# Hyperbolic geometry for 3d gravity

## 1. Introduction and motivations

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# Motivations

Classical gravity, empty space = 4-dim Einstein metric with Lorentz signature. $ric_g = \Lambda g$. Difficult.

Toy model : 3-dim analog. In dim 3, Einstein metrics have constant curvature. Much easyer (finite dimensional spaces of metrics). $\Lambda = -1$ : AdS (anti-de Sitter) metrics. In particular GHMC.

AdS manifolds are strongly related to hyperbolic surfaces and Teichmüller theory.

GHMC AdS manifolds have Riemannian analogs : quasifuchsian hyperbolic 3-manifolds. Those quasifuchsian manifolds actually appear in the physical theory. They can also be described in terms of Teichmüller theory.

Teichmüller theory is rich but fairly well understood ; possible quantization.

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Designed for physicists : more statements than proofs. Goal : get a broad idea of what is going on. Then read further !

- def of Teichmüller space. (1)
- the hyperbolic plane. (2)
- Fenchel-Nielsen coordinates on Teichmüller space. (2)
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- "hyperbolic" Teichmüller theory : measured laminations, earthquakes, etc. (3)
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Closed surfaces : compact, no boundary.
Oriented if $\exists$ non-vanishing area form.
Or : if one can choose a "left-hand" side.
The Möbius strip is not oriented.
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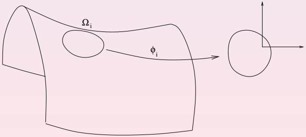

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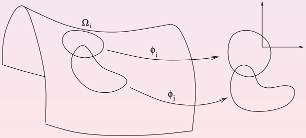

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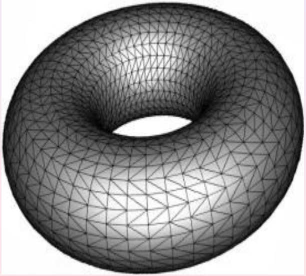

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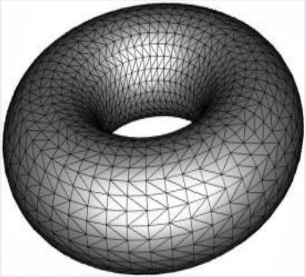

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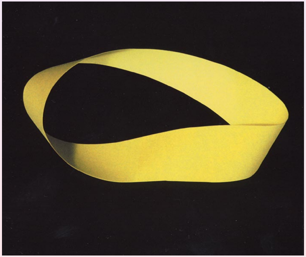

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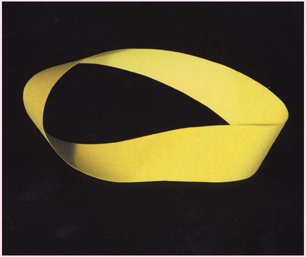

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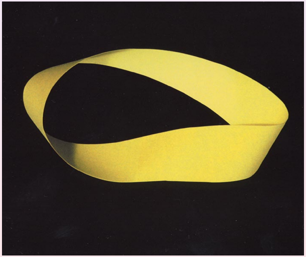

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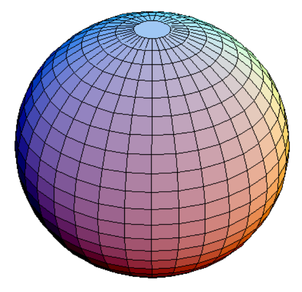

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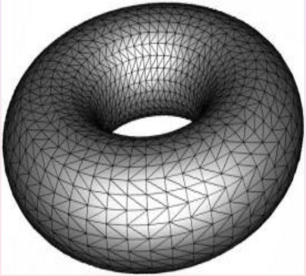

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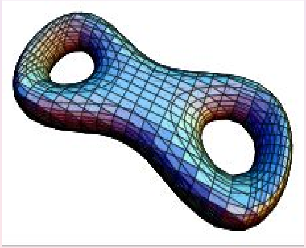

# Riemann surfaces

A *Riemann surface* is a closed surface with a *complex structure*.
Def : charts now in $\mathbb{C}$, $\phi_j \circ \phi_i^{-1}$ is holomorphic.
Riemann surfaces are always oriented (multiply by $i$ for left-hand side).
Other possible def : by $(S, J)$, where, $\forall x \in S, J : T_x S \to T_x S$ is such that $J^2 = -I$. $J$ defines a *complex structure* on $S$.
**Question :** understand all possible complex structures on a surface.
To answer it, we will use *hyperbolic metrics* on $S$.

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To answer it, we will use *hyperbolic metrics* on $S$.

# Riemann surfaces

A *Riemann surface* is a closed surface with a *complex structure*.
Def : charts now in $\mathbb{C}$, $\phi_j \circ \phi_i^{-1}$ is holomorphic.
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# Diffeos, isotopies

Two possible notions of equivalent complex structures on $S$.

Def : a *diffeomorphism* $f : S \to S$ is a smooth, one-to-one map, such that $df : T_xS \to T_{f(x)}S$ is an isomorphism at each point. Form a group, $\mathcal{D}_S$.
$J, J'$ can be considered to be equivalent if $\exists$ a diffeo $f : S \to S$ sending $J$ to $J'$ :

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The space of complex structures on $S$ up to diffeomorphism is the *moduli space* $\mathcal{M}_S$.
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## Teichmüller vs moduli space

Def : $\mathcal{D}/\mathcal{D}_0$ is the *mapping-class group* of $S$.

Examples :

- $S = S^2 : \mathcal{D} = \mathcal{D}_0$.
- $S = T^2 = \mathbb{R}^2/\mathbb{Z}^2 : \mathcal{D}/\mathcal{D}^0 = SL(2, \mathbb{Z})$. Any $M \in M(2, \mathbb{Z})$ with $\det(M) \neq 0$ defines a diffeo $\mathbb{R}^2 \to \mathbb{R}^2$ sending $\mathbb{Z}^2$ to $\mathbb{Z}^2$, hence a map $m : T^2 \to T^2$. If $M \in SL(2, \mathbb{Z})$ then $m$ is one-to-one. Any diffeo $f : T^2 \to T^2$ is isotopic to such a map.
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# Dehn twists

A simple example of diffeo not isotopic to the identity.

Start with a closed surface $S$, choose a simple closed curve $\gamma$. Open $S$ along $\gamma$, turn the right-hand side by $2\pi$, and glue back alongn $\gamma$.

This defines a diffeo of $S$, not isotopic to the identity, a *Dehn twist*.

Thm : Dehn twists generate $\mathcal{D}_+/\mathcal{D}_0$.

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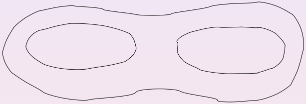

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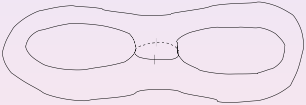

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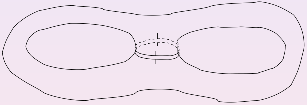

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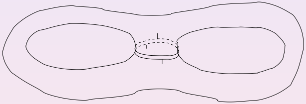

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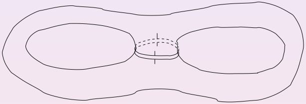

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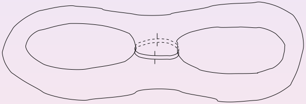

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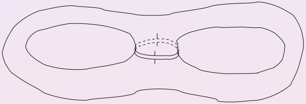

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# Riemannian metrics on surfaces

A geometer's viewpoint. Riemannian metric : symmetric bilinear form $g_x$ on $T_xS$, $\forall x \in S$.
The Levi-Cività connection $\nabla$ : associates to two vector fields $u, v$ a vector field $\nabla_u v$, such that

- connection : $\nabla_u(fv) = du(f)v + f\nabla_u v$,
- compatible with $g$ : $u.g(v, w) = g(\nabla_u v, w) + g(v, \nabla_u w)$,
- torsion-free : $\nabla_u v - \nabla_v u = [u, v]$.

The curvature operator : $R_{u,v}w = \nabla_u\nabla_v w - \nabla_v\nabla_u w - \nabla_{[u,v]}w$. Thm : $g(R_{u,v}w, z)$ at $x \in S$ only depends of $u, v, w, z$ at $x$. Moreover, antisymetric in $u, v$ and $w, z$ and symmetric in $(u, v), (w, z)$.
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# The hyperbolic plane

The hyperbolic plane is an analogue of $S^2$...

but in the Minkowski 3-dim space, $\mathbb{R}^{2,1}$.
$H^2 := \{x \in \mathbb{R}^{2,1} \mid \langle x, x \rangle = -1 \wedge x_0 > 0\}$
$H^2$ is complete, with constant curvature $-1$. Its geodesics are the intersections with the planes containing 0.

Its (orientation-preserving) isometry group is $SO_+(2,1) = PSL(2,\mathbb{R})$
Same construction in higher dim, in dim 3 the isometry group is $SO(3,1) = PSL(2,\mathbb{C})$.

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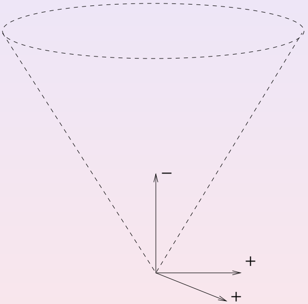


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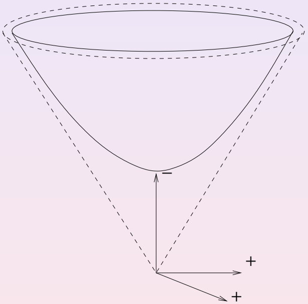


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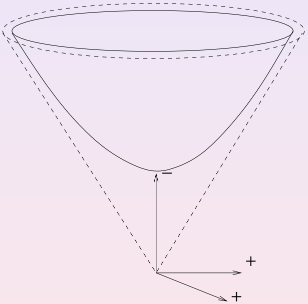


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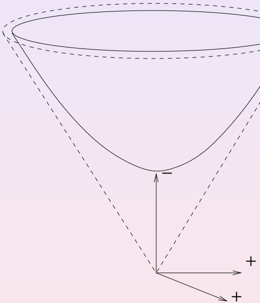


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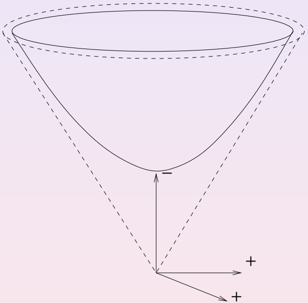


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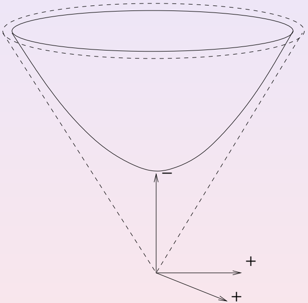


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Obtained by projecting each $x \in H^2$ on $z = 1$ in the direction of 0.
The images of the geodesics of $H^2$ are the segments.

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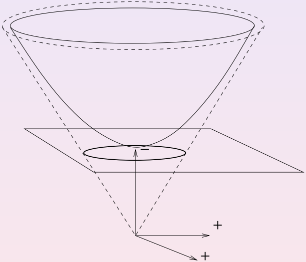


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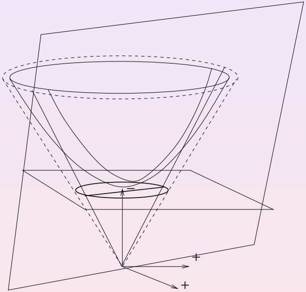


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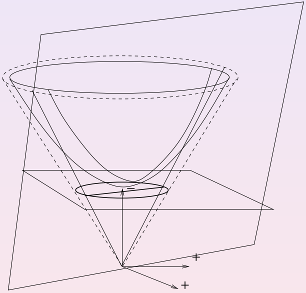

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This model is *conformal*, angles are preserved. Geodesics are sent to circles arcs orthogonal to the boundary. Def : "boundary at infinity". The circles tangent to the boundary are the *horocycles*, they are at "constant distance" from a point at infinity.

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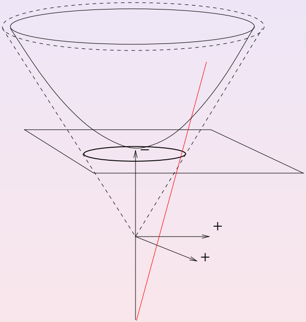

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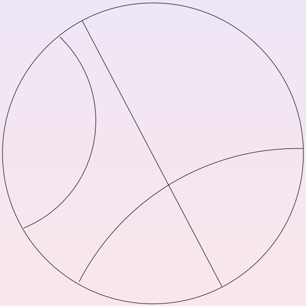

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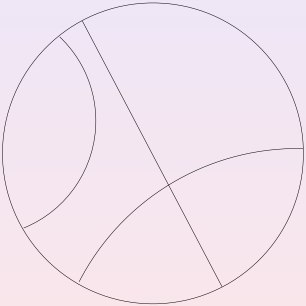

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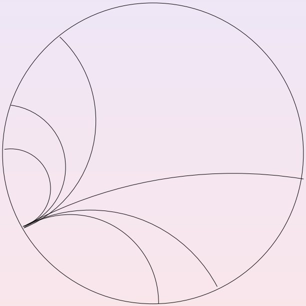

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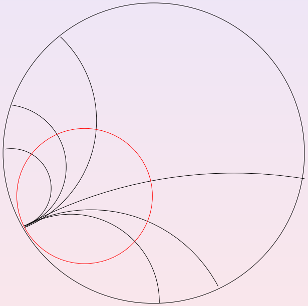

# The PoincarÃľ half-plane model

Apply to the Poincaré disk model the transformation $z \mapsto 1/(z-i) - i$ (conforme). Yields the "Poincaré half-plane".

The boundary at infinity is identified with the real line, plus a point "at infinity". The geodesics are the half-circles centered on $\mathbb{R}$ and the vertical half-lines, the horocycles are the horizontal lines and the circles tangent to $\mathbb{R}$.

The metric is : $(dx^2+dy^2)/y^2$. The identification $SO_+(2,1) = PSL(2,R)$ is best seen in this model : projective action on the real line.

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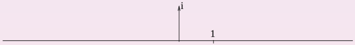


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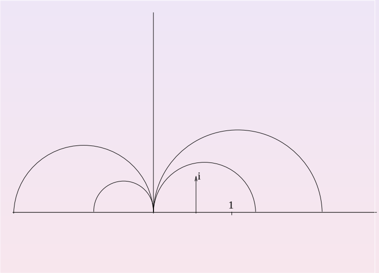


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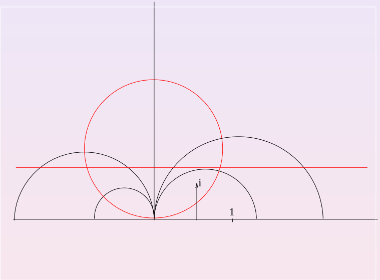


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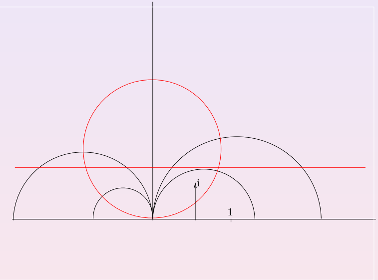


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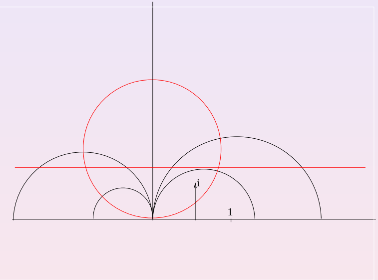


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# Topology and Riemannian geometry

1. Show that the two defs of a Riemann surface – in terms of a map to $\mathbb{C}$ and in terms of $J$ – are equivalent.
2. Show that $S^2$ is diffeomorphic to $\mathbb{C}P^1$.
3. Show (properly) that the mapping-class group of the torus is $SL(2,\mathbb{Z})$.
4. Show that the Levi-Civitá connection of $g$ is uniquely determined.
5. Show that $g(R_{u,v}w,z)$ is anti-symmetric in $u,v$.
6. Show that $g(R_{u,v}w,z)$ is anti-symmetric in $w,z$.
7. Show that $g(R_{u,v}w,z)$ is symmetric in $(u,v),(w,z)$.