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Functional Renormalization Group Equations, Asymptotic Safety, and Quantum Einstein Gravity

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Abstract

These lecture notes provide a pedagogical introduction to a specific continuum implementation of the Wilsonian renormalization group, the effective average action. Its general properties and, in particular, its functional renormalization group equation are explained in a simple scalar setting. The approach is then applied to Quantum Einstein Gravity (QEG). The possibility of constructing a fundamental theory of quantum gravity in the framework of Asymptotic Safety is discussed and the supporting evidence is summarized.

1 Introduction

After the introduction of a functional renormalization group equation for gravity [1] detailed investigations of the non-perturbative renormalization group (RG) behavior of Quantum Einstein Gravity have become possible [1]-[16]. The exact RG equation underlying this approach defines a Wilsonian RG flow on a theory space which consists of all diffeomorphism invariant functionals of the metric $g_{\mu\nu}$. The approach turned out to be ideal for investigating the asymptotic safety scenario in gravity [17, 18, 19] and, in fact, substantial evidence was found for the non-perturbative renormalizability of Quantum Einstein Gravity. The theory emerging from this construction (henceforth denoted "QEG") is not a quantization of classical General Relativity. Instead, its bare action corresponds to a non-trivial fixed point of the RG flow and is a prediction therefore. Independent support for the asymptotic safety conjecture comes from a two-dimensional symmetry reduction of the gravitational path-integral [20].

The approach of [1] employs the effective average action [21, 22, 23, 24] which has crucial advantages as compared to other continuum implementations of the Wilsonian RG flow [25]. In particular it is closely related to the standard effective action and defines a family of effective field theories $\{\Gamma_k[g_{\mu\nu}], 0 \leq k < \infty\}$ labeled by the coarse graining scale k. The latter property opens the door to a rather direct extraction of physical information from the RG flow, at least in single-scale cases: If the physical process under consideration involves a single typical momentum scale p_0 only, it can be described by a tree-level evaluation of $\Gamma_k[g_{\mu\nu}]$, with $k = p_0$.[1]

The effective field theory techniques proved useful for an understanding of the

[1] The precision which can be achieved by this effective field theory description depends on the size of the fluctuations relative to mean values. If they turn out large, or if more than one scale is involved, it might be necessary to go beyond the tree-level analysis.

scale dependent geometry of the effective QEG spacetimes [26, 27, 28]. In particular it has been shown [3, 5, 28] that these spacetimes have fractal properties, with a fractal dimension of 2 at small, and 4 at large distances. The same dynamical dimensional reduction was also observed in numerical studies of Lorentzian dynamical triangulations [29, 30, 31]; in [32] A. Connes et al. speculated about its possible relevance to the non-commutative geometry of the standard model.

As for possible physics implications of the RG flow predicted by QEG, ideas from particle physics, in particular the "RG improvement", have been employed in order to study the leading quantum gravity effects in black hole and cosmological spacetimes [33]-[43]. Among other results, it was found [33] that the quantum effects tend to decrease the Hawking temperature of black holes, and that their evaporation process presumably stops completely once the black holes mass is of the order of the Planck mass.

These notes are intended to provide the background necessary for understanding these developments. In the next section we introduce the general idea of the effective average action and its associated functional renormalization group equation (FRGE) by means of a simple scalar example [21, 23], before reviewing the corresponding construction for gravity [1] in section 3. In all practical calculations based upon this approach which have been performed to date the truncation of theory space has been used as a non-perturbative approximation scheme. In section 3 we explain the general ideas and problems behind this method, and in section 4 we illustrate it explicitly in a simple context, the so-called Einstein-Hilbert truncation. Section 5 introduces the concept of asymptotic safety while section 6 contains a summary of the results obtained using truncated flow equations, with an emphasis on the question as to whether there exists a non-trivial fixed point for the average action's RG flow. If so, QEG could be established as a fundamental theory of quantum gravity which is non-perturbatively renormalizable and "asymptotically safe" from unphysical divergences.

2 Introducing the effective average action

In this section we introduce the concept of the effective average action [21, 23, 22, 24] in the simplest context: scalar field theory on flat d-dimensional Euclidean space $\mathbb{R}^d$.

2.1 The basic construction for scalar fields

We start by considering a single-component real scalar field χ: $\mathbb{R}^d \rightarrow \mathbb{R}$ whose dynamics is governed by the bare action $S[\chi]$. Typically the functional S has the structure $S[\chi] = \int d^d x \left\{ \frac{1}{2}(\partial_\mu \chi)^2 + \frac{1}{2} m^2 \chi^2 + \text{interactions} \right\}$, but we shall not need to assume any specific form of S in the following. After coupling $\chi(x)$ to a source $J(x)$ we can write down an a priori formal path integral representation for the generating functional of the connected Green's functions: $W[J] = \ln \int \mathcal{D}\chi \exp\{-S[\chi] + \int d^d x \, \chi(x) J(x)\}$. By definition, the (conventional) effective action $\Gamma[\phi]$ is the Legendre transform of $W[J]$. It depends on the field expectation value $\phi \equiv \langle \chi \rangle = \delta W[J]/\delta J$ and generates all 1-particle irreducible Greens functions of the theory by multiple functional differentiation with respect to $\phi(x)$ and setting $\phi = \phi[J = 0]$ thereafter. In order to make the functional integral well-defined a UV cutoff is needed; for example one could replace $\mathbb{R}^d$ by a d-dimensional lattice $\mathbb{Z}^d$. The functional integral $\mathcal{D}\chi$ would then read $\prod_{x \in \mathbb{Z}^d} d\chi(x)$. In the following we implicitly assume such a UV regularization but leave the details unspecified and use continuum notation for the fields and their Fourier transforms.

The construction of the effective average action [21] starts out from a modified form, $W_k[J]$, of the functional $W[J]$ which depends on a variable mass scale k. This scale is used to separate the Fourier modes of χ into "short wave length" and "long wave length", depending on whether or not their momentum square $p^2 \equiv p_\mu p^\mu$ is larger or smaller than k^2. By construction, the modes with $p^2 > k^2$ contribute without any suppression to the functional integral defining $W_k[J]$, while those with $p^2 < k^2$ contribute only with a reduced weight or are suppressed altogether, depending on which variant of the formalism is used. The new functional $W_k[J]$ is obtained from the conventional one by adding a "cutoff action" $\Delta_k S[\chi]$ to the bare action $S[\chi]$:

$$\exp\{W_k[J]\} = \int \mathcal{D}\chi \exp\Big\{ - S[\chi] - \Delta_k S[\chi] + \int d^d x\, \chi(x) J(x) \Big\} . \tag{2.1}$$

The factor $\exp\{-\Delta_k S[\chi]\}$ serves the purpose of suppressing the "IR modes" having $p^2 < k^2$. In momentum space the cutoff action is taken to be of the form

$$\Delta_k S[\chi] \equiv \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \, \mathcal{R}_k(p^2) \, |\widehat{\chi}(p)|^2 , \tag{2.2}$$

where $\widehat{\chi}(p) = \int d^d x \, \chi(x) \exp(-ipx)$ is the Fourier transform of $\chi(x)$. The precise shape of the function $\mathcal{R}_k(p^2)$ is arbitrary to some extent; what matters is its limiting behavior for $p^2 \gg k^2$ and $p^2 \ll k^2$ only. In the simplest case[2] we require that

$$\mathcal{R}_k(p^2) \approx \begin{cases} k^2 & \text{for } \; p^2 \ll k^2 , \\ 0 & \text{for } \; p^2 \gg k^2 . \end{cases} \tag{2.3}$$

The first condition leads to a suppression of the small momentum modes by a soft mass-like IR cutoff, the second guarantees that the large momentum modes are integrated out in the usual way. Adding $\Delta_k S$ to the bare action $S[\chi]$ leads to

$$S + \Delta_k S = \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \Big[p^2 + m^2 + \mathcal{R}_k(p^2) \Big] \, |\widehat{\chi}(p)|^2 + \text{interactions} . \tag{2.4}$$

Obviously the cutoff function $\mathcal{R}_k(p^2)$ has the interpretation of a momentum dependent mass square which vanishes for $p^2 \gg k^2$ and assumes the constant value

[2] We shall discuss a slight generalization of these conditions at the end of this section.

k^2 for $p^2 \ll k^2$. How $\mathcal{R}_k(p^2)$ is assumed to interpolate between these two regimes is a matter of calculational convenience. In practical calculations one often uses the exponential cutoff $\mathcal{R}_k(p^2) = p^2[\exp(p^2/k^2) - 1]^{-1}$, but many other choices are possible [23, 44]. One could also think of suppressing the $p^2 < k^2$ modes completely. This could be achieved by allowing $\mathcal{R}_k(p^2)$ to diverge for $p^2 \ll k^2$ so that $\exp\{-\Delta_k S[\chi]\} \to 0$ for modes with $p^2 \ll k^2$. While this behavior of $\mathcal{R}_k(p^2)$ seems most natural from the viewpoint of a Kadanoff-Wilson type coarse graining, its singular behavior makes the resulting generating functional problematic to deal with technically. For this reason, and since it still allows for the derivation of an exact RG equation, one usually prefers to work with a smooth cutoff satisfying (2.3). At the non-perturbative path integral level it suppresses the long wavelength modes by a factor $\exp\{-\frac{1}{2}k^2 \int |\widehat{\chi}|^2\}$. In perturbation theory, according to eq. (2.4), the $\Delta_k S$ term leads to the modified propagator $[p^2 + m^2 + \mathcal{R}_k(p^2)]^{-1}$ which equals $[p^2 + m^2 + k^2]^{-1}$ for $p^2 \ll k^2$. Thus, when computing loops with this propagator, k^2 acts indeed as a conventional IR cutoff if $m^2 \ll k^2$. (It plays no role in the opposite limit $m^2 \gg k^2$ in which the physical particle mass cuts off the p-integration.) We note that by replacing p^2 with $-\partial^2$ in the argument of $\mathcal{R}_k(p^2)$ the cutoff action can be written in a way which makes no reference to the Fourier decomposition of χ:

$$\Delta_k S[\chi] = \frac{1}{2} \int d^d x \, \chi(x) \mathcal{R}_k(-\partial^2) \chi(x) \,. \tag{2.5}$$

The next steps towards the definition of the effective average action are similar to the usual procedure. One defines the (now k-dependent) field expectation value $\phi(x) \equiv \langle \chi(x) \rangle = \delta W_k[J]/\delta J(x)$, assumes that the functional relationship $\phi = \phi[J]$ can be inverted to yield $J = J[\phi]$, and introduces the Legendre transform of W_k,

$$\widetilde{\Gamma}_k[\phi] \equiv \int d^d x \, J(x) \phi(x) - W_k[J] \,, \tag{2.6}$$

where $J = J[\phi]$. The actual effective average action, denoted by $\Gamma_k[\phi]$, is obtained from $\widetilde{\Gamma}_k$ by subtracting $\Delta_k S[\phi]$:

$$\Gamma_k[\phi] \equiv \widetilde{\Gamma}_k[\phi] - \frac{1}{2}\int dx\, \phi(x)\mathcal{R}_k(-\partial^2)\phi(x)\,. \tag{2.7}$$

The rationale for this definition becomes clear when we look at the list of properties enjoyed by the functional Γ_k:

(1) The scale dependence of Γ_k is governed by the FRGE

$$k\frac{\partial}{\partial k}\Gamma_k[\phi] = \frac{1}{2}\mathrm{Tr}\Big[k\frac{\partial}{\partial k}\mathcal{R}_k\left(\Gamma_k^{(2)}[\phi] + \mathcal{R}_k\right)^{-1}\Big]\,. \tag{2.8}$$

Here the RHS uses a compact matrix notation. In a position space representation $\Gamma_k^{(2)}$ has the matrix elements $\Gamma_k^{(2)}(x,y) \equiv \delta^2\Gamma_k/\delta\phi(x)\delta\phi(y)$, i.e., it is the Hessian of the average action, $\mathcal{R}_k(x,y) \equiv \mathcal{R}_k(-\partial_x^2)\delta(x-y)$, and the trace Tr corresponds to an integral $\int d^dx$. In (2.8) the implicit UV cutoff can be removed trivially. This is most easily seen in the momentum representation where $k\frac{\partial}{\partial k}\mathcal{R}_k(p^2)$, considered a function of p^2, is significantly different from zero only in the region centered around $p^2 = k^2$. Hence the trace receives contributions from a thin shell of momenta $p^2 \approx k^2$ only and is therefore well convergent both in the UV and IR.

The RHS of (2.8) can be rewritten in a style reminiscent of a one-loop expression:

$$k\frac{\partial}{\partial k}\Gamma_k[\phi] = \frac{1}{2}\frac{D}{D\ln k}\mathrm{Tr}\ln\left(\Gamma_k^{(2)}[\phi] + \mathcal{R}_k\right)\,. \tag{2.9}$$

Here the scale derivative $D/D\ln k$ acts only on the k-dependence of $\mathcal{R}_k$, not on $\Gamma_k^{(2)}$. The $\mathrm{Tr}\ln(\cdots) = \ln\det(\cdots)$ expression in (2.9) differs from a standard one-loop determinant in two ways: it contains the Hessian of the actual effective action rather than that of the bare action S and it has a built in IR regulator $\mathcal{R}_k$. These modifications make (2.9) an *exact* equation. In a sense, solving it amounts to solving the complete theory.

The derivation of (2.8) proceeds as follows [21]. Taking the k-derivative of (2.6) with (2.1) and (2.5) inserted one finds

$$k\frac{\partial}{\partial k}\widetilde{\Gamma}_k[\phi] = \frac{1}{2}\int d^dx d^dy\, \langle\chi(x)\chi(y)\rangle\, k\frac{\partial}{\partial k}\mathcal{R}_k(x,y)\,, \tag{2.10}$$

with $\langle A \rangle \equiv e^{-W_k} \int \mathcal{D}\chi \, A \exp\{-S - \Delta_k S - \int J\phi\}$ defining the J and k dependent expectation values. Next it is convenient to introduce the connected 2-point function $G_{xy} \equiv G(x,y) \equiv \delta^2 W_k[J]/\delta J(x)\delta J(y)$ and the Hessian of $\widetilde{\Gamma}_k$: $(\widetilde{\Gamma}_k^{(2)})_{xy} \equiv \delta^2\widetilde{\Gamma}_k[J]/\delta\phi(x)\delta\phi(y)$. Since W_k and $\widetilde{\Gamma}_k$ are related by a Legendre transformation one shows in the usual way that G and $\widetilde{\Gamma}^{(2)}$ are mutually inverse matrices: $G\widetilde{\Gamma}^{(2)} = 1$. Furthermore, taking two J-derivatives of (2.1) one obtains $\langle \chi(x)\chi(y)\rangle = G(x,y) + \phi(x)\phi(y)$. Substituting this expression for the two-point function into (2.10) we arrive at

$$\partial_t \widetilde{\Gamma}_k[\phi] = \frac{1}{2}\mathrm{Tr}[\partial_t \mathcal{R}_k G] + \frac{1}{2}\int d^d x \, \phi(x) \, \partial_t \mathcal{R}_k(-\partial^2) \, \phi(x) \,, \tag{2.11}$$

where $t \equiv \ln(k/k_0)$. In terms of Γ_k, the effective average action proper, this becomes $\partial_t \Gamma_k[\phi] = \frac{1}{2}\mathrm{Tr}[\partial_t \mathcal{R}_k G]$. The cancellation of the $\frac{1}{2}\int \phi \mathcal{R}_k \phi$ term is a first motivation for the definition (2.7) where this term is subtracted from the Legendre transform $\widetilde{\Gamma}_k$. The derivation is completed by noting that $G = [\widetilde{\Gamma}^{(2)}]^{-1} = (\Gamma_k^{(2)} + \mathcal{R}_k)^{-1}$, where the second equality follows by differentiating (2.7): $\Gamma_k^{(2)} = \widetilde{\Gamma}_k^{(2)} - \mathcal{R}_k$.

(2) The effective average action satisfies the following integro-differential equation:

$$\exp\{-\Gamma_k[\phi]\} = \int \mathcal{D}\chi \, \exp\Big\{ - S[\chi] + \int d^d x \, (\chi - \phi) \frac{\delta \Gamma_k[\phi]}{\delta \phi} \Big\} \times$$
$$\times \exp\Big\{ - \int d^d x \, (\chi - \phi) \mathcal{R}_k(-\partial^2)(\chi - \phi) \Big\} \,. \tag{2.12}$$

This equation is easily derived by combining eqs. (2.1), (2.6) and (2.7), and by using the effective field equation $\delta\widetilde{\Gamma}_k/\delta\phi = J$, which is 'dual' to $\delta W_k/\delta J = \phi$. (Note that it is $\widetilde{\Gamma}_k$ which appears here, not Γ_k.)

(3) For $k \to 0$ the effective average action approaches the ordinary effective action, $\lim_{k\to 0} \Gamma_k = \Gamma$, and for $k \to \infty$ the bare action $\Gamma_{k\to\infty} = S$. The $k \to 0$ limit is a consequence of (2.3), $\mathcal{R}_k(p^2)$ vanishes for all $p^2 > 0$ when $k \to 0$. The derivation of the $k \to \infty$ limit makes use of the integro-differential equation (2.12). A formal version the argument is as follows. Since $\mathcal{R}_k(p^2)$ approaches k^2 for $k \to \infty$, the second exponential on the RHS of (2.12) becomes $\exp\{-k^2 \int dx(\chi-\phi)^2\}$, which, up to a normalization factor, approaches a delta-functional $\delta[\chi-\phi]$. The χ integration can be performed trivially then and one ends up with $\lim_{k\to\infty}\Gamma_k[\phi] = S[\phi]$. In a more careful treatment [21] one shows that the saddle point approximation of the functional integral in (2.12) about the point $\chi = \phi$ becomes exact in the limit $k \to \infty$. As a result, $\lim_{k\to\infty}\Gamma_k$ and S differ at most by the infinite mass limit of a one-loop determinant, which we suppress here since it plays no role in typical applications (see [45] for a more detailed discussion).

(4) The FRGE (2.8) is independent of the bare action S which enters only via the initial condition $\Gamma_\infty = S$. In the FRGE approach the calculation of the path integral for W_k is replaced by integrating the RG equation from $k = \infty$, where the initial condition $\Gamma_\infty = S$ is imposed, down to $k = 0$, where the effective average action equals the ordinary effective action Γ, the object which we actually would like to know.

2.2 Theory space

The arena in which the Wilsonian RG dynamics takes place is the "theory space". Albeit a somewhat formal notion it helps in visualizing various concepts related to functional renormalization group equations, see fig. 1. To describe it, we shall be slightly more general than in the previous subsection and consider an arbitrary set of fields $\phi(x)$. Then the corresponding theory space consists of all (action) functionals $A : \phi \mapsto A[\phi]$ depending on this set, possibly subject to certain symmetry requirements (a $\mathbb{Z}_2$-symmetry for a single scalar, or diffeomorphism invariance if ϕ denotes the spacetime metric, for instance). So the theory space $\{A[\,\cdot\,]\}$ is fixed once the field content and the symmetries are fixed. Let us assume we can find a set of "basis functionals" $\{P_\alpha[\,\cdot\,]\}$ so that every point of theory space has an expansion of the form [18]

$$A[\phi] = \sum_{\alpha=1}^{\infty} \bar{u}_\alpha \, P_\alpha[\phi] \tag{2.13}$$

The basis $\{P_\alpha[\,\cdot\,]\}$ will include both local field monomials and non-local invariants and we may use the "generalized couplings" $\{\bar{u}_\alpha, \alpha = 1, 2, \cdots\}$ as local coordinates. More precisely, the theory space is coordinatized by the subset of "essential couplings", i.e., those coordinates which cannot be absorbed by a field reparameterization.

Geometrically speaking the FRGE for the effective average action, eq. (2.8) or its generalization for an arbitrary set of fields, defines a vector field $\vec{\beta}$ on theory space. The integral curves along this vector field are the "RG trajectories" $k \mapsto \Gamma_k$ parameterized by the scale k. They start, for $k \to \infty$, at the bare action S (up to the correction term mentioned earlier) and terminate at the ordinary effective action at $k = 0$. The natural orientation of the trajectories is from higher to lower scales k, the direction of increasing "coarse graining". Expanding Γ_k as in (2.13),

$$\Gamma_k[\phi] = \sum_{\alpha=1}^{\infty} \bar{u}_\alpha(k) \, P_\alpha[\phi] \, , \tag{2.14}$$

the trajectory is described by infinitely many "running couplings" $\bar{u}_\alpha(k)$. Inserting (2.14) into the FRGE we obtain a system of infinitely many coupled differential equations for the $\bar{u}_\alpha$'s:

$$k\partial_k \, \bar{u}_\alpha(k) = \overline{\beta}_\alpha(\bar{u}_1, \bar{u}_2, \cdots; k) \, , \quad \alpha = 1, 2, \cdots \, . \tag{2.15}$$

Here the "beta functions" $\overline{\beta}_\alpha$ arise by expanding the trace on the RHS of the FRGE in terms of $\{P_\alpha[\,\cdot\,]\}$, i.e., $\frac{1}{2}\mathrm{Tr}\,[\cdots] = \sum_{\alpha=1}^{\infty} \overline{\beta}_\alpha(\bar{u}_1, \bar{u}_2, \cdots; k) P_\alpha[\phi]$. The expansion coefficients $\overline{\beta}_\alpha$ have the interpretation of beta functions similar to those of perturbation theory, but not restricted to relevant couplings. In standard field theory jargon one would refer to $\bar{u}_\alpha(k = \infty)$ as the "bare" parameters and to $\bar{u}_\alpha(k = 0)$ as the "renormalized" or "dressed" parameters.

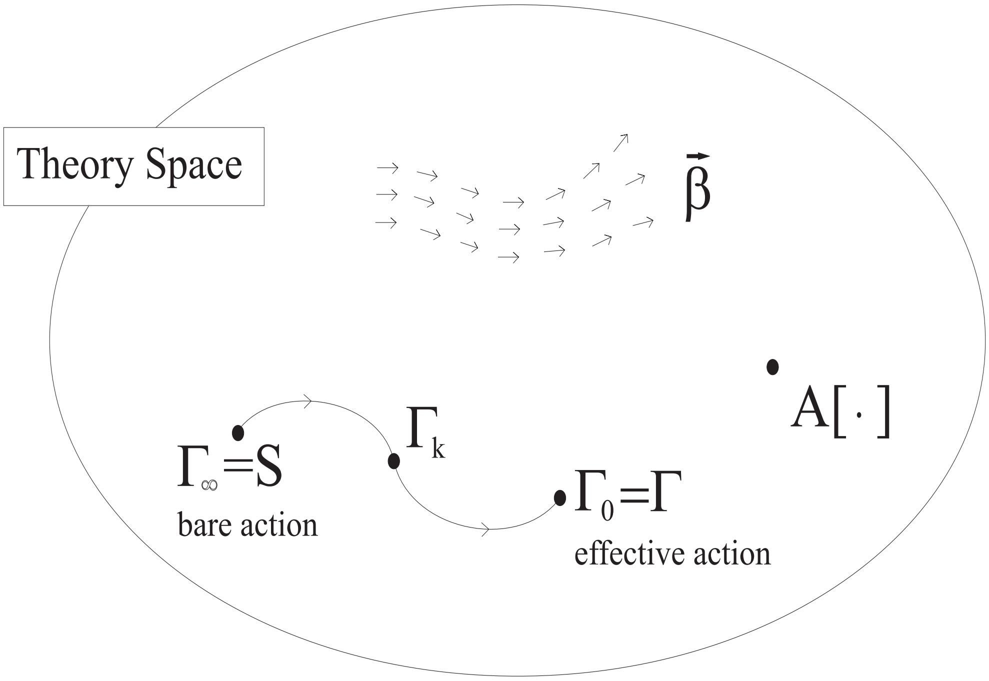


Figure 1: The points of theory space are the action functionals $A[\,\cdot\,]$. The RG equation defines a vector field $\vec{\beta}$ on this space; its integral curves are the RG trajectories $k \mapsto \Gamma_k$. They start at the bare action S and end at the standard effective action Γ.

The notation with the bar on $\bar{u}_\alpha$ and $\overline{\beta}_\alpha$ is to indicate that we are still dealing with dimensionful couplings. Usually the flow equation is reexpressed in terms of the dimensionless couplings $u_\alpha \equiv k^{-d_\alpha}\bar{u}_\alpha$, where d_α is the canonical mass dimension of $\bar{u}_\alpha$. Correspondingly the essential u_α's are used as coordinates of theory space. The resulting RG equations

$$k\partial_k u_\alpha(k) = \beta_\alpha(u_1, u_2, \cdots) \tag{2.16}$$

are a coupled system of autonomous differential equations. The β_α's have no explicit k-dependence and define a "time independent" vector field on theory space.

Fig. 1 gives a schematic summary of the theory space and its structures. It should be kept in mind, though, that only the essential couplings are coordinates on theory space, and that Γ_∞ and S might differ by a simple, explicitly known functional.

2.3 Non-perturbative approximations through truncations

Up to this point our discussion did not involve any approximation. In practice, however, it is usually impossible to find exact solutions to the flow equation. As a way out, one could evaluate the trace on the RHS of the FRGE by expanding it with respect to some small coupling constant, for instance, thus recovering the familiar perturbative beta functions. A more interesting option which gives rise to non-perturbative approximate solutions is to truncate the theory space $\{A[\,\cdot\,]\}$. The basic idea is to project the RG flow onto a finite dimensional subspace of theory space. The subspace should be chosen in such a way that the projected flow encapsulates the essential physical features of the exact flow on the full space.

Concretely the projection onto a truncation subspace is performed as follows. One makes an ansatz of the form $\Gamma_k[\phi] = \sum_{i=1}^{N} \bar{u}_i(k) P_i[\phi]$, where the k-independent functionals $\{P_i[\,\cdot\,], i = 1, \cdots, N\}$ form a 'basis' on the subspace selected. For a scalar field, say, examples include pure potential terms $\int d^dx \phi^m(x)$, $\int d^dx \phi^n(x) \ln \phi^2(x)$, $\cdots$, a standard kinetic term $\int d^dx (\partial\phi)^2$, higher order derivative terms $\int d^dx\, \phi\, (\partial^2)^n\, \phi$, $\int d^dx\, f(\phi)\, (\partial^2)^n\, \phi\, (\partial^2)^m\, \phi$, $\cdots$, and non-local terms like $\int d^dx\, \phi \ln(-\partial^2)\phi$, $\cdots$. Even if $S = \Gamma_\infty$ is simple, a standard ϕ^4 action, say, the evolution from $k = \infty$ downwards will generate such terms, a priori only constrained by symmetry requirements. The difficult task in practical RG applications consists in selecting a set of P_i's which, on the one hand, is generic enough to allow for a sufficiently precise description of the physics one is interested in, and which, on the other hand, is small enough to be computationally manageable.

The projected RG flow is described by a set of ordinary (if $N < \infty$) differential equations for the couplings $\bar{u}_i(k)$. They arise as follows. Let us assume we expand the ϕ-dependence of $\frac{1}{2}\mathrm{Tr}[\cdots]$ (with the ansatz for $\Gamma_k[\phi]$ inserted) in a basis $\{P_\alpha[\,\cdot\,]\}$ of the *full* theory space which contains the P_i's spanning the truncated space as a subset:

$$\frac{1}{2}\mathrm{Tr}[\cdots] = \sum_{\alpha=1}^{\infty} \overline{\beta}_\alpha(\bar{u}_1, \cdots, \bar{u}_N; k)\, P_\alpha[\phi] = \sum_{i=1}^{N} \overline{\beta}_i(\bar{u}_1, \cdots, \bar{u}_N; k)\, P_i[\phi] + \mathrm{rest}\,. \quad (2.17)$$

Here the "rest" contains all terms outside the truncated theory space; the approximation consists in neglecting precisely those terms. Thus, equating (2.17) to the LHS of the flow equation, $\partial_t \Gamma_k = \sum_{i=1}^{N} \partial_t \bar{u}_i(k) P_i$, the linear independence of the P_i's implies the coupled system of ordinary differential equations

$$\partial_t \bar{u}_i(k) = \overline{\beta}_i(\bar{u}_1, \cdots, \bar{u}_N; k)\,, \quad i = 1, \cdots, N\,. \quad (2.18)$$

Solving (2.18) one obtains an *approximation* to the exact RG trajectory projected onto the chosen subspace. Note that this approximate trajectory does, in general, not coincide with the projection of the exact trajectory, but if the subspace is well chosen, it will not be very different from it. In fact, the most non-trivial problem in using truncated flow equations is to find and justify a truncation subspace which should be as low dimensional as possible to make the calculations feasible, but at the same time large enough to describe at least qualitatively the essential physics. We shall return to the issue of testing the quality of a given truncation later on.

As a simple example of a truncation we mention the 'local potential approximation' [23]. The corresponding subspace consists of functionals containing a standard kinetic term plus arbitrary non-derivative terms:

$$\Gamma_k[\phi] \equiv \int d^d x \left\{ \frac{1}{2}(\partial\phi(x))^2 + U_k(\phi(x)) \right\}. \quad (2.19)$$

In this case N is infinite, the coordinates $\bar{u}_i$ on truncated theory space being the infinitely many parameters characterizing an arbitrary potential function $\phi \mapsto U(\phi)$. The infinitely many component equations (2.18) amount to a partial differential equation for the running potential $U_k(\phi)$. It is obtained by inserting (2.19) into the FRGE and projecting the trace onto functionals of the form (2.19). This is most easily done by inserting a *constant* field $\phi = \varphi = const$ into both sides of the equation since this gives a non-vanishing value precisely to the non-derivative P_i's. Since $\Gamma_k^{(2)} = -\partial^2 + U_k''(\varphi)$ with $U'' \equiv d^2U_k/d\phi^2$ has no explicit x-dependence the trace is easily evaluated in momentum space. This leads to the following partial differential equation:

$$k\partial_k U_k(\varphi) = \frac{1}{2}\int \frac{d^dp}{(2\pi)^d} \frac{k\partial_k \mathcal{R}_k(p^2)}{p^2 + \mathcal{R}_k(p^2) + U_k''(\varphi)}\,. \tag{2.20}$$

It describes how the classical (or microscopic) potential $U_\infty = V_{\rm class}$ evolves into the standard effective potential $U_0 = V_{\rm eff}$. Remarkably, the limit $\lim_{k\to 0} U_k$ is automatically a convex function of φ, and there is no need to perform the Maxwell construction 'by hand', in the case of spontaneous symmetry breaking. For a detailed discussion of this point we refer to [23].

One can continue the truncation process and make a specific ansatz for the φ-dependence of the running potential, $U_k(\varphi) = \frac{1}{2}\overline{m}(k)^2\varphi^2 + \frac{1}{12}\overline{\lambda}(k)\varphi^4$, say. Then, upon inserting $U_k''(\varphi) = \overline{m}(k)^2 + \overline{\lambda}(k)\varphi^2$ into the RHS of (2.19) and expanding to $O(\varphi^4)$ one can equate the coefficients of φ^2 and φ^4 to obtain the flow equations on a 2-dimensional subspace: $k\partial_k\overline{m}^2 = \overline{\beta}_{\overline{m}^2}$, $k\partial_k\overline{\lambda} = \overline{\beta}_{\overline{\lambda}}$.

If one wants to go beyond the local potential approximation (2.19) the first step is to allow for a (ϕ independent in the simplest case) wave function renormalization, i.e., a running prefactor of the kinetic term: $\Gamma_k = \int d^dx\,\{\frac{1}{2}Z_k(\partial\phi)^2 + U_k\}$. Using truncations of this type one should employ a slightly different normalization of $\mathcal{R}_k(p^2)$, namely $\mathcal{R}_k(p^2) \approx Z_k k^2$ for $p^2 \ll k^2$. Then $\mathcal{R}_k$ combines with $\Gamma_k^{(2)}$ to the inverse propagator $\Gamma_k^{(2)} + \mathcal{R}_k = Z_k(p^2 + k^2) + \cdots$, as it is necessary if the IR cutoff is to give rise to a (mass)^2 of size k^2 rather than k^2/Z_k. In particular in more complicated theories with more than one field it is important that all fields are cut off at precisely the same k^2. This is achieved by a cutoff function of the form

$$\mathcal{R}_k(p^2) = \mathcal{Z}_k \, k^2 \, R^{(0)}(p^2/k^2) \,, \tag{2.21}$$

where $R^{(0)}$ is normalized such that $R^{(0)}(0) = 1$ and $R^{(0)}(\infty) = 0$. In general the factor $\mathcal{Z}_k$ is a matrix in field space. In the sector of modes with inverse propagator $Z_k^{(i)} p^2 + \cdots$ this matrix is chosen diagonal with entries $\mathcal{Z}_k = Z_k^{(i)}$.

3 The effective average action for gravity

We saw that the FRGE of the effective average action does not depend on the bare action S. Given a theory space, the form of the FRGE and, as a result, the vector field $\vec{\beta}$ are completely fixed. To define a theory space $\{A[\,\cdot\,]\}$ one has to specify on which types of fields the functionals A are supposed to depend, and what their symmetries are. This is the only input data needed for finding the RG flow.

In the case of QEG the theory space consists, by definition, of functionals $A[g_{\mu\nu}]$ depending on a symmetric tensor field, the metric, in a diffeomorphism invariant way. Unfortunately it is not possible to straightforwardly apply the constructions of the previous section to this theory space. Diffeomorphism invariance leads to two types of complications one has to deal with [1].

The first one is not specific to the RG approach. It occurs already in the standard functional integral quantization of gauge or gravity theories, and is familiar from Yang-Mills theories. If one gauge-fixes the functional integral with an ordinary (covariant) gauge fixing condition like $\partial^\mu A_\mu^a = 0$, couples the (non-abelian) gauge field A_μ^a to a source, and constructs the ordinary effective action the resulting functional $\Gamma[A_\mu^a]$ is *not* invariant under the gauge transformations of A_μ^a, $A_\mu^a \mapsto A_\mu^a + D_\mu^{ab}(A)\,\omega^b$. Only at the level of physical quantities constructed from $\Gamma[A_\mu^a]$, S-matrix elements for instance, gauge invariance is recovered.

The second problem is related to the fact that in a gauge theory a "coarse graining" based on a naive Fourier decomposition of $A_\mu^a(x)$ is not gauge covariant and hence not physical. In fact, if one were to gauge transform a slowly varying $A_\mu^a(x)$ using a parameter function $\omega^a(x)$ with a fast x-variation, a gauge field with a fast x-variation would arise which, however, still describes the same physics. In a non-gauge theory the coarse graining is performed by expanding the field in terms of eigenfunctions of the (positive) operator $-\partial^2$ and declaring its eigenmodes 'long' or 'short' wavelength depending on whether the corresponding eigenvalue p^2 is smaller or larger than a given k^2. In a gauge theory the best one can do in installing this procedure is to expand with respect to the *covariant* Laplacian or a similar operator, and then organize the modes according to the size of their eigenvalues. While gauge covariant, this approach sacrifices to some extent the intuition of a Fourier coarse graining in terms of slow and fast modes. Analogous remarks apply to theories of gravity covariant under general coordinate transformations.

The key idea which led to a solution of both problems was the use of the background field method. In fact, it is well kown [46, 47] that the background gauge fixing method leads to an effective action which depends on its arguments in a gauge invariant way. As it turned out [22, 1] this technique also lends itself for implementing a covariant IR cutoff, and it is at the core of the effective average action for Yang-Mills theories [22, 24] and for gravity [1]. In the following we briefly review the effective average action for gravity which has been introduced in ref. [1].

The ultimate goal is to give meaning to an integral over 'all' metrics $\gamma_{\mu\nu}$ of the form $\int \mathcal{D}\gamma_{\mu\nu} \exp\{-S[\gamma_{\mu\nu}] + \text{source terms}\}$ whose bare action $S[\gamma_{\mu\nu}]$ is invariant under general coordinate transformations,

$$\delta\gamma_{\mu\nu} = \mathcal{L}_v \gamma_{\mu\nu} \equiv v^\rho \partial_\rho \gamma_{\mu\nu} + \partial_\mu v^\rho \gamma_{\rho\nu} + \partial_\nu v^\rho \gamma_{\rho\mu}\,, \tag{3.1}$$

where $\mathcal{L}_v$ is the Lie derivative with respect to the vector field $v^\mu \partial_\mu$. To start with we consider $\gamma_{\mu\nu}$ to be a Riemannian metric and assume that $S[\gamma_{\mu\nu}]$ is positive definite. Heading towards the background field formalism, the first step consists in decomposing the variable of integration according to $\gamma_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$, where $\bar{g}_{\mu\nu}$ is a fixed background metric. Note that we are not implying a perturbative expansion here, $h_{\mu\nu}$ is not supposed to be small in any sense. After the background split the measure $\mathcal{D}\gamma_{\mu\nu}$ becomes $\mathcal{D}h_{\mu\nu}$ and the gauge transformations which we have to gauge-fix read

$$\delta h_{\mu\nu} = \mathcal{L}_v \gamma_{\mu\nu} = \mathcal{L}_v(\bar{g}_{\mu\nu} + h_{\mu\nu}) \, , \quad \delta \bar{g}_{\mu\nu} = 0 \, . \tag{3.2}$$

Picking an a priori arbitrary gauge fixing condition $F_\mu(h; \bar{g}) = 0$ the Faddeev-Popov trick can be applied straightforwardly [46]. Upon including an IR cutoff as in the scalar case we are lead to the following k-dependent generating functional W_k for the connected Green functions:

$$\begin{aligned} \exp\left\{W_k[t^{\mu\nu}, \sigma^\mu, \bar{\sigma}_\mu; \bar{g}_{\mu\nu}]\right\} = \int \mathcal{D}h_{\mu\nu} \mathcal{D}C^\mu \mathcal{D}\bar{C}_\mu \, \exp\Big\{ - S[\bar{g} + h] - S_{\mathrm{gf}}[h; \bar{g}] \\ - S_{\mathrm{gh}}[h, C, \bar{C}; \bar{g}] - \Delta_k S[h, C, \bar{C}; \bar{g}] - S_{\mathrm{source}} \Big\} \, . \end{aligned} \tag{3.3}$$

Here S_{gf} denotes the gauge fixing term

$$S_{\mathrm{gf}}[h; \bar{g}] = \frac{1}{2\alpha} \int d^d x \sqrt{\bar{g}} \, \bar{g}^{\mu\nu} F_\mu F_\nu \, , \tag{3.4}$$

and S_{gh} is the action for the corresponding Faddeev–Popov ghosts C^μ and $\bar{C}_\mu$:

$$S_{\mathrm{gh}}[h, C, \bar{C}; \bar{g}] = -\kappa^{-1} \int d^d x \, \bar{C}_\mu \, \bar{g}^{\mu\nu} \, \frac{\partial F_\nu}{\partial h_{\alpha\beta}} \, \mathcal{L}_C \left(\bar{g}_{\alpha\beta} + h_{\alpha\beta} \right) \, . \tag{3.5}$$

The Faddeev–Popov action S_{gh} is obtained along the same lines as in Yang–Mills theory: one applies a gauge transformation (3.2) to F_μ and replaces the parameters v^μ by the ghost field C^μ. The integral over C^μ and $\bar{C}_\mu$ exponentiates the Faddeev-Popov determinant $\det[\delta F_\mu / \delta v^\nu]$. In (3.3) we coupled $h_{\mu\nu}$, C^μ and $\bar{C}_\mu$ to sources $t^{\mu\nu}$, $\bar{\sigma}_\mu$ and σ^μ, respectively: $S_{\mathrm{source}} = -\int d^d x \, \sqrt{\bar{g}} \Big\{ t^{\mu\nu} h_{\mu\nu} + \bar{\sigma}_\mu C^\mu + \sigma^\mu \bar{C}_\mu \Big\}$. The k and source dependent expectation values of $h_{\mu\nu}$, C^μ and $\bar{C}_\mu$ are then given by

$$\bar{h}_{\mu\nu} = \frac{1}{\sqrt{\bar{g}}} \frac{\delta W_k}{\delta t^{\mu\nu}} \qquad , \qquad \xi^\mu = \frac{1}{\sqrt{\bar{g}}} \frac{\delta W_k}{\delta \bar{\sigma}_\mu} \qquad , \qquad \bar{\xi}_\mu = \frac{1}{\sqrt{\bar{g}}} \frac{\delta W_k}{\delta \sigma^\mu} \, . \tag{3.6}$$

As usual we assume that one can invert the relations (3.6) and solve for the sources $(t^{\mu\nu}\,,\,\sigma^{\mu}\,,\,\bar{\sigma}_{\mu})$ as functionals of $(\bar{h}_{\mu\nu}\,,\,\xi^{\mu}\,,\,\bar{\xi}_{\mu})$ and, parameterically, of $\bar{g}_{\mu\nu}$. The Legendre transform $\widetilde{\Gamma}_k$ of W_k reads

$$\widetilde{\Gamma}_k[\bar{h},\xi,\bar{\xi};\bar{g}] = \int d^d x\,\sqrt{\bar{g}}\,\left\{t^{\mu\nu}\bar{h}_{\mu\nu} + \bar{\sigma}_{\mu}\xi^{\mu} + \sigma^{\mu}\bar{\xi}_{\mu}\right\} - W_k[t,\sigma,\bar{\sigma};\bar{g}]\,. \tag{3.7}$$

This functional inherits a parametric $\bar{g}_{\mu\nu}$-dependence from W_k.

As mentioned earlier for a generic gauge fixing condition the Legendre transform (3.7) is not a diffeomorphism invariant functional of its arguments since the gauge breaking under the functional integral is communicated to $\widetilde{\Gamma}_k$ via the sources. While $\widetilde{\Gamma}_k$ does indeed describe the correct 'on-shell' physics satisfying all constraints coming from BRST invariance, it is not invariant off-shell [46, 47]. The situation is different for the class of gauge fixing conditions of the background type. While – as any gauge fixing condition must – they break the invariance under (3.2) they are chosen to be invariant under the so-called background gauge transformations

$$\delta h_{\mu\nu} = \mathcal{L}_v h_{\mu\nu}\,, \qquad \delta\bar{g}_{\mu\nu} = \mathcal{L}_v \bar{g}_{\mu\nu}\,. \tag{3.8}$$

The complete metric $\gamma_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$ transforms as $\delta\gamma_{\mu\nu} = \mathcal{L}_v\gamma_{\mu\nu}$ both under (3.8) and under (3.2). The crucial difference is that the ('quantum') gauge transformations (3.2) keep $\bar{g}_{\mu\nu}$ unchanged so that the entire change of $\gamma_{\mu\nu}$ is ascribed to $h_{\mu\nu}$. This is the point of view one adopts in a standard perturbative calculation around flat space where one fixes $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$ and allows for no variation of the background. In the present construction, instead, we leave $\bar{g}_{\mu\nu}$ unspecified but insist on covariance under (3.8). This will lead to a completely background covariant formulation.

Clearly there exist many possible gauge fixing terms $S_{\rm gf}[h;\bar{g}]$ of the form (3.4) which break (3.2) and are invariant under (3.8). A convenient choice which has been employed in practical calculations is the background version of the harmonic coordinate condition [46]:

$$F_{\mu} = \sqrt{2}\kappa\left[\delta^{\beta}_{\mu}\bar{g}^{\alpha\gamma}\bar{D}_{\gamma} - \frac{1}{2}\bar{g}^{\alpha\beta}\bar{D}_{\mu}\right] h_{\alpha\beta}\,. \tag{3.9}$$

The covariant derivative $\bar{D}_\mu$ involves the Christoffel symbols $\bar{\Gamma}^\rho_{\mu\nu}$ of the background metric. Note that (3.9) is linear in the quantum field $h_{\alpha\beta}$. On a flat background with $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$ the condition $F_\mu = 0$ reduces to the familiar harmonic coordinate condition, $\partial^\mu h_{\mu\nu} = \frac{1}{2}\partial_\nu h_\mu^{\ \mu}$. In eqs. (3.9) and (3.5) κ is an arbitrary constant with the dimension of a mass. We shall set $\kappa \equiv (32\pi\bar{G})^{-1/2}$ with $\bar{G}$ a constant reference value of Newton's constant. The ghost action for the gauge condition (3.9) reads

$$S_{\rm gh}[h, C, \bar{C}; \bar{g}] = -\sqrt{2}\int d^dx \, \sqrt{\bar{g}}\, \bar{C}_\mu \mathcal{M}[g, \bar{g}]^\mu{}_\nu C^\nu \tag{3.10}$$

with the Faddeev–Popov operator

$$\mathcal{M}[g, \bar{g}]^\mu{}_\nu = \bar{g}^{\mu\rho}\bar{g}^{\sigma\lambda}\bar{D}_\lambda(g_{\rho\nu}D_\sigma + g_{\sigma\nu}D_\rho) - \bar{g}^{\rho\sigma}\bar{g}^{\mu\lambda}\bar{D}_\lambda g_{\sigma\nu}D_\rho \, . \tag{3.11}$$

It will prove crucial that for every background-type choice of F_μ, $S_{\rm gh}$ is invariant under (3.8) together with

$$\delta C^\mu = \mathcal{L}_v C^\mu \, , \qquad \delta \bar{C}_\mu = \mathcal{L}_v \bar{C}_\mu \, . \tag{3.12}$$

The essential piece in eq. (3.3) is the IR cutoff for the gravitational field $h_{\mu\nu}$ and for the ghosts. It is taken to be of the form

$$\Delta_k S = \frac{\kappa^2}{2}\int d^dx \, \sqrt{\bar{g}}\, h_{\mu\nu}\mathcal{R}_k^{\rm grav}[\bar{g}]^{\mu\nu\rho\sigma}h_{\rho\sigma} + \sqrt{2}\int d^dx\, \sqrt{\bar{g}}\, \bar{C}_\mu \mathcal{R}_k^{\rm gh}[\bar{g}]C^\mu \, . \tag{3.13}$$

The cutoff operators $\mathcal{R}_k^{\rm grav}$ and $\mathcal{R}_k^{\rm gh}$ serve the purpose of discriminating between high–momentum and low–momentum modes. Eigenmodes of $-\bar{D}^2$ with eigenvalues $p^2 \gg k^2$ are integrated out without any suppression whereas modes with small eigenvalues $p^2 \ll k^2$ are suppressed. The operators $\mathcal{R}_k^{\rm grav}$ and $\mathcal{R}_k^{\rm gh}$ have the structure $\mathcal{R}_k[\bar{g}] = \mathcal{Z}_k k^2 R^{(0)}(-\bar{D}^2/k^2)$, where the dimensionless function $R^{(0)}$ interpolates between $R^{(0)}(0) = 1$ and $R^{(0)}(\infty) = 0$. A convenient choice is, e.g., the exponential cutoff $R^{(0)}(w) = w[\exp(w) - 1]^{-1}$ where $w = p^2/k^2$. The factors $\mathcal{Z}_k$ are different for the graviton and the ghost cutoff. For the ghost $\mathcal{Z}_k \equiv Z_k^{\rm gh}$ is a pure number, whereas for the metric fluctuation $\mathcal{Z}_k \equiv \mathcal{Z}_k^{\rm grav}$ is a tensor, constructed only from the background metric $\bar{g}_{\mu\nu}$, which must be fixed along the lines described at the end of section 2.

A feature of $\Delta_k S$ which is essential from a practical point of view is that the modes of $h_{\mu\nu}$ and the ghosts are organized according to their eigenvalues with respect to the *background* Laplace operator $\bar{D}^2 = \bar{g}^{\mu\nu}\bar{D}_\mu \bar{D}_\nu$ rather than $D^2 = g^{\mu\nu}D_\mu D_\nu$, which would pertain to the full quantum metric $\bar{g}_{\mu\nu} + h_{\mu\nu}$. Using $\bar{D}^2$ the functional $\Delta_k S$ is quadratic in the quantum field $h_{\mu\nu}$, while it becomes extremely complicated if D^2 is used instead. The virtue of a quadratic $\Delta_k S$ is that it gives rise to a flow equation which contains only *second* functional derivatives of Γ_k but no higher ones. The flow equations resulting from the cutoff operator D^2 are prohibitively complicated and can hardly be used for practical computations. A second property of $\Delta_k S$ which is crucial for our purposes is that it is invariant under the background gauge transformations (3.8) with (3.13).

Having specified all the ingredients which enter the functional integral (3.3) for the generating functional W_k we can write down the final definition of the effective average action Γ_k. It is obtained from the Legendre transform $\widetilde{\Gamma}_k$ by subtracting the cutoff action $\Delta_k S$ with the classical fields inserted:

$$\Gamma_k[\bar{h}, \xi, \bar{\xi}; \bar{g}] = \widetilde{\Gamma}_k[\bar{h}, \xi, \bar{\xi}; \bar{g}] - \Delta_k S[\bar{h}, \xi, \bar{\xi}; \bar{g}]\,. \tag{3.14}$$

It is convenient to define the expectation value of the quantum metric $\gamma_{\mu\nu}$,

$$g_{\mu\nu}(x) \equiv \bar{g}_{\mu\nu}(x) + \bar{h}_{\mu\nu}(x)\,, \tag{3.15}$$

and consider Γ_k as a functional of $g_{\mu\nu}$ rather than $\bar{h}_{\mu\nu}$:

$$\Gamma_k[g_{\mu\nu}, \bar{g}_{\mu\nu}, \xi^\mu, \bar{\xi}_\mu] \equiv \Gamma_k[g_{\mu\nu} - \bar{g}_{\mu\nu}, \xi^\mu, \bar{\xi}_\mu; \bar{g}_{\mu\nu}]\,. \tag{3.16}$$

So, what did we gain going through this seemingly complicated background field construction, eventually ending up with an action functional which depends on *two* metrics even? The main advantage of this setting is that the corresponding functionals $\widetilde{\Gamma}_k$, and as a result Γ_k, are invariant under general coordinate transformations where all its arguments transform as tensors of the corresponding rank:

$$\Gamma_k[\Phi + \mathcal{L}_v \Phi] = \Gamma_k[\Phi] \,, \qquad\qquad \Phi \equiv \left\{ g_{\mu\nu}, \bar{g}_{\mu\nu}, \xi^\mu, \bar{\xi}_\mu \right\} \,. \tag{3.17}$$

Note that in (3.17), contrary to the "quantum gauge transformation" (3.2), also the background metric transforms as an ordinary tensor field: $\delta \bar{g}_{\mu\nu} = \mathcal{L}_v \bar{g}_{\mu\nu}$. Eq. (3.17) is a consequence of

$$W_k \left[\mathcal{J} + \mathcal{L}_v \mathcal{J} \right] = W_k \left[\mathcal{J} \right] \,, \qquad \mathcal{J} \equiv \left\{ t^{\mu\nu}, \sigma^\mu, \bar{\sigma}_\mu ; \, \bar{g}_{\mu\nu} \right\} \,. \tag{3.18}$$

This invariance property follows from (3.3) if one performs a compensating transformation (3.8), (3.13) on the integration variables $h_{\mu\nu}$, C^μ and $\bar{C}_\mu$ and uses the invariance of $S[\bar{g} + h]$, $S_{\rm gf}$, $S_{\rm gh}$ and $\Delta_k S$. At this point we assume that the functional measure in (3.3) is diffeomorphism invariant.

Since the $\mathcal{R}_k$'s vanish for $k = 0$, the limit $k \to 0$ of $\Gamma_k[g_{\mu\nu}, \bar{g}_{\mu\nu}, \xi^\mu, \bar{\xi}_\mu]$ brings us back to the standard effective action functional which still depends on two metrics, though. The "ordinary" effective action $\Gamma[g_{\mu\nu}]$ with one metric argument is obtained from this functional by setting $\bar{g}_{\mu\nu} = g_{\mu\nu}$, or equivalently $\bar{h}_{\mu\nu} = 0$ [46, 47]:

$$\Gamma[g] \equiv \lim_{k \to 0} \Gamma_k[g, \bar{g} = g, \xi = 0, \bar{\xi} = 0] = \lim_{k \to 0} \Gamma_k[\bar{h} = 0, \xi = 0, \bar{\xi} = 0; g = \bar{g}] \,. \tag{3.19}$$

This equation brings about the "magic property" of the background field formalism: a priori the 1PI n-point functions of the metric are obtained by an n-fold functional differentiation of $\Gamma_0[\bar{h}, 0, 0; \bar{g}_{\mu\nu}]$ with respect to $\bar{h}_{\mu\nu}$. Hereby $\bar{g}_{\mu\nu}$ is kept fixed; it acts simply as an externally prescribed function which specifies the form of the gauge fixing condition. Hence the functional Γ_0 and the resulting *off-shell* Green functions do depend on $\bar{g}_{\mu\nu}$, but the *on-shell* Green functions, related to observable scattering amplitudes, do not depend on $\bar{g}_{\mu\nu}$. In this respect $\bar{g}_{\mu\nu}$ plays a role similar to the gauge parameter α in the standard approach. Remarkably, the same on-shell Green functions can be obtained by differentiating the functional $\Gamma[g_{\mu\nu}]$ of (3.19) with respect to $g_{\mu\nu}$, or equivalently $\Gamma_0[\bar{h}=0, \xi=0, \bar{\xi}=0; \bar{g}=g]$, with respect to its $\bar{g}$ argument. In this context, 'on-shell' means that the metric satisfies the effective field equation $\delta\Gamma_0[g]/\delta g_{\mu\nu}=0$.

With (3.19) and its k-dependent counterpart

$$\bar{\Gamma}_k[g_{\mu\nu}] \equiv \Gamma_k[g_{\mu\nu}, g_{\mu\nu}, 0, 0] \tag{3.20}$$

we succeeded in constructing a diffeomorphism invariant generating functional for gravity: thanks to (3.17) $\Gamma[g_{\mu\nu}]$ and $\bar{\Gamma}_k[g_{\mu\nu}]$ are invariant under general coordinate transformations $\delta g_{\mu\nu} = \mathcal{L}_v g_{\mu\nu}$. However, there is a price to be paid for their invariance: the simplified functional $\bar{\Gamma}_k[g_{\mu\nu}]$ does not satisfy an exact RG equation, basically because it contains insufficient information. The actual RG evolution has to be performed at the level of the functional $\Gamma_k[g, \bar{g}, \xi, \bar{\xi}\,]$. Only *after* the evolution one may set $\bar{g}=g$, $\xi=0, \bar{\xi}=0$. As a result, the actual theory space of QEG, $\{A[g, \bar{g}, \xi, \bar{\xi}\,]\}$, consists of functionals of all four variables, $g_{\mu\nu}, \bar{g}_{\mu\nu}, \xi^\mu, \bar{\xi}_\mu$, subject to the invariance condition (3.17).

The derivation of the FRGE for Γ_k is analogous to the scalar case. Following exactly the same steps one arrives at

$$\begin{aligned}\partial_t \Gamma_k[\bar{h}, \xi, \bar{\xi}; \bar{g}] =& \frac{1}{2}\mathrm{Tr}\left[\left(\Gamma_k^{(2)} + \widehat{\mathcal{R}}_k\right)^{-1}_{\bar{h}\bar{h}} \left(\partial_t \widehat{\mathcal{R}}_k\right)_{\bar{h}\bar{h}}\right] \\ & - \frac{1}{2}\mathrm{Tr}\left[\left\{\left(\Gamma_k^{(2)} + \widehat{\mathcal{R}}_k\right)^{-1}_{\bar{\xi}\xi} - \left(\Gamma_k^{(2)} + \widehat{\mathcal{R}}_k\right)^{-1}_{\xi\bar{\xi}}\right\}\left(\partial_t \widehat{\mathcal{R}}_k\right)_{\bar{\xi}\xi}\right].\end{aligned} \tag{3.21}$$

Here $\Gamma_k^{(2)}$ denotes the Hessian of Γ_k with respect to the dynamical fields $\bar{h}$, ξ, $\bar{\xi}$ at fixed $\bar{g}$. It is a block matrix labeled by the fields $\varphi_i \equiv \{\bar{h}_{\mu\nu},\, \xi^\mu, \bar{\xi}_\mu\}$:

$$\Gamma_k^{(2)\,ij}(x,y) \equiv \frac{1}{\sqrt{\bar{g}(x)\bar{g}(y)}}\,\frac{\delta^2 \Gamma_k}{\delta\varphi_i(x)\delta\varphi_j(y)}\,. \tag{3.22}$$

(In the ghost sector the derivatives are understood as left derivatives.) Likewise, $\widehat{\mathcal{R}}_k$ is a block diagonal matrix with entries $(\widehat{\mathcal{R}}_k)_{\bar{h}\bar{h}}^{\mu\nu\rho\sigma} \equiv \kappa^2(\mathcal{R}_k^{\mathrm{grav}}[\bar{g}])^{\mu\nu\rho\sigma}$ and $\widehat{\mathcal{R}}_{\bar{\xi}\xi} =$

$\sqrt{2}\mathcal{R}_k^{\rm gh}[\bar{g}]$. Performing the trace in the position representation it includes an integration $\int d^dx\sqrt{\bar{g}(x)}$ involving the background volume element. For any cutoff which is qualitatively similar to the exponential cutoff the traces on the RHS of eq. (3.21) are well convergent, both in the IR and the UV. By virtue of the factor $\partial_t\widehat{\mathcal{R}}_k$, the dominant contributions come from a narrow band of generalized momenta centered around k. Large momenta are exponentially suppressed.

Besides the FRGE the effective average action also satisfies an exact integro-differential equation similar to (2.12) in the scalar case. By the same argument as there it can be used to find the $k \to \infty$ limit of the average action:

$$\Gamma_{k\to\infty}[\bar{h},\xi,\bar{\xi};\bar{g}] = S[\bar{g}+\bar{h}] + S_{\rm gf}[\bar{h};\bar{g}] + S_{\rm gh}[\bar{h},\xi,\bar{\xi};\bar{g}]\,. \tag{3.23}$$

Note that the 'initial value' $\Gamma_{k\to\infty}$ includes the gauge fixing and ghost actions. At the level of the functional $\bar{\Gamma}_k[g]$, eq. (3.23) boils down to $\bar{\Gamma}_{k\to\infty}[g] = S[g]$. However, as $\Gamma_k^{(2)}$ involves derivatives with respect to $\bar{h}_{\mu\nu}$ (or equivalently $g_{\mu\nu}$) at fixed $\bar{g}_{\mu\nu}$ it is clear that the evolution cannot be formulated entirely in terms of $\bar{\Gamma}_k$ alone.

The background gauge invariance of Γ_k, expressed in eq. (3.17), is of enormous practical importance. It implies that if the initial functional does not contain non-invariant terms, the flow will not generate such terms. Very often this reduces the number of terms to be retained in a reliable truncation ansatz quite considerably. Nevertheless, even if the initial action is simple, the RG flow will generate all sorts of local and non-local terms in Γ_k which are consistent with the symmetries.

Let us close this section by remarking that, at least formally, the construction of the effective average action can be repeated for Lorentzian signature metrics. In this case one deals with oscillating exponentials e^{iS}, and for arguments like the one leading to (3.23) one has to employ the Riemann-Lebesgue lemma. Apart from the obvious substitutions $\Gamma_k \to -i\Gamma_k$, $\mathcal{R}_k \to -i\mathcal{R}_k$ the evolution equation remains unaltered.

4 Truncated flow equations

Solving the FRGE (3.21) subject to the initial condition (3.23) is equivalent to (and in practice as difficult as) calculating the original functional integral over $\gamma_{\mu\nu}$. It is therefore important to devise efficient approximation methods. The truncation of theory space is the one which makes maximum use of the FRGE reformulation of the quantum field theory problem at hand.

As for the flow on the theory space $\{A[g, \bar{g}, \xi, \bar{\xi}]\}$ a still very general truncation consists of neglecting the evolution of the ghost action by making the ansatz

$$\Gamma_k[g, \bar{g}, \xi, \bar{\xi}] = \bar{\Gamma}_k[g] + \widehat{\Gamma}_k[g, \bar{g}] + S_{\rm gf}[g - \bar{g}; \bar{g}] + S_{\rm gh}[g - \bar{g}, \xi, \bar{\xi}; \bar{g}] \,, \tag{4.1}$$

where we extracted the classical $S_{\rm gf}$ and $S_{\rm gh}$ from Γ_k. The remaining functional depends on both $g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$. It is further decomposed as $\bar{\Gamma}_k + \widehat{\Gamma}_k$ where $\bar{\Gamma}_k$ is defined as in (3.20) and $\widehat{\Gamma}_k$ contains the deviations for $\bar{g} \neq g$. Hence, by definition, $\widehat{\Gamma}_k[g, g] = 0$, and $\widehat{\Gamma}_k$ contains in particular quantum corrections to the gauge fixing term which vanishes for $\bar{g} = g$, too. This ansatz satisfies the initial condition (3.23) if

$$\bar{\Gamma}_{k\to\infty} = S \qquad \text{and} \qquad \widehat{\Gamma}_{k\to\infty} = 0 \,. \tag{4.2}$$

Inserting (4.1) into the exact FRGE (3.21) one obtains an evolution equation on the truncated space $\{A[g, \bar{g}]\}$:

$$\begin{aligned} \partial_t \Gamma_k[g, \bar{g}] \;\; = \;\; & \frac{1}{2} \mathrm{Tr} \left[\left(\kappa^{-2} \Gamma_k^{(2)}[g, \bar{g}] + \mathcal{R}_k^{\rm grav}[\bar{g}] \right)^{-1} \partial_t \mathcal{R}_k^{\rm grav}[\bar{g}] \right] \\ & - \mathrm{Tr} \left[\left(-\mathcal{M}[g, \bar{g}] + \mathcal{R}_k^{\rm gh}[\bar{g}] \right)^{-1} \partial_t \mathcal{R}_k^{\rm gh}[\bar{g}] \right] \,. \end{aligned} \tag{4.3}$$

This equation evolves the functional

$$\Gamma_k[g, \bar{g}] \equiv \bar{\Gamma}_k[g] + S_{\rm gf}[g - \bar{g}; \bar{g}] + \widehat{\Gamma}_k[g, \bar{g}] \,. \tag{4.4}$$

Here $\Gamma_k^{(2)}$ denotes the Hessian of $\Gamma_k[g, \bar{g}]$ with respect to $g_{\mu\nu}$ at fixed $\bar{g}_{\mu\nu}$.

The truncation ansatz (4.1) is still too general for practical calculations to be easily possible. The first truncation for which the RG flow has been found [1] is the "Einstein-Hilbert truncation" which retains in $\bar{\Gamma}_k[g]$ only the terms $\int d^dx\,\sqrt{g}$ and $\int d^dx\,\sqrt{g}R$, already present in the in the classical action, with k-dependent coupling constants, and includes only the wave function renormalization in $\widehat{\Gamma}_k$:

$$\Gamma_k[g,\bar{g}] = 2\kappa^2 Z_{Nk}\int d^dx\,\sqrt{g}\,\{-R(g)+2\bar{\lambda}_k\} + \frac{Z_{Nk}}{2\alpha}\int d^dx\,\sqrt{\bar{g}}\,\bar{g}^{\mu\nu}F_\mu F_\nu\,. \tag{4.5}$$

In this case the truncation subspace is 2-dimensional. The ansatz (4.5) contains two free functions of the scale, the running cosmological constant $\bar{\lambda}_k$ and Z_{Nk} or, equivalently, the running Newton constant $G_k \equiv \bar{G}/Z_{Nk}$. Here $\bar{G}$ is a fixed constant, and $\kappa \equiv (32\pi\bar{G})^{-1/2}$. As for the gauge fixing term, F_μ is given by eq. (3.9) with $\bar{h}_{\mu\nu} \equiv g_{\mu\nu} - \bar{g}_{\mu\nu}$ replacing $h_{\mu\nu}$; it vanishes for $g = \bar{g}$. The ansatz (4.5) has the general structure of (4.1) with $\widehat{\Gamma}_k = (Z_{Nk}-1)S_{\mathrm{gf}}$. Within the Einstein-Hilbert approximation the gauge fixing parameter α is kept constant. Here we shall set $\alpha = 1$ and comment on generalizations later on.

Upon inserting the ansatz (4.5) into the flow equation (4.3) it boils down to a system of two ordinary differential equations for Z_{Nk} and $\bar{\lambda}_k$. Their derivation is rather technical, so we shall focus on the conceptual aspects here. In order to find $\partial_t Z_{Nk}$ and $\partial_t\bar{\lambda}_k$ it is sufficient to consider (4.3) for $g_{\mu\nu} = \bar{g}_{\mu\nu}$. In this case the LHS of the flow equation becomes $2\kappa^2\int d^dx\sqrt{g}[-R(g)\partial_t Z_{Nk} + 2\partial_t(Z_{Nk}\bar{\lambda}_k)]$. The RHS is assumed to admit an expansion in terms of invariants $P_i[g_{\mu\nu}]$. In the Einstein-Hilbert truncation only two of them, $\int d^dx\,\sqrt{g}$ and $\int d^dx\,\sqrt{g}R$, need to be retained. They can be extracted from the traces in (4.3) by standard derivative expansion techniques. Equating the result to the LHS and comparing the coefficients of $\int d^dx\sqrt{g}$ and $\int d^dx\sqrt{g}R$, a pair of coupled differential equations for Z_{Nk} and $\bar{\lambda}_k$ arises. It is important to note that, on the RHS, we may set $g_{\mu\nu} = \bar{g}_{\mu\nu}$ only *after* the functional derivatives of $\Gamma_k^{(2)}$ have been obtained since they must be taken at fixed $\bar{g}_{\mu\nu}$.

In principle this calculation can be performed without ever considering any specific metric $g_{\mu\nu} = \bar{g}_{\mu\nu}$. This reflects the fact that the approach is background covariant. The RG flow is universal in the sense that it does not depend on any specific metric. In this respect gravity is not different from the more traditional applications of the renormalization group: the RG flow in the Ising universality class, say, has nothing to do with any specific spin configuration, it rather reflects the statistical properties of very many such configurations.

While there is no conceptual necessity to fix the background metric, it nevertheless is sometimes advantageous from a computational point of view to pick a specific class of backgrounds. Leaving $\bar{g}_{\mu\nu}$ completely general, the calculation of the functional traces is very hard work usually. In principle there exist well known derivative expansion and heat kernel techniques which could be used for this purpose, but their application is an extremely lengthy and tedious task usually. Moreover, typically the operators $\Gamma_k^{(2)}$ and $\mathcal{R}_k$ are of a complicated non-standard type so that no efficient use of the tabulated Seeley coefficients can be made. However, often calculations of this type simplify if one can assume that $g_{\mu\nu} = \bar{g}_{\mu\nu}$ has specific properties. Since the beta functions are background independent we may therefore restrict $\bar{g}_{\mu\nu}$ to lie in a conveniently chosen class of geometries which is still general enough to disentangle the invariants retained and at the same time simplifies the calculation.

For the Einstein-Hilbert truncation the most efficient choice is a family of d-spheres $S^d(r)$, labeled by their radius r. For those geometries, $D_\alpha R_{\mu\nu\rho\sigma} = 0$, so they give a vanishing value to all invariants constructed from $g = \bar{g}$ containing covariant derivatives acting on curvature tensors. What remains (among the local invariants) are terms of the form $\int \sqrt{g} P(R)$, where P is a polynomial in the Riemann tensor with arbitrary index contractions. To linear order in the (contractions of the) Riemann tensor the two invariants relevant for the Einstein-Hilbert truncation are discriminated by the S^d metrics as the latter scale differently with the radius of the sphere: $\int\sqrt{g} \sim r^d$, $\int\sqrt{g}R \sim r^{d-2}$. Thus, in order to compute the beta functions of $\bar{\lambda}_k$ and Z_{Nk} it is sufficient to insert an S^d metric with arbitrary r and to compare the coefficients of r^d and r^{d-2}. If one wants to do better and include the three quadratic invariants $\int R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$, $\int R_{\mu\nu}R^{\mu\nu}$, and $\int R^2$, the family $S^d(r)$ is not general enough to separate them; all scale like r^{d-4} with the radius.

Under the trace we need the operator $\Gamma_k^{(2)}[\bar{h};\bar{g}]$. It is most easily calculated by Taylor expanding the truncation ansatz, $\Gamma_k[\bar{g}+\bar{h},\bar{g}] = \Gamma_k[\bar{g},\bar{g}] + O(\bar{h}) + \Gamma_k^{\text{quad}}[\bar{h};\bar{g}] + O(\bar{h}^3)$, and stripping off the two $\bar{h}$'s from the quadratic term, $\Gamma_k^{\text{quad}} = \frac{1}{2}\int \bar{h}\Gamma_k^{(2)}\bar{h}$. For $\bar{g}_{\mu\nu}$ the metric on $S^d(r)$ one obtains

$$\begin{aligned}\Gamma_k^{\text{quad}}[\bar{h};\bar{g}] \;=\; & \frac{1}{2}Z_{Nk}\kappa^2\int d^dx\left\{\widehat{h}_{\mu\nu}\left[-\bar{D}^2 - 2\bar{\lambda}_k + C_T\bar{R}\right]\widehat{h}^{\mu\nu}\right. \\ & \left. -\left(\frac{d-2}{2d}\right)\phi\left[-\bar{D}^2 - 2\bar{\lambda}_k + C_S\bar{R}\right]\phi\right\}, \end{aligned} \tag{4.6}$$

with $C_T \equiv (d(d-3)+4)/(d(d-1))$, $C_S \equiv (d-4)/d$. In order to partially diagonalize this quadratic form $\bar{h}_{\mu\nu}$ has been decomposed into a traceless part $\widehat{h}_{\mu\nu}$ and the trace part proportional to ϕ: $\bar{h}_{\mu\nu} = \widehat{h}_{\mu\nu} + d^{-1}\bar{g}_{\mu\nu}\phi$, $\bar{g}^{\mu\nu}\widehat{h}_{\mu\nu} = 0$. Further, $\bar{D}^2 = \bar{g}^{\mu\nu}\bar{D}_\mu\bar{D}_\nu$ is the covariant Laplace operator corresponding to the background geometry, and $\bar{R} = d(d-1)/r^2$ is the numerical value of the curvature scalar on $S^d(r)$.

At this point we can fix the constants $\mathcal{Z}_k$ which appear in the cutoff operators $\mathcal{R}_k^{\text{grav}}$ and $\mathcal{R}_k^{\text{gh}}$ of (3.13). They should be adjusted in such a way that for every low–momentum mode the cutoff combines with the kinetic term of this mode to $-\bar{D}^2+k^2$ times a constant. Looking at (4.6) we see that the respective kinetic terms for $\widehat{h}_{\mu\nu}$ and ϕ differ by a factor of $-(d-2)/2d$. This suggests the following choice:

$$(\mathcal{Z}_k^{\text{grav}})^{\mu\nu\rho\sigma} = \left[(\mathbb{1}-P_\phi)^{\mu\nu\rho\sigma} - \frac{d-2}{2d}P_\phi^{\mu\nu\rho\sigma}\right]Z_{Nk}\,. \tag{4.7}$$

Here $(P_\phi)_{\mu\nu}{}^{\rho\sigma} = d^{-1}\bar{g}_{\mu\nu}\bar{g}^{\rho\sigma}$ is the projector on the trace part of the metric. For the traceless tensor (4.7) gives $\mathcal{Z}_k^{\text{grav}} = Z_{Nk}\mathbb{1}$, and for ϕ the different relative normalization is taken into account. (See ref. [1] for a detailed discussion of the subtleties related to this choice.) Thus we obtain in the $\widehat{h}$ and the ϕ-sector, respectively:

$$\begin{aligned}\left(\kappa^{-2}\Gamma_k^{(2)}[g,g]+\mathcal{R}_k^{\rm grav}\right)_{\widehat{h}\widehat{h}} &= Z_{Nk}\left[-D^2+k^2R^{(0)}(-D^2/k^2)-2\bar{\lambda}_k+C_TR\right], \qquad (4.8)\\ \left(\kappa^{-2}\Gamma_k^{(2)}[g,g]+\mathcal{R}_k^{\rm grav}\right)_{\phi\phi} &= -\frac{d-2}{2d}Z_{Nk}\left[-D^2+k^2R^{(0)}(-D^2/k^2)-2\bar{\lambda}_k+C_SR\right]\end{aligned}$$

From now on we may set $\bar{g}=g$ and for simplicity we have omitted the bars from the metric and the curvature. Since we did not take into account any renormalization effects in the ghost action we set $Z_k^{\rm gh}\equiv 1$ in $\mathcal{R}_k^{\rm gh}$ and obtain

$$-\mathcal{M}+\mathcal{R}_k^{\rm gh}=-D^2+k^2R^{(0)}(-D^2/k^2)+C_VR\,, \qquad (4.9)$$

with $C_V\equiv -1/d$. At this point the operator under the first trace on the RHS of (4.3) has become block diagonal, with the $\widehat{h}\widehat{h}$ and $\phi\phi$ blocks given by (4.8). Both block operators are expressible in terms of the Laplacian D^2, in the former case acting on traceless symmetric tensor fields, in the latter on scalars. The second trace in (4.3) stems from the ghosts; it contains (4.9) with D^2 acting on vector fields.

It is now a matter of straightforward algebra to compute the first two terms in the derivative expansion of those traces, proportional to $\int d^dx\sqrt{g}\sim r^d$ and $\int d^dx\sqrt{g}R\sim r^{d-2}$. Considering the trace of an arbitrary function of the Laplacian, $W(-D^2)$, the expansion up to second order derivatives of the metric is given by

$$\begin{aligned}{\rm Tr}[W(-D^2)] &= (4\pi)^{-d/2}{\rm tr}(I)\Bigg\{Q_{d/2}[W]\int d^dx\,\sqrt{g}\\ &\qquad +\frac{1}{6}Q_{d/2-1}[W]\int d^dx\,\sqrt{g}R+O(R^2)\Bigg\}. \qquad (4.10)\end{aligned}$$

The Q_n's are defined as

$$Q_n[W]=\frac{1}{\Gamma(n)}\int_0^\infty dz\,z^{n-1}W(z)\,, \qquad (4.11)$$

for $n>0$, and $Q_0[W]=W(0)$ for $n=0$. The trace ${\rm tr}(I)$ counts the number of independent field components. It equals 1, d, and $(d-1)(d+2)/2$, for scalars, vectors, and symmetric traceless tensors, respectively. The expansion (4.10) is easily derived using standard heat kernel and Mellin transform techniques [1].

Using (4.10) it is easy to calculate the traces in (4.3) and to obtain the RG equations in the form $\partial_t Z_{Nk} = \cdots$ and $\partial_t(Z_{Nk}\bar{\lambda}_k) = \cdots$. We shall not display them here since it is more convenient to rewrite them in terms of the dimensionless running cosmological constant and Newton constant, respectively:

$$\lambda_k \equiv k^{-2}\bar{\lambda}_k\,, \qquad g_k \equiv k^{d-2}G_k \equiv k^{d-2}Z_{Nk}^{-1}\bar{G}\,. \tag{4.12}$$

Recall that the dimensionful running Newton constant is given by $G_k = Z_{Nk}^{-1}\bar{G}$. In terms of the dimensionless couplings g and λ the RG equations become a system of autonomous differential equations:

$$\partial_t g_k = \left[d-2+\eta_N(g_k,\lambda_k)\right] g_k \equiv \beta_g(g_k,\lambda_k)\,, \tag{4.13a}$$

$$\partial_t \lambda_k = \beta_\lambda(g_k,\lambda_k)\,. \tag{4.13b}$$

Here $\eta_N \equiv -\partial_t \ln Z_{Nk}$ is the anomalous dimension of the operator $\sqrt{g}R$,

$$\eta_N(g_k,\lambda_k) = \frac{g_k\,B_1(\lambda_k)}{1-g_k\,B_2(\lambda_k)}\,, \tag{4.14}$$

with the following functions of λ_k:

$$\begin{aligned} B_1(\lambda_k) \;&\equiv\; \frac{1}{3}(4\pi)^{1-d/2}\Big[d(d+1)\Phi^1_{d/2-1}(-2\lambda_k) - 6d(d-1)\Phi^2_{d/2}(-2\lambda_k) \\ &\qquad -4d\Phi^1_{d/2-1}(0) - 24\Phi^2_{d/2}(0)\Big] \\ B_2(\lambda_k) \;&\equiv\; -\frac{1}{6}(4\pi)^{1-d/2}\left[d(d+1)\widetilde{\Phi}^1_{d/2-1}(-2\lambda_k) - 6d(d-1)\widetilde{\Phi}^2_{d/2}(-2\lambda_k)\right]. \end{aligned} \tag{4.15}$$

The beta function for λ is given by a similar expression:

$$\begin{aligned} \beta_\lambda(g_k,\lambda_k) \;=\; &-(2-\eta_N)\lambda_k + \frac{1}{2}g_k(4\pi)^{1-d/2}\,\cdot \\ &\cdot\left[2d(d+1)\Phi^1_{d/2}(-2\lambda_k) - 8d\Phi^1_{d/2}(0) - d(d+1)\eta_N\widetilde{\Phi}^1_{d/2}(-2\lambda_k)\right]. \end{aligned} \tag{4.16}$$

The "threshold functions" Φ and $\widetilde{\Phi}$ appearing in (4.15) and (4.16) are certain integrals involving the normalized cutoff function $R^{(0)}$:

$$
\begin{aligned}
\Phi^p_n(w) &\equiv \frac{1}{\Gamma(n)} \int_0^\infty dz\, z^{n-1} \frac{R^{(0)}(z) - zR^{(0)\prime}(z)}{[z + R^{(0)}(z) + w]^p}\,, \\
\widetilde{\Phi}^p_n(w) &\equiv \frac{1}{\Gamma(n)} \int_0^\infty dz\, z^{n-1} \frac{R^{(0)}(z)}{[z + R^{(0)}(z) + w]^p}\,.
\end{aligned}
\tag{4.17}
$$

They are defined for positive integers p, and $n > 0$.

With the derivation of the system (4.13) we managed to find an approximation to a two-dimensional projection of the RG flow. Its properties, and in particular the domain of applicability and reliability of the Einstein-Hilbert truncation will be discussed in the following section.

While there are (few) aspects of the truncated RG flow which are independent of the cutoff scheme, i.e., independent of the function $R^{(0)}$, the explicit solution of the flow equation requires a specific choice of this function. As we discussed already, the normalized cutoff function $R^{(0)}(w)$, $w = p^2/k^2$, describes the "shape" of $\mathcal{R}_k(p^2)$ in the transition region where it interpolates between the prescribed behavior for $p^2 \ll k^2$ and $p^2 \gg k^2$, respectively, and is referred to as the "shape function" therefore. In the literature various forms of $R^{(0)}$'s have been employed. Easy to handle, but disadvantageous for high precision calculations is the sharp cutoff [4] defined by $\mathcal{R}_k(p^2) = \lim_{\hat{R}\to\infty} \hat{R}\, \theta(1 - p^2/k^2)$, where the limit is to be taken after the p^2 integration. This cutoff allows for an evaluation of the Φ and $\widetilde{\Phi}$ integrals in closed form. Taking $d = 4$ as an example, eqs. (4.13) boil down to the following simple system of equations:[3]

$$\partial_t \lambda_k = -(2-\eta_N)\lambda_k - \frac{g_k}{\pi}\Big[5\ln(1-2\lambda_k) - 2\zeta(3) + \frac{5}{2}\eta_N\Big]\,, \tag{4.18a}$$

$$\partial_t g_k = (2+\eta_N)\,g_k\,, \tag{4.18b}$$

$$\eta_N = -\frac{2\,g_k}{6\pi + 5\,g_k}\Big[\frac{18}{1-2\lambda_k} + 5\ln(1-2\lambda_k) - \zeta(2) + 6\Big]\,. \tag{4.18c}$$

Also the "optimized cutoff" [44] with $R^{(0)}(w) = (1-w)\theta(1-w)$ allows for an analytic evaluation of the integrals [14]. In order to check the scheme (in)dependence of the results it is desirable to perform the calculation for a whole class of $R^{(0)}$'s. For this purpose the following one parameter family of exponential cutoffs has been used [8, 3, 5]:

$$R^{(0)}(w;s) = \frac{sw}{e^{sw}-1}\,. \tag{4.19}$$

The precise form of the cutoff is controlled by the "shape parameter" s. For $s=1$, (4.19) coincides with the standard exponential cutoff. The exponential cutoffs are suitable for precision calculations, but the price to be paid is that their Φ and $\widetilde{\Phi}$ integrals can be evaluated only numerically. The same is true for a one-parameter family of shape functions with compact support which was used in [3, 5].

Above we illustrated the general ideas and constructions underlying gravitational RG flows by means of the simplest example, the Einstein-Hilbert truncation. In the literature various extensions have been investigated. The derivation and analysis of these more general flow equations, corresponding to higher dimensional truncation subspaces, is an extremely complex and calculationally demanding problem in general. For this reason we cannot go into the technical details here and just mention some further developments.

(1) The natural next step beyond the Einstein-Hilbert truncation consists in generalizing the functional $\bar{\Gamma}_k[g]$, while keeping the gauge fixing and ghost sector classical,

[3]To be precise, (4.18) corresponds to the sharp cutoff with $s=1$, see [4].

as in (4.1). During the RG evolution the flow generates all possible diffeomorphism invariant terms in $\bar{\Gamma}_k[g]$ which one can construct from $g_{\mu\nu}$. Both local and non-local terms are induced. The local invariants contain strings of curvature tensors and covariant derivatives acting upon them, with any number of tensors and derivatives, and of all possible index structures. The first truncation of this class which has been worked out completely [5, 6] is the "R^2-truncation" defined by (4.1) with the same $\widehat{\Gamma}_k$ as before, and the (curvature)^2 action

$$\bar{\Gamma}_k[g] = \int d^dx \sqrt{g} \Big\{ (16\pi G_k)^{-1} [-R(g) + 2\bar{\lambda}_k] + \bar{\beta}_k R^2(g) \Big\} \,. \tag{4.20}$$

In this case the truncated theory space is 3-dimensional. Its natural (dimensionless) coordinates are (g, λ, β), where $\beta_k \equiv k^{4-d} \bar{\beta}_k$, and g and λ defined in (4.12). Even though (4.20) contains only one additional invariant, the derivation of the corresponding RG equations is far more complicated than in the Einstein-Hilbert case. We shall summarize the results obtained with (4.20) [5, 6] in the next section.

(2) As for generalizing the ghost sector of the truncation beyond (4.1) no results are available yet, but there is a partial result concerning the gauge fixing term. Even if one makes the ansatz (4.5) for $\Gamma_k[g, \bar{g}]$ in which the gauge fixing term has the classical (or more appropriately, bare) structure one should treat its prefactor as a running coupling: $\alpha = \alpha_k$. The beta function of α has not been determined yet from the FRGE, but there is a simple argument which allows us to bypass this calculation.

In non-perturbative Yang-Mills theory and in perturbative quantum gravity $\alpha = \alpha_k = 0$ is known to be a fixed point for the α evolution. The following reasoning suggests that the same is true within the non-perturbative FRGE approach to gravity. In the standard functional integral the limit $\alpha \to 0$ corresponds to a sharp implementation of the gauge fixing condition, i.e., $\exp(-S_{\rm gf})$ becomes proportional to $\delta[F_\mu]$. The domain of the $\int \mathcal{D}h_{\mu\nu}$ integration consists of those $h_{\mu\nu}$'s which satisfy the gauge fixing condition exactly, $F_\mu = 0$. Adding the IR cutoff at k amounts to suppressing some of the $h_{\mu\nu}$ modes while retaining the others. But since all of them satisfy $F_\mu = 0$, a variation of k cannot change the domain of the $h_{\mu\nu}$ integration. The delta functional $\delta[F_\mu]$ continues to be present for any value of k if it was there originally. As a consequence, α vanishes for all k, i.e., $\alpha = 0$ is a fixed point of the α evolution [48].

Thus we can mimic the dynamical treatment of a running α by setting the gauge fixing parameter to the constant value $\alpha = 0$. The calculation for $\alpha = 0$ is more complicated than at $\alpha = 1$, but for the Einstein-Hilbert truncation the α-dependence of β_g and β_λ, for arbitrary constant α has been found in [49, 3]. The R^2-truncations could be analyzed only in the simple $\alpha = 1$ gauge, but the results from the Einstein-Hilbert truncation suggest the UV quantities of interest do not change much between $\alpha = 0$ and $\alpha = 1$ [3, 5].

(3) Up to now we considered pure gravity. As for as the general formalism, the inclusion of matter fields is straightforward. The structure of the flow equation remains unaltered, except that now $\Gamma_k^{(2)}$ and $\mathcal{R}_k$ are operators on the larger Hilbert space of both gravity and matter fluctuations. In practice the derivation of the projected RG equations can be quite a formidable task, however, the difficult part being the decoupling of the various modes (diagonalization of $\Gamma_k^{(2)}$) which in most calculational schemes is necessary for the computation of the functional traces. Various matter systems, both interacting and non-interacting (apart from their interaction with gravity) have been studied in the literature [2, 50, 51]. A rather detailed analysis has been performed by Percacci et al. In [2, 12] arbitrary multiplets of free (massless) fields with spin $0, 1/2, 1$ and $3/2$ were included. In [12] an interacting scalar theory coupled to gravity in the Einstein-Hilbert approximation was analyzed, and a possible solution to the triviality and the hierarchy problem [16] was proposed in this context.

(4) Finally we mention another generalization of the simplest case reviewed above which is of a more technical nature [3]. In order to facilitate the calculation of the functional traces it is helpful to employ a transverse-traceless (TT) decomposition of the metric: $h_{\mu\nu} = h^T_{\mu\nu} + \bar{D}_\mu V_\nu + \bar{D}_\nu V_\mu + \bar{D}_\mu \bar{D}_\nu \sigma - d^{-1}\bar{g}_{\mu\nu}\bar{D}^2\sigma + d^{-1}\bar{g}_{\mu\nu}\phi$. Here $h^T_{\mu\nu}$ is a transverse traceless tensor, V_μ a transverse vector, and σ and ϕ are scalars. In this framework it is natural to formulate the cutoff in terms of the component fields appearing in the TT decomposition: $\Delta_k S \sim \int h^T_{\mu\nu}\mathcal{R}_k h^{T\mu\nu} + \int V_\mu \mathcal{R}_k V^\mu + \cdots$. This cutoff is referred to as a cutoff of "type B", in contradistinction to the "type A" cutoff described above, $\Delta_k S \sim \int h_{\mu\nu}\mathcal{R}_k h^{\mu\nu}$. Since covariant derivatives do not commute the two cutoffs are not exactly equal even if they contain the same shape function. Thus, comparing type A and type B cutoffs is an additional possibility for checking scheme (in)dependence [3, 5].

5 Asymptotic Safety

In intuitive terms, the basic idea of asymptotic safety can be understood as follows. The boundary of theory space depicted in fig. 1 is meant to separate points with coordinates $\{u_\alpha, \alpha = 1, 2, \cdots\}$ with all the essential couplings u_α well defined, from points with undefined, divergent couplings. The basic task of renormalization theory consists in constructing an "infinitely long" RG trajectory which lies entirely within this theory space, i.e., a trajectory which neither leaves theory space (that is, develops divergences) in the UV limit $k \to \infty$ nor in the IR limit $k \to 0$. Every such trajectory defines one possible quantum theory.

The idea of asymptotic safety is to perform the UV limit $k \to \infty$ at a fixed point $\{u^*_\alpha, \alpha = 1, 2, \cdots\} \equiv u^*$ of the RG flow. The fixed point is a zero of the vector field $\vec{\beta} \equiv (\beta_\alpha)$, i.e., $\beta_\alpha(u^*) = 0$ for all $\alpha = 1, 2, \cdots$. The RG trajectories, solutions of $k\partial_k u_\alpha(k) = \beta_\alpha(u(k))$, have a low "velocity" near a fixed point because the β_α's are small there and directly at the fixed point the running stops completely. As a result, one can "use up" an infinite amount of RG time near/at the fixed point if one bases the quantum theory on a trajectory which runs into such a fixed point for $k \to \infty$. This is the key idea of asymptotic safety: If in the UV limit the trajectory ends at a fixed point, an "inner point" of theory space giving rise to a well behaved action functional, we can be sure that, for $k \to \infty$, the trajectory does not escape from theory space, i.e., does not develop pathological properties such as divergent couplings. For $k \to \infty$ the resulting quantum theory is "asymptotically safe" from unphysical divergences. In the context of gravity, Weinberg [17] proposed to use a non-Gaussian fixed point (NGFP) for letting $k \to \infty$. By definition, not all of its coordinates u_α^* vanish.[4]

Recall from section 2.2 that the coordinates u_α are the *dimensionless* essential couplings related to the dimensionful ones $\bar{u}_\alpha$ by $u_\alpha \equiv k^{-d_\alpha}\bar{u}_\alpha$. Hence the running of the $\bar{u}$'s is given by

$$\bar{u}_\alpha(k) = k^{d_\alpha}\, u_\alpha(k)\,. \tag{5.1}$$

Therefore, even directly at a NGFP where $u_\alpha(k) \equiv u_\alpha^*$, the dimensionful couplings keep running according to a power law involving their canonical dimensions d_α:

$$\bar{u}_\alpha(k) = u_\alpha^*\, k^{d_\alpha}\,. \tag{5.2}$$

Furthermore, non-essential dimensionless couplings are not required to attain fixed point values.

Given a NGFP, an important concept is its *UV critical hypersurface* $\mathcal{S}_{\text{UV}}$, or synonymously, its *unstable manifold*. By definition, it consists of all points of theory space which are pulled into the NGFP by the inverse RG flow, i.e., for *in*creasing k. Its dimensionality $\dim(\mathcal{S}_{\text{UV}}) \equiv \Delta_{\text{UV}}$ is given by the number of attractive (for

[4]In contrast, $u_\alpha^* = 0, \forall\alpha = 1, 2, \cdots$ is a so-called Gaussian fixed point (GFP). In a sense standard perturbation theory takes the $k \to \infty$ limit at the GFP; see [18] for a detailed discussion.

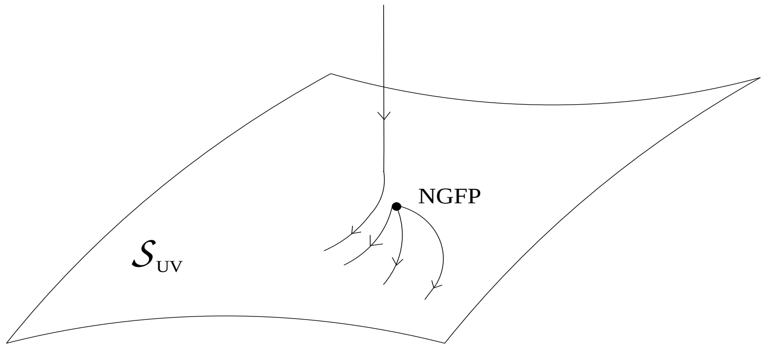


Figure 2: Schematic picture of the UV critical hypersurface $\mathcal{S}_{\text{UV}}$ of the NGFP. It is spanned by RG trajectories emanating from the NGFP as the RG scale k is lowered. Trajectories not in the surface are attracted towards $\mathcal{S}_{\text{UV}}$ as k decreases. (The arrows point in the direction of decreasing k, from the "UV" to the "IR".)

*in*creasing cutoff k) directions in the space of couplings.

Writing the RG equations as $k\,\partial_k u_\alpha = \beta_\alpha(u_1, u_2, \cdots)$, the linearized flow near the fixed point is governed by the Jacobi matrix $\mathbf{B} = (B_{\alpha\gamma})$, $B_{\alpha\gamma} \equiv \partial_\gamma \beta_\alpha(u^*)$:

$$k\,\partial_k\, u_\alpha(k) = \sum_\gamma B_{\alpha\gamma}\, \left(u_\gamma(k) - u_\gamma^*\right) \,. \tag{5.3}$$

The general solution to this equation reads

$$u_\alpha(k) = u_\alpha^* + \sum_I C_I\, V_\alpha^I\, \left(\frac{k_0}{k}\right)^{\theta_I} \tag{5.4}$$

where the V^I's are the right-eigenvectors of $\mathbf{B}$ with eigenvalues $-\theta_I$, i.e., $\sum_\gamma B_{\alpha\gamma}\, V_\gamma^I = -\theta_I\, V_\alpha^I$. Since $\mathbf{B}$ is not symmetric in general the θ_I's are not guaranteed to be real. We assume that the eigenvectors form a complete system though. Furthermore, k_0 is a fixed reference scale, and the C_I's are constants of integration.

If $u_\alpha(k)$ is to describe a trajectory in $\mathcal{S}_{\text{UV}}$, $u_\alpha(k)$ must approach u_α^* in the limit $k \to \infty$ and therefore we must set $C_I = 0$ for all I with $\operatorname{Re}\theta_I < 0$. Hence the dimensionality Δ_{UV} equals the number of $\mathbf{B}$-eigenvalues with a negative real part, i.e., the number of θ_I's with $\operatorname{Re}\theta_I > 0$. The corresponding eigenvectors span the tangent space to $\mathcal{S}_{\text{UV}}$ at the NGFP.

If $u_\alpha(k)$ describes a generic trajectory with all C_I nonzero and we *lower* the cutoff, only $\Delta_{\rm UV}$ "relevant" parameters corresponding to the eigendirections tangent to $\mathcal{S}_{\rm UV}$ grow ($\mathrm{Re}\,\theta_I > 0$), while the remaining "irrelevant" couplings pertaining to the eigendirections normal to $\mathcal{S}_{\rm UV}$ decrease ($\mathrm{Re}\,\theta_I < 0$). Thus near the NGFP a generic trajectory is attracted towards $\mathcal{S}_{\rm UV}$, see fig. 2.

Coming back to the asymptotic safety construction, let us now use this fixed point in order to take the limit $k \to \infty$. The trajectories which define an infinite cutoff limit for QEG are special in that all irrelevant couplings are set to zero: $C_I = 0$ if $\mathrm{Re}\,\theta_I < 0$. These conditions place the trajectory exactly on $\mathcal{S}_{\rm UV}$. There is a $\Delta_{\rm UV}$-parameter family of such trajectories, and the experiment must decide which one is realized in Nature. Therefore the predictive power of the theory increases with decreasing dimensionality of $\mathcal{S}_{\rm UV}$, i.e., number of UV attractive eigendirections of the NGFP. (If $\Delta_{\rm UV} < \infty$, the quantum field theory thus constructed is comparable to and as predictive as a perturbatively renormalizable model with $\Delta_{\rm UV}$ "renormalizable couplings", i.e., couplings relevant at the GFP.)

The quantities θ_I are referred to as critical exponents since when the renormalization group is applied to critical phenomena (second order phase transitions) the traditionally defined critical exponents are related to the θ_I's in a simple way [23]. In fact, one of the early successes of the RG ideas was an explanation of the universality properties of critical phenomena, i.e., the fact that systems at the critical point seem to "forget" the precise form of their microdynamics and just depend on the universality class, characterized by a set of critical exponents, they belong to.

In the present context, "universality" means that certain, very special, quantities related to the RG flow are independent of the precise form of the cutoff and, in particular, its shape function $R^{(0)}$. Universal quantities are potentially measurable or at least closely related to observables. The θ_I's are examples of universal quantities, while the coordinates of the fixed point, u^*_α, are not, even in an exact calculation.

Quantities independently known to be universal provide an important tool for testing the reliability or accuracy of *approximate* RG calculations and of truncations in particular. Since they are known to be $R^{(0)}$ independent in an exact treatment, we can determine the degree of their $R^{(0)}$-dependence within the truncation and use it as a measure for the quality of the truncated calculation.

For a more detailed and formal discussion of asymptotic safety and, in particular, its relation to perturbation theory we refer to the review [18].

6 Average Action approach to Asymptotic Safety

Our discussion of the asymptotic safety construction in the previous section was at the level of the exact (untruncated) RG flow. In this section we are going to implement these ideas in the context of explicitly computable approximate RG flows on truncated theory spaces. We shall mostly concentrate on the Einstein-Hilbert ("R–") and the R^2–truncation of pure gravity in $d = 4$. The corresponding d-dimensional flow equations were derived in refs. [1] and [5], respectively.

6.1 The phase portrait of the Einstein-Hilbert truncation

In [4] the RG equations (4.13) implied by the Einstein-Hilbert truncation have been analyzed in detail, using both analytical and numerical methods. In particular all RG trajectories of this system have been classified, and examples have been computed numerically. The most important classes of trajectories in the phase portrait on the g-λ−plane are shown in fig. 3. The trajectories were obtained by numerically solving the system (4.18) for a sharp cutoff; using a smooth one all qualitative features remain unchanged. The RG flow is found to be dominated by two fixed points (g^*, λ^*): the GFP at $g^* = \lambda^* = 0$, and a NGFP with $g^* > 0$

and $\lambda^* > 0$. There are three classes of trajectories emanating from the NGFP: trajectories of Type Ia and IIIa run towards negative and positive cosmological constants, respectively, and the single trajectory of Type IIa ("separatrix") hits the GFP for $k \to 0$. The high momentum properties of QEG are governed by the NGFP; for $k \to \infty$, in fig. 3 all RG trajectories on the half–plane $g > 0$ run into this point. Note that near the NGFP the dimensionful Newton constant vanishes for $k \to \infty$ according to $G_k \equiv g_k/k^2 \approx g^*/k^2 \to 0$, while the cosmological constant diverges: $\bar{\lambda}_k \equiv \lambda_k k^2 \approx \lambda^* k^2 \to \infty$.

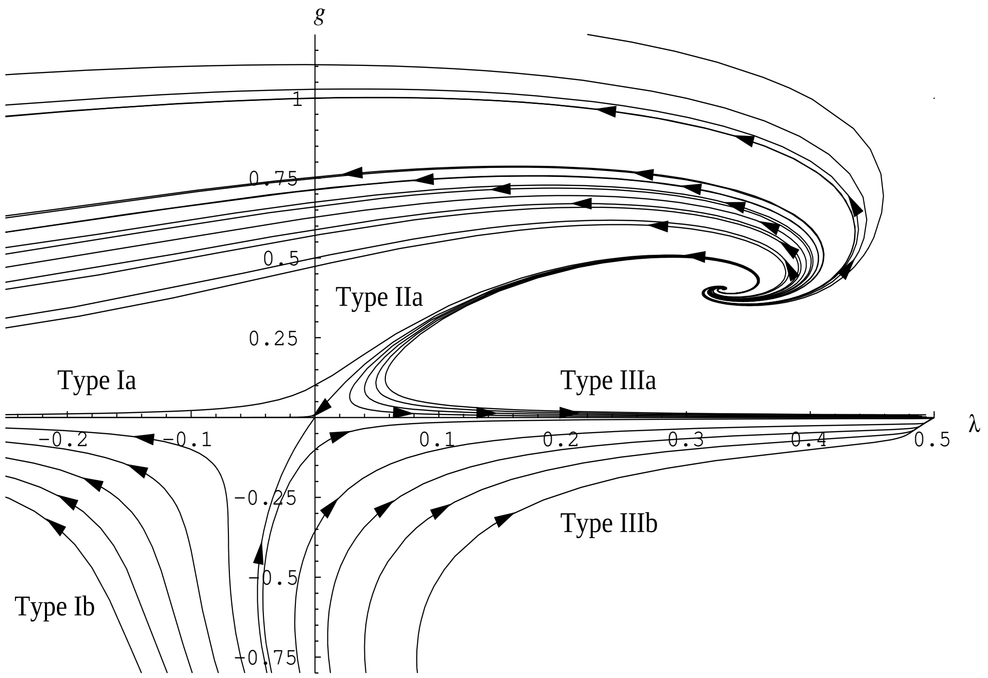


Figure 3: RG flow in the g-λ−plane. The arrows point in the direction of increasing coarse graining, i.e., of decreasing k. (From [4].)

So, the Einstein-Hilbert truncation does indeed predict the existence of a NGFP with exactly the properties needed for the asymptotic safety construction. Clearly the crucial question to be analyzed now is whether the NGFP found is the projection of a fixed point in the exact theory on the untruncated theory space or whether it is merely the artifact of an insufficient approximation.

6.2 Testing the Einstein-Hilbert truncation

We mentioned already that the residual $R^{(0)}$-dependence of universal quantities is a measure for the quality of a truncation. This test has been applied to the Einstein-Hilbert truncation in [8, 3]. We shall display the results in the next subsection. In accordance with the general theory the coordinates of the fixed point (g^*, λ^*) are not universal. However, it can be argued that they should give rise to a universal combination, the product $g^*\lambda^*$ which can be measured in principle [3]. While k and, at a fixed value of k, G_k and $\bar{\lambda}_k$ cannot be measured separately, we may invert the function $k \mapsto G_k$ and insert the result $k = k(G)$ into $\bar{\lambda}_k$. This leads to an in principle experimentally testable relationship $\bar{\lambda} = \bar{\lambda}(G)$ between Newton's constant and the cosmological constant. Here $\bar{\lambda}$ and G should be determined in experiments involving similar scales. In the fixed point regime this relationship reads $\bar{\lambda}(G) = g^*\lambda^*/G$. So, even if this is quite difficult in practice, one can determine the product $g^*\lambda^*$ experimentally. As a consequence in any reliable calculation $g^*\lambda^*$ should be approximately $R^{(0)}$ independent.

The ultimate justification of a given truncation consists in checking that if one adds further terms to it, its physical predictions remain robust. The first step towards testing the robustness of the Einstein-Hilbert truncation near the NGFP against the inclusion of other invariants has been taken in refs. [5, 6] where the R^2–truncation of eq. (4.20) has been analyzed. The corresponding beta functions for the three generalized couplings g, λ and β have been derived, but they are too complicated to be reproduced here. Suffice it to say that on the 3-dimensional (g, λ, β) space, too, a NGFP has been found which generalizes the one from the pure R–calculation. This allows for a comparison of the fixed point results for the

R^2– and the Einstein-Hilbert truncation, and for a check of the approximate $R^{(0)}$ independence of universal quantities in the 3-dimensional setting. For the Einstein-Hilbert truncation the universality analysis has been performed for an arbitrary constant gauge parameter α, including the 'physical' value $\alpha = 0$ [3]. Because of its algebraic complexity the R^2–analysis [5] has been carried out in the simpler $\alpha = 1$ gauge.

6.3 Evidence for Asymptotic Safety

We now summarize the results concerning the NGFP which were obtained with the R– (items (1)-(5)) and R^2–truncation (items (6)-(9)), respectively [3, 4, 5, 6]. All properties mentioned below are independent pieces of evidence pointing in the direction that QEG is indeed asymptotically safe in four dimensions. Except for point (5) all results refer to $d = 4$.

(1) Universal existence: Both for type A and type B cutoffs the non-Gaussian fixed point exists for all shape functions $R^{(0)}$. (This generalizes earlier results in [8].) It seems impossible to find an admissible cutoff which destroys the fixed point in $d = 4$. This result is highly non-trivial since in higher dimensions ($d \gtrsim 5$) the existence of the NGFP depends on the cutoff chosen [4].

(2) Positive Newton constant: While the position of the fixed point is scheme dependent, all cutoffs yield *positive* values of g^* and λ^*. A negative g^* might have been problematic for stability reasons, but there is no mechanism in the flow equation which would exclude it on general grounds.

(3) Stability: For any cutoff employed the NGFP is found to be UV attractive in both directions of the λ-g−plane. Linearizing the flow equation according to eq. (5.3) we obtain a pair of complex conjugate critical exponents $\theta_1 = \theta_2^*$ with positive real part θ' and imaginary parts $\pm\theta''$. In terms of $t = \ln(k/k_0)$ the general solution to the linearized flow equations reads

$$\begin{aligned}(\lambda_k, g_k)^{\mathbf{T}} &= (\lambda^*, g^*)^{\mathbf{T}} + 2\Big\{ [\mathrm{Re}\, C\, \cos(\theta''\, t) + \mathrm{Im}\, C\, \sin(\theta''\, t)]\, \mathrm{Re}\, V \\ &\qquad + [\mathrm{Re}\, C\, \sin(\theta''\, t) - \mathrm{Im}\, C\, \cos(\theta''\, t)]\, \mathrm{Im}\, V \Big\} e^{-\theta' t}\,. \end{aligned} \tag{6.1}$$

with $C \equiv C_1 = (C_2)^*$ an arbitrary complex number and $V \equiv V^1 = (V^2)^*$ the right-eigenvector of $\mathbf{B}$ with eigenvalue $-\theta_1 = -\theta_2^*$. Eq. (5.3) implies that, due to the positivity of θ', all trajectories hit the fixed point as t is sent to infinity. The non-vanishing imaginary part θ'' has no impact on the stability. However, it influences the shape of the trajectories which spiral into the fixed point for $k \to \infty$. Thus, the fixed point has the stability properties needed in the asymptotic safety scenario.

Solving the full, non-linear flow equations [4] shows that the asymptotic scaling region where the linearization (6.1) is valid extends from $k = \infty$ down to about $k \approx m_{\mathrm{Pl}}$ with the Planck mass defined as $m_{\mathrm{Pl}} \equiv G_0^{-1/2}$. Here m_{Pl} plays a role similar to Λ_{QCD} in QCD: it marks the lower boundary of the asymptotic scaling region. We set $k_0 \equiv m_{\mathrm{Pl}}$ so that the asymptotic scaling regime extends from about $t = 0$ to $t = \infty$.

(4) Scheme- and gauge dependence: Analyzing the cutoff scheme dependence of θ', θ'', and $g^*\lambda^*$ as a measure for the reliability of the truncation, the critical exponents were found to be reasonably constant within about a factor of 2. For $\alpha = 1$ and $\alpha = 0$, for instance, they assume values in the ranges $1.4 \lesssim \theta' \lesssim 1.8$, $2.3 \lesssim \theta'' \lesssim 4$ and $1.7 \lesssim \theta' \lesssim 2.1$, $2.5 \lesssim \theta'' \lesssim 5$, respectively. The universality properties of the product $g^*\lambda^*$ are even more impressive. Despite the rather strong scheme dependence of g^* and λ^* separately, their product has almost no visible s-dependence for not too small values of s. Its value is

$$g^*\lambda^* \approx \begin{cases} 0.12 \text{ for } \alpha = 1 \\ 0.14 \text{ for } \alpha = 0\,. \end{cases} \tag{6.2}$$

The difference between the "physical" (fixed point) value of the gauge parameter, $\alpha = 0$, and the technically more convenient $\alpha = 1$ are at the level of about 10 to 20 percent.

(5) Higher and lower dimensions: The beta functions implied by the FRGE are continuous functions of the spacetime dimensionality and it is instructive to analyze them for $d \neq 4$. In ref. [1] it has been shown that for $d = 2 + \epsilon$, $|\epsilon| \ll 1$, the FRGE reproduces Weinberg's [17] fixed point for Newton's constant, $g^* = \frac{3}{38}\epsilon$, and also supplies a corresponding fixed point value for the cosmological constant, $\lambda^* = -\frac{3}{38}\Phi^1_1(0)\epsilon$, with the treshold function given in (4.17). For arbitrary d and a generic cutoff the RG flow is quantitatively similar to the 4-dimensional one for all d smaller than a certain critical dimension $d_{\rm crit}$, above which the existence or non-existence of the NGFP becomes cutoff-dependent. The critical dimension is scheme dependent, but for any admissible cutoff it lies well above $d = 4$. As d approaches $d_{\rm crit}$ from below, the scheme dependence of the universal quantities increases drastically, indicating that the R-truncation becomes insufficient near $d_{\rm crit}$.

In fig. 4 we show the d-dependence of g^*, λ^*, θ', and θ'' for two versions of the sharp cutoff (with $s = 1$ and $s = 30$, respectively) and for the exponential cutoff with $s = 1$. For $2 + \epsilon \leq d \leq 4$ the scheme dependence of the critical exponents is rather weak; it becomes appreciable only near $d \approx 6$ [4]. Fig. 4 suggests that the Einstein-Hilbert truncation in $d = 4$ performs almost as well as near $d = 2$. Its validity can be extended towards larger dimensionalities by optimizing the shape function [14].

(6) Position of the fixed point (R^2)**:** Also with the generalized truncation the NGFP is found to exist for all admissible cutoffs. Fig. 5 shows its coordinates $(\lambda^*, g^*, \beta^*)$ for the family of shape functions (4.19) and the type B cutoff. For every shape parameter s, the values of λ^* and g^* are almost the same as those obtained with the Einstein-Hilbert truncation. In particular, the product $g^*\lambda^*$ is constant with a

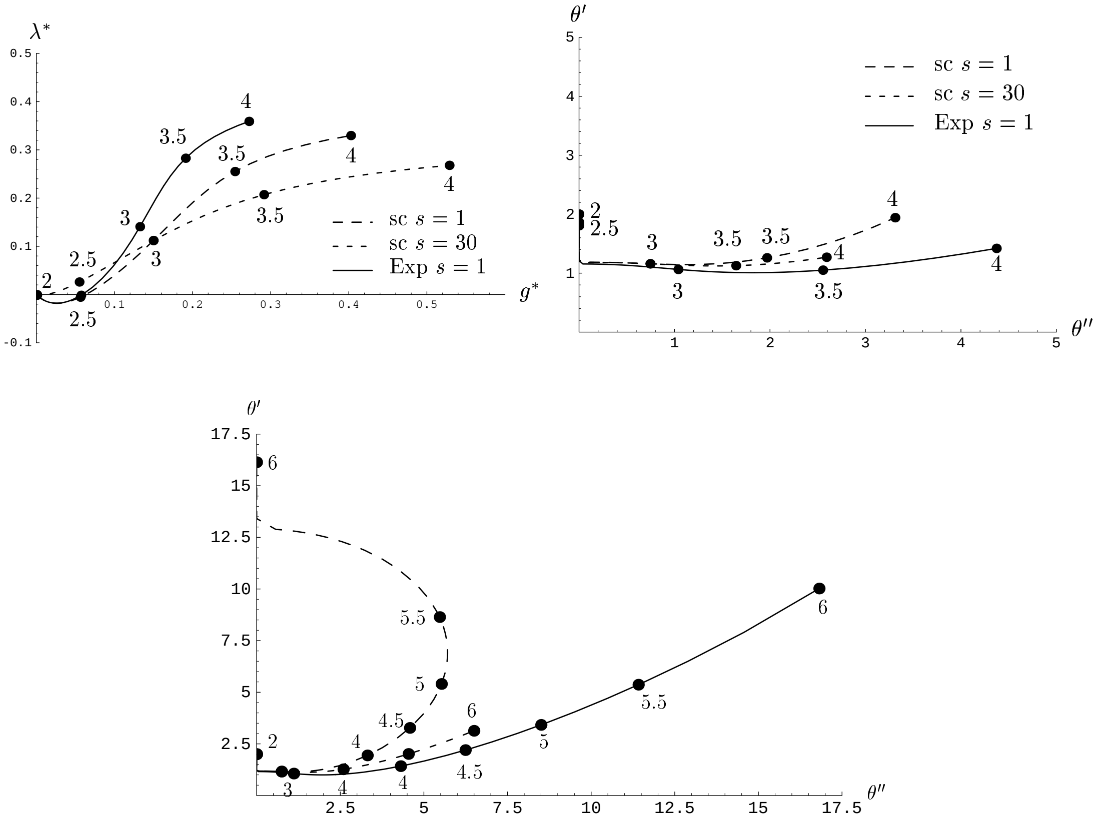


Figure 4: Comparison of λ^*, g^*, θ' and θ'' for different cutoff functions in dependence of the dimension d. Two versions of the sharp cutoff (sc) and the exponential cutoff with $s = 1$ (Exp) have been employed. The upper line shows that for $2 + \epsilon \leq d \leq 4$ the cutoff scheme dependence of the results is rather small. The lower diagram shows that increasing d beyond about 5 leads to a significant difference in the results for θ', θ'' obtained with the different cutoff schemes. (From [4].)

very high accuracy. For $s = 1$, for instance, one obtains $(\lambda^*, g^*) = (0.348, 0.272)$ from the Einstein-Hilbert truncation and $(\lambda^*, g^*, \beta^*) = (0.330, 0.292, 0.005)$ from the generalized truncation. It is quite remarkable that β^* is always significantly smaller than λ^* and g^*. Within the limited precision of our calculation this means that in the 3-dimensional parameter space the fixed point practically lies on the λ-g−plane with $\beta = 0$, i.e., on the parameter space of the pure Einstein-Hilbert truncation.

(7) Eigenvalues and -vectors (R^2)**:** The NGFP of the R^2-truncation proves to be UV attractive in any of the three directions of the (λ, g, β) space for all cutoffs

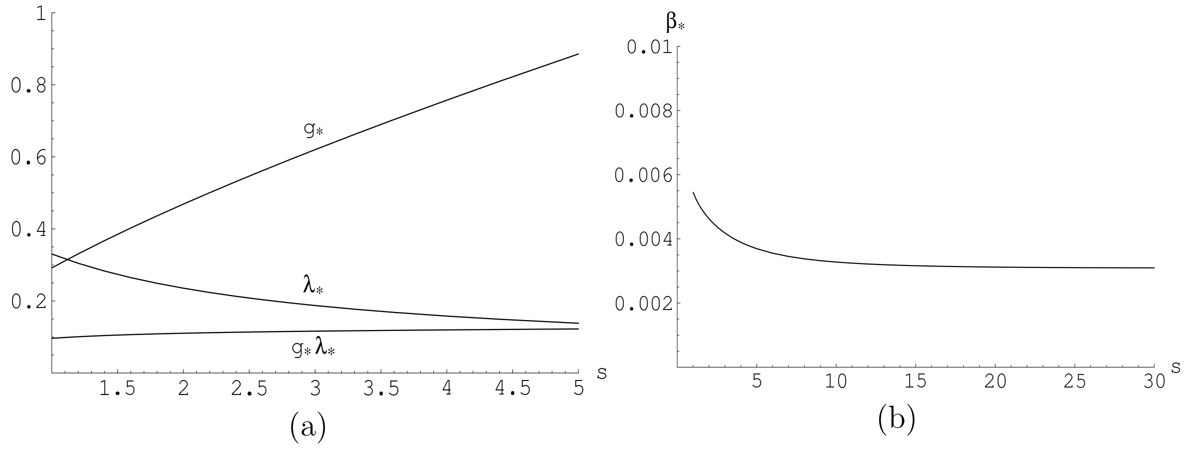


Figure 5: (a) g^*, λ^*, and $g^*\lambda^*$ as functions of s for $1 \leq s \leq 5$, and (b) β^* as a function of s for $1 \leq s \leq 30$, using the family of exponential shape functions (4.19). (From ref. [6].)

used. The linearized flow in its vicinity is always governed by a pair of complex conjugate critical exponents $\theta_1 = \theta' + \mathrm{i}\theta'' = \theta_2^*$ with $\theta' > 0$ and a single real, positive critical exponent $\theta_3 > 0$. It may be expressed as

$$
\begin{aligned}
(\lambda_k, g_k, \beta_k)^{\mathbf{T}} \;&=\; (\lambda^*, g^*, \beta^*)^{\mathbf{T}} + 2\Big\{ [\mathrm{Re}\, C\, \cos(\theta''\, t) + \mathrm{Im}\, C\, \sin(\theta''\, t)]\, \mathrm{Re}\, V \\
&\quad + [\mathrm{Re}\, C\, \sin(\theta''\, t) - \mathrm{Im}\, C\, \cos(\theta''\, t)]\, \mathrm{Im}\, V \Big\}\, e^{-\theta' t} + C_3 V^3\, e^{-\theta_3 t} \quad (6.3)
\end{aligned}
$$

with arbitrary complex $C \equiv C_1 = (C_2)^*$ and real C_3, and with $V \equiv V^1 = (V^2)^*$ and V^3 the right-eigenvectors of the stability matrix $(B_{ij})_{i,j\in\{\lambda,g,\beta\}}$ with eigenvalues $-\theta_1 = -\theta_2^*$ and $-\theta_3$, respectively. Clearly the conditions for UV stability are $\theta' > 0$ and $\theta_3 > 0$. They are indeed satisfied for all cutoffs. For the exponential shape function with $s = 1$, for instance, we find $\theta' = 2.15$, $\theta'' = 3.79$, $\theta_3 = 28.8$, and $\mathrm{Re}\, V = (-0.164, 0.753, -0.008)^{\mathbf{T}}$, $\mathrm{Im}\, V = (0.64, 0, -0.01)^{\mathbf{T}}$, $V^3 = -(0.92, 0.39, 0.04)^{\mathbf{T}}$. (The vectors are normalized such that $\|V\| = \|V^3\| = 1$.) The trajectories (6.3) comprise three independent normal modes with amplitudes proportional to $\mathrm{Re}\, C$, $\mathrm{Im}C$ and C_3, respectively. The first two are again of the spiral type while the third one is a straight line.

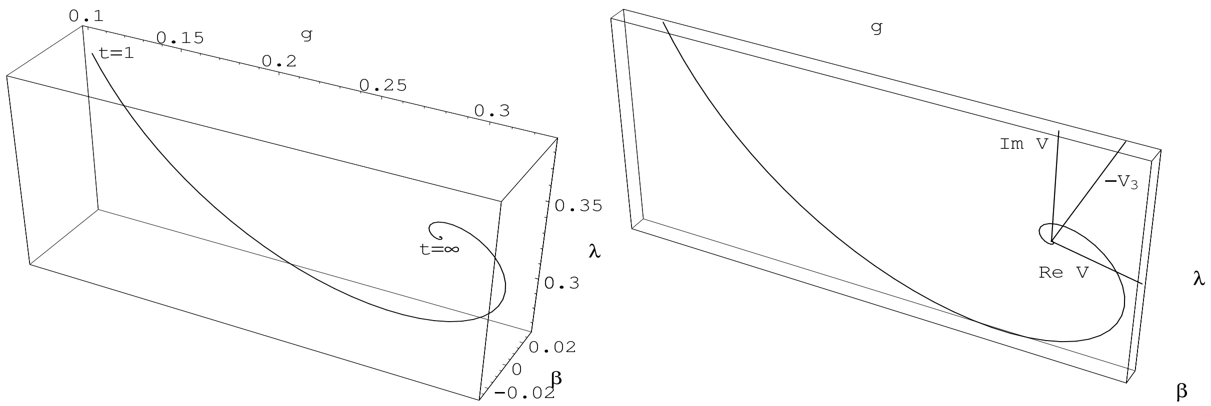
For any cutoff, the numerical results have several quite remarkable properties. They all indicate that, close to the NGFP, the RG flow is rather well approximated by the pure Einstein-Hilbert truncation.

(a) The β-components of $\mathrm{Re}\, V$ and $\mathrm{Im}\, V$ are very tiny. Hence these two vectors span a plane which virtually coincides with the g-λ–subspace at $\beta = 0$, i.e., with the parameter space of the Einstein-Hilbert truncation. As a consequence, the $\mathrm{Re}\, C$– and $\mathrm{Im} C$–normal modes are essentially the same trajectories as the "old" normal modes already found without the R^2–term. Also the corresponding θ'– and θ''–values coincide within the scheme dependence.

(b) The new eigenvalue θ_3 introduced by the R^2–term is significantly larger than θ'. When a trajectory approaches the fixed point from below ($t \to \infty$), the "old" normal modes $\propto \mathrm{Re}\, C, \mathrm{Im}\, C$ are proportional to $\exp(-\theta' t)$, but the new one is proportional to $\exp(-\theta_3 t)$, so that it decays much quicker. For every trajectory running into the fixed point, i.e., for every set of constants $(\mathrm{Re}\, C, \mathrm{Im}\, C, C_3)$, we find therefore that once t is sufficiently large the trajectory lies entirely in the $\mathrm{Re}\, V$-$\mathrm{Im}\, V$–subspace, i.e., the $\beta = 0$-plane practically.

Due to the large value of θ_3, the new scaling field is very "relevant". However, when we start at the fixed point ($t = \infty$) and lower t it is only at the low energy scale $k \approx m_{\mathrm{Pl}}$ ($t \approx 0$) that $\exp(-\theta_3 t)$ reaches unity, and only then, i.e., far away from the fixed point, the new scaling field starts growing rapidly.

(c) Since the matrix $\mathbf{B}$ is not symmetric its eigenvectors have no reason to be orthogonal. In fact, one finds that V^3 lies almost in the $\mathrm{Re}\, V$-$\mathrm{Im}\, V$–plane. For the angles between the eigenvectors given above we obtain $\sphericalangle(\mathrm{Re}\, V, \mathrm{Im}\, V) = 102.3°$, $\sphericalangle(\mathrm{Re}\, V, V^3) = 100.7°$, $\sphericalangle(\mathrm{Im}\, V, V^3) = 156.7°$. Their sum is $359.7°$ which confirms that $\mathrm{Re}\, V$, $\mathrm{Im}\, V$ and V^3 are almost coplanar. This implies that when we lower t



(a) (b)

Figure 6: Trajectory of the linearized flow equation obtained from the R^2–truncation for $1 \leq t = \ln(k/k_0) < \infty$. In (b) we depict the eigendirections and the "box" to which the trajectory is confined. (From ref. [6].)

and move away from the fixed point so that the V^3–scaling field starts growing, it is again predominantly the $\int d^dx \sqrt{g}$ and $\int d^dx \sqrt{g}R$ invariants which get excited, but not $\int d^dx \sqrt{g}R^2$ in the first place.

Summarizing the three points above, we can say that very close to the fixed point the RG flow seems to be essentially two-dimensional, and that this two-dimensional flow is well approximated by the RG equations of the Einstein-Hilbert truncation. In fig. 6 we show a typical trajectory which has all three normal modes excited with equal strength ($\mathrm{Re}\, C = \mathrm{Im}\, C = 1/\sqrt{2}$, $C_3 = 1$). All the way down from $k = \infty$ to about $k = m_{\mathrm{Pl}}$ it is confined to a very thin box surrounding the $\beta = 0$–plane.

(8) Scheme Dependence (R^2)**:** The scheme dependence of the critical exponents and of the product $g^*\lambda^*$ turns out to be of the same order of magnitude as in the case of the Einstein-Hilbert truncation. Fig. 7 shows the cutoff dependence of the critical exponents, using the family of shape functions (4.19). For the cutoffs employed θ' and θ'' assume values in the ranges $2.1 \lesssim \theta' \lesssim 3.4$ and $3.1 \lesssim \theta'' \lesssim 4.3$, respectively. While the scheme dependence of θ'' is weaker than in the case of the

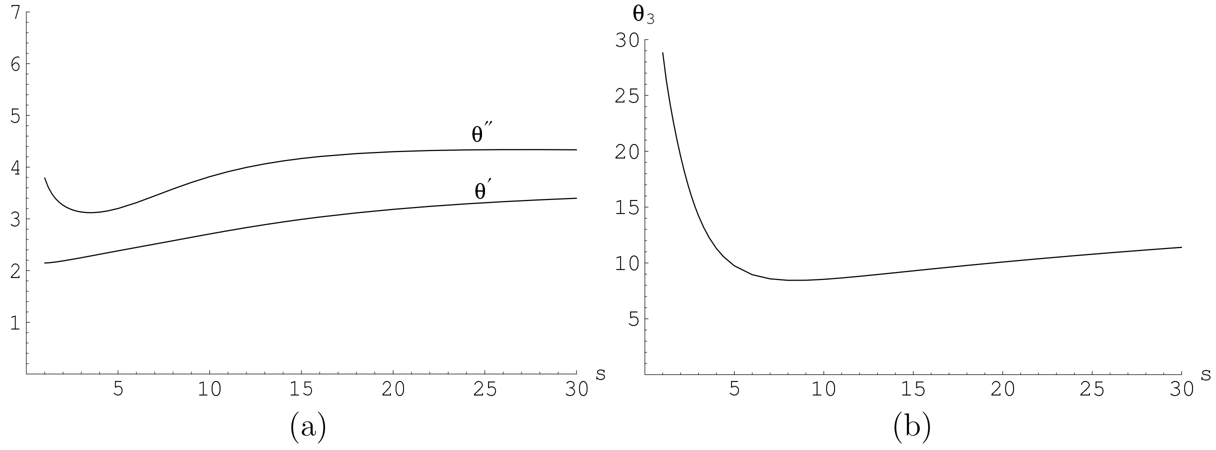


Figure 7: (a) $\theta' = \mathrm{Re}\,\theta_1$ and $\theta'' = \mathrm{Im}\,\theta_1$, and (b) θ_3 as functions of s, using the family of exponential shape functions (4.19). (From [5].)

Einstein-Hilbert truncation one finds that it is slightly larger for θ'. The exponent θ_3 suffers from relatively strong variations as the cutoff is changed, $8.4 \lesssim \theta_3 \lesssim 28.8$, but it is always significantly larger than θ'. The product $g^*\lambda^*$ again exhibits an extremely weak scheme dependence. Fig. 5(a) displays $g^*\lambda^*$ as a function of s. It is impressive to see how the cutoff dependences of g^* and λ^* cancel almost perfectly. Fig. 5(a) suggests the universal value $g^*\lambda^* \approx 0.14$. Comparing this value to those obtained from the Einstein-Hilbert truncation we find that it differs slightly from the one based upon the same gauge $\alpha = 1$. The deviation is of the same size as the difference between the $\alpha = 0$– and the $\alpha = 1$–results of the Einstein-Hilbert truncation.

As for the universality of the critical exponents we emphasize that the qualitative properties listed above (e.g., $\theta', \theta_3 > 0$, $\theta_3 \gg \theta'$, etc.) obtained for all cutoffs. The θ's have a much stronger scheme dependence than $g^*\lambda^*$, however. This is most probably due to neglecting further relevant operators in the truncation so that the **B**-matrix we are diagonalizing is too small still.

(9) Dimensionality of $\mathcal{S}_{\rm UV}$: According to the canonical dimensional analysis, the (curvature)^n-invariants in 4 dimensions are classically marginal for $n = 2$ and irrelevant for $n > 2$. The results for θ_3 indicate that there are large non-classical contributions so that there might be relevant operators perhaps even beyond $n = 2$. With the present approach it is clearly not possible to determine their number Δ_{UV}. However, as it is hardly conceivable that the quantum effects change the signs of arbitrarily large (negative) classical scaling dimensions, Δ_{UV} should be finite [17].

A first confirmation of this picture comes from the R^2-calculation which has also been performed in $d = 2 + \varepsilon$ where, at least canonically, the dimensional count is shifted by two units. In this case we find indeed that the third scaling field is irrelevant, $\theta_3 < 0$. Therefore the dimensionality of $\mathcal{S}_{\text{UV}}$ could be as small as $\Delta_{\text{UV}} = 2$, but this is not a proof, of course. If so, the quantum theory would be characterized by only two free parameters, the renormalized Newton constant and cosmological constant, respectively.

7 Discussion and Conclusion

On the basis of the above results we believe that the non-Gaussian fixed point occurring in the Einstein-Hilbert truncation is not a truncation artifact but rather the projection of a fixed point in the exact theory space. The fixed point and all its qualitative properties are stable against variations of the cutoff and the inclusion of a further invariant in the truncation. It is particularly remarkable that within the scheme dependence the additional R^2–term has essentially no impact on the fixed point. We interpret the above results and their mutual consistency as quite non-trivial indications supporting the conjecture that 4-dimensional QEG indeed possesses a RG fixed point with precisely the properties needed for its non-perturbative renormalizability and asymptotic safety.

Recently this picture has been beautifully confirmed by Codello, Percacci and

Rahmede [52] who, in $d = 4$, considered truncations of the form

$$\bar{\Gamma}_k[g] = \int d^4x \sqrt{g} \sum_{n=0}^{N} \bar{u}_n(k) \, R^n \, . \tag{7.1}$$

In the most advanced case the highest power of the curvature scalar was as large as $N = 7$. An important result obtained with these truncations is that going beyond the R^2 truncation the new eigendirections at the NGFP are all UV repulsive ($\mathrm{Re}\,\theta_I < 0$), indicating that Δ_{UV} is indeed likely to be a small finite number. Increasing the order N of the curvature polynomial the values of the universal quantities show a certain degree of convergence, in particular $g^*\lambda^*$ agrees with the Einstein-Hilbert result (6.2) to within 10 or 20 percent for any $N = 2, \cdots, 7$. It is quite amazing how well the RG flow near the NGFP is approximated by the Einstein-Hilbert truncation; the reason for this is not yet fully understood.

In these notes we focused on the average action approach to QEG. For a detailed discussion including evidence for asymptotic safety from other approaches we refer to [18].

Before closing, some further comments might be helpful here.

(1) The construction of an effective average action for gravity as introduced in [1] represents a *background independent* approach to quantum gravity. Somewhat paradoxically, this background independence is achieved by means of the background field formalism: One fixes an arbitrary background, quantizes the fluctuation field in this background, and afterwards adjusts $\bar{g}_{\mu\nu}$ in such a way that the expectation value of the fluctuation vanishes: $\bar{h}_{\mu\nu} = 0$. In this way the background gets fixed dynamically.

(2) The combination of the effective average action with the background field method has been successfully tested within conventional field theory. In QED and Yang-Mills type gauge theories it reproduces the known results and extends them into the non-perturbative domain [22, 24].

(3) The coexistence of asymptotic safety and perturbative non-renormalizability is well understood. In particular upon fixing $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$ and expanding the trace on its RHS in powers of G the FRGE reproduces the divergences of perturbation theory; see ref. [18] for a detailed discussion of this point.

(4) It is to be emphasized that in the average action framework the RG flow, i.e., the vector field $\vec{\beta}$, is completely determined once a theory space is fixed. As a consequence, the choice of theory space determines the set of fixed points Γ^* at which asymptotically safe theories can be defined. Therefore, in the asymptotic safety scenario the bare action $S = \Gamma^*$ is a *prediction* of the theory rather than an ad hoc postulate as usually in quantum field theory. (Ambiguities could arise only if there is more than one suitable NGFP.)

(5) According to the results available to date, the Einstein-Hilbert action of classical General Relativity seems not to play any distinguished role in the asymptotic safety context, at least not at the conceptual level. The only known NGFP on the theory space of QEG has the structure $\Gamma^* =$ Einstein-Hilbert action+ "more" where "more" stands for both local and non-local corrections. So it seems that the Einstein-Hilbert action is only an approximation to the true fixed point action, albeit an approximation which was found to be rather reliable for many purposes.

(6) Any quantum theory of gravity must reproduce the successes of classical General Relativity. As for QEG, it cannot be expected that this will happen for all RG trajectories in $\mathcal{S}_{\text{UV}}$, but it should happen for some or at least one of them. Within the Einstein-Hilbert truncation it has been shown [41] that there actually do exist trajectories (of type IIIa) which have an extended classical regime and are consistent with all observations.

(7) In the classical regime mentioned above the spacetime geometry is non-dynamical to a very good approximation. In this regime the familiar methods of quantum field theory in curved classical spacetimes apply, and it is clear therefore that effects such as Hawking radiation or cosmological particle production are reproduced by the general framework of QEG with matter.

(8) Coupling free massless matter fields to gravity, it turned out [12] that the fixed point continues to exist under very weak conditions concerning the number of various types of matter fields (scalars, fermions, etc.). No fine tuning with respect to the matter multiplets is necessary. In particular asymptotic safety does not seem to require any special constraints or symmetries among the matter fields such as supersymmetry, for instance.

(9) Since the NGFP seems to exist already in pure gravity it is likely that a widespread prejudice about gravity may be incorrect: its quantization seems not to require any kind of unification with the other fundamental interactions.

Given the situation that by now the asymptotic safety of QEG hardly can be questioned any more, future work will have to focus on its physics implications. The effective average action is an ideal framework for investigations of this sort since, contrary to other exact RG schemes, it provides a family of scale dependent *effective* (rather than *bare*) actions, $\{\Gamma_k[\,\cdot\,], 0 \leq k < \infty\}$. Dealing with phenomena involving typical scales k, a tree–level evaluation of Γ_k is sufficient for finding the leading quantum gravity effects. The investigations already performed in this direction employed the following methods.

(a) *RG improvement*: In refs. [33] and [35], respectively, a first study of the asymptotic safety-based "phenomenology" of black hole and cosmological spacetimes has been carried out by "RG improving" the classical field equations or their solutions. Hereby k is identified with a fixed, geometrically motivated scale. Using the same method, modified dispersion relations of point particles were discussed in [42].

(b) *Scale dependent geometry:* In the spirit of the gravitational average action, a spacetime manifold can be visualized as a fixed differentiable manifold equipped with infinitely many metric structures $\{\langle g_{\mu\nu}\rangle_k, 0 \leq k < \infty\}$ where $\langle g_{\mu\nu}\rangle_k$ is a solution to the effective field equation implied by Γ_k. Comparable to the situation in fractal geometry the metric, and therefore all distances, depend on the resolution of the experiment by means of which spacetime is probed. A general discussion of the geometrical issues involved (scale dependent diffeomorphisms, symmetries, causal structures, etc.) was given in [27], and in [26] these ideas were applied to show that QEG can generate a minimum length dynamically. In [3, 5] it has been pointed out that the QEG spacetimes should have fractal properties, with a fractal dimension equal to 4 on macroscopic and 2 on microscopic scales. This picture was confirmed by the computation of their spectral dimension in [28]. Quite remarkably, the same dynamical dimensional reduction from 4 to 2 has also been observed in Monte-Carlo simulations using the causal triangulation approach [29, 30, 31]. It is therefore intriguing to speculate that this discrete approach and the gravitational average action actually describe the same underlying theory.

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