ON QUANTUM ITÔ ALGEBRAS AND THEIR DECOMPOSITIONS.

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Abstract. A simple axiomatic characterization of the noncommutative Itô algebra is given and a pseudo-Euclidean fundamental representation for such algebra is described. It is proved that every quotient Itô algebra has a faithful representation in a Minkowski space and is canonically decomposed into the orthogonal sum of quantum Brownian (Wiener) algebra and quantum Lévy (Poisson) algebra. In particular, every quantum thermal noise of a finite number of degrees of freedom is the orthogonal sum of a quantum Wiener noise and a quantum Poisson noise as it is stated by the Lévy-Khinchin theorem in the classical case. Two basic examples of non-commutative Itô finite group algebras are considered.

1. Introduction

The classical differential calculus for the infinitesimal increments $\mathrm{d}x = x(t+\mathrm{d}t) - x(t)$ became generally accepted only after Newton gave a very simple algebraic rule $(\mathrm{d}t)^2 = 0$ for the formal computations of first order differentials for smooth trajectories $t \mapsto x(t)$ in a phase space. The linear space of the differentials $\mathrm{d}x = \alpha \mathrm{d}t$ for a (complex) trajectory became treated at each $x = x(t) \in \mathbb{C}$ as a one-dimensional algebra $\mathfrak{a} = \mathbb{C}đ$ of the elements $a = \alpha đ$ with involution $a^\star = \bar{\alpha} đ$ given by the complex conjugation $\alpha \mapsto \bar{\alpha}$ of the derivative $\alpha = \mathrm{d}x/\mathrm{d}t \in \mathbb{C}$ and the nilpotent multiplication $a \cdot a^\star = 0$ corresponding to the multiplication table $đ \cdot đ^\star = 0$ for the basic nilpotent element $đ = đ^\star$, the abstract notation of $\mathrm{d}t$. Note that the nilpotent $\star$-algebra $\mathfrak{a}$ of abstract infinitesimals $\alpha đ$ has no realization in complex numbers, as well as no operator representation $\alpha Đ$ with a Hermitian nilpotent $Đ = Đ^\dagger$ in a Euclidean (complex pre-Hilbert) space, but it can be represented by the algebra of complex nilpotent 2×2 matrices $\hat{a} = \alpha \hat{đ}$, where $\hat{đ} = \frac{1}{2}(\hat{\sigma}_3 + i\hat{\sigma}_1) = \hat{đ}^\dagger$ with respect to the standard Minkowski metric $(\mathrm{x}|\mathrm{x}) = |\zeta|^2 - |\eta|^2$ for $\mathrm{x} = \zeta \mathrm{e}_+ + \eta \mathrm{e}_-$ in $\mathbb{C}^2$. The complex pseudo-Hermitian nilpotent matrix $\hat{đ}$, $\hat{đ}^2 = 0$, representing the multiplication $đ^2 = đ \cdot đ = 0$, has the canonical triangular form

$$
(1.1) \qquad Đ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad Đ^\dagger = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} Đ^* \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = Đ, \quad Đ^* = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}
$$

in the basis $\mathrm{k}_\pm = (\mathrm{e}_+ \pm \mathrm{e}_-)/\sqrt{2}$ in which $(\mathbf{x}Đ|\mathbf{x}) = \overline{(\mathbf{x}Đ|\mathbf{x})}$ for all $\mathbf{x} = (\xi_-, \xi_+)$ with respect to the pseudo-Euclidean scalar product $(\mathbf{x}|\mathbf{x}) = \xi_- \xi^- + \xi_+ \xi^+$, where $\xi^\pm = (\zeta \pm \eta)/\sqrt{2} = \bar{\xi}_\mp \in \mathbb{C}$.

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The Newton's formal computations can be generalized to non-smooth paths to include the calculus of first order forward differentials $\mathrm{d}y \simeq (\mathrm{d}t)^{1/2}$ of continuous diffusions $y(t) \in \mathbb{R}$ which have no derivative at any t, and the forward differentials $\mathrm{d}n \in \{0,1\}$ of left continuous counting trajectories $n(t) \in \mathbb{Z}_+$ which have zero derivative for almost all t (except the points of discontinuity when $\mathrm{d}n = 1$). The first is usually done by adding the rules

$$(\mathrm{d}w)^2 = \mathrm{d}t, \quad \mathrm{d}w\mathrm{d}t = 0 = \mathrm{d}t\mathrm{d}w \tag{1.2}$$

in formal computations of continuous trajectories having the first order forward differentials $\mathrm{d}x = \alpha\mathrm{d}t + \beta\mathrm{d}w$ with the diffusive part given by the increments of standard Brownian paths $w(t)$. The second can be done by adding the rules

$$(\mathrm{d}m)^2 = \mathrm{d}m + \mathrm{d}t, \quad \mathrm{d}m\mathrm{d}t = 0 = \mathrm{d}t\mathrm{d}m \tag{1.3}$$

in formal computations of left continuous and smooth for almost all t trajectories having the forward differentials $\mathrm{d}x = \alpha\mathrm{d}t + \gamma\mathrm{d}m$ with jumping part $\mathrm{d}z \in \{\gamma, -\gamma\mathrm{d}t\}$ given by the increments of standard Lévy paths $m(t) = n(t) - t$. These rules, well known since the beginning of this century, were formalized by Itô [1] into the form of a stochastic calculus: the first one is now known as the multiplication rule for the forward differential of the standard Wiener process $w(t)$, and the second one is the multiplication rule for the forward differential of the standard Poisson process $n(t)$, compensated by its mean value t.

The linear span of $\mathrm{d}t$ and $\mathrm{d}w$ forms a two-dimensional differential Itô algebra $\mathfrak{b} = \mathbb{C}đ+\mathbb{C}d_w$ for the complex Brownian motions $x(t) = \int \alpha\mathrm{d}t + \int \eta\mathrm{d}w$, where $d_w = d_w^\star$ is a nilpotent of second order element, representing the real increment $\mathrm{d}w$, with multiplication table $d_w^2 = đ$, $d_w \cdot đ = 0 = đ \cdot d_w$, while the linear span of $\mathrm{d}t$ and $\mathrm{d}m$ forms a two-dimensional differential Itô algebra $\mathfrak{c} = \mathbb{C}đ+\mathbb{C}d_m$ for the complex Lévy motions $x = \int \alpha\mathrm{d}t + \int \zeta\mathrm{d}m$, where $d_m = d_m^\star$ is a basic element, representing the real increment $\mathrm{d}m$, with multiplication table $d_m^2 = d_m + đ$, $d_m \cdot đ = 0 = đ \cdot d_m$. As in the case of the Newton algebra, the Itô $\star$-algebras $\mathfrak{b}$ and $\mathfrak{c}$ have no Euclidean operator realization, but they can be represented by the algebras of triangular matrices $\mathrm{B} = \alpha\mathrm{Đ}+\eta\mathrm{D}_w$, $\mathrm{C} = \alpha\mathrm{Đ}+\zeta\mathrm{D}_m$ with pseudo-Hermitian basis elements

$$\mathrm{Đ} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathrm{D}_w = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathrm{D}_m = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \tag{1.4}$$

$$\mathrm{Đ}^\dagger = \mathrm{Đ}, \quad \mathrm{D}_w^\dagger = \mathrm{D}_w, \quad \mathrm{D}_m^\dagger = \mathrm{D}_m$$

where $\left(\mathbf{x}\mathrm{B}^\dagger|\mathbf{x}\right) = \overline{(\mathbf{x}\mathrm{B}|\mathbf{x})}$ for all $\mathbf{x} = \left(\xi_-, \xi_\circ, \xi_+\right) \in \mathbb{C}^3$ in the complex three-dimensional Minkowski space with respect to the indefinite metric $(\mathbf{x}|\mathbf{x}) = \xi_-\xi^- + \xi_\circ\xi^\circ + \xi_+\xi^+$, where $\xi^\mu = \bar{\xi}_{-\mu}$ with $-(-,\circ,+) = (+,\circ,-)$.

Note that according to the Lévy-Khinchin theorem, every stochastic process $x(t)$ with independent increments can be canonically decomposed into a smooth, Wiener and Poisson parts as in the mixed case of one-dimensional complex motion $x(t) = \int \alpha\mathrm{d}t + \int \eta\mathrm{d}w + \int \zeta\mathrm{d}m$ given by the orthogonal and thus commutative increments $\mathrm{d}w\mathrm{d}m = 0 = \mathrm{d}m\mathrm{d}w$. In fact such generalized commutative differential calculus applies not only to the stochastic integration with respect to the processes with independent increments; these formal algebraic rules, or their multidimensional versions, can be used for formal computations of forward differentials for any classical trajectories decomposed into the smooth, diffusive and jumping parts.

Two natural questions arise: are there other then these two commutative differential algebras which could be useful, in particular, for formal computations of the noncommutative differentials in quantum theory, and if there are, is it possible to characterize them by simple axioms and to give a generalized version of the Lévy-Khinchin decomposition theorem? The first question has been already positively answered since the well known differential realization of the simplest non-commutative table $d_w d_w^\star = \rho_+ đ$, $d_w^\star d_w = \rho_- đ$ for $\rho_+ > \rho_- \geq 0$ was given in the mid of 60-th in terms of the annihilators $\hat{w}(t)$ and creators $\hat{w}^\dagger(t)$ of a quantum Brownian thermal noise [2]. This paper gives a systematic answer on the second question, the first part of which has been in principle positively resolved in our papers [4, 5].

Although the orthogonality condition $d_w \cdot d_m = 0 = d_w \cdot d_m$ for the classical independent increments $\mathrm{d}w$ and $\mathrm{d}m$ can be realized only in a higher, at least four, dimensional Minkowski space, it is interesting to make sense of the non-commutative $\star$-algebra, generated by three dimensional non-orthogonal matrix representations (1.4) of these differentials with $d_w \cdot d_m \neq d_w \cdot d_m$:

$$\mathrm{D}_w \mathrm{D}_m = (\mathrm{D}_m \mathrm{D}_w)^\dagger = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = (\mathrm{D}_w \mathrm{D}_m)^\dagger = \mathrm{D}_m \mathrm{D}_w.$$

This is the four-dimensional $\star$-algebra $\mathfrak{a} = \mathbb{C}\text{Đ} + \mathbb{C}\mathrm{E}_- + \mathbb{C}\mathrm{E}^+ + \mathbb{C}\mathrm{E}$ of triangular matrices $\mathrm{A} = \alpha + z^- \mathrm{E}_- + z_+ \mathrm{E}^+ + z\mathrm{E}$,

$$\mathrm{E}_- = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ \mathrm{E}^+ = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \ \mathrm{E} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where $\mathrm{E}^+ = \mathrm{G}\mathrm{E}_-^* \mathrm{G} = \mathrm{E}_-^\dagger$, $\mathrm{E} = \mathrm{E}^\dagger$ with respect to the Minkowski metric tensor G in the canonical basis. given by the algebraic combinations

$$\mathrm{E}_- = \mathrm{D}_w \mathrm{D}_m - , \quad \mathrm{E}^+ = \mathrm{D}_m \mathrm{D}_w - , \quad \mathrm{E} = \mathrm{D}_m - \mathrm{D}_w$$

of three matrices (1.4). It realizes the multiplication table

$$\begin{aligned} e_- \cdot e^+ &= đ, \quad e_- \cdot e = e_-, \\ e \cdot e^+ &= e^+, \quad e \cdot e = e \end{aligned}$$

with the products for all other pairs being zero, unifying the commutative tables (1.2), (1.3). It is well known HP (Hudson-Parthasarathy) table of the vacuum quantum stochastic calculus [3]

$$\begin{aligned} \mathrm{d}\Lambda_- \mathrm{d}\Lambda^+ &= I \mathrm{d}t, \quad \mathrm{d}\Lambda_- \mathrm{d}\Lambda = \mathrm{d}\Lambda_-, \\ \mathrm{d}\Lambda \mathrm{d}\Lambda^+ &= \mathrm{d}\Lambda^+, \quad \mathrm{d}\Lambda \cdot \mathrm{d}\Lambda = \mathrm{d}\Lambda, \end{aligned}$$

with zero products for all other pairs, for the multiplication of the canonical number $\mathrm{d}\Lambda$, creation $\mathrm{d}\Lambda^+$, annihilation $\mathrm{d}\Lambda_-$, and preservation $\mathrm{d}\Lambda_+^- = I\mathrm{d}t$ differentials in Fock space over the Hilbert space $L^2(\mathbb{R}_+)$ of square-integrable complex functions $f(t), t \in \mathbb{R}_+$.

Note that any two-dimensional Itô $\star$-algebra $\mathfrak{a}$ is commutative as $đa = 0 = ađ$ for any other element $a \neq đ$ of the basis $\{a, đ\}$ in $\mathfrak{a}$. Moreover, each such algebra is either of the Wiener or of the Poisson type, as it is either second order nilpotent, or contains a unital one-dimensional subalgebra, as the cases of the subalgebras $\mathfrak{b}, \mathfrak{c}$. Other two-dimensional sub-algebras containing $đ$, are generated by either Wiener

$d_w = \bar{\xi}e_- + \xi e^+$ or Poisson $d_m = e + \lambda d_w$ element with the special case $d_m = e$, corresponding to the only non-faithful Itô algebra of the Poisson process with zero intensity $\lambda^2 = 0$. However there is only one three dimensional $\star$-subalgebra of the four-dimensional HP algebra with $đ$, namely the noncommutative subalgebra of vacuum Brownian motion, generated by the creation e^+ and annihilation e_- differentials. Thus our results on the classification of noncommutative Itô $\star$-algebras will be nontrivial only in the higher dimensions of $\mathfrak{a}$.

The well known Lévy-Khinchin classification of the classical noise can be reformulated in purely algebraic terms as the decomposability of any commutative Itô algebra into Wiener (Brownian) and Poisson (Lévy) orthogonal components. In the general case we shall show that every Itô $\star$-algebra is also decomposable into a quantum Brownian, and a quantum Lévy orthogonal components.

Thus classical stochastic calculus developed by Itô, and its quantum stochastic analog, given by Hudson and Parthasarathy in [3], was unified in our $\star$-algebraic approach to the operator integration in Fock space [4], in which the classical and quantum calculi become represented as two extreme commutative and noncommutative cases of a generalized Itô calculus.

In the next section we remind the definition of the general Itô algebra given in [4], and show that every such algebra can be embedded as a $\star$-subalgebra into an infinite dimensional vacuum Itô algebra as it was first proved in [5].

2. Representations of Itô $\star$-algebras

The generalized Itô algebra was defined in [4] as a linear span of the differentials

$$\mathrm{d}\Lambda(t,a) = \Lambda(t+\mathrm{d}t,a) - \Lambda(t,a), \quad a \in \mathfrak{a}$$

for a family $\{\Lambda(a) : a \in \mathfrak{a}\}$ of operator-valued integrators $\Lambda(t,a)$ on a pre-Hilbert space, satisfying for each $t \in \mathbb{R}_+$ the $\star$-semigroup conditions

$$\mathrm{d}\Lambda(t, a \cdot b) = \mathrm{d}\Lambda(t,a)\,\mathrm{d}\Lambda(t,b), \tag{2.1}$$

$$\Lambda(t, a^\star) = \Lambda(t,a)^\dagger, \quad \Lambda(t, đ) = tI, \tag{2.2}$$

with mean values $\langle \mathrm{d}\Lambda(t,a)\rangle = l(a)\,\mathrm{d}t$ in a given vector state $\langle\cdot\rangle$, absolutely continuous with respect to $\mathrm{d}t$. Here $\Lambda(t,a)^\dagger$ means the Hermitian conjugation of the (unbounded) operator $\Lambda(t,a)$, which is defined on the pre-Hilbert space for each $t \in \mathbb{R}_+$ as the operator $\Lambda(t,a^\star)$,

$$\mathrm{d}\Lambda(t,a)\,\mathrm{d}\Lambda(t,b) = \mathrm{d}\left(\Lambda(t,a)\,\Lambda(t,b)\right) - \mathrm{d}\Lambda(t,a)\,\Lambda(t,b) - \Lambda(t,a)\,\mathrm{d}\Lambda(t,b),$$

and $\mathrm{d}t$ is embedded into the family of the operator-valued differentials as $\mathrm{d}\Lambda(t,đ)$ with the help of a special element $đ = đ^\star$ of the parametrizing $\star$-semigroup $\mathfrak{a}$.

Assuming that the parametrization is exact such that $\mathrm{d}\Lambda(t,a) = 0 \Rightarrow a = 0$, where $0 = a đ$ for any $a \in \mathfrak{a}$, we can always identify $\mathfrak{a}$ with the linear span,

$$\sum \lambda_i \mathrm{d}\Lambda(t,a_i) = \mathrm{d}\Lambda\left(t, \sum \lambda_i a_i\right), \quad \forall \lambda_i \in \mathbb{C}, a_i \in \mathfrak{a},$$

and consider it as a complex associative $\star$-algebra, having the death $đ \in \mathfrak{a}$, a $\star$-invariant annihilator $\mathfrak{a}\cdot đ = \{0\}$ corresponding to $\mathrm{d}\Lambda(t,\mathfrak{a})\,\mathrm{d}t = \{0\}$. The derivative l of the differential expectations $a \mapsto l(a)\,\mathrm{d}t$ with respect to the Lebesgue measure $\mathrm{d}t$, called the Itô algebra state, is a linear positive $\star$-functional

$$l : \mathfrak{a} \to \mathbb{C}, \quad l(a \cdot a^\star) \geq 0, \quad l(a^\star) = \overline{l(a)}, \quad \forall a \in \mathfrak{a},$$

normalized as $l(\not{d}) = 1$ correspondingly to the determinism $\langle I\mathrm{d}t\rangle = \mathrm{d}t$ of $\mathrm{d}\Lambda(t,\not{d})$. We shall identify the Itô algebra $(\mathrm{d}\Lambda(\mathfrak{a}), l\mathrm{d}t)$ and the parametrizing algebra $(\mathfrak{a}, l)$ and assume that it is faithful in the sense that the $\star$-ideal

$$\mathfrak{i} = \{b \in \mathfrak{a} : l(b) = l(b\cdot c) = l(a\cdot b) = l(a\cdot b\cdot c) = 0 \quad \forall a, c \in \mathfrak{a}\} \tag{2.3}$$

is trivial, $\mathfrak{i} = \{0\}$, otherwise $\mathfrak{a}$ should be factorized with respect to this ideal. Note that the associativity of the algebra $\mathfrak{a}$ as well as the possibility of its noncommutativity is inherited from the associativity and noncommutativity of the operator product $\Delta\Lambda(t,a)\Delta\Lambda(t,b)$ on the pre-Hilbert space.

Now we can study the representations of the Itô algebra $(\mathfrak{a}, l)$. Because any Itô algebra contains the Newton nilpotent subalgebra $(\mathbb{C}\not{d}, l)$, it has no identity and cannot be represented by operators in a Euclidean space even if it is finite-dimensional $\star$-algebra. Thus we have to consider the operator representations of $\mathfrak{a}$ in a pseudo-Euclidean space, and we shall find such representations in a Krein space, including the simplest one, a complex Minkowski space.

Let $\mathbb{K}$ be a complex pseudo-Euclidean space with respect to a separating indefinite metric $(\mathrm{x}|\mathrm{x})$, and $\mathrm{k}\in\mathbb{K}$ be a non-zero vector. We denote by $\mathcal{L}(\mathbb{K})$ the $\dagger$-algebra of all operators $\mathrm{A} : \mathbb{K} \to \mathbb{K}$ with $\mathrm{A}^\dagger\mathbb{K} \subseteq \mathbb{K}$, where $\mathrm{A}^\dagger$ is defined as the kernel of the Hermitian adjoint sesquilinear form $\left(\mathrm{x}|\mathrm{A}^\dagger\mathrm{x}\right) = \overline{(\mathrm{x}|\mathrm{A}\mathrm{x})}$. A linear map $\mathrm{i} : \mathfrak{a} \to \mathcal{L}(\mathbb{K})$ is a representation of the Itô $\star$-algebra $(\mathfrak{a}, l)$ on $(\mathbb{K}, \mathrm{k})$ if

$$\mathrm{i}(a^\star) = \mathrm{i}(a)^\dagger, \quad \mathrm{i}(a\cdot b) = \mathrm{i}(a)\,\mathrm{i}(b), \quad (\mathrm{k}|\mathrm{i}(a)\,\mathrm{k}) = l(a) \quad \forall a, b \in \mathfrak{a}. \tag{2.4}$$

We can always assume that $(\mathrm{k}|\mathrm{k}) = 0$, otherwise k should be replaced by the vector $\mathrm{k}_+ = \mathrm{k} - \frac{1}{2}(\mathrm{k}|\mathrm{k})\,\mathrm{k}_-$, where $\mathrm{k}_- = \mathrm{i}(\not{d})\,\mathrm{k}$, with the same result

$$(\mathrm{k}_+|\mathrm{i}(a)\,\mathrm{k}_+) = l(a) - \frac{1}{2}(\mathrm{k}|\mathrm{k})\left(\mathrm{k}|\mathrm{i}\left(\not{d}a + a\not{d} - \frac{1}{2}(\mathrm{k}|\mathrm{k})\,\not{d}a\not{d}\right)\mathrm{k}\right) = l(a). \tag{2.5}$$

Proposition 1. *Every operator representation* $(\mathbb{K},\mathrm{i},\mathrm{k})$ *of any Itô algebra* $(\mathfrak{a}, l)$ *is equivalent to the triangular-matrix representation* $\mathbf{i} = [i^\mu_\nu]^{\mu=-,\circ,+}_{\nu=-,\circ,+}$ *with* $i^\mu_\nu(a) = 0$ *if* $\mu = +$ *or* $\nu = -$ *and* $i^-_+(a) = l(a)$ *for all* $a \in \mathfrak{a}$. *Here* $a^\mu_\nu = i^\mu_\nu(a)$ *are linear operators* $\mathbb{K}_\nu \to \mathbb{K}_\mu$ *on a pseudo-Hilbert (Euclidean if minimal) space* $\mathbb{K}_\circ$ *and on* $\mathbb{K}_+ = \mathbb{C} = \mathbb{K}_-$, *having the adjoints* $a^{\mu\dagger}_\nu : \mathbb{K}_\mu \to \mathbb{K}_\nu$, *which define the pseudo-Hermitian involution* $\mathbf{a} \mapsto \mathbf{a}^\dagger$ *by* $a^{\star\mu}_{-\nu} = a^{\nu\dagger}_{-\mu}$, *where* $-(-,\circ,+) = (+,\circ,-)$. *Moreover, if the representation is minimal, then* $\mathbb{K}_\circ$ *is a Euclidean space and* $i^\mu_\nu(\not{d}) = \delta^\mu_-\delta^+_\nu$.

Proof. In the matrix notation $i^\mu_\nu(a) = \mathrm{k}^\mu\mathrm{i}(a)\,\mathrm{k}_\nu$, where $\mathrm{k}^- = \mathrm{k}^\dagger_+$, $\mathrm{k}^+ = \mathrm{k}^\dagger_-$ are defined by $\mathrm{k}^\dagger\mathrm{x} = (\mathrm{k}|\mathrm{x})$ for all $\mathrm{k},\mathrm{x} \in \mathbb{K}$, (2.5) can be written as $i^-_+(a) = l(a)$, and $i^+_+(a) = 0 = i^-_-(a)$ and $i^+_-(a) = 0$ as

$$\mathrm{i}(a)\,\mathrm{k}_- = \mathrm{i}(a\not{d})\,\mathrm{k} = 0 = \mathrm{k}^\dagger\mathrm{i}(\not{d}a) = \mathrm{k}^+\mathrm{i}(a) \quad \forall a \in \mathfrak{a}.$$

Moreover, due to the pseudo-orthogonality

$$(\mathrm{x}|\mathrm{x}) = \xi_-\xi^- + (\mathrm{x}_\circ|\mathrm{x}_\circ) + \xi_+\xi^+ \equiv (\mathbf{x}|\mathbf{x}),$$

of the decomposition $\mathrm{x}=\xi^-\mathrm{k}_- + \mathrm{x}_\circ + \xi^+\mathrm{k}_+$, where $\xi^- = \mathrm{k}^-\mathrm{x} = \bar{\xi}_+$, $\xi^+ = \mathrm{k}^+\mathrm{x} = \bar{\xi}_-$, $\mathbf{x} = \left(\xi_-, \mathrm{x}^\dagger_\circ, \xi_+\right)$, the representation of the Itô $\star$-algebra $(\mathfrak{a}, l)$ is defined by the homomorphism $\mathbf{i} : a \mapsto [i^\mu_\nu(a)]$ into the space of triangular block-matrices $\mathbf{a} = [a^\mu_\nu]^{\mu=-,\circ,+}_{\nu=-,\circ,+}$ with $a^\mu_\nu = 0$ if $\mu = +$ or $\nu = -$.

If the representation is minimal in the sense that $\mathbb{K}=\mathrm{i}\,(\mathfrak{a})\,\mathrm{k}$, and k has zero length, it is pseudo-unitary equivalent to the triangular representation on the complex Minkowski space $\mathbb{C}\oplus\mathbb{K}_\circ\oplus\mathbb{C}$, as it can be easily seen in the basis $\mathrm{k}_+ = \mathrm{k}$, $\mathrm{k}_- = \mathrm{i}\,(\text{đ})\,\mathrm{k}$. Indeed, the pseudo-orthogonal to the zero length vectors $\mathrm{k}_-, \mathrm{k}_+$ space $\mathbb{K}_\circ$ in this case is the complex Euclidean space $\mathbb{K}_\circ = \{\mathrm{i}\,(a)\,\mathrm{k} : l\,(a) = 0\}$ as

$$\xi^+ = \mathrm{k}^+\mathrm{i}\,(a)\,\mathrm{k} = 0, \quad \xi^- = \mathrm{k}^-\mathrm{i}\,(a)\,\mathrm{k} = l\,(a) \quad \forall a \in \mathfrak{a},$$

and $(\mathrm{x}_\circ|\mathrm{x}_\circ) = l\,(a^\star \cdot a) \geq 0$ for all $\mathrm{x}_\circ = \mathrm{i}\,(a)\,\mathrm{k} - \xi^-\mathrm{k}_- = \mathrm{i}\,(a - l\,(a)\,\text{đ})\,\mathrm{k}$. Moreover, in the minimal representation $i_\circ^-\,(\text{đ}) = 0 = i_+^\circ\,(\text{đ})$ and $i_\circ^\circ\,(\text{đ}) = 0$ as

$$\mathrm{i}\,(\text{đ})\,\mathrm{x}_\circ = \mathrm{i}\,(\text{đ}a)\,\mathrm{k} = 0 = \mathrm{i}\,(a^\star\text{đ})\,\mathrm{k} = \mathrm{x}_\circ^\dagger\mathrm{i}\,(\text{đ}) \quad \forall \mathrm{x}_\circ \in \mathbb{K}_\circ.$$

Thus, the only nonzero matrix element of $\mathrm{i}\,(\text{đ})$ is $i_+^-\,(\text{đ}) = 1$. □

Note that the matrix representation is also defined as the right representation $\mathbf{x} \mapsto \mathbf{xa}$ on all raw-vectors $\mathbf{x} = (\xi_-, \xi, \xi_+)$, $\xi \in \mathbb{K}_\circ^\dagger$, into the dual space $\mathbb{K}^* = \mathbb{C}\times\mathbb{K}_\circ^*\times\mathbb{C} \supseteq \mathbb{K}^\dagger$ with the invariant $\mathbb{K}^\dagger = \{\mathrm{x}^\dagger : \mathrm{x} \in \mathbb{K}\}$ such that $a_+^- = (\mathrm{k}^-\mathbf{a}|\mathrm{k}^-) = l\,(a)$, where $\mathrm{k}^- = (1, 0, 0)$. We shall call the triangular matrix representation on a Minkowski space $\mathbb{C}\oplus\mathcal{K}\oplus\mathbb{C}$ canonical if $\mathcal{K}$ is a minimal pre-Hilbert space. Thus we have proved the second part of the following

Theorem 2. *Every Itô $\star$-algebra $(\mathfrak{a}, l)$ can be canonically realized in a complex Minkowski space. Moreover, every minimal closed pseudo-Euclidean representation is equivalent to the canonical one in the Minkowski space.*

Proof. Now we construct a faithful canonical operator representation for any Itô algebra $(\mathfrak{a}, l)$. The functional l defines for each $a \in \mathfrak{a}$ the canonical quadruple

$$a_\circ^\circ = i\,(a)\,, \quad a_+^\circ = k\,(a)\,, \quad a_\circ^- = k^\dagger\,(a)\,, \quad a_+^- = l\,(a)\,, \tag{2.6}$$

where $i\,(a) = i\,(a^\star)^\dagger$ is the GNS representation $k\,(ab) = i\,(a)\,k\,(b)$ of $\mathfrak{a}$ in the pre-Hilbert space $\mathbb{K}_\circ \ni k\,(b)$, $b \in \mathfrak{a}$ of the Kolmogorov decomposition $l\,(a\cdot b) = k^\dagger\,(a)\,k\,(b)$, and $k^\dagger\,(a) = k\,(a^\star)^\dagger$. Such quadrupole representation $\boldsymbol{i} : a \mapsto \boldsymbol{a} = (a_\nu^\mu)_{\nu=+,\circ}^{\mu=-,\circ}$ of $\mathfrak{a}$ is multiplicative, $\boldsymbol{i}\,(a\cdot b) = (a_\circ^\mu b_\nu^\circ)_{\nu=+,\circ}^{\mu=-,\circ}$ with respect to the product given by the convolution of the components a_ν and b^μ over the common index values $\mu = \circ = \nu$:

$$\begin{aligned} i\,(a)\,i\,(b) &= i\,(a\cdot b)\,, \quad k^\dagger\,(a)\,i\,(b) = k^\dagger\,(a\cdot b) \\ i\,(a)\,k\,(b) &= k\,(a\cdot b)\,, \quad k^\dagger\,(a)\,k\,(b) = l\,(a\cdot b)\,. \end{aligned}$$

It is faithful because of the triviality of the ideal (2.3). One can also use the convenience $a_-^\mu = 0 = a_\nu^+$ of the tensor notations (2.6), extending the quadruples $\boldsymbol{a} = \boldsymbol{i}\,(a)$ to the triangular matrices $\mathbf{a} = [a_\nu^\mu]_{\nu=-,\circ,+}^{\mu=-,\circ,+}$, in which (3.2) is simply given by $\mathbf{i}\,(a\cdot b) = \mathbf{ab}$ in terms of the usual product of the matrices $\mathbf{a} = \mathbf{i}\,(a)$ and $\mathbf{b} = \mathbf{i}\,(b)$. However the involution $a \mapsto a^\star$, which is given by the Hermitian conjugation $\boldsymbol{i}\,(a^\star) = \left(a_{-\mu}^{-\nu\dagger}\right)_{\nu=+,\circ}^{\mu=-,\circ}$ of the quadruples $\boldsymbol{a}$, where $-(-) = +$, $-\circ = \circ$, $-(+) = -$, is represented by the adjoint matrix $\mathbf{a}^\dagger = \mathrm{G}\mathbf{a}^*\mathrm{G}$ w.r.t. the pseudo-Euclidean (complex Minkowski) metric tensor $\mathrm{G} = \left[\delta_{-\nu}^\mu\right]_{\nu=-,\circ,+}^{\mu=-,\circ,+}$. Thus, we have constructed the faithful canonical representation

$$\mathbf{i}(a) = \begin{bmatrix} 0 & k^{\dagger}(a) & l(a) \\ 0 & i(a) & k(a) \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{i}(a\cdot b) = \mathbf{i}(a)\,\mathbf{i}(b), \tag{2.7}$$

$$\mathbf{i}(a^{\star}) = \mathrm{G}\mathbf{i}(a)^{*}\mathrm{G}, \qquad \mathrm{G} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & I & 0 \\ 1 & 0 & 0 \end{bmatrix} \tag{2.8}$$

in the Minkowski space $\mathbb{C}\oplus\mathcal{K}\oplus\mathbb{C}$ with $\mathcal{K} = k(\mathfrak{a})$ and $\mathrm{k}^{-} = (1,0,0)$. □

3. Decomposition of Itô ⋆-algebras

This was already noted in [4, 5] that every (classical or quantum) stochastic noise described by a process $t \in \mathbb{R}_{+} \mapsto \Lambda(t,a), a \in \mathfrak{a}$ with independent increments $\mathrm{d}\Lambda(t,a) = \Lambda(t+\mathrm{d}t,a) - \Lambda(t,a)$ forming an Itô †-algebra, can be represented in the Fock space $\mathfrak{F}$ over the space of $\mathcal{K}$-valued square-integrable functions on $\mathbb{R}_{+}$ with the vacuum vector state. This representation is given by $\Lambda(t,a) = a^{\mu}_{\nu}\Lambda^{\nu}_{\mu}(t)$, where

$$a^{\mu}_{\nu}\Lambda^{\nu}_{\mu}(t) = a^{\circ}_{\circ}\Lambda^{\circ}_{\circ}(t) + a^{\circ}_{+}\Lambda^{+}_{\circ}(t) + a^{-}_{\circ}\Lambda^{\circ}_{-}(t) + a^{-}_{+}\Lambda^{+}_{-}(t), \tag{3.1}$$

is the canonical decomposition of Λ into the exchange $\Lambda^{\circ}_{\circ}$, creation $\Lambda^{+}_{\circ}$, annihilation Λ°_{-} and preservation (time) $\Lambda^{+}_{-} = t\mathrm{I}$ operator-valued processes of the vacuum quantum stochastic calculus, having the mean values $\langle\Lambda^{\nu}_{\mu}(t)\rangle = t\delta^{\nu}_{+}\delta^{-}_{\mu}$, and $a^{\mu}_{\nu} = i^{\mu}_{\nu}(a)$ are the matrix elements of the canonical representation associated with the Itô algebra state l. If the Itô algebra $\mathfrak{a}$ is faithful in the sense of the triviality of the ideal (2.3), the constructed canonical representation (2.7) is obviously also faithful. Thus we can identify the faithful algebra $\mathfrak{a}$ with the family of quadrupoles $\boldsymbol{a} = (i^{\mu}_{\nu}(a))^{\mu=-,\circ}_{\nu=+,\circ}$, and the state l on $\mathfrak{a}$ with $l(\boldsymbol{a}) = i^{-}_{+}(a)$, by saying that the Itô algebra is given in its fundamental representation.

Definition 1. *Let $\mathcal{K}$ be a pre-Hilbert space, and $\mathfrak{l}(\mathcal{K})$ be the associated ⋆-algebra of all quadrupoles $A = (a^{\mu}_{\nu})^{\mu=-,\circ}_{\nu=+,\circ}$, where a^{μ}_{ν} are linear operators $\mathbb{K}_{\nu} \to \mathbb{K}_{\mu}$ with $\mathbb{K}_{\circ} = \mathcal{K}$, $\mathbb{K}_{+} = \mathbb{C} = \mathbb{K}_{-}$, having the adjoints $a^{\mu\dagger}_{\nu} : \mathbb{K}_{\mu} \to \mathbb{K}_{\nu}$, with the product and involution*

$$A\cdot B = (a^{\mu}_{\circ}b^{\circ}_{\nu})^{\mu=-,\circ}_{\nu=+,\circ}, \quad A^{*} = \left(a^{-\nu\dagger}_{-\mu}\right)^{\mu=-,\circ}_{\nu=+,\circ}. \tag{3.2}$$

It is an Itô algebra with respect to $l(\boldsymbol{a}) = a^{-}_{+}$ and the death $Đ = \left(\delta^{\mu}_{-}\delta^{+}_{\nu}\right)^{\mu=-,\circ}_{\nu=+,\circ} = Đ^{*}$, $A\cdot Đ = 0$, $\forall A \in \mathfrak{l}(\mathcal{K})$, called the vacuum, or HP (Hudson-Parthasarathy) algebra associated with the space $\mathcal{K}$. The fundamental representation of an Itô algebra $(\mathfrak{a}, l)$ is given by the constructed canonical homomorphism $\boldsymbol{i} : \mathfrak{a} \to \mathfrak{l}(\mathcal{K})$

$$\begin{aligned} \boldsymbol{i}(a) &= \begin{pmatrix} l(a) & k^{\dagger}(a) \\ k(a) & i(a) \end{pmatrix}, \ \boldsymbol{i}(a^{\star}) = \begin{pmatrix} l(a^{\star}) & k(a)^{\dagger} \\ k(a^{\star}) & i(a)^{\dagger} \end{pmatrix}, \\ \boldsymbol{i}(a\cdot b) &= \boldsymbol{i}(a)\cdot\boldsymbol{i}(b), \quad \boldsymbol{i}(a^{\star}) = \boldsymbol{i}(a)^{*} \end{aligned}$$

into the HP algebra, associated with the space $\mathcal{K}$ of its canonical representation. An Itô algebra is called vacuum B^{*}-algebra if $\mathfrak{n}^{\perp}_{+} = \mathfrak{n}^{-}$, where

$$\mathfrak{n}_{+} = \{c \in \mathfrak{a} : k(c) = 0\}, \quad \mathfrak{n}^{-} = \left\{b \in \mathfrak{a} : k^{\dagger}(b) = 0\right\}, \tag{3.3}$$

and $\mathfrak{n}_+^\perp$ is the right (and left) orthogonal complement to $\mathfrak{n}_+$, and it is called thermal algebra if $\mathfrak{n}_+ = \mathbb{C}đ = \mathfrak{n}^-$ and the involution $\star$ is left (or right) closable on the pre-Hilbert space $\mathcal{D} = \mathfrak{a}/\mathbb{C}đ$.

A subalgebra of $\mathfrak{l}(\mathcal{K})$ is a vacuum Itô algebra iff from orthogonality of $k(a)$ to all $k(c)$ with $k^\dagger(c) = 0$ it follows that $k(a) = 0$. This means the maximality $\mathfrak{n}_+ = \mathfrak{k}^-$ of the left ideal $\mathfrak{n}_+ = k^{-1}(0) \subseteq \mathfrak{k}^-$, where

$$\mathfrak{k}^- = \left\{a \in \mathfrak{a} : \langle a|b\rangle_+ = 0, \forall b \in \mathfrak{n}^-\right\} = \mathfrak{k}_+^\star \tag{3.4}$$

or $\mathfrak{n}^- = \mathfrak{k}_+$ as the right null ideal $\mathfrak{n}^- = \mathfrak{n}_+^\star$ for the map $k^\dagger$ in terms of the right orthogonal complement $\mathfrak{k}_+ = \mathfrak{n}_+^\perp$. It follows from the canonical construction of $\mathcal{K}$ as the quotient space $\mathfrak{a}/\mathfrak{n}_+$. Note that due to the orthogonality of $\mathfrak{k}_+$ and $\mathfrak{k}^-$ in vacuum Itô algebras, the involution $\star$ is never defined in $\mathfrak{k}_+$ or in $\mathfrak{k}^-$ except on the jointly null ideal $\mathfrak{n}^- \cap \mathfrak{n}_+$.

In the case of thermal Itô algebras the ideals $\mathfrak{n}_+$ (and $\mathfrak{n}^-$) are minimal, and the involution $\star$ is defined into $\mathfrak{k}_+$ on the whole $\mathfrak{k}_+ = \mathfrak{a} = \mathfrak{k}^-$, and thus on the pre-Hilbert space $\mathcal{D} = \mathfrak{a}/\mathbb{C}đ$ identified with $\{x = a - l(a)\, đ : a \in \mathfrak{a}\}$, by $x^\star = a^\star - l(a^\star)\, đ$. So the subalgebra of $\mathfrak{l}(\mathcal{K})$ is a thermal algebra iff the involution is closable on the dense domain $\mathcal{D} = k(\mathfrak{a})$ of the GNS space $\mathcal{K}$, as it is in the case of tracial Itô algebras, when the involution is isometric. The involution $\star$ onto $\mathcal{D}$ has densely defined left and right adjoints in $\mathcal{D}$ (coinciding with it in the tracial case) iff it is closable.

We shall call an Itô algebra $\mathfrak{a}$ the Brownian algebra if $i(\mathfrak{a}) = 0$, and the Lévy algebra in the opposite case, when $i(\mathfrak{a})$ is non-degenerated on $k(\mathfrak{a})$ and thus $i(\mathfrak{a})$ has an identity operator $I \in i(\mathfrak{a})$ in the finite dimensional case. We shall say that an Itô algebra has a quotient identity $e \in \mathfrak{a}$ if $E = i(e) = E^\dagger$ is the identity for the operator algebra $\mathcal{A} = i(\mathfrak{a})$. The following theorem proves that every Itô algebra is an orthogonal sum of a Brownian algebra and of a Lévy algebra at least in the finite dimensional case, as it states the famous Lévy-Khinchin theorem in the commutative case. A general infinite dimensional non-commutative version of the Lévy-Khinchin decomposition theorem is also true and will be published elsewhere.

Theorem 3. *Let $\mathfrak{a}$ be an Itô algebra with a quotient identity. Then it is an orthogonal sum $\mathfrak{b} + \mathfrak{c}$, $\mathfrak{b} \cdot \mathfrak{c} = 0$ of a quantum Brownian algebra $\mathfrak{b}$ and a quantum Lévy algebra $\mathfrak{c}$.*

Proof. We assume that the quotient algebra $\mathfrak{a}/\mathfrak{n}$ with respect to the null $\star$-ideal $\mathfrak{n} = \{a \in \mathfrak{a} : i(a) = 0\}$ has an identity, which defines the supporting ortho-projector $E = i(e)$ for the operator representation $\mathcal{A} = i(\mathfrak{a}) \simeq \mathfrak{a}/\mathfrak{n}$ on a pre-Hilbert space $\mathcal{K}$. This means that there exists an element $e = e^\star \in \mathfrak{a}$ such that $aec = ac$ for all $a, c \in \mathfrak{a}$, where

$$ac = a \cdot c - l(a \cdot c)\, đ$$

is the associative factor-product of the $\star$-algebra $\mathfrak{a}/\mathbb{C}đ \simeq \{a \in \mathfrak{a} : l(a) = 0\}$ for the zero mean elements $a - l(a)\, đ$. Ideed, it is so in the canonical representation (2.7) as $i(ae) = i(a) = i(ea)$ for all $a \in \mathfrak{a}$ because $i(đ) = 0$ and $i(e) = E$,

$$k(aec) = i(ae)\, k(c) = k(ac), \quad k^\dagger(aec) = k^\dagger(c)\, i(ea) = k^\dagger(ac),$$

and $l(aec) = 0 = l(ac)$. We assume that e is an idempotent, otherwise it should be replaced by e^2.

We can easily then define the required orthogonal decomposition $\mathfrak{a} = \mathfrak{b} + \mathfrak{c}$ by

$$a = b + c, \quad c = ae + ea - eae.$$

Here b is an element of the quantum Brownian algebra $\mathfrak{b} = \{b \in \mathfrak{a} : be = 0 = eb\} \subseteq \mathfrak{n}$ which is orthogonal to the subalgebra $\mathfrak{a}\mathfrak{a}$ as

$$b \cdot a = bea + l(b \cdot a)\text{đ} = l\,(b \cdot a)\,\text{đ}, \quad a \cdot b = aeb + l\,(a \cdot b)\,\text{đ} = l\,(a \cdot b)\,\text{đ}$$

for all $b \in \mathfrak{b}$, and hence

$$b \cdot aa^\star = ba \cdot a^\star = 0 = a^\star \cdot ab = a^\star a \cdot b$$

for all $a \in \mathfrak{a}$. And c is an element of a quantum Lévy algebra $\mathfrak{c}$, the closure of $\mathfrak{a}\mathfrak{a}$ in $\mathfrak{a}$ which coincides with all the algebraic combinations c as $c^\star c = a^\star a$ for all $a \in \mathfrak{a}$. Thus $a = b + c$, $b \cdot c = 0$ for all $a \in \mathfrak{a}$, where $b \in \mathfrak{b}$ is in a Brownian algebra with the fundamental representation $\mathfrak{b} \subseteq \mathbb{C} \oplus \mathcal{G}_+ \oplus \mathcal{G}^-$, where $\mathcal{G}_+ = Pk\,(\mathfrak{a})$, $P = I - E$, $\mathcal{G}^- = k^\dagger\,(\mathfrak{a})\,P$, and $c \in \mathfrak{c}$ is in a Lévy algebra, having the fundamental representation $\mathfrak{c} \subseteq \mathbb{C} \oplus \mathcal{E}_+ \oplus \mathcal{E}^- \oplus \mathcal{A}$ with non-degenerated operator algebra $\mathcal{A} = i\,(\mathfrak{a})$, left and right represented on $\mathcal{E}_+ = Ek\,(\mathfrak{a})$ and $\mathcal{E}^- = k^\dagger\,(\mathfrak{a})\,E$. □

4. Vacuum and Thermal Itô algebras

Here we consider the two extreme cases of Itô algebras as sub-algebras of the vacuum algebra $\mathfrak{l}\,(\mathcal{K})$ associated with a pre-Hilbert space $\mathcal{K}$. The first case corresponds to a pure state l on $\mathfrak{a}$ as it is in the case of a quantum noise of zero temperature, and the second case corresponds to a completely mixed l as in the case of a quantum noise of a finite temperature.

4.1. **Vacuum noise $\star$-algebra.** Let $\mathcal{K}$ be a pre-Hilbert space of ket-vectors ζ with scalar product $(\zeta|\zeta)$ and $\mathcal{A} \subseteq \mathcal{L}\,(\mathcal{K})$ be a $\star$-algebra, represented on $\mathcal{K}$ by the operators $\mathcal{A} \ni A : \zeta \mapsto A\zeta$ with the adjoints $\left(A^\dagger\zeta|\xi\right) = (\zeta|A\xi)$, $A^\dagger\mathcal{K} \subseteq \mathcal{K}$. We denote by $\mathcal{K}^\dagger$ the dual space of bra-vectors $\eta = \zeta^\dagger$, $\zeta \in \mathcal{K}$ with the scalar product $\left(\eta|\xi^\dagger\right) = \eta\xi = \left(\eta^\dagger|\xi\right)$, $\xi \in \mathcal{K}$ given by inverting anti-linear isomorphism $\mathcal{K}^\dagger \ni \eta \mapsto \eta^\dagger \in \mathcal{K}$, and the dual representation of $\mathcal{A}$ as the right representation $A' : \eta \mapsto \eta A$, $\eta \in \mathcal{K}^\dagger$, given by $(\eta A)\,\zeta = \eta\,(A\zeta)$ such that $\left(\eta A^\dagger|\eta\right) = (\eta|\eta A)$ on $\mathcal{K}^\dagger$. Then the direct sum $\mathcal{K} \oplus \mathcal{K}^\dagger$ of $\xi = \zeta \oplus \eta$ becomes a two-sided $\mathcal{A}$-module

$$A\,(\zeta \oplus \eta) = A\zeta, \quad (\zeta \oplus \eta)\,A = \eta A, \quad \forall \zeta \in \mathcal{K}, \eta \in \mathcal{K}^\dagger, \tag{4.1}$$

with the flip-involution $\xi^\star = \eta^\dagger \oplus \zeta^\dagger$ and two scalar products

$$\left\langle \zeta \oplus \eta' | \zeta' \oplus \eta \right\rangle_+ = \left(\zeta|\zeta'\right), \quad \left\langle \zeta \oplus \eta' | \zeta' \oplus \eta \right\rangle^- = (\eta'|\eta)\,. \tag{4.2}$$

The space $\mathfrak{a} = \mathbb{C} \oplus \mathcal{K} \oplus \mathcal{K}^\dagger \oplus \mathcal{A}$ of triples $a = (\alpha, \xi, A)$ becomes an Itô $\star$-algebra with respect to the non-commutative product

$$a^\star \cdot a = \left(\langle\xi|\xi\rangle_+, \xi^\star A + A^\dagger\xi, A^\dagger A\right), \quad a \cdot a^\star = \left(\langle\xi|\xi\rangle^-, A\xi^\star + \xi A^\dagger, AA^\dagger\right), \tag{4.3}$$

where $(\alpha, \xi, A)^\star = \left(\bar{\alpha}, \xi^\star, A^\dagger\right)$, with death $\text{đ} = (1, 0, 0)$ and $l\,(\alpha, \xi, A) = \alpha$. Obviously $a^\star \cdot a \neq a \cdot a^\star$ if $\|\xi\|_+ = \|\zeta\| \neq \|\eta\| = \|\xi\|^-$ even if the operator algebra $\mathcal{A}$ is commutative, $A^\dagger A = AA^\dagger$.

We shall call such Itô algebra the vacuum algebra as $l\,(a^\star \cdot a) = 0$ for any $a \in \mathfrak{a}$ with $\xi \in \mathcal{K}^\dagger$ (the Hudson-Parthasarathy algebra $\mathfrak{a} = \mathfrak{l}\,(\mathcal{K})$ if $\mathcal{A} = \mathcal{L}\,(\mathcal{K})$). Every Itô algebra is a subalgebra $\mathfrak{a} \subseteq \mathfrak{l}\,(\mathcal{K})$ of the HP algebra $\mathfrak{l}\,(\mathcal{K})$ for a pre-Hilbert space $\mathcal{K}$ with the operator factor-algebra $\mathcal{A}$ represented on $\mathcal{K}$.

If the algebra $\mathcal{A}$ is completely degenerated on $\mathcal{K}$, $\mathcal{A} = \{0\}$, the Itô algebra $\mathfrak{a}$ is nilpotent of second order, and contains only two-dimensional subalgebras of Wiener type $\mathfrak{b} = \mathbb{C} \oplus \mathbb{C} \oplus \{0\}$ generated by an $a = (\alpha, \zeta \oplus \eta, 0)$ with $\|\zeta\| = \|\eta\|$. Every Itô subalgebra $\mathfrak{b} \subseteq \mathfrak{a}$ of the HP algebra $\mathfrak{a} = \mathfrak{l}(\mathcal{K})$ is called the Itô algebra of a vacuum Brownian motion if it is defined by a $\star$-invariant direct sum $\mathcal{G} \oplus \mathcal{G}^\dagger$ given by a subspace $\mathcal{G} \subseteq \mathcal{K}$ and $\mathcal{A} = \{0\}$.

In the case $I \in \mathcal{A}$ the algebra $\mathcal{A}$ is not degenerated and contains also the vacuum Poisson subalgebra $\mathbb{C} \oplus \{0\} \oplus \mathbb{C}I$ of the total quantum number on $\mathcal{K}$, and other Poisson two-dimensional subalgebras, generated by $a = (\alpha, \zeta \oplus \eta, I)$ with $\eta = e^{i\theta}\zeta^\dagger$. We shall call a closed Itô subalgebra $\mathfrak{c} \subseteq \mathfrak{a}$ of the HP algebra $\mathfrak{a} = \mathfrak{l}(\mathcal{K})$ the algebra of a vacuum Lévy motion if it is given by a direct sum $\mathcal{E} \oplus \mathcal{E}^\dagger$ and a $\star$-subalgebra $\mathcal{A} \subseteq \mathcal{L}(\mathcal{K})$ non-degenerated on the subspace $\mathcal{E} \subseteq \mathcal{K}$.

Our decomposition theorem for the vacuum algebras can be reformulated as follows

Theorem 4. *Every vacuum algebra $\mathfrak{a}$ having the unital factor-algebra $\mathcal{A}$ can be decomposed into an orthogonal sum $\mathfrak{a} = \mathfrak{b} + \mathfrak{c}$, $\mathfrak{b} \cdot \mathfrak{c} = \{0\}$ of the Brownian vacuum algebra $\mathfrak{b}$ and the Lévy vacuum algebra $\mathfrak{c}$.*

Proof. This decomposition is uniquely defined for all $a = (\alpha, \xi, A)$ by $a = \alpha \text{đ} + b + c$, with $b = (0, \eta, 0)$, $c = (0, \zeta, A)$, $\eta = P\xi \oplus \xi P \in \mathcal{G}$, $\zeta = \xi - \eta \in \mathcal{E}$, where $P = I - E = P^\dagger$ is the maximal projector in $\mathcal{K}$, for which $\mathcal{A}P = \{0\}$, $\mathcal{G} = P\mathcal{K}$, and $\mathcal{E} = \mathcal{G}^\perp$ is the range of the identity orthoprojector $E \in \mathcal{A}$. □

4.2. **Thermal noise $\star$-algebra.** Let $\mathcal{D}$ be a left $\star$-algebra [6] with respect to a Hilbert norm $\|\xi\|_+ = 0 \Rightarrow \xi = 0$, and thus a right pre-Hilbert $\star$-algebra with respect to $\|\xi\|^- = \|\xi^\star\|_+$. This means that $\mathcal{D}$ is a complex Euclidean space with left (right) multiplications $C : \zeta \mapsto \xi\zeta$ ($C' : \eta \mapsto \eta\xi$) w.r.t. $\|\cdot\|_+$ (w.r.t. $\|\cdot\|^-$) of the elements $\zeta, \eta \in \mathcal{D}$ respectively, defined by an associative product in $\mathcal{D}$, and the involution $\mathcal{D} \ni \xi \mapsto \xi^\star \in \mathcal{D}$ such that

$$(4.4) \qquad \langle \eta\zeta^\star | \xi \rangle^- = \langle \eta | \xi\zeta \rangle^-, \quad \langle \eta^\star\zeta | \xi \rangle_+ = \langle \zeta | \eta\xi \rangle_+ \quad \forall \xi, \zeta, \eta \in \mathcal{D},$$

were $\langle \eta | \xi^\star \rangle^- = \langle \eta^\star | \xi \rangle_+$ is the right scalar product. The involution is assumed to have the adjoints

$$(4.5) \qquad \langle \eta | \xi^\star \rangle^- = \left\langle \xi | \eta^\sharp \right\rangle^-, \quad \langle \zeta | \xi^\star \rangle_+ = \left\langle \xi | \zeta^\flat \right\rangle_+ \quad \forall \eta \in \mathcal{D}^-, \zeta \in \mathcal{D}_+,$$

where $\mathcal{D}_+ = \mathcal{D}_+^\flat$ is a dense domain for the left adjoint involution $\zeta \mapsto \zeta^\flat$, $\zeta^{\flat\flat} = \zeta$, and $\mathcal{D}^- = \mathcal{D}_+^\star$ is the dense domain for the right adjoint involution $\eta \mapsto \eta^\sharp$, $\left(\eta^\sharp\eta\right)^\sharp = \eta^\sharp\eta$ such that $\zeta^{\flat\star} = \zeta^{\star\natural}$, $\eta^{\sharp\star} = \eta^{\star\flat}$.

Note that we do not require the sub-space $\mathcal{D}\mathcal{D} \subseteq \mathcal{D}$ of all products $\{\eta\zeta : \eta, \zeta \in \mathcal{D}\}$ to be dense in $\mathcal{D}$ w.r.t. any of two Hilbert norms on $\mathcal{D}$. Hence the operator factor-algebra $\mathcal{C} = \{C : \mathcal{D} \ni \zeta \mapsto \xi\zeta | \xi \in \mathcal{D}\}$ w.r.t. the left scalar product, which is also represented on the $\mathcal{D} \ni \eta$ equipped with $\langle \cdot | \cdot \rangle^-$ by the right multiplications $\eta C = \eta\xi$, $\xi \in \mathcal{D}$, can be degenerated on $\mathcal{D}$.

Thus the direct sum $\mathfrak{a} = \mathbb{C} \oplus \mathcal{D}$ of pairs $a = (\alpha, \xi)$ becomes an Itô $\star$-algebra with the product

$$(4.6) \qquad a^\star a = \left(\langle \xi | \xi \rangle_+, \xi^\star\xi\right), \quad aa^\star = \left(\langle \xi | \xi \rangle^-, \xi\xi^\star\right),$$

where $(\alpha, \xi)^\star = (\bar{\alpha}, \xi^\star)$, with death $\text{đ} = (1, 0)$ and $l(\alpha, \xi) = \alpha$. Obviously $a^\star a \neq aa^\star$ if the involution $a \mapsto a^\star$ is not isometric w.r.t. any of two Hilbert norms even if the algebra $\mathcal{D}$ is commutative.

We shall call such Itô algebra the thermal Itô algebra as $l(a^\star a) = \|\xi\|_+^2 \neq 0$ for any $a \in \mathfrak{a}$ with $\xi \neq 0$. If $\zeta\eta = 0$ for all $\zeta, \eta \in \mathcal{D}$, it is the Itô algebra of a thermal Brownian motion. A thermal subalgebra $\mathfrak{b} \subseteq \mathfrak{a}$ with such trivial product is given by any involutive pre-Hilbert $\star$-invariant two-normed subspace $\mathcal{G} \subseteq \mathcal{D}$. We shall call such Brownian algebra $\mathfrak{b} = \mathbb{C} \oplus \mathcal{G}$ the quantum (if $\|\cdot\|_+ \neq \|\cdot\|^-$) Wiener algebra associated with the space $\mathcal{G}$.

In the opposite case, if $\mathcal{D}\mathcal{D} = \{\zeta\eta : \zeta, \eta \in \mathcal{D}\}$ is dense in $\mathcal{D}$, it has non-degenerated operator representation $\mathcal{C}$ on $\mathcal{D}$. Any involutive sub-algebra $\mathcal{E} \subseteq \mathcal{D}$ which is non-degenerated on $\mathcal{E}$ defines an Itô algebra $\mathfrak{c} = \mathbb{C} \oplus \mathcal{E}$ of thermal Lévy motion. We shall call such Itô algebra the quantum (if $\mathcal{E}$ is non-commutative) Poisson algebra.

Our decomposition theorem for the thermal algebras can be reformulated as follows

Theorem 5. *Every thermal Itô algebra $\mathfrak{a}$ having the unital subalgebra $\mathcal{D}\mathcal{D}$ is an orthogonal sum $\mathfrak{a} = \mathfrak{b} + \mathfrak{c}$, $\mathfrak{b}\mathfrak{c} = \{0\}$ of the Wiener algebra $\mathfrak{b}$ and the Poisson algebra $\mathfrak{c}$.*

Proof. The orthogonal decomposition $a = \alpha đ + b + c$ for all $a = (\alpha, \xi) \in \mathfrak{a}$, uniquely given by the decomposition $\xi = \eta + \zeta$ w.r.t. any of two scalar products in $\mathcal{D}$, where $\eta = P\xi = \xi P$ is the orthogonal projection onto $\mathcal{G} \perp \mathcal{D}\mathcal{D}$ w.r.t. any of two Hilbert norms, and $\zeta = \xi - \eta$.

Indeed, if $\xi \in \mathcal{D}$ is left orthogonal to $\mathcal{D}\mathcal{D}$, then it is also right orthogonal to $\mathcal{D}\mathcal{D}$ and vice versa:

$$\begin{aligned} \langle \eta\zeta^\star | \xi \rangle^- &= \langle \zeta\eta^\star | \xi^\star \rangle_+ = \left\langle \xi | \eta^{\sharp\star} \zeta^\flat \right\rangle_+ = 0, \quad \forall \eta \in \mathcal{D}^-, \zeta \in \mathcal{D}_+, \\ \langle \eta^\star\zeta | \xi \rangle_+ &= \langle \zeta^\star\eta | \xi^\star \rangle^- = \left\langle \xi | \eta^\sharp \zeta^{\flat\star} \right\rangle^- = 0, \quad \forall \eta \in \mathcal{D}^-, \zeta \in \mathcal{D}_+. \end{aligned}$$

From these and (4.4) equations it follows that $\eta\xi = 0 = \xi\zeta$ for all $\zeta, \eta \in \mathcal{D}$ if ξ is (right or left) orthogonal to $\mathcal{D}\mathcal{D}$, and so $i(\xi) = 0$ for such ξ and vice versa. Thus the orthogonal subspace $\mathcal{G} = \{\xi \in \mathcal{D} : i(\xi) = 0\}$ is the range of the orthoprojector $P = I - E$, where E is the identity orthoprojector of $\mathcal{C}$, representing the unity $\varepsilon = \varepsilon^\star$ of $\mathcal{D}\mathcal{D}$ such that $\varepsilon\xi = \zeta = \xi\varepsilon \in \mathcal{D}$ is in $\mathcal{E} = \mathcal{D}\mathcal{D}$ and $\eta = \xi - \zeta \in \mathcal{G}$. □

Example 1. *The commutative multiplication table $\mathrm{d}b_i \mathrm{d}\bar{b}_k = \delta_k^i \mathrm{d}t$ for the complex Itô differentials $\mathrm{d}b_k = \mathrm{d}\bar{b}_{-k}$, $k = 0, \pm 1, \ldots, \pm K$ of the complex amplitudes*

$$b_k(t) = \sum_{n=1}^{N} e^{jk\theta_n} w_n(t), \quad \mathrm{d}w_i(t)\, \mathrm{d}w_n(t) = \frac{1}{N} \delta_n^i \mathrm{d}t$$

for N independent Wiener processes $w_n, n = 1, \ldots, N$ with $N \geq 2K + 1, \theta_n = \frac{2\pi}{N} n - \pi$ can be generalized in the following way.

Let $\rho_k > 0$, $k = 0, \pm 1, \ldots, \pm K$ be a self-inverse family of spectral eigen-values $\rho_{-k} = \rho_k^{-1}$. The generalized multiplication table $d_i \cdot d_k^\star = \rho_k \delta_k^i đ$ for abstract infinitesimals $d_k = d_{-k}^\star$, $k \in \mathbb{Z}$ is obviously non-commutative for all k with $\rho_k \neq 1$. The $\star$-semigroup $\{0, đ, d_k : |k| \leq K\}$ generates a $2(K+1)$-dimensional Itô algebra $(\mathfrak{b}, l)$ of a quantum Wiener periodic motion on $[-\pi, \pi]$ as it is the second order nilpotent algebra $\mathfrak{a}$ of $a = \alpha đ + b$, $y = \sum \eta^k d_k$ with $y^\star = \sum \eta_k d_k^\star$ and $l(y) = 0$ for all $\eta^k = \bar{\eta}_k \in \mathbb{C}$. It is a Brownian algebra with closed involution on the complex space $\mathcal{D}$ of all η given by all complex sequences $\eta = (\eta_k)_{|k| \leq K}$. The operator representation of

$\mathfrak{a}$ *in Fock space is defined by the forward differentials of* $\Lambda(t,a) = \alpha t\mathrm{I} + \sum \eta^k \hat{w}_k(t)$, *where* $\hat{w}_k(t) = \hat{v}_k^-(t) + \hat{v}_k^+(t)$,

$$\hat{v}_k^-(t) = \left(\frac{\rho_{-k}}{N}\right)^{\frac{1}{2}} \sum_{n=1}^{N} e^{jk\theta_n}\Lambda_-^n(t), \quad \hat{v}_k^+(t) = \left(\frac{\rho_k}{N}\right)^{\frac{1}{2}} \sum_{n=1}^{N} e^{jk\theta_n}\Lambda_n^+(t)$$

are given by the annihilation and creation measures in Fock space over square integrable functions on $\mathbb{R}_+ \times [-\pi,\pi]$ *with the standard multiplication table*

$$\mathrm{d}\Lambda_-^n(t)\,\mathrm{d}\Lambda_m^+(t) = \delta_m^n I\mathrm{d}t, \quad \mathrm{d}\Lambda_m^+(t)\,\mathrm{d}\Lambda_-^n(t) = 0.$$

Example 2. *The commutative multiplication table* $\mathrm{d}c_i\mathrm{d}\bar{c}_k = \delta_k^i\mathrm{d}t + \mathrm{d}c_{i-k}$ *for the complex Itô differentials* $\mathrm{d}c_k = \mathrm{d}\bar{c}_{-k}$, $k = 0, \pm1, \ldots, \pm K$ *of the complex amplitudes* $c_k(t) = \sum_{n=1}^{N} e^{jk\theta_n} m_n(t)$, $\theta_n = \theta_n = \frac{2\pi}{N}n - \pi$, *given by* $N \geq 2K+1$ *compensated Poisson processes* $m_n(t)$ *with*

$$\mathrm{d}m_i(t)\,\mathrm{d}m_n(t) = \frac{1}{N}\delta_n^i\mathrm{d}t + \mathrm{d}m_n(t)\,\delta_n^i, \quad i,n = 1,\ldots,N,$$

can be generalized in the following way.

Let G *be a finite group, and* $G \ni g \mapsto \lambda_g \in \mathbb{C}$ *be a positive-definite function,* $\lambda_{g^{-1}} = \bar{\lambda}_g$, *which is self-inverse in the convolutional sense* $\left[\bar{\lambda} * \lambda\right]_g = \sum_h \bar{\lambda}_{gh^{-1}}\lambda_h = \delta_g^1$. *The generalized multiplication table for abstract infinitesimals*

$$d_g = d_{-g}^\star, \quad d_g \cdot d_h^\star = \lambda_{gh^{-1}}\,đ + d_{gh^{-1}}, \quad g,h \in G$$

is obviously associative and commutative if G *is Abelian as in the above case, but it is non-commutative for non Abelian* G *even if* $\lambda_k = \delta_k^1$ *as in the above case. The* $\star$*-semigroup* $\{0, đ, d_g : g \in G\}$ *generates finite dimensional Itô algebra* $(\mathfrak{a}, l)$ *of a quantum compensated Poisson motion on the spectrum* Ω *of the group* G *as* $\mathfrak{a}$ *is the sum of* $\mathbb{C}đ$ *and the unital group algebra* $\mathcal{D}$ *of* $z = \sum \zeta^g d_g$ *with involution* $z^\star = \sum \zeta_g d_g^\star$ *and* $l(z) = 0$ *for all* $\zeta^g = \bar{\zeta}_g \in \mathbb{C}$. *It is a Lévy algebra with closed involution on the complex space* $\mathcal{D}$ *of all complex sequences* $\zeta = \left(\zeta_g\right)_{g\in G}$ *with*

$$\|\zeta\|^- = \left(\sum \left(\bar{\zeta} * \tilde{\zeta}\right)_g^2 \lambda_g\right)^{1/2} = \|\zeta^\star\|_+,$$

where $\tilde{\zeta} = \left(\zeta_{g^{-1}}\right)_{g\in G}$. *The operator representation of* $\mathfrak{a}$ *in Fock space is defined over the Hilbert space of square integrable function on* $\mathbb{R}_+$ *with values in the direct sum* $\mathcal{K} = \oplus_{n\in\hat{G}}\mathcal{K}(n)\,\mathrm{d}_n$ *of finite dimensional Euclidean spaces* $\mathcal{K}(n)$ *for unitary irreducible representations* $U_g(n)$ *of spectrum* $\hat{G}$ *of the group* G *with the Plansherel measure* $\mathrm{d}_n, n \in \hat{G}$. *It is given by the forward differentials of* $\Lambda(t,a) = \alpha t\mathrm{I} + \sum \zeta^g \hat{m}_g(t)$, *where*

$$\hat{m}_g(t) = \sum_{n\in N} \mathrm{Tr}_{\mathcal{K}(n)}\left\{U_g(n)\left[\rho_n^{1/2}\Lambda_-^n(t) + \Lambda_n^+(t)\,\rho_n^{1/2} + \Lambda_n^n(t)\right]\right\},$$

with the standard annihilation, creation and exchange operators in this Fock space, and a family $\left(\rho_n\right)_{n\in\hat{G}}$ *of positive operators in* $\mathcal{K}(n)$ *with the traces* $\mathrm{Tr}\rho_n$, *defining the spectral decomposition*

$$\lambda_g = \sum_{n\in\hat{G}} \mathrm{Tr}\left[\rho_n U_g(n)\right]\mathrm{d}_n.$$

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