\langle z^\star|x\rangle = \langle z^\star|x\rangle_+ + \langle x|z^\star\rangle^- = \langle z|x^\star\rangle^- + \langle x^\star|z\rangle_+ = \langle x^\star|z\rangle .$$

Let us prove that the symmetric orthogonal projection $\langle y|\pi(x)\rangle = \langle y|x\rangle$ of $x \in \mathfrak{x}$ onto $\mathfrak{y} \ni y$ is defined as in (4.9). Indeed,

$$\begin{aligned}\langle y|x\rangle &= \langle y|x\rangle_+ + \langle x|y\rangle^- = \langle y|Px\rangle_+ + \langle xP|y\rangle^- \\ &= \langle y|\pi(x)\rangle_+ + \langle \pi(x)|y\rangle^- = \langle y|\pi(x)\rangle ,\end{aligned}$$

where P is the left orthoprojector $\langle y|Px\rangle_+ = \langle y|x\rangle_+$ in $\mathcal{H}$ onto $\mathcal{G}_+ = \mathfrak{y}\rangle_+$, defining the right orthoprojector P' onto $\mathcal{G}^- = \mathcal{G}_+^\dagger$ by $\mathcal{H}^\dagger \ni \langle x \mapsto \langle xP$. The $\star$-projection $\pi(x^\star) = \pi(x)^\star$ is the homorphism as $\pi(xz) = 0$ due to $\langle y|xz\rangle = 0$ for all $x, z \in \mathfrak{x}$, $y \in \mathfrak{y}$, and $\varepsilon(x) = x - \pi(x)$, defined in (4.9) is also a $\star$-homomorphism, $\varepsilon(xz) = xz$, for all $x, z \in \mathfrak{x}$ as its kernel $\varepsilon^{-1}(0) = \{x \in \mathfrak{x} : \varepsilon(x) = 0\}$ is the $\star$-ideal $\mathfrak{y}$. Thus the range $\mathfrak{z} = \varepsilon(\mathfrak{x})$ is a closed $\star$-subalgebra, which is left and right orthogonal to $\mathfrak{y}$:

$$\langle y|\varepsilon(x)\rangle_+ = \langle y|Ex\rangle_+ = 0 = \langle xE|y\rangle^- = \langle \varepsilon(x)|y\rangle^- , \quad \forall x \in \mathfrak{x}, y \in \mathfrak{y}.$$

Due to $\mathcal{E}_+ = E\mathfrak{x}\rangle_+$ contains $\mathfrak{aa}\rangle_+ = \mathcal{A}\mathfrak{x}\rangle_+$ as a dense part in the Hilbert space $\mathcal{H}$, the algebra $\mathfrak{z}$ contains $\mathfrak{aa}$ as a left (right) dense part with respect to the left (right) Hilbert seminorm $\|\cdot\|_+$ ($\|\cdot\|^-$.) Obviously $\mathfrak{z} \subseteq \mathfrak{x}$ is uniquely defined as the maximal such $\star$-subalgebra, and ε is uniquely defined as the representation $\mathfrak{z}$ of the quotient algebra $\mathfrak{x}/\mathfrak{y}$.

Thus $a = b + c$, $bc = 0$ for all $a \in \mathfrak{a}$, where $b = \beta đ + y \in \mathfrak{b}$ define the Brownian B*-subalgebra with the fundamental representation $\mathfrak{b} \subseteq \mathbb{C} \oplus \mathcal{G}_+ \oplus \mathcal{G}^-$, where $\mathcal{G}_+ = Pk(\mathfrak{a})$, $\mathcal{G}^- = k^\dagger(\mathfrak{a})P$, and $c = (\alpha - \beta)đ + z \in \mathfrak{c}$ define the Lévy B*-subalgebra, having the fundamental representation $\mathfrak{c} \subseteq \mathbb{C} \oplus \mathcal{E}_+ \oplus \mathcal{E}^- \oplus \mathcal{A}$ with non-degenerated operator algebra $\mathcal{A} = i(\mathfrak{a})$, left and right represented on $\mathcal{E}_+ = Ek(\mathfrak{a})$ and $\mathcal{E}^- = k^\dagger(\mathfrak{a})E$. □

Example 4.1. *The commutative multiplication table* $\mathrm{d}w_i\mathrm{d}\bar{w}_k = \delta^i_k\mathrm{d}t$ *for the complex Itô differentials* $\mathrm{d}w_k = \mathrm{d}\bar{w}_{-k}$, $k = \mathbb{Z}$ *of the Fourier amplitudes*

$$w_k(t) = \int_{-\pi}^{\pi} e^{jk\theta} W(t, d\theta),$$

$$\mathrm{d}W(t, \Delta)\,\mathrm{d}W(t, \Delta') = \frac{1}{2\pi}|\Delta' \cap \Delta|\,\mathrm{d}t$$

for an orthogonal Wiener measure $W(t, \Delta)$ *on* $\Delta \subseteq [-\pi, \pi] \ni \theta$ *of the normalized intensity can be generalized in the following way.*

Let $\rho_k > 0$, $k \in \mathbb{Z}$ be a self-inverse family of spectral eigen-values $\rho_{-k} = \rho_k^{-1}$ for a positive-definite (generalized) periodic function

$$\lambda(\theta) = \sum_{k=-\infty}^{\infty} \rho_k e^{jk\theta},$$

$$\left[\bar{\lambda} * \lambda\right](\theta) := \frac{1}{2\pi}\int_{-\pi}^{\pi} \bar{\lambda}(\theta - \phi)\,\lambda(\phi)\,\mathrm{d}\phi = \delta(\theta).$$

The generalized multiplication table $d_i \cdot d_k^\star = \rho_k \delta_k^i đ$ for abstract infinitesimals $d_k = d_{-k}^\star$, $k \in \mathbb{Z}$ is obviously non-commutative for all k with $\lambda_k \neq 1$. The $\star$-semigroup $\{0, đ, d_k : k \in \mathbb{Z}\}$ generates an infinite dimensional Itô algebra $(\mathfrak{b}, l)$ of a quantum Wiener periodic motion on $[-\pi, \pi]$ as it is the second order nilpotent algebra $\mathfrak{a}$ of $a = \alpha đ + y$, $y = \sum \eta^k d_k$ with $y^\star = \sum \eta_k d_k^\star$ and $l(y) = 0$ for all $\eta^k = \bar{\eta}_k \in \mathbb{C}$. It is a Brownian B^*-algebra with closed involution on the complex space $\mathcal{D}$ of all η given by all complex sequences $\eta = (\eta_k)_{k\in\mathbb{Z}}$ with

$$\|\eta\|^- = \left(\sum |\eta_k|^2 \rho_k\right)^{1/2} = \|\eta^\star\|_+ < \infty.$$

The operator representation of $\mathfrak{a}$ in Fock space is defined by the forward differentials of $\Lambda(t, a) = \alpha t \mathrm{I} + \sum \eta^k \hat{w}_k(t)$, where

$$\hat{w}_k(t) = \hat{v}_k^-(t) + \hat{v}_k^+(t),$$

$$\hat{v}_k^-(t) = \rho_{-k}^{1/2} \int_{-\pi}^{\pi} e^{jk\theta} \Lambda_-(t, d\theta), \quad \hat{v}_k^+(t) = \rho_k^{1/2} \int_{-\pi}^{\pi} e^{jk\theta} \Lambda^+(t, d\theta)$$

are given by the annihilation and creation measures in Fock space over square integrable functions on $\mathbb{R}_+ \times [-\pi, \pi]$ with the standard multiplication table

$$\mathrm{d}\Lambda_-(t, \Delta)\,\mathrm{d}\Lambda^+(t, \Delta') = \frac{1}{2\pi}|\Delta' \cap \Delta|\, I\mathrm{d}t,$$

$$\mathrm{d}\Lambda^+(t, \Delta')\,\mathrm{d}\Lambda_-(t, \Delta) = 0.$$

Example 4.2. *The commutative multiplication table $\mathrm{d}m_i \mathrm{d}\bar{m}_k = \delta_k^i \mathrm{d}t + \mathrm{d}m_{i-k}$ for the complex Itô differentials $\mathrm{d}m_k = \mathrm{d}\bar{m}_{-k}$, $k = \mathbb{Z}$ of the Fourier amplitudes $m_k(t) = \int_{-\pi}^{\pi} e^{jk\theta} M(t, d\theta)$,*

$$\mathrm{d}M(t, \Delta') = \frac{1}{2\pi}|\Delta' \cap \Delta|\,\mathrm{d}t + \mathrm{d}M(t, \Delta' \cap \Delta), \quad \Delta, \Delta' \subseteq [-\pi, \pi]$$

for the standard compensated Poisson measure $M(t, \Delta)$ can be generalized in the following way.

Let G be a discrete locally compact group, and $G \ni g \mapsto \lambda_g \in \mathbb{C}$ be a positive-definite summable function, $\lambda_{g^{-1}} = \bar{\lambda}_g$, which is self-inverse in the convolutional sense $\left[\bar{\lambda} * \lambda\right]_g = \sum_h \bar{\lambda}_{gh^{-1}} \lambda_h = \delta_g^1$. The generalized multiplication table for abstract infinitesimals

$$d_g = d_{-g}^\star, \quad d_g \cdot d_h^\star = \lambda_{gh^{-1}} đ + d_{gh^{-1}}, \quad g, h \in G$$

is obviously associative and commutative if G is Abelian, $G \simeq \mathbb{Z}$, but it is non-commutative for non Abelian G even if $\lambda_k = \delta_k^1$ as in the above case. The $\star$-semigroup $\{0, đ, d_g : g \in G\}$ generates an infinite dimensional Itô algebra $(\mathfrak{a}, l)$ of a quantum compensated Poisson motion on the spectrum Ω of the group G as $\mathfrak{a}$ is the sum of $\mathbb{C}đ$ and the unital group algebra $\mathcal{D}$ of $z = \sum \zeta^g d_g$ with involution

$z^\star = \sum \zeta_g d_g^\star$ and $l(z) = 0$ for all $\zeta^g = \bar{\zeta}_g \in \mathbb{C}$. It is a Lévy B^*-algebra with closed involution on the complex space $\mathcal{D}$ of all complex sequences $\zeta = \left(\zeta_g\right)_{g\in G}$ with

$$\|\zeta\|^- = \left(\sum \left(\bar{\zeta} * \tilde{\zeta}\right)_g^2 \lambda_g\right)^{1/2} = \|\zeta^\star\|_+ < \infty,$$

where $\tilde{\zeta} = \left(\zeta_{g^{-1}}\right)_{g\in G}$. The operator integral representation of $\mathfrak{a}$ is defined in Fock space over the Hilbert space of square integrable function on $\mathbb{R}_+$ with values in the direct integral $\mathcal{H} = \int_\Omega^\oplus \mathcal{H}(\omega)\,\mathrm{d}\pi_\omega$ of Hilbert spaces $\mathcal{H}(\omega)$ for the spectral decomposition

$$\lambda_g = \int_\Omega \mathrm{Tr}\left[\rho_\omega U_g(\omega)\right] \mathrm{d}\pi_\omega$$

w.r.t. the unitary irreducible representations $U_g(\omega)$ of G and the Plansherel measure $\mathrm{d}\pi_\omega$. It is given by the forward differentials of $\Lambda(t,a) = \alpha t\mathrm{I} + \sum \zeta^g \hat{m}_g(t)$, with

$$\hat{m}_g(t) = \int_\Omega \mathrm{Tr}_{\mathcal{H}(\omega)}\left\{U_g(\omega)\left[\rho_\omega^{1/2}\Lambda_-(t,d\omega) + \Lambda^+(t,d\omega)\,\rho_\omega^{1/2} + \Lambda(t,d\omega)\right]\right\}$$

defined by the standard annihilation, creation and exchange operator-valued measures in Fock space with a measurable family $\left(\rho_\omega\right)_{\omega\in\Omega}$ of positive operators in $\mathcal{H}(\omega)$ having the integrable w.r.t. $\mathrm{d}\pi_\omega$ traces $\mathrm{Tr}\rho_\omega < \infty$.

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