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INTRODUCTION TO REGGE CALCULUS

PGQG1, Zakopane, March 2007

Lecture 1 Basics of the theory

Motivation

Why do we want to consider discrete space-times in general? (broader question)

- Fundamental reason: space-time may be discrete at the smallest scales. eg Wheeler's "space-time foam" - large fluctuations in topology at the Planck scale.
- Practical reasons:
  1. (Regge's original motivation) approximation scheme in (classical) numerical relativity
  2. Reduces the number of variables (to a countable set?)
  3. Provides a cut-off (eg minimum edge length) in quantum gravity or some other way of regularizing.
  4. Can make use of discrete techniques developed in other areas of quantum physics eg lattice gauge theories.

Why Regge calculus?

- Concerned with structure of space-time (∴ could not be more basic - although Regge often said to me that we should move on to triangulating group manifolds!)
- In principle, provides way of solving Einstein's
- Coordinate-independent approach (title of Regge's paper) - predictions must be coords. indep.

equations for systems without a large degree of symmetry (just tweak edge-lengths here and there!)

- Provides way of representing complicated topologies
- Helps with visualization of solutions – very geometric.
- There is a considerable body of mathematical literature dealing with piecewise linear spaces, the theory of intrinsic curvature on polyhedra etc – provides rigorous proof of what Regge seemed to know intuitively! Also provides way of relating it to the continuum

A brief history

1961. Regge's paper. Ideas taken up by Wheeler. Mainly used for finding classical solutions, often numerically, except

1968 Ponzano and Regge – an aside in a paper on the asymptotic values of 6j-symbols (much studied by nuclear physicists) pointed out the relation between a state sum for a 3-d manifold and the Feynman path integral with the Regge action. ie Ponzano-Regge model

70's & 80's – numerical relativity, weak field 4-d quantum gravity, numerical simulation of QG, quantum cosmology (simplicial minisuperspace). Dynamical triangulation. Ponzano-Regge model virtual ignored until

1991 Turaev-Viro Model – regularization of PR → spin foam models.

Work continued on RC, but fair to say that main progress not on RC as such, but spin foam models, & dynamical triangulation

A good understanding of RC is important for these!

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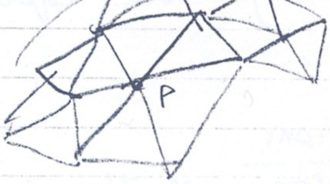
Essentials of the theory

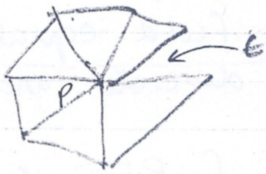
Rather than considering space(time)s with continuously varying curvature, consider ones with the curvature restricted to subspaces of co-dimension.

1. Take a collection of flat n-simplices (simplices are best because their shapes are uniquely determined by the specification of their edge-lengths).
2. "Glue" them together by identifying their flat (n-1)-d faces
3. The curvature lies on the (n-2)-d subspaces known as hinges or bones. (Why?)

It is easiest to get a feel for this by starting in 2-d and working up. We use the basic idea that there is no curvature at a subspace which can be projected into flat space without gaps or distortion.

2-d: network of flat triangles eg geodes. dome. Take triangles meeting at P. Project onto plane. No curvature on triangles or at edges, but only at vertices. The deficit angle ϵ is





given by

$$\epsilon = 2\pi - \sum(\text{vertex angles})$$

and is a measure of the curvature.
(δ large - very spiky, δ negative - negatively curve)

3-d - tessellate with flat tetrahedra. Consider the tetrahedra meeting at an edge

$$\epsilon = 2\pi - \sum \text{dihedral angles at edge}$$

4-d: use 4-simplices (5 vertices all connected)
∴ hinges are triangles
⋮
and so on - easy to generalize to any dimension
(For more mathematical view, see P4')

So far we have a collection of simplices which may be good for visualization but as yet have nothing to do with general relativity. We need to find a way of deciding whether our piecewise linear space is an Einstein space, so we need some variables. as a start, we take the fundamental variables to be the edge-lengths (or their squares), which are an obvious analogue of the continuum metric. What is the analogue of Einstein's equations for these edge lengths? We construct the analogue of the Einstein action from these edge lengths then use the principle of stationary action, which is a standard way of deriving the equations in the continuum.

The action and field equations

We want the discrete analogue of the Einstein-Hilbert action

$$I = \frac{1}{16\pi G}\int R\sqrt{-g}\, d^n x$$

in n dim, where R is the scalar curvature. There are various ways of obtaining this. In his original paper, Regge put forward a heuristic argument (for the benefit of physicists with limited mathematical knowledge?)

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Insertion on P4

What was Regge doing?

2-d: Gaussian integral curvature of a (spherical) triangle $\varepsilon_t = \alpha + \beta + \gamma - \pi$

Local Gaussian curvature at P

$$K(P) = \lim_{A_t \to 0} \frac{\varepsilon_t}{A_t} \qquad (t \text{ shrinks to point } P).$$

Can define this for polyhedra (think of tet)

$K(P) = 0$ if P is not a vertex.

$\sum$ internal angles $= 2\pi - \varepsilon_v$ as I wrote down

Total curvature $= \sum_v \varepsilon_v$

For compact finite closed M,

$$\sum_v \varepsilon_v = 2\pi\,(2 - N) \qquad \text{- Gauss-Bonnet}$$

N: genus

Higher-d: first express 2-d in terms of ϵ cones

$$ds^2 = d\rho^2 + \rho^2 d\theta^2$$

- identify θ differing by $2\pi - \epsilon$ rather than 2π

Then take $\mathbb{R}^{n-2} \times \epsilon$-cone

$$ds^2 = dx_1^2 \ldots + dx_{n-2}^2 + d\rho^2 + \rho^2 d\theta^2.$$

1. Interior of n-simplices flat
2. Boundary of n-simplex decomposable into (n+1) flat (n-1)-simplices
3. (n-2)-simplices have geom. of ε cone.

Use parallel transport. Decompose vector into part parallel to hinge and 2-d vector $\perp$r to it.

Rotation angle $\varepsilon(a) = \int_a K(P)\, dA$

Obtain $\varepsilon(t) = \varepsilon_t$

Angle is additive function of loop.

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Since in the simplicial space the curvature is restricted to the (n-2)-d, subspaces or hinges,

$$I_R = \sum_{\text{hinges } i} F_i .$$

The hinges are homogeneous, so F_i is proportional to the volume of the hinge

$$F_i = \underset{\text{volume}}{|\sigma_i|}\ \underset{\text{deficit angle}}{f(\epsilon_i)}$$

Since a hinge can be considered as the superposition of 2 hinges of the same shape and size and such that $\epsilon_i = \epsilon_i^{(1)} + \epsilon_i^{(2)}$, the

$$f(\epsilon_i) = f(\epsilon_i^{(1)}) + f(\epsilon_i^{(2)})$$

and f is linear in ϵ_i. The final result is

$$I_R = \frac{1}{8\pi G} \sum_{\text{hinges}} |\sigma_i|\, \epsilon_i$$

Alternatively we can consider parallel transport of a vector A_μ about a collection of hinges with deficit angle ϵ and density ρ say in 3-d: U_ν a unit vector parallel to the hinges and Σ_μ the vector area of the loop,

$$\delta A_\mu = \rho\, \epsilon\, \epsilon_{\mu\gamma\sigma} U^\gamma A^\sigma$$

Comparing this with the standard formula for parallel transport,

$$\delta A_\mu = R^{\ \ \sigma}_{\mu\ \alpha\beta}\, \Sigma^{\alpha\beta} A_\sigma$$

with $\Sigma^{\alpha\beta} = \epsilon^{\alpha\beta\gamma} \Sigma_\gamma$ etc,

it can be shown that

$$R_{\rho\sigma\alpha\beta} = \rho\, \epsilon\, U_{\rho\sigma} U_{\alpha\beta}$$

where $U_{\rho\sigma}$ is a bivector normal to the hinge, given by

$$U_{\rho\sigma} = \epsilon_{\rho\sigma\lambda} U^{\lambda}$$

By suitable contraction, the expression for I_R is again obtained

To obtain the field equations, we now vary the action wrt the edge lengths, the discrete analogues of the metric tensor:

$$\frac{\partial I_R}{\partial l_p} = \sum_{\text{hinges}} \left(\frac{\partial |\sigma_i|}{\partial l_p} \varepsilon_i + |\sigma_i| \frac{\partial \varepsilon_i}{\partial l_p} \right)$$

Remarkably the second term is zero. Recall that

$$\varepsilon_i = 2\pi - \sum_{\substack{\text{simplices} \\ q \text{ meeting} \\ \text{on hinge } i}} \theta_i^q$$

where θ_i^q is the dihedral angle between the two faces of simplex q meeting on hinge i. Then

$$\sum_{\text{hinges } i} |\sigma_i| \frac{\partial \varepsilon_i}{\partial l_p} = -\sum_i \sum_q |\sigma_i| \frac{\partial \theta_i^q}{\partial l_p}$$

$$= -\sum_{\substack{\text{simplices} \\ q}} \left(\sum_{\substack{\text{hinges } i \\ \text{of simplex} \\ q}} |\sigma_i| \frac{\partial \theta_i^q}{\partial l_p} \right)$$

and it is the sum within each simplex which is actually zero. For a proof see the appendix of Regge's paper. It is based on the simple fact that the flux of a constant vector through a closed surface is zero! (Stokes theorem)

So finally the classical field equations are

$$\sum_{\text{hinges } i} \frac{\partial |\sigma_i|}{\partial l_p} \varepsilon_i = 0$$

There appears to be one equation for each unknown so it looks as though we shall be able to solve for the edge lengths and obtain a simplicial Einstein Space. But.... see later.

Simplex practicalities (Euclidean case)

For the formalism to be of any use, we are going to have to calculate deficit angles and volumes of simplices. It is simple to include a cosmological constant term:

$$\int \lambda \sqrt{g}\, d^n x \longrightarrow \lambda \sum_{\text{simplices } q} |\sigma_q|$$

so we need volumes of both the n- and $(n-2)$-d simplices.

Volume The easiest formula to remember is

$$V_n^2 = \frac{(-1)^{\frac{n+1}{2}}}{n!\, 2^{n/2}} \left| \begin{array}{cccccc} 0 & 1 & 1 & 1 & 1 \ldots & 1 \\ 1 & 0 & s_{01} & s_{02} & & s_{0n} \\ 1 & s_{01} & 0 & s_{12} & \ldots & s_{1n} \\ 1 & s_{02} & s_{12} & 0 & s_{23} \ldots & s_{2n} \\ 1 & s_{03} & & s_{23} & 0 & \\ 1 & & & \vdots & & \\ 1 & & & & & \end{array} \right|$$

where $s_{ij} = l_{ij}^2$, the squared edge length between vertices i and j, with the n-simplex vertices labelled $0, 1, \ldots n$.

Alternatively

$$V_n^2 = \frac{1}{(n!)^2} \det(e_i \cdot e_j)$$

where $e_i \cdot e_j = \frac{1}{2}(s_{0i} + s_{0j} - s_{ij})$ (This also gives the metric within the simplex in coords where the vertices are (00,..0),(00..01) etc)

(so $e_i \cdot e_i = s_{0i}$)

Dihedral angles

For the dihedral angle at a hinge with volume V_{n-2} between 2 faces with volumes V_{n-1}, V'_{n-1}, in a simplex with volume V_n, the simplest formula is

$$\sin\theta = \frac{n}{(n-1)} \frac{V_{n-2} V_n}{V_{n-1} V'_{n-1}}$$

The problem with this nice formula is that in a convex simplex, a dihedral angle lies between 0 and π, and this formula does not distinguish between θ and $\pi - \theta$. So sometimes it is necessary to work out $\cos\theta$. The formula directly in terms of edge lengths is very complicated by a beautiful version is

$$\cos\theta = \frac{\omega_{n-1} \cdot \omega'_{n-1}}{V_{n-1} V'_{n-1}}$$

where $\omega_n = e_1 \wedge e_2 \wedge \ldots \wedge e_n$, the volume n-form associated with an n-simplex, with $e_1, \ldots e_n$ being the vectors from vertex 0 to the remaining n vertices

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Simplicial Manifolds and the Dehn-Sommerville relations

A simplicial complex is not neccesarily a simplicial manifold – there is an important restriction. The <u>star</u> of a vertex in a simplicial complex is the collection of simplices of all dimensions which share that vertex. The <u>link</u> of a vertex is the star <u>minus</u> the simplices or subsimplices which contain that vertex. The condition for an n-d simp. complex to be a manifold is that the link of each vertex is homeomorphic to an (n−1)-d sphere.

eg 2-d

For simplicial manifolds, there are lots of formula connecting numbers of simplices of each dimension, deficit angles etc. eg Dehn-Sommerville relations

$$N_p^{(2n)} = \sum_{i=1}^{2n} (-1)^i \binom{i+1}{p+1} N_i^{(2n)}$$

where $N_i^{(2n)}$ is the number of simplices of dimension i in a $(2n)$-d manifold. For example, for a 4-d manifold, this gives

$$2N_3 = 5N_4$$

(taking $p=2$)

Also the Euler character is given by

$$\chi^{(2n)} = \sum_{i=0}^{2n} (-1)^i N_i^{(2n)}$$

These relations are very useful – eg for simplifying the action for dynamical triangulations

(Can also obtain formulae in terms of numbers of even dimensional subsimplices or in terms of dihedral angles. Coefficients related to Bernoulli numbers)

Bianchi identities

I need to explain my cautious statement about the match between the number of equations and variables in Regge calculus. The "But" is because Regge calculus has its own Bianchi identities, which means that all the equations are not independent.

Regge already described the BIs in his paper and showed that they have a beautiful topological interpretation. It is easiest to see how this works in 3-d, but the generalization to higher dimensions is straightforward (in 2-d, there are no BI in RC or in the continuum). Recall that if we parallel transport a vector on a path enclosing an edge, it rotates; if the path does not enclose an edge, nothing happens. Consider the path shown – each of the edges is enclosed once but it is topologically trivial – it can be deformed into a path which encloses no edge. Therefore the product of the rotation matrices for the edges is the identity matrix, which means a relation between the deficit angles for edges meeting at a vertex

$$\prod_{\substack{\text{edges } i \\ \text{at chosen vertex}}} \exp\left(\epsilon_i U^{(i)}_{\alpha\beta}\right) = 1$$

In the limit of small deficit angles (ie just keeping terms of order ϵ in the expansion of the exponentials), this reduces to the continuum Bianchi identities. Similarly in higher dimensions.

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So what about finding classical solutions if the equations are not all independent? This is analogous to the gauge freedom in the continuum where (in the 3+1 ADM formalism) one can freely specify the lapse and shift vector; we are free to specify an appropriate set of edge lengths (eg time-like ones cf lapse)

Diffeomorphisms

In the continuum, the existence of diffeomorphisms is closely related to the Bianchi identities. No-one has been able to make this ^explicit^ correspondence for Regge calculus. There is not even agreement on how to define diffeos in RC - the two main suggestions are as follows

1. Diffeos are transformations of the edge lengths which leave the geometry invariant (GR viewpoint!) Then diffeos exist only in flat space and correspond to changes in the edge lengths as the vertices move around in the flat space. If the space is almost flat, then one can define approximate diffeos.
2. Diffeos are transformations of the edge lengths which leave the action invariant and this is less restrictive - one can imagine making changes in the edge lengths in one region and compensating changes in another region or it could even be done locally. (Particle physics viewpoint!)

[It is even possible in 3-d to construct transformations which are exact invariances of the action — this relies on the uniqueness of the embedding of the star of a vertex in flat 4-d space : the number of dof (ie the edge lengths of the star) equals the number of coordinates for its embedding in 4-d. So the diffeos are a 3-parameter family of motions of the point in the flat 4-d space which leave the action invariant. This method does not work in 4-d as there is no unique embedding of a 4-d star].

A so-called "gauge-fixed" version of RC was constructed by Römer and Zähringer. Their argument was that one needs only one representation of each geometry in the path integral, so they restricted their simplicial complexes to be equilateral — a fore-runner of dynamical triangulations

Conformal transformations

These have been defined only infinitesimally for RC. Define a scalar function ϕ at each vertex and require the edge length joining vertices i and j to transform as

$$l'_{ij} = \phi_i \phi_j l_{ij}$$

Recall that the edges of a simplicial complex must satisfy the triangle inequalities and their higher dimensional analogues. It can be shown that the product of two transformations which satisfy these constraints will in general violate them. It is only infinitesimal transformations that satisfy the required group property

The continuum limit

A natural question to ask is how closely RC is connected with continuum GR. The most profound work on this was done by Cheeger, Müller and Schrader who considered analogue of the Lipschitz-Killing curvatures of smooth manifolds for piecewise linear spaces. As a special case they showed that the Regge action converges to the continuum action in the sense of measures (when integrated over a finite region) provided a condition on the "fatness" of the simplices is satisfied (they have to be fat enough, not too elongated and spiky!)

Even if the actions are related, it does not necessarily mean a similar relation for the field equations as the derivations may not be equivalent. John Barrett studied the relation between the Regge equations and the Einstein equations, and set up a convergence criterion in the linearized case only ie for solutions of the linearized Regge equations to converge to solutions of the linearized Einstein equation.

Lecture 2!! Regge calculus in low(ish) dimensions

Regge calculus in 2-d

In 2 dimensions, the Regge action is

$$I_R = \frac{1}{8\pi G} \sum_i \varepsilon_i$$

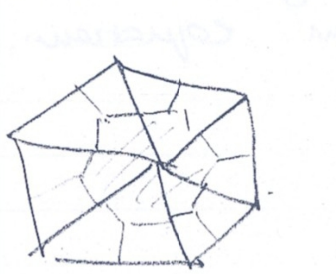
(the "volume" of a vertex is defined to be 1!) This is actually a topological invariant, the Euler character.

$$I_R = \frac{\chi}{4G},$$

which makes it sound as though 2-d RC is totally trivial. However it becomes less so if one includes a cosmological constant term, and possibly a higher derivation one too (I want to discuss the details of that in the 4-d context). A possible action is

$$I = \sum_{\text{hinges } i} \left(\lambda A_i - 2k\, \varepsilon_i + 4a \frac{\varepsilon_i^2}{A_i} \right)$$

where A_i is the area associated with a hinge (using the Voronoi construction or some such prescription). This action has been the basis of numerical simulations in which the phase diagram is studied – the behaviour differs in variation regions determined by the coupling constants. These have been compared with similar calculations using dynamic triangulations and with analytic work based on the Polyakov string. The Regge results for the Ising model agree with the flat space values, whereas the DT ones agree with the analytic values



<u>Regge calculus in 3-d</u>

We know that GR in 3-d is very different from 4-d. What about 3-d RC? The action for pure gravity is

$$I_R = \frac{1}{8\pi G} \sum_{\text{edges}} l_i \varepsilon_i ,$$

so variation wrt l_j leads to the field equation

$$\varepsilon_j = 0 .$$

ie flat space! Now include a matter contribution

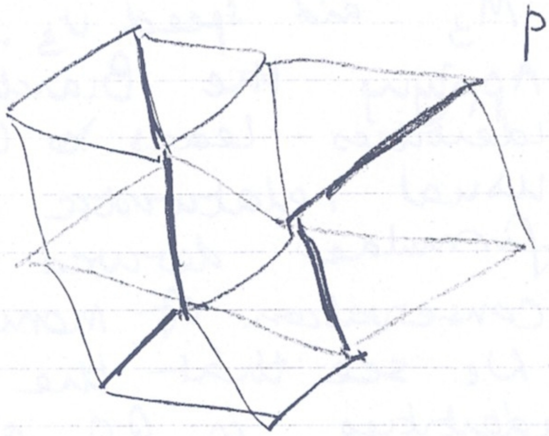
$$I_R \to \frac{1}{8\pi G} \sum l_i \varepsilon_i - \int \mathcal{L}_{\text{matter}} \sqrt{-g}\, d^3x .$$

and suppose that the matter consists of point particles with mass M_i moving along some subset of the hinges:

$$\therefore I_R = \frac{1}{8\pi G} \sum l_i \varepsilon_i - \sum l_i M_i$$

Then variation wrt l_j gives

$$\varepsilon_j = \begin{cases} 0 & \text{for a hinge without a particle} \\ 8\pi G M_i & \text{" " " on which a particle of mass } M_i \text{ moves} \end{cases}$$



The picture is of world lines of particles consisting of linked sequences of time-like edges. (Deficit angles at space-like edges would correspond to tachyons in this interpretation.)

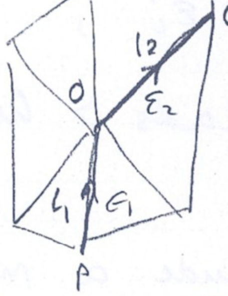
Recall the 3-d Bianchi identities

$$\prod_{\substack{\text{edges} \\ i \text{ meeting} \\ \text{at a vertex}}} \exp(\varepsilon_i U^{(i)}) = 1$$

where the rotation generator for edge l_i is

$$(U^{(i)})_{\mu\nu} = \frac{1}{l_i} \epsilon_{\mu\nu\rho} l_i^{\rho}$$

We will now apply this to two scenarios

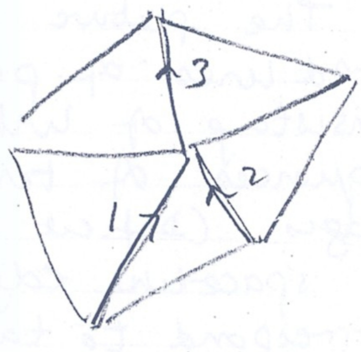
1. 

Suppose two time-like edges PO, OQ with deficit angles ε_1, ε_2, meet at O, with zero deficits at all the other edges at O. If we set up local coordinates, assume the first particle is at rest and take the trace of $\exp(\varepsilon_1 U^{(1)}) \exp(\varepsilon_2 U^{(2)}) = 1$ we obtain

$$\varepsilon_1 = \varepsilon_2$$
$$v_2 = 0$$

ie a particle at rest, with mass M, will continue to be at rest with that mass (ie conservation of momentum for a single particle)

2. Consider now a collision process – a particle with mass M_1 at rest is hit by a particle of mass M_2, speed v_2 and they coalesce to form a third particle of mass M_3 and speed v_3.



Applying the Bianchi identities leads to the usual relativistic collision formulae derived from conservation of momentum. We see that the Bianchi identities in RC are equivalent to conservation of momentum in a system of gravitating point particles in (2+1)-d.

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<u>(2+1)-d RC and 't Hooft's approach</u>

't Hooft developed a formalism for (2+1)-d gravity partly as a response to a paper by Gott claiming acausality for (2+1)-d gravity coupled to point particles. It is not RC and I will describe it only briefly, but there are a number of similarities.

The basic idea is to tessellate 2-d space-like hypersurfaces with polygons and then to allow these polygons to evolve in time subject to certain rules. For simplicity the vertices are all taken to be trivalent and at some of them there are particles. A Lorentz frame is attached to each frame and it is assumed that all polygons have a common clock (≡ partial fixing of gauge). The consequences of the basic assumptions are

i) lengths of adjacent edges measured in neighbouring frames are equal
ii) the velocity of each edge is perpendicular to that edge
iii) velocities of adjacent edges are equal.

Evolution occurs when the tessellation changes because an edge shrinks to zero or a vertex hits an edge. As in GR, there are constraints on the initial data which follow from considering parallel transport around vertices. Particles produce conical singularities as in RC.

't Hooft stated and proved his "crunch and bang theorem" which says that if at a certain time, all polygons are contracting this, will hold at all moments of time. The time reversal of this also holds. He did calculations on a rather ancient computer and found how the types of evolution depended on the initial topology. His formulae can be interpreted in terms of hyperbolic geometry but there is no time to discuss that.

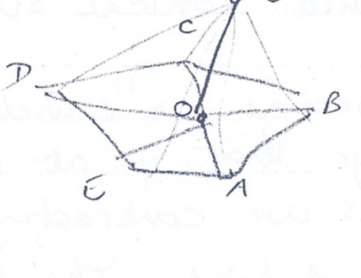
In (2+1)-d Regge calculus, similar calculations can be performed; analytically for very symmetric initial conditions and numerically otherwise. We have yet to see whether all the results are the same.

't Hooft used his model as a basis for the quantization of (2+1)-d gravity and seems to have ended up with non-commuting coordinates (CQG 13 (1996) 1023-39).

(3+1)-d Regge calculus

(a) Sorkin evolution

Let me first mention a property of the Regge equations which is very important in classical calculations. Since the equation for an edge length involves all the surrounding edge lengths, one might think one has to solve all the equations for an evolved hypersurface simultaneously. However this is not the case. It's actually easiest to see in (2+1)-d. Consider a spacelike hypersurface in which all edge lengths are known.



Focus on the region around vertex O, and imagine building a tent on this - first the tent pole, the time-like edge from O to O'

Then join O' to A, B, C, D, E - The new edge-lengths are those of $O'O$, $O'A$, $O'B$, $O'C$, $O'D$, $O'E$, and the equations available are those from the variation of OO', OA, OB, OC, OD, OE which are all interior edges. These equations involve only known edges, plus the new ones, and we see that we have a match between equations and unknown variables and so can solve locally before going on to another tent.

(b) Lund-Regge approach

It is natural to want to construct a $(3+1)$-formalism of RC, analogous to the ADM formalism in the continuum. It turns out that this is a non-trivial task. Lund and Regge made the first progress here. They first developed a calculus on simplices as follows:

Given a symmetric tensor T_{ab} and an n-simplex (with $\frac{1}{2}n(n+1)$ edges), define the quantities T_i by

$$T_i \equiv T_{ab}\, l_i^a l_i^b \qquad i = 1, \ldots \tfrac{1}{2}n(n+1).$$

$$\text{eg } g_i = g_{ab}\, l_i^a l_i^b = l_i^2 = s_i$$

To find the trace of T_{ab}, use

$$\mathrm{Tr}\, T = \sum_i T_i \frac{\partial(\log V^2)}{\partial s_i}$$

$$\text{eg } \mathrm{Tr}\, g = \sum s_i \frac{\partial(\log V^2)}{\partial s_i} = n$$

using the fact that V^2 is a homogeneous function of s_i of degree n.

Similarly $\mathrm{Tr}(T^2) = -\sum_{ij} T_i T_j \dfrac{\partial^2 \log V^2}{\partial s_i \partial s_j}$

Then $(\mathrm{Tr}\,T)^2 - \mathrm{Tr}(T^2) = \sum_{i,j} T_i T_j \left(\frac{\partial \log V^2}{\partial s_i} \frac{\partial \log V^2}{\partial s_j} + \frac{\partial^2 (\log V^2)}{\partial s_i \partial s_j} \right)$

$$= \frac{1}{V^2} \sum_{i,j} T_i T_j \frac{\partial^2 V}{\partial s_i \partial s_j} \quad \text{(check!)}$$

Recall that in the continuum, the action in its (3+1)-form is

$$I \propto \int dt \; d^3x \; N\sqrt{g}\left(\mathrm{Tr}(K^2) - (\mathrm{Tr}\,K)^2 + {}^3R\right)$$

where $K_{ij} = -\frac{1}{2N}\left(\frac{\partial g_{ij}}{\partial t} - N_{i|j} - N_{j|i}\right)$

is the extrinsic curvature on a hypersurface of constant time. N and N_i are the lapse and shift functions as usual.

For homogeneous spaces the momentum constraints are identically satisfied and so the shift need never appear in the formalism (since their rôle is as Lagrange multipliers). Now consider the simplicial form of such a space. We assume that all the tetrahedra in our 3-d spatial hypersurface have identical values. We have

$$K_i = -\frac{1}{2N} \dot{s}_i$$

$$\mathrm{Tr}\,K^2 - (\mathrm{Tr}\,K)^2 = -\frac{1}{4N^2 V^2} \sum_{i,j} \dot{s}_i \dot{s}_j \frac{\partial^2 V^2}{\partial s_i \partial s_j}$$

leading to

$$I_R = \frac{1}{8\pi G} \int dt \left(-\frac{1}{4N} \frac{N_4}{V} \sum_{i,j} \dot{s}_i \dot{s}_j \frac{\partial^2 V^2}{\partial s_i \partial s_j} + 2N \sum l_i \varepsilon_i \right)$$

R^3 term

It is then possible to solve for the edge lengths using the principle of stationary action (or the E-L equations)

In the more general case, there has to be a shift vector and momentum constraints Friedman and Jack included these but they had problems with closure of the algebra of constraints. They applied the Dirac procedure and were able to satisfy the constraint equations by solving them for the lapse and shift functions.

The (3+1) - formalism of RC has never been widely used, probably because quantization attempts have tended to use the path integral approach rather than canonical quantization.

Lecture 3 Regge calculus in 4 dimensions (Mainly Euclidean).

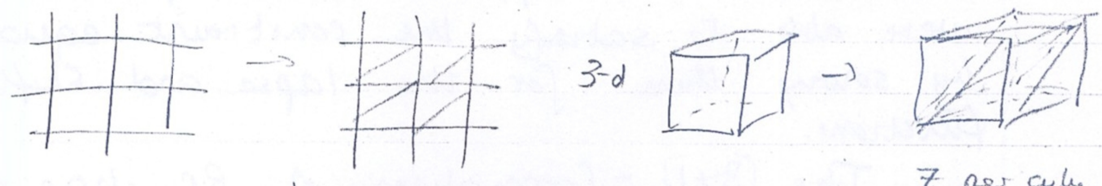
Perturbation theory

Although Ponzano and Regge produced their model of 3-d quantum Regge calculus in 1968, it was a long time before anyone did any 4-d quantum RC, partly because it was hard to see how the 3-d result could be extended. The difficulty of doing analytic work in 4-d in the general case prompted detailed investigations of the weak field case in the early 80s. The idea is to do perturbation theory about a classical background and for example to compare the Regge propagator in the weak field limit and the continuum propagator. I shall explain this in some detail as it is the basis for quite a lot of work in quantum RC.

We are going to consider an infinite space - a tessellation of $\mathbb{R}^4$, so there had better be a lot of symmetry or it will be impossible. We consider a lattice of unit hypercubes, (ie we are going to perturb about a Euclidean flat space) with vertices labelled by coordinates (n_1, n_2, n_3, n_4) where $n_i \in \mathbb{Z}$. Each hypercube is divided into 24 4-simplices by drawing in appropriate forward-going diagonals. It's easier to visualize in lower dimensions:

2-d

3-d



1 per square

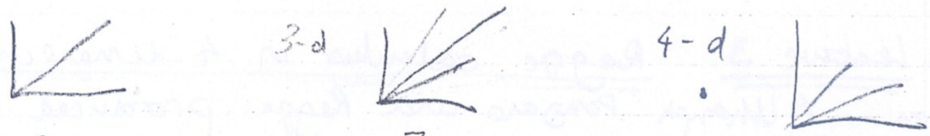
7 per cube.

The whole lattice can be generated by the translation of a set of edges based at the origin

2-d

3-d

4-d



3

7

15

$(2^n - 1)$

We interpret the coordinates of vertices neighbouring the origin as binary numbers (eg (0,1,0,0) is vertex 4, (1,1,1,1) is 15). The edges emanating from the origin in the positive direction join it to
vertices 1, 2, 4, 8 ("coordinate edges")
" 3, 5, 6, 9, 10, 12 ("face diagonals")
" 7, 11, 13, 14 ("body diagonals")
vertex 15 ("hyperbody diagonal")

Small perturbations are then made about the flat space edge lengths, so that
$$L_j^{(i)} = l_j\left(1 + \delta_j^{(i)}\right)$$
where the superscript i denotes the base point, the subscript j denotes the direction (1, 2, ... 15) and l_j is the

unperturbed edge length ($1, \sqrt{2}, \sqrt{3}$ or 2). Thus for example, $\delta^{(1)}_{14}$ would be the perturbation in the length of the edge from vertex 1 in the 14 – direction ie to vertex 15. The δ's are assumed to be small compared with 1.

The Regge action

$$I_R = \frac{1}{8\pi G} \sum_{\text{triangles}} A_i \epsilon_i$$

is then calculated for the hypercube at the origin (it is enough to keep first order terms in δ in both A and ϵ) and then obtained for all others by translation. Now

$$I_R(\delta) = \underbrace{I_R(0)}_{\substack{0 \text{ for} \\ \text{flat space}}} + \underbrace{\delta_i \frac{\partial I_R}{\partial \delta_i}}_{\substack{0 \\ \text{since flat space is a} \\ \text{solution of the} \\ \text{Regge equations} \\ \text{(stationary point} \\ \text{of the action)}}} + \frac{1}{2!} \delta_i \delta_j \frac{\partial^2 I_R}{\partial \delta_i \partial \delta_j} + \ldots$$

Therefore we write symbolically

$$I_R = \sum \underline{\delta}^T M \underline{\delta} + O(\delta^3)$$

where $\underline{\delta}$ is an infinite-dimensional column vector with 15 components / point and M is an infinite-dimensional sparse matrix. All the entries corresponding to the fluctuation of the hyperbody diagonal are zero, so these form a one-parameter family of zero eigenmodes. Physical translations of the vertices which leave the space flat also form a (4-param) family of eigenmodes. These are the exact diffeos in this case.

Nobody likes an infinite-dimensional matrix! It is block diagonalized by Fourier transformation or expansion in periodic modes. We set

$$\delta_j^{(a,b,c,d)} = (\omega_1)^d (\omega_2)^c (\omega_4)^b (\omega_8)^a \, \delta_j^{(0)}$$

(remember that the action only links edges in adjacent simplices), where $\omega_k = e^{2\pi i/n_k}$, $k = 1, 2, 4, 8$. Acting on these periodic modes, M becomes a matrix with 15×15 blocks M_ω along the diagonal, with

$$M_\omega = \begin{pmatrix} A_{10} & B & 0 \\ B^+ & 18 I_4 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where the 10×10 matrix A and the 4×10 matrix B have entries which are functions of the ω's.

M_ω is then itself block diagonalized by a non-unitary but uni-modular similarity transformation, giving diagonal blocks

$$Z = A_{10} - \frac{1}{18} B B^+, \quad 18 I_4, \quad 0$$

The 4×4 block $18 I_4$ means that the fluctuations δ_j for $j = 7, 11, 13, 14$, have been decoupled – they are constrained to vanish by the equations of motion. Note that rather amazingly the number of degrees of freedom/point has been reduced from 15 to the 10 we expect in the continuum.

We now compare this with the continuum, where we work with trace reversed metric fluctuations

ie $g_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu}$

and $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\delta_{\mu\nu} h$

It turns out that Z corresponds exactly to L_{sym} where

$$L_{sym} = L + \frac{1}{2} C_\mu^2 ,$$

with $L = -\frac{1}{2} \partial_\lambda h_{\alpha\beta} V_{\alpha\beta\mu\nu} \partial_\lambda h_{\mu\nu}$,

$$V_{\alpha\beta\mu\nu} = \frac{1}{2}\delta_{\alpha\mu}\delta_{\beta\nu} - \frac{1}{4}\delta_{\alpha\beta}\delta_{\mu\nu},$$

and $C_\mu = \partial_\nu h_{\mu\nu} - \frac{1}{2}\partial_\mu h$

C_μ is the gauge-breaking term. To make this correspondence, the long wave-length (or weak-field) limit has been taken by expanding the ω's in powers of the momentum k, with

$$k_i = -\frac{2\pi}{a n_i}, \quad a \text{ the lattice spacing}$$

and so $\omega_i = 1 - i a k_i - \frac{a^2 k_i^2}{2} \ldots$

In this limit, we have exact agreement of the propagators, which gives us confidence that at least in this régime, Regge calculus is equivalent to the weak field continuum theory.

Boundary terms

So far we have ignored the possibility of manifolds having boundaries but this is obviously not always the case. Hartle and Sorkin derived the RC equivalent of the Gibbons-Hawking boundary term in the following way.

In obtaining the Regge field equations we made use of the identity

$$\sum_{\text{hinges } i} |\sigma^i| \frac{\partial \theta_i^q}{\partial l_p} = 0$$

in simplex ...

for each simplex q, to show that the variation of the deficit angle does not contribute to the equations. If an edge on the boundary is fixed, the cancellation is not complete, so the H-S boundary term was constructed so that the usual Regge equations hold for all interior edges, while the boundary edges are held fixed, corresponding to specifying the metric on the boundary in the continuum. The resulting boundary term is

$$\sum_{\substack{\text{hinges}\\ i \text{ on}\\ \text{boundary}}} |\sigma^i| \psi_i$$

where $\psi_i = \pi - \sum_{\text{at hinge}} \text{dihedral angles}$

which is precisely what one would expect. It was shown to correspond to the Gibbons-Hawking term $\int K \sqrt{h}\, d^3x$ in the continuum limit.

In quantum applications, it is often the connection, or some combination of the metric and the connection, which is held fixed on the boundary and it is not known what the appropriate Regge boundary term is in those cases

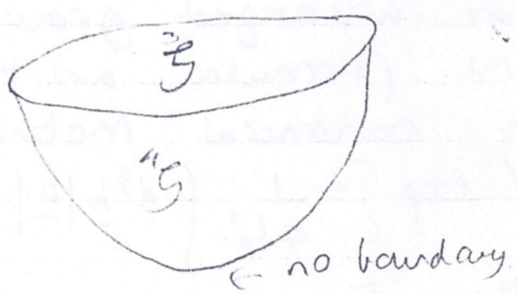
Simplicial minisuperspace and quantum cosmology

Quantum cosmology aims, amongst other things, to calculate the wave function of the universe. According to the Hartle-Hawking prescription,

$$\Psi({}^3\mathcal{G}) = \int d[{}^4\mathcal{G}] \exp i S({}^4\mathcal{G})$$

given 3-geom

all 4-geom with ${}^3\mathcal{G}$ as boundary



For both technical and practical reasons, this is impossible to do in its full generality, so one might hope to integrate over 4-geometries which dominate the sum over histories. This has led to the idea of minisuperspace models involving the use of a single 4-geometry or perhaps several. Even this is complicated enough unless there is an enormous amount of symmetry and Hartle introduced the idea of summing over simplicial 4-geometries, which can be described by a finite number of edge lengths. There are still technical difficulties - the unboundedness (from below) of the Einstein action persists in RC, leading to convergence problems for the integral, so it is necessary to rotate the integration contour in the complex plane to fix this.

The sum over 4-geometries should in principle also include a sum over 4-manifolds with different topologies, and we then run into the

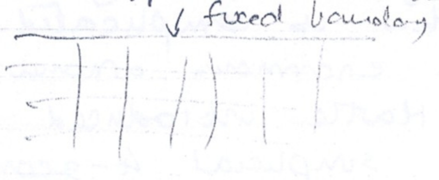
In 2-d, these fail to be manifolds (28) at a finite number of vertices, and Hartle showed how the generalization worked – the action is still dominated by manif... However pseudo-manifolds are not satisfactory in higher dimensions.

classification problem for 4-manifolds. This led Hartle to suggest integrating over the more general class of "unruly topologies". Schleich and Witt have explored using conifolds in 4d, which differ from manifolds at a finite number of points. However a sum over topologies is still very far from implementation.

Hartle calculated the ground state wave function for linearized gravity, obtaining the same formula as Kuchař had derived from canonical methods:

$$\Psi_0[h^{TT}_{ij}, t] = \mathcal{N} \exp\left\{-\frac{1}{4l_p^2}\int d^3\underline{k}\,|\underline{k}|\,h^{TT}_{ij}(\underline{k})\bar{h}^{TT}_{ij}(\underline{k}\right.$$

where $h^{TT}_{ij}(\underline{k})$ is the Fourier transform of the transverse traceless part of the metric deviation from flatness. This calculation was repeated in Regge calculus – clearly the RC boundary term was needed here, and the tessellation based on hypercubes



was put to use again.

Matter fields in RC and the measure

We have mainly discussed pure gravity (ie vacuum space-times) so far but it is obviously very important to have a way of coupling gravity to all types of matter. In the earliest work (eg the Friedmann universe) the matter was taken to be pressureless dust with uniform density

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In early work, fluids were defined using velocity potentials in each simplex, so the contribution to the action involves products of densities and volumes.
More generally, it is conventional for scalar fields to be defined at vertices in lattice theories, and gauge fields on edges. On the other hand, fermions need to be defined within simplices, or rather on the sites of the dual lattice, with their coupling defined by way of the LT relating the frames in neighbouring simplices.

Since most of the quantum applications of RC involve the path integral approach, the definition of the measure is very important. There was some formal work on this in the early days, but it is still rather controversial. One's attitude depends mainly on whether integrating over edge-lengths is thought to overcount each diffeomorphically-equivalent contribution. In his numerical Regge calculus, Hamber (more details later) has found no problem of overcounting and uses a simple measure

$$\int \prod_k |\sigma_k^4|^\sigma \prod_{ij} dl_{ij}^2 \, \Theta(l_{ij}^2)$$

imposes the triangle inequalities

- lattice analogue of the DeWitt measure

$$\int \prod_x \left(\sqrt{g(x)}\right)^\sigma \prod_{\mu \geq \nu} dg_{\mu\nu}(x)$$

- this is derived from the DeWitt supermetric on the space of metrics, and the lattice version can be derived from the RC version of the supermetric (which is equivalent to the Lund-Regge metric). Often simple scale invariant (dl/l) measures are used, or $l dl$ which seems more like $dg_{\mu\nu}$.

The opposite view is that at least for weak field perturbation theory, it is necessary to divide through by the volume of the diffeomorphism group using the Fadeev-Popov determinant. Menotti and Peirano have derived a very complicated formula for this in 2-d and it is not clear how to extend this calculation to higher dimensions. But perhaps we need not worry because as Martin has pointed out, what we usually want to calculate is the expectation value of some operator A, which is given by

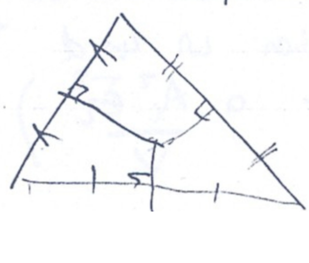
$$\langle A \rangle = \frac{\int d[g]\, A(g)\, \exp(-I[g])}{\int d[g]\, \exp(-I[g])}$$

and formally something very useful happens. Two diffeomorphic manifolds contribute identically in both numerator and denominator since A and I are both invariant under diffeos. Each integral is therefore the volume of the diffeo group times a sum over physically distinct metrics, and since the volume of the diffeo group is infinite, each integral diverges. The divergent factor formally cancels. But he adds that to implement the cancellation in a practical way requires the techniques of gauge fixing and ghosts.

Numerical simulations of discrete quantum gravity

There is no time to give a complete survey of this so I will concentrate on the work with which I am most familiar. The basic idea of this is to start with a Regge lattice for, say, flat space and allow it to evolve using a Monte Carlo algorithm. Random fluctuations are made in the edge lengths and the new configuration rejected if it increases the action, and accepted with a certain probability if it decreases the action. The system evolves to some equilibrium configuration, about which it makes quantum fluctuations and then the expectation values of various operators can be calculated. It is also possible to study the phase diagram and search for phase transitions, which can determine whether or not the theory has a continuum limit.

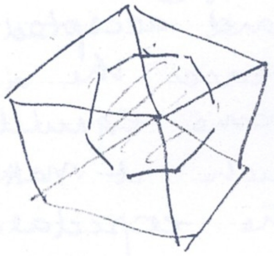
The use of a higher derivative term in the action (Hamber et al) was motivated partly by the problem of the unboundedness of the action below. Let me say a little about how the lattice form of the term was arrived at. Since the Regge curvature is restricted to the $(n-2)$-d hypers, it looks at first as though a curvature-squared term would involve the product of δ-functions, so we need to find a procedure which avoids this. First consider the construction of the dual lattice. In 2-d, we construct the $\perp$r bisectors of the sides of the triangles to divide



a triangle into 3 regions, each consisting of points nearer to its vertex than any other point

The generalization to higher dimension is straightforward – this is known as the Voronoi construction, and one can write down formulae for all the volumes thereby constructed. Suppose that V_i is the volume of the n-simplices meeting at hinge i, which is associated with that hinge. Then we may write the total volume as

2-d



$$\sum_{n\text{-simplices } q} |\sigma_q|$$

or equivalently

$$\sum_{\text{hinges } i} V_i$$

Then $\int d^n x \sqrt{g} \longleftrightarrow \sum_{\text{hinges } i} V_i$

We now regard the scalar curvature R at each hinge as being represented by $2|\sigma_i|\varepsilon_i / V_i$ which is then consistent with

$$\int d^n x \sqrt{g}\, R \longrightarrow \sum_i V_i \frac{2|\sigma_i|\varepsilon_i}{V_i} = 2\sum_i |\sigma_i|\varepsilon_i$$

closely related to definition of sectional curvature

It follows that

$$\int d^n x \sqrt{g}\, R^n \longrightarrow \sum_i V_i \left(\frac{2|\sigma_i|\varepsilon_i}{V_i}\right)^n$$

This form gives a good approximation to the continuum value for regular tessellations of S^3 and S^4 (it is exact for S^2)

Based on this, the action in the numerical simulations is taken in 4-d to be

$$I_R = \sum \left(\lambda V_i - k A_i \varepsilon_i + a \frac{A_i^2 \varepsilon_i^2}{V_i} \right)$$

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in analogy with the continuum action

$$I = \int d^4x \sqrt{g}\left(\lambda - \frac{k}{2}R + \frac{a}{4}R^2\right)$$

with $k = \frac{1}{8\pi G}$: One can see that $\left[\begin{array}{l}\text{or } R_{\mu\nu}R^{\mu\nu} \\ \text{or } R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}\end{array}\right.$

the Regge action is positive definite provided the coefficients satisfy a simple inequality. In practice it was found that a, the coefficient of the higher derivative term, could be taken to be arbitrarily small without any noticeable problems.

Simulations along these lines have been done for the last 20 years. I'll mention just one result: for a lattice with topology T^4, the system developed a small negative average curvature at strong coupling. For details and other results see the review by Loll (1998).

RC in a large number of dimensions

Many of the formulae I have written down are valid in arbitrary dimension, so one can go to large n and apply the results of mean field theory, which exploits the fact that in large n, each point is typically surrounded by many neighbours, whose action can be either treated exactly or included as some sort of local average. It is quite challenging to calculate deficit angles in arbitrary dimension, so perturbation theory has been done about an equilateral Tessellation. For large n, the dihedral angle in an equilateral n-simplex is approximately $\pi/2$, so to obtain a space with small deficit angles, we need a simplicial complex in which 4 n-simplices meet at each $(n-2)$-simplex. This is provided by the surface of a cross polytope in $(n+1)$-d, an object of dimension d. It corresponds to a triangulated manifold with no boundary, homeomorphic to the sphere. It can be visualized as a set of $2n+2$ vertices arranged in a circle, with each vertex joined to every other one, except the one opposite it. (It corresponds to the octahedron in $n=2$) One can use all this to calculate the partition function for pure gravity in arbitrary dimension and calculate the average edge length fluctuation.

Lecture 4 Area Regge calculus

Motivation

In the Ponzano-Regge model of quantum gravity, the basic variables are the edge lengths or rather the representations of $SU(2)$ with which the edges are labelled. This corresponds to the usual choice of variables in RC. In generalisation of P-R to 4-d, it seems usually to be the triangles (or 2-d dual faces) to which representations are assigned. This led Rovelli to suggest that these models might be related to a modified version of RC, in 4-d, in which the variables are the triangle areas rather than the edge-lengths. This idea fitted in with earlier work by Mäkelä who used triangle areas as variables, based on a modification of the Ashtekar variables. He noted that there are constraints amongst the triangle areas (because they are functions of a (generally) smaller number of edge lengths). He took account of these constraints so his formalism really reduced to using edge-length variables. The alternative is to take the triangle areas as fundamental and independent variables and we shall explore this a bit now.

Formalism and problems
A 4-simplex has both 10 triangles and 10 edges which is very suggestive and we would expect to be able to invert the relationship between these variables A_t and s_i to express the Regge action

$$I_R = \frac{1}{8\pi G} \sum_t A_t(s_i)\, \epsilon_t(s_i)$$

in terms of the areas A_t. Variation of the action wrt A_t should lead to simplicial Einstein spaces – or does it? There are a number of snags here.

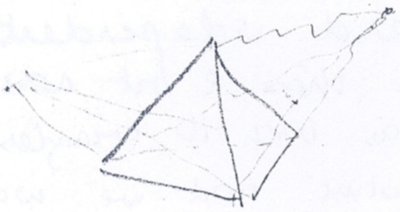
1. Inverting the relationship between areas and edge lengths: for a triangle with vertices ijk,

$$A^2_{ijk} = \left[2(s_{ij}s_{jk} + s_{ik}s_{jk} + s_{ij}s_{ik}) - s_{ij}^2 - s_{ik}^2 - s_{jk}^2\right]/16$$

(inverting is certainly not easy!)

Tuckey: there are pairs of 4-simplices with squared edge lengths $(1, 1, \ldots, 1, a)$ which have the same triangle areas but with different values of a either side of 2. (A^2 is quadratic in a and are all a maximum at $a = 2$) This corresponds to a geometry in which some of the triangles are right angled. There needs to be a restriction on the class of edge lengths considered, to avoid such singular points of the Jacobian. This can be done by restricting to some neighbourhood of an equilateral 4-simplex

2 The match between numbers of variables for a single 4-simplex does not hold for collections of 4-simplices eg consider 2 4-simplices meeting on a common tetrahedral face. This complex has 14 edges but 16 triangles. For any closed 4-manifold,



$$N_2 \geq \frac{4}{3} N_1$$

(use Dehn-Sommerville!) This can give rise to complicated constraints among the areas, or else some apparent paradox: Consider 2 4-simplices joined in this way, with areas specified, and assume we are in a region where the mapping between areas and edge-lengths is (1-1). Solve for the edge-lengths in one 4-simplex (well-defined) then in the other. Since the common tetrahedron has 6 edges and 4 triangles, the areas do not determine the edge-lengths of the tetrahedron uniquely, so we can imagine the bizarre situation where the edge-lengths of the common tetrahedron, have different values, depending on which 4-simplex they are considered part of! More of that later.

What type of solutions are there in area RC? (We restrict ourselves to the class of "simplices for which $J \neq 0$, then the defect angles are determined unambiguously by the triangle areas. Then the action is

$$I_{AR} = \sum_{\text{triangles } t} A_t \, \epsilon_t(A_s)$$

Variation wrt A_u gives

$$0 = \epsilon_u(A_s) + \sum_t A_t \frac{\partial \epsilon_t}{\partial A_u}$$

$$= \epsilon_u(A_s) + \sum_t A_t \frac{\partial}{\partial A_u}\left(2\pi - \sum_q \theta_t^q\right)$$

where θ_t^q is the internal dihedral angle at triangle t in simplex q. Use the chain rule and interchange the orders of summation over triangles and over 4-simplices to obtain

$$0 = \epsilon_u(A_s) - \sum_q \sum_t A_t \sum_i \frac{\partial \theta_t^q}{\partial s_i} \frac{\partial s_i}{\partial A_u}.$$

where t now runs over the triangles in simplex q and i runs over the edges in q. Recall that

$$\sum_t A_t \frac{\partial \theta_t^q}{\partial s_i} = 0$$

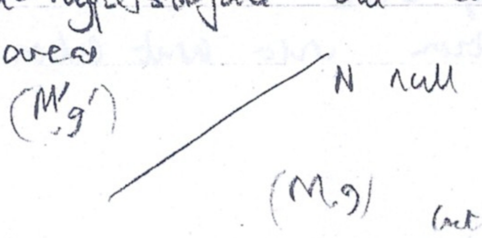
for each 4-simplex q (Regge), and $\partial s_i / \partial A_u$ is non-singular. $\therefore$ we have

$$\epsilon_u = 0 \quad \forall u.$$

In conventional RC, this would mean that space is locally flat but here the deficit angle ϵ does not necessarily measure the holonomy of a connection, so the conclusion is not so clear.

Interpretation: gravitational waves and discontinuous metrics

Barrett defined "refractive gravitational waves" a generalization of impulsive gravitational waves, in which the metric is discontinuous across a null-hypersurface but there is a matching condition on areas



Impulsive wave – metrics induced on N by g, g' match

Refractive wave – metrics disjoint but areas in N [illegible]

Barrett showed that this can be a singular solution of Einstein's equation (linearized approx, continuous sandwich wave).

This seems reminiscent of the situation in area RC, and the details were discussed by Wainwright. The basic idea is to consider a triangulated manifold M, divided by a hypersurface Σ into $M^\pm$. With areas as basic variables, the field equations just impose zero deficit angles. Demand that the areas are chosen to give well-defined edge-lengths in $M^\pm$. Combined with $\varepsilon_u = 0 \ \forall u$, this means that $M^\pm$ are both flat. For simplicity, assume that Σ is embedded in $M^\pm$ with no extrinsic curvature. Then the interior dihedral angles on Σ will sum to π in both $M^\pm$. Assume each lattice site is identical to its neighbours except across Σ.

In practice, we proceed as follows.

- Take a flat 3-lattice in ∂M^-
- Transform the coords of the lattice points, (keeping the lattice flat) such that the areas of the triangles are unchanged. This gives the lattice on ∂M^+
- Extend the 3-lattices on $\partial M^\pm$ to 4-lattices in $M^\pm$. Indentify lattices on $\partial M^\pm$ (since triangle areas agree)

Look at different classes of hypersurface

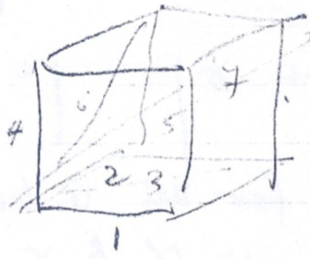
(1) Σ spacelike

Work in coords (t, x, y, z) with Σ at $t=0$

Then on Σ

$$ds^2 = dx^2 + dy^2 + dz^2$$

We use the tessellation into unit cubes and label the edges with the usual binary notation. Since the hypersurface is flat, the edges 3, 5, 6, 7 are determined by edges 1, 2, 4.



Let x_i be a vector along the ith coord edge, then a transformation of the lattice is determined by a trans^n of x_1, x_2, x_3 given by the matrix $T \in GL(3)$

Since each lattice site is identical there are only 6 indep. triangle areas

$$A_{13},\ A_{15}\ \ A_{17}\ \ A_{26}\ \ A_{27}\ \ A_{37}$$

(A_{ij} = area of triangle with i, j as edges).

We require this area to be invariant under $x_i \to x_i' = T x_i$. To see what transformation satisfy this, decompose T as follow

$$T = \underbrace{O}_{\in O(3)}\underbrace{P}_{\text{positive definite symmetric}}$$

$$\underbrace{Q}_{\in O(3)}\underbrace{D}_{\text{diagonal (positive entries)}}Q^T$$

$$\therefore\ T = O\,Q\,D\,Q^T.$$

O, Q orthog and do not change the areas

$\therefore$ just require that D doesn't. Suppose

$$D = \mathrm{diag}(a, b, c) \qquad a, b, c > 0$$

Consider a triangle based at the origin, with its other vertices at $(X_A\ Y_A\ Z_A)$ and (X_B, Y_B, Z_B). Its area is given by

$$A^2 = \frac{1}{4}\left(\begin{vmatrix} Y_A & Z_A \\ Y_B & Z_B\end{vmatrix}^2 + \begin{vmatrix} Z_A & X_A \\ Z_B & X_B\end{vmatrix}^2 + \begin{vmatrix} X_A & Y_A \\ X_B & Y_B\end{vmatrix}^2\right)$$

and under D, this goes to

$$A'^2 = \frac{1}{4}\left(b^2c^2 \left|\quad\right|^2 + a^2c^2\left|\quad\right|^2 + a^2b^2\left|\quad\right|^2\right)$$

We require $A' = A$. Putting $\alpha = b^2c^2 - 1$, $\beta = a^2c^2 - 1$, $\gamma = a^2b^2 - 1$, we have

$$\alpha \left| \quad \right|^2 + \beta \left| \quad \right|^2 + \gamma \left| \quad \right|^2 = 0$$

This has to be satisfied for all 6 triangles, giving 6 linear equations in α, β, γ. Let $\underline{v}^T = (\alpha, \beta, \gamma)$. Then

$$G\underline{v} = 0$$

where G is a 6×3 matrix. It can be argued that G has rank 3, and so the only solution is $\underline{v} = 0$, which implies $a = b = c = 1$. Therefore only orthogonal transformations leave the triangle areas independent. A similar argument can be made for time-like hypersurfaces.

(11) $\underline{\Sigma \text{ null}}$

Work in double null coords (u, v, y, z) with Σ at $u = 0$. Then

$$ds^2 = dy^2 + dz^2 \qquad (\text{since } v \text{ null})$$

The areas of the triangles do not depend on v, and we decompose the transformation matrix $\tilde{T}$ as

$$\begin{pmatrix} a & \underline{b}^T \\ \underline{c} & E \end{pmatrix} = \begin{pmatrix} a - \underline{b}^T E^{-1} \underline{c} & \underline{b}^T E^{-1} \\ 0 & \underline{1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \underline{c} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & E \end{pmatrix}$$

$E \in GL(2)$, $\underline{b}, \underline{c}$ 2 cpt vectors.

$$E = \tilde{O}\hat{P} = \tilde{O}\hat{Q}\tilde{D}\tilde{Q}^T.$$

Then $\tilde{T} = \begin{pmatrix} * & * \\ 0 & \tilde{O}\hat{Q} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \underline{d} & \tilde{D} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \tilde{Q}^T \end{pmatrix}$

with $\underline{d} = (\hat{O}\hat{Q})^T \underline{c}$

A triangle based at the origin with other vertices $(V_A Y_A Z_A)$, $(V_B Y_B Z_B)$ has area

$$A = \frac{1}{2} \begin{vmatrix} Y_A & Z_A \\ Y_B & Z_B \end{vmatrix}$$

Only the middle matrix can alter this

Let $\underline{d}^T = (c, d)$, $\hat{D} = \mathrm{diag}(a, b)$ $a, b > 0$
Then we require the area to be invariant under
$$y' = ay + cv \quad , \quad z' = bz + dv$$
This gives the condition
$$(ab-1)\begin{vmatrix} Y_A & Z_A \\ Y_B & Z_B \end{vmatrix} - bc\begin{vmatrix} Z_A & V_A \\ Z_B & V_B \end{vmatrix} - ad\begin{vmatrix} V_A & Y_A \\ V_B & Y_B \end{vmatrix} = 0$$
The invariance of the 6 triangles areas gives 6 equations for $\hat{\underline{v}}^T = (ab-1, -bc, -ad)$
$$\hat{G}\,\hat{\underline{v}} = 0$$
Again $\hat{G}$ has rank 3, so the only solution is $\hat{v} = 0$. This gives $ab = 1$, $c = d = 0$ and we have $\hat{D} = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ $a > 0$

We now look at how these solutions could be related to metric discontinuities. Consider a lattice in Σ with metric g. The edge lengths (squared) are defined by
$$s = \underline{x}^T g\, \underline{x}$$
Under T, $s \to s' = (Tx)^T g\, Tx$ and so the new metric is
$$g' = T^T g\, T$$
For Σ <u>spacelike</u>, the initial metric was $g = \mathbb{1}_3$ and T had to be orthogonal
$$\therefore \quad g' = T^T \mathbb{1}_3 T = \mathbb{1}_3$$
$\therefore$ there is no difference in metrics across Σ and edge-lengths will be continuous.
For Σ <u>null</u>, the initial metric was
$$g = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix}$$
and we found the required form of T.
Then $g' = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{Q}\hat{D}^2\tilde{Q}^T \end{pmatrix}$

with $\tilde{Q}$ being a rotation through θ, this becomes

$$g' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a^2\cos^2\theta + a^{-2}\sin^2\theta & (a^2 - a^{-2})\cos\theta\sin\theta \\ 0 & (a^2-a^{-2})\cos\theta\sin\theta & a^2\sin^2\theta + a^{-2}\cos^2\theta \end{pmatrix}$$

which is precisely the metric of Barrett's refractive wave space time

These solutions can be related to linearized general relativity (where distributional solutions allowed) but not to the full nonlinear theory as then the curvature tensor would not be defined

In principle, it is possible to combine refractive waves and construct geodesics which start off parallel but get distorted by them. One has to face questions like whether it is possible to tessellate a manifold with null triangles (no) and even more importantly of how to interpret area Regge calculus at non-null triangles.

Dynamics

In spite of these difficulties, it seems that area RC has some dynamical content. We performed the perturbation of the lattice of unit hypercubes using area variables - in this case we have to use a distorted lattice to get rid of the right angles which can lead to vanishing of J. (Each hypercube is squashed along the hyperbody diagonal so that it has length 1 rather than 2. The variation in triangle areas are defined by

$$A_i = A_i^{(0)}(1 + \Delta_i) \qquad \Delta_i \ll 1$$

The second variation of the action is given by

$$I_{AR} = \Delta_i N_{ij} \Delta_j$$

where again N_{ij} is a sparse infinite dimensional matrix. Fourier transformation reduces N to a block diagonal matrix with 50x50 blocks N_ω. The size of N_ω makes it necessary to investigate it numerically and surprisingly the number of dynamical modes is the same as in the edge length case. There are again 4 zero modes and a further 6 scaling with k^2. The remaining 40 modes enter non-dynamically (they are massive + do not scale w. momentum) and they are constrained to vanish by their equations of motion.

We see that at least in the weak field case, area Regge calculus has the same dynamic content as the original RC. However there is still lots to understand about it

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