ASPHERICITY OF LENGTH FOUR RELATIVE GROUP PRESENTATIONS

Abstract

We consider the relative group presentation $\mathcal{P} = \langle G, \boldsymbol{x} | \boldsymbol{r} \rangle$ where $\boldsymbol{x} = \{x\}$ and $\boldsymbol{r} = \{xg_1xg_2xg_3x^{-1}g_4\}$. We show modulo a small number of exceptional cases exactly when $\mathcal{P}$ is aspherical. If $H = \langle g_1^{-1}g_2, g_1^{-1}g_3g_1, g_4 \rangle \leq G$ then the exceptional cases occur when H is isomorphic to one of C_5, C_6, C_8 or $C_2 \times C_4$.
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1 Introduction

A *relative group presentation* is a presentation of the form $\mathcal{P} = \langle G, \boldsymbol{x} | \boldsymbol{r} \rangle$ where G is a group, $\boldsymbol{x}$ a set disjoint from G and $\boldsymbol{r}$ is a set of cyclically reduced words in the free product $G * \langle \boldsymbol{x} \rangle$ where $\langle \boldsymbol{x} \rangle$ denotes the free group on $\boldsymbol{x}$. If $G(\mathcal{P})$ denotes the group defined by $\mathcal{P}$ then $G(\mathcal{P})$ is the quotient group $G * \langle \boldsymbol{x} \rangle / N$ where N denotes the normal closure in $G * \langle \boldsymbol{x} \rangle$ of $\boldsymbol{r}$. A relative presentation is defined in [2] to be *aspherical* if every spherical picture over it contains a *dipole*. If $\mathcal{P}$ is aspherical then statements about $G(\mathcal{P})$ can be deduced and the reader is referred to [2] for a discussion of these; in particular torsion in $G(\mathcal{P})$ can easily be described.

We will consider the case when both $\boldsymbol{x}$ and $\boldsymbol{r}$ consists of a single element. Thus $\boldsymbol{r} = \{r\}$ where $r = x^{\varepsilon_1}g_1 \ldots x^{\varepsilon_k}g_k$ where $g_i \in G$, $\varepsilon_i = \pm 1$ and $g_i = 1$ implies $\varepsilon_i + \varepsilon_{i+1} \neq 0$ for $1 \leq i \leq k$, subscripts mod k. If $k \leq 3$ or if $r \in \{xg_1xg_2xg_3xg_4, xg_1xg_2xg_3xg_4xg_5\}$ then, modulo some open cases, a complete classification of when $\mathcal{P}$ is aspherical has been obtained in [1], [2], [7] and [10]. The case $r = (xg_1)^p(xg_2)^q(xg_3)^r$ for $p, q, r > 1$ was studied in [11] and $r = x^n g_1 x^{-1} g_2$ $(n \geq 4)$ was studied in [5]. The authors of [9] used results from [7] in which $r = xg_1xg_2x^{-1}g_3$ to prove asphericity for certain LOG groups. In this paper we continue the study of asphericity and consider $r = xg_1xg_2xg_3x^{-1}g_4$. Observe that $r = 1$ if and only if $x^{-1}g_2^{-1}x^{-1}g_1^{-1}x^{-1}g_4^{-1}xg_3^{-1} = 1$ so replacing x^{-1} by x it follows that we can work modulo $g_1 \leftrightarrow g_2^{-1}$ and $g_3 \leftrightarrow g_4^{-1}$. A standard approach is to make the transformation $t = xg_1$ and then consider the subgroup H of G generated by the resulting coefficients. In our case r becomes $t^2g_1^{-1}g_2tg_1^{-1}g_3g_1t^{-1}g_4$ and so $H = \langle g_1^{-1}g_2, g_1^{-1}g_3g_1, g_4 \rangle$. One then usually proceeds according to either when H is non-cyclic or when H is cyclic. (Note that $\langle g_1^{-1}g_2, g_1^{-1}g_3g_1, g_4 \rangle$ is cyclic if and only if $\langle g_2g_1^{-1}, g_2g_4^{-1}g_2^{-1}, g_3^{-1} \rangle$ is cyclic.) The latter case seems to be the more complicated – indeed the open cases referred to in the above paragraph almost all involve H being cyclic. Our results reflect this difference in difficulty. When H is non-cyclic we obtain a complete answer except for the following case (modulo $g_1 \leftrightarrow g_2^{-1}$, $g_3 \leftrightarrow g_4^{-1}$) in which $H \cong C_2 \times C_4$:

(**E**) $|g_3| = 2$; $|g_4| = 4$; $g_1^{-1}g_2 = 1$; $g_1^{-1}g_3g_1g_4 = g_4g_1^{-1}g_3g_1$.

Theorem 1.1 *Let $\mathcal{P}$ be the relative presentation*

$$\mathcal{P} = \langle G, x | xg_1xg_2xg_3x^{-1}g_4 \rangle,$$

*where $g_i \in G$ $(1 \le i \le 4)$, $g_3 \ne 1$, $g_4 \ne 1$ and $x \notin G$. Let $H = \langle g_1^{-1}g_2, g_1^{-1}g_3g_1, g_4 \rangle$ and assume that H is non-cyclic and the exceptional case (**E**) does not hold. Then $\mathcal{P}$ is aspherical if and only if (modulo $g_1 \leftrightarrow g_2^{-1}$, $g_3 \leftrightarrow g_4^{-1}$) none of the following conditions holds:*

(i) $|g_4| < \infty$ *and* $g_3g_1g_2^{-1} = 1$;

(ii) $|g_1^{-1}g_2| < \infty$, $|g_3| = |g_4| = 2$ *and* $g_1^{-1}g_3g_1g_4 = g_2g_4^{-1}g_2^{-1}g_3^{-1} = 1$;

(iii) $\dfrac{1}{|g_1^{-1}g_2|} + \dfrac{1}{|g_3|} + \dfrac{1}{|g_4|} + \dfrac{1}{|g_2g_4g_1^{-1}g_3^{-1}|} > 2.$

Now let H be cyclic. Before stating the theorem we make a list of exceptions (modulo $g_1 \leftrightarrow g_2^{-1}$, $g_3 \leftrightarrow g_4^{-1}$).

(**E1**) $|g_4| = 5$; $g_1^{-1}g_2 = g_4^2$; $g_1^{-1}g_3g_1 = g_4^3$.

(**E2**) $|g_4| = 6$; $g_1^{-1}g_2 = 1$; $g_1^{-1}g_3g_1 = g_4^2$.

(**E3**) $|g_4| = 6$; $g_1^{-1}g_2 = 1$; $g_1^{-1}g_3g_1 = g_4^4$.

(**E4**) $|g_4| = 8$; $g_1^{-1}g_2 = 1$; $g_1^{-1}g_3g_1 = g_4^4$.

Observe that (**E1**) implies $H \cong C_5$; (**E2**) and (**E3**) imply $H \cong C_6$; and (**E4**) implies $H \cong C_8$.

Theorem 1.2 *Let $\mathcal{P}$ be the relative presentation*

$$\mathcal{P} = \langle G, x | xg_1xg_2xg_3x^{-1}g_4 \rangle$$

*where $g_i \in G$ $(1 \le i \le 4)$, $g_3 \ne 1$, $g_4 \ne 1$ and $x \notin G$. Let $H = \langle g_1^{-1}g_2, g_1^{-1}g_3g_1, g_4 \rangle$ be a cyclic group. Suppose that none of the exceptional conditions (**E1**)–(**E4**) holds. Then $\mathcal{P}$ is aspherical if and only if either H is infinite or H is finite and (modulo $g_1 \leftrightarrow g_2^{-1}$, $g_3 \leftrightarrow g_4^{-1}$) none of the following conditions holds:*

(i) $g_3g_1g_2^{-1} = 1$;

(ii) $g_3^{-1}g_1g_2^{-1} = g_2g_4^{-1}g_1^{-1}g_3^{-1} = 1$;

(iii) $g_3^{-1}g_1g_2^{-1} = g_4g_2^{-1}g_1 = 1$;

(iv) $|g_3| = 2$; $|g_4| = 2$;

(v) $|g_3| = 2$; $g_1^{-1}g_3g_2g_4 = g_1^{-1}g_2g_4^{-2} = 1$;

(vi) $|g_3| = 2$; $g_1^{-1}g_3g_2g_4 = (g_1^{-1}g_2)^2g_4^{-1} = 1$;

(vii) $g_1^{-1}g_2 = 1$; $g_1^{-1}g_3g_1g_4^{\pm 1} = 1$;

(viii) $g_1^{-1}g_2 = 1$; $|g_3| = 2$; $|g_4| = 3$;

(ix) $g_1^{-1}g_2 = 1$; $4 \le |g_3| \le 5$; $g_1^{-1}g_3^2g_1g_4$;

(x) $g_1^{-1}g_2 = 1$; $|g_3| = 6$; $g_1^{-1}g_3^3g_1g_4$.

In Section 2 we discuss pictures and curvature; in Section 3 there are some preliminary results; Theorem 1.1 and Theorem 1.2 are proved in Sections 4 and 5.

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2 Pictures

The definitions of this section are taken from [2]. The reader should consult [2] and [1] for more details.

A *picture* $\boldsymbol{P}$ is a finite collection of pairwise disjoint discs $\{\Delta_1, \ldots, \Delta_m\}$ in the interior of a disc D^2, together with a finite collection of pairwise disjoint simple arcs $\{\alpha_1, \ldots, \alpha_n\}$ embedded in the closure of $D^2 - \bigcup_{i=1}^m \Delta_i$ in such a way that each arc meets $\partial D^2 \cup \bigcup_{i=1}^m \Delta_i$ transversely in its end points. The *boundary* of $\boldsymbol{P}$ is the circle ∂D^2, denoted $\partial \boldsymbol{P}$. For $1 \leq i \leq m$, the *corners* of Δ_i are the closures of the connected components of $\partial\Delta_i - \bigcup_{j=1}^n \alpha_j$, where $\partial\Delta_i$ is the boundary of Δ_i. The *regions* of $\boldsymbol{P}$ are the closures of the connected components of $D^2 - \left(\bigcup_{i=1}^m \Delta_i \cup \bigcup_{j=1}^n \alpha_j\right)$.

An *inner region* of $\boldsymbol{P}$ is a simply connected region of $\boldsymbol{P}$ that does not meet $\partial\boldsymbol{P}$. The picture $\boldsymbol{P}$ is *non-trivial* if $m \geq 1$, is *connected* if $\bigcup_{i=1}^m \Delta_i \cup \bigcup_{j=1}^n \alpha_j$ is connected, and is *spherical* if it is non-trivial and if none of the arcs meets the boundary of D^2. The number of edges in a region Δ is called the *degree* of Δ and is denoted by $d(\Delta)$. If $\boldsymbol{P}$ is a spherical picture, the number of different discs to which a disc Δ_i is connected is called the *degree* of Δ_i, denoted by $\deg(\Delta_i)$.

With $\mathcal{P} = \langle G, \boldsymbol{x} | \boldsymbol{r} \rangle$ define the following labelling: each arc α_j is equipped with a normal orientation, indicated by a short arrow meeting the arc transversely, and labelled by an element of $\boldsymbol{x} \cup \boldsymbol{x}^{-1}$. Each corner of $\boldsymbol{P}$ is oriented *anticlockwise* (with respect to D^2) and labelled by an element of G. If κ is a corner of a disc Δ_i of $\boldsymbol{P}$, then $W(\kappa)$ is the word obtained by reading in an anticlockwise order the labels on the arcs and corners meeting $\partial\Delta_i$ beginning with the label on the first arc we meet as we read the anticlockwise corner κ. If we cross an arc labelled x in the direction of its normal orientation, we read x, otherwise we read x^{-1}.

A picture $\boldsymbol{P}$ is called a *picture over the relative presentation* $\mathcal{P}$ if the above labelling satisfies the following conditions.

(1) For each corner κ of $\boldsymbol{P}$, $W(\kappa) \in \boldsymbol{r}^*$, the set of all cyclic permutations of the members of $\boldsymbol{r} \cup \boldsymbol{r}^{-1}$ which begin with a member of $\boldsymbol{x}$.

(2) If $g_1, \ldots, g_l$ is the sequence of corner labels encountered in a *clockwise* traversal of the boundary of an inner region Δ of $\boldsymbol{P}$, then the product $g_1 \ldots g_l = 1$ in G. We say that $g_1 \ldots g_l$ is the *label* of Δ.

A *dipole* in a labelled picture $\boldsymbol{P}$ over $\mathcal{P}$ consists of corners κ and κ' of $\boldsymbol{P}$ together with an arc joining the two corners such that κ and κ' belong to the same region and such that if $W(\kappa) = Sg$ where $g \in G$ and S begins and ends with a member of $\boldsymbol{x} \cup \boldsymbol{x}^{-1}$, then $W(\kappa') = S^{-1}h^{-1}$. The picture $\boldsymbol{P}$ is *reduced* if it does not contain a dipole. A relative presentation $\mathcal{P}$ is called *aspherical* if every connected spherical picture over $\mathcal{P}$ contains a dipole.

The *star graph* $\mathcal{P}^{\mathrm{st}}$ of a relative presentation $\mathcal{P}$ is a graph whose vertex set is $\boldsymbol{x} \cup \boldsymbol{x}^{-1}$ and edge set is $\boldsymbol{r}^*$. For $R \in \boldsymbol{r}^*$, write $R = Sg$ where $g \in G$ and S begins and ends with a member of $\boldsymbol{x} \cup \boldsymbol{x}^{-1}$. The initial and terminal functions are given as follows: $\iota(R)$ is the first symbol of S, and $\tau(R)$ is the inverse of the last symbol of S. The labelling function on the edges is defined by $\lambda(R) = g^{-1}$ and is extended to paths in the usual way. A non-empty cyclically reduced cycle (closed path) in $\mathcal{P}^{\mathrm{st}}$ will be called *admissible* if it has trivial label in G. Each inner region of a reduced picture over $\mathcal{P}$ supports an admissible cycle in $\mathcal{P}^{\mathrm{st}}$.

As described in the introduction we will consider spherical pictures over $\mathcal{P} = \langle G, t | r \rangle$ where $r = t^2 g_1^{-1} g_2 t g_1^{-1} g_3 g_1 t^{-1} g_4$. For ease of presentation we introduce the following notation: $a = 1$, $b = g_1^{-1} g_2$, $c = g_1^{-1} g_3 g_1$ and $d = g_4$ and consider $tatbtct^{-1}d$. Exception (**E**) and conditions (i)–(iii) of Theorem 1.1 can then be re-written as

(**E**) $|c| = 2$; $|d| = 4$; $b = 1$; $cd = dc$.

(i) $|d| < \infty$ and $cab^{-1} = 1$;

(ii) $|a^{-1}b| < \infty$, $|c| = |d| = 2$ and $a^{-1}cad = bd^{-1}b^{-1}c^{-1} = 1$;

(iii) $\dfrac{1}{|a^{-1}b|} + \dfrac{1}{|c|} + \dfrac{1}{|d|} + \dfrac{1}{|bda^{-1}c^{-1}|} > 2.$

The exceptions (**E1**)–(**E4**) and conditions (i)–(x) of Theorem 1.2 can be rewritten as

(**E1**) $|d| = 5$; $b = d^2$; $c = d^3$;

(**E2**) $|d| = 6$; $b = 1$; $c = d^2$;

(**E3**) $|d| = 6$; $b = 1$; $c = d^4$;

(**E4**) $|d| = 8$; $b = 1$; $c = d^4$;

(i) $cab^{-1} = 1$;
(ii) $c^{-1}ab^{-1} = cadb^{-1} = 1$;
(iii) $c^{-1}ab^{-1} = db^{-1}a = 1$;
(iv) $|c| = |d| = 2$;
(v) $|c| = 2$; $cbda^{-1} = a^{-1}bd^{-2} = 1$;
(vi) $|c| = 2$; $cbda^{-1} = (a^{-1}b)^2 d^{-1} = 1$;
(vii) $a^{-1}b = 1$; $cad^{\pm 1}a^{-1} = 1$;
(viii) $a^{-1}b = 1$; $|c| = 2$; $|d| = 3$;
(ix) $a^{-1}b = 1$; $4 \leq |c| \leq 5$; $c^2ada^{-1} = 1$;
(x) $a^{-1}b = 1$; $|c| = 6$; $c^3ada^{-1} = 1$.

Figure 2.1

Let $\boldsymbol{P}$ be a reduced connected spherical picture over $\mathcal{P}$. Then the vertices (discs) of $\boldsymbol{P}$ are given by Figure 2.1(i) and (ii); and the star graph Γ is given by Figure 2.1(iii).

We make the following assumptions.

(**A1**) $\boldsymbol{P}$ has a minimum number of vertices.

(**A2**) If $|c| = 2$ then, subject to (**A1**), $\boldsymbol{P}$ has a maximum number of regions of degree 2 with label $c^{\pm 2}$.

Observe that (**A1**) implies that $c^{\varepsilon_1} w c^{\varepsilon_2}$, $d^{\varepsilon_1} w d^{\varepsilon_2}$ where $\varepsilon_1 = -\varepsilon_2 = \pm 1$ and $w = 1$ in G cannot occur as sublabels of a region. For otherwise a sequence of bridge moves [4] can be applied to produce a dipole which can then be deleted to obtain a picture with fewer vertices. Moreover if $|c| = 2$ then (**A2**) implies that $c^{\pm 2}$ cannot be a proper sublabel and $c^{\varepsilon} w c^{\varepsilon}$ where $\varepsilon = \pm 1$, $w = 1$ in G cannot be a sublabel of a region in $\boldsymbol{P}$. For otherwise bridge moves can be applied to increase the number of regions labelled $c^{\pm 2}$ while leaving the number of vertices unchanged.

To prove asphericity we adopt the approach of [6]. Let each corner in every region of $\boldsymbol{P}$ be given an angle. The *curvature* of a vertex is defined to be 2π less the sum of the angles at that vertex. The curvature $c(\Delta)$ of a k-gonal region Δ of $\boldsymbol{P}$ is the sum of all the angles of the corners of this region less $(k-2)\pi$. Our method of associating angles is to give each corner at a vertex of degree d an angle $2\pi/d$. This way the vertices have zero curvature and we need consider only the regions. Thus if Δ is a k-gonal region of $\boldsymbol{P}$ (a k-gon), denoted by $d(\Delta) = k$, and the degree of the vertices of Δ are d_i $(1 \leq i \leq k)$ then

$$c(\Delta) = c(d_1, d_2, \ldots, d_k) = (2-k)\pi + 2\pi \sum_{i=1}^{k} (1/d_i).$$

In fact since each $d_i = 4$ $(1 \leq i \leq k)$ we obtain

$$c(\Delta) = \pi(2 - k/2)$$

so if $d(\Delta) \geq 4$ then $c(\Delta) \leq 0$.

It follows from the fundamental curvature formula that $\sum c(\Delta) = 4\pi$ is where the sum is taken over all the regions Δ of $\boldsymbol{P}$. Our strategy to show asphericity will be to show that the positive curvature that exists in $\boldsymbol{P}$ can be sufficiently compensated by the negative curvature. To this end, as a first step, we located the regions Δ of $\boldsymbol{P}$ satisfying $c(\Delta) > 0$, that is, of positive curvature. For each such Δ we distribute all of $c(\Delta)$ to near regions $\hat{\Delta}$ of Δ. For such regions $\hat{\Delta}$ of $\boldsymbol{P}$ define $c^*(\hat{\Delta})$ to equal $c(\hat{\Delta})$ plus all the positive curvature $\hat{\Delta}$ receives in the distribution procedure mentioned above with the understanding that if $\hat{\Delta}$ receives no positive curvature then $c^*(\hat{\Delta}) = c(\hat{\Delta})$. Observe then that the total curvature of $\boldsymbol{P}$ is at most $\sum(c^*(\hat{\Delta}))$ where the sum is taken over all regions $\hat{\Delta}$ of D that are not positively curved regions. Therefore to prove $\mathcal{P}$ is aspherical it suffices to show that $c^*(\hat{\Delta}) \leq 0$ for each $\hat{\Delta}$.

Using the star graph Γ of Figure 2.1(iii) we can list the possible labels of regions of small degree (up to cyclic permutation and inversion).

$$d(\Delta) = 2 \Rightarrow l(\Delta) \in S_2 = \{c^2, d^2, a^{-1}b\}$$

$$d(\Delta) = 3 \Rightarrow l(\Delta) \in S_3 = \{c^3, cab^{-1}, c^{-1}ab^{-1}, d^3, db^{-1}a, d^{-1}b^{-1}a\}$$

Allowing each element in $S_2 \cup S_3$ to be either trivial or non-trivial yields 512 possibilities. This number can be reduced without any loss as follows.

1. Work modulo *T-equivalence*, that is, $a \leftrightarrow b^{-1}$, $c \leftrightarrow d^{-1}$.
2. Delete any combination that implies $c = 1$ or $d = 1$.
3. Delete any combination that yields a contradiction (for example $c^2 = 1$, $cab^{-1} = 1$, $c^{-1}ab^{-1} \neq 1$.
4. Delete any combination that yields $|d| < \infty$ and $cab^{-1} = 1$ or $|c| < \infty$ and $d^{-1}b^{-1}a = 1$ (see Lemma 3.1(i)).
5. When $H = \langle b, c, d \rangle$ is cyclic it can be assumed that H is finite (see Lemma 3.4(i)).

It can be readily verified that there remain 23 cases partitioned according to the existence in $\boldsymbol{P}$ of regions of degree 2 and are listed below.

Case A: There are no regions of degree two.

(**A0**) $|c| > 3$, $|d| > 3$, $a^{-1}b \neq 1$, $c^{\pm 1}ab^{-1} \neq 1$, $d^{\pm 1}b^{-1}a \neq 1$.
(**A1**) $|c| = 3$, $|d| > 3$, $a^{-1}b \neq 1$, $c^{\pm 1}ab^{-1} \neq 1$, $d^{\pm 1}b^{-1}a \neq 1$.
(**A2**) $|c| = 3$, $|d| = 3$, $a^{-1}b \neq 1$, $c^{\pm 1}ab^{-1} \neq 1$, $d^{\pm 1}b^{-1}a \neq 1$.
(**A3**) $|c| > 3$, $|d| > 3$, $a^{-1}b \neq 1$, $cab^{-1} = 1$, $c^{-1}ab^{-1} \neq 1$, $d^{\pm 1}b^{-1}a \neq 1$.
(**A4**) $|c| > 3$, $|d| > 3$, $a^{-1}b \neq 1$, $cab^{-1} \neq 1$, $c^{-1}ab^{-1} = 1$, $d^{\pm 1}b^{-1}a \neq 1$.
(**A5**) $|c| = 3$, $|d| > 3$, $a^{-1}b \neq 1$, $cab^{-1} = 1$, $c^{-1}ab^{-1} \neq 1$, $d^{\pm 1}b^{-1}a \neq 1$.
(**A6**) $|c| = 3$, $|d| > 3$, $a^{-1}b \neq 1$, $cab^{-1} \neq 1$, $c^{-1}ab^{-1} = 1$, $d^{\pm 1}b^{-1}a \neq 1$.
(**A7**) $|c| = 3$, $|d| > 3$, $a^{-1}b \neq 1$, $c^{\pm 1}ab^{-1} \neq 1$, $db^{-1}a = 1$, $d^{-1}b^{-1}a \neq 1$.
(**A8**) $|c| = 3$, $|d| = 3$, $a^{-1}b \neq 1$, $cab^{-1} \neq 1$, $c^{-1}ab^{-1} = 1$, $d^{\pm 1}b^{-1}a \neq 1$.
(**A9**) $|c| = 3$, $|d| = 3$, $a^{-1}b \neq 1$, $cab^{-1} \neq 1$, $c^{-1}ab^{-1} = 1$, $db^{-1}a = 1$, $d^{-1}b^{-1}a \neq 1$.
(**A10**) $|c| > 3$, $|d| > 3$, $a^{-1}b \neq 1$, $cab^{-1} \neq 1$, $c^{-1}ab^{-1} = 1$, $db^{-1}a = 1$, $d^{-1}b^{-1}a \neq 1$.

Case B: Regions of degree two are possible.

(**B1**) $|c| = 2$, $|d| > 3$, $a^{-1}b \neq 1$, $c^{\pm 1}ab^{-1} \neq 1$, $d^{\pm 1}b^{-1}a \neq 1$.
(**B2**) $|c| = 2$, $|d| = 2$, $a^{-1}b \neq 1$, $c^{\pm 1}ab^{-1} \neq 1$, $d^{\pm 1}b^{-1}a \neq 1$.
(**B3**) $|c| = 2$, $|d| = 3$, $a^{-1}b \neq 1$, $c^{\pm 1}ab^{-1} \neq 1$, $d^{\pm 1}b^{-1}a \neq 1$.
(**B4**) $|c| = 2$, $|d| > 3$, $a^{-1}b \neq 1$, $c^{\pm 1}ab^{-1} = 1$, $d^{\pm 1}b^{-1}a \neq 1$.
(**B5**) $|c| = 2$, $|d| > 3$, $a^{-1}b \neq 1$, $c^{\pm 1}ab^{-1} \neq 1$, $db^{-1}a = 1$, $d^{-1}b^{-1}a \neq 1$.
(**B6**) $|c| = 2$, $|d| = 3$, $a^{-1}b \neq 1$, $c^{\pm 1}ab^{-1} \neq 1$, $db^{-1}a = 1$, $d^{-1}b^{-1}a \neq 1$.
(**B7**) $|c| > 3$, $|d| > 3$, $a^{-1}b = 1$, $c^{\pm 1}ab^{-1} \neq 1$, $d^{\pm 1}b^{-1}a \neq 1$.
(**B8**) $|c| = 2$, $|d| > 3$, $a^{-1}b = 1$, $c^{\pm 1}ab^{-1} \neq 1$, $d^{\pm 1}b^{-1}a \neq 1$.
(**B9**) $|c| = 3$, $|d| > 3$, $a^{-1}b = 1$, $c^{\pm 1}ab^{-1} \neq 1$, $d^{\pm 1}b^{-1}a \neq 1$.

(**B10**) $|c| = 2$, $|d| = 2$, $a^{-1}b = 1$, $c^{\pm 1}ab^{-1} \neq 1$, $d^{\pm 1}b^{-1}a \neq 1$.
(**B11**) $|c| = 2$, $|d| = 3$, $a^{-1}b = 1$, $c^{\pm 1}ab^{-1} \neq 1$, $d^{\pm 1}b^{-1}a \neq 1$.
(**B12**) $|c| = 3$, $|d| = 3$, $a^{-1}b = 1$, $c^{\pm 1}ab^{-1} \neq 1$, $d^{\pm 1}b^{-1}a \neq 1$.

Figure 3.1

3 Preliminary results

We first exhibit spherical diagrams corresponding to some of the conditions of Theorems 1.1 and 1.2. **Note**: when drawing figures the discs (vertices) will often be represented by points; the edge arrows shown in Figure 2.1 will be omitted; and regions with label $c^{\pm 2}, d^{\pm 2}$ will be labelled simply by $c^{\pm 1}, d^{\pm 1}$.

Lemma 3.1

(a) *If any of the following conditions holds then $\mathcal{P}$ fails to be aspherical:*

- (i) $|d| < \infty$ *and* $cab^{-1} = 1$;
- (ii) $|d| < \infty$ *and* $c^{-1}ab^{-1} = bd^{-1}a^{-1}c^{-1} = 1$;
- (iii) $|d| < \infty$ and $a^{-1}b = cad^{-1}b^{-1} = 1$.

(b) *If* $bda^{-1}c^{-1} = 1$ *and any of the following conditions holds then $\mathcal{P}$ fails to be aspherical.*

- (i) $(2, 2, < \infty)$;
- (ii) $(< \infty, 2, 2,)$;
- (iii) $(2, 3, k)$ $(3 \leq k \leq 5)$;
- (iv) $(3, 2, l)$ $(4 \leq l \leq 5)$;
- (v) $(k, 2, 3)$ $(3 \leq k \leq 5)$;

where $(f_1, f_2, f_3) = (|a^{-1}b|, |c|, |d|)$.

(c) *If* $a^{-1}b = 1$ *and any of the following conditions holds then $\mathcal{P}$ fails to be aspherical.*

- (i) $(2, 2, < \infty)$;
- (ii) $(2, < \infty, 2)$;
- (iii) $(2, l, 3)$ $(4 \leq l \leq 5)$;
- (iv) $(3, k, 2)$ $(3 \leq k \leq 5)$;

(v) $(2,3,k)$ $(3 \leq k \leq 5)$,

where $(f_1, f_2, f_3) = (|c|, |d|, |bda^{-1}c^{-1}|)$.

Figure 3.2

Proof

(a) (i) If $|d| = n < \infty$ there is a spherical picture over $\mathcal{P}$ consisting of two n-gons labelled d^n, d^{-n} together with $2n$ 3-gons labelled cab^{-1}. The sphere for $n = 5$ is shown in Figure 3.1(i).

(ii) If $|d| = n < \infty$ then there is a spherical picture over $\mathcal{P}$ consisting of two regions with labels d^n, d^{-n} and between them two layers. Each layer has n 4-gons and n 3-gons with labels $cadb^{-1}$, cba^{-1} and $bd^{-1}a^{-1}c^{-1}$, $c^{-1}ab^{-1}$. The sphere for $n = 6$ is shown in Figure 3.1(ii).

(iii) If $|d| = n < \infty$ then there is a spherical picture over $\mathcal{P}$ consisting of a region labelled d^n, a region labelled c^{-n} and a layer of $2n$ regions between them. The layer consists of n 2-gons labelled $a^{-1}b$ and n 4-gons labelled $cad^{-1}b^{-1}$. The sphere for $n = 6$ is given by Figure 3.1(iii).

(b) (i) If $|d| = n < \infty$ then there is a spherical picture over $\mathcal{P}$ consisting of two regions with label d^n and between them a layer consisting of n 4-gons with label $(a^{-1}b)^2$, n 2-gons with label c^{-2} and $2n$ 4-gons with label $ad^{-1}b^{-1}c$. The case $n = 5$ is shown in Figure 3.1(iv).

(ii) If $|a^{-1}b| = n < \infty$ then there is a spherical picture over $\mathcal{P}$ consisting of two regions with label $(a^{-1}b)^n$ and between them a layer consisting of n 2-gons with label c^{-2}, n 2-gons with label d^2 and $2n$ 4-gons with label $ad^{-1}b^{-1}c$. The case $n = 3$ is shown in Figure 3.1(v).

(iii) If $k = 3$ a spherical picture is given by Figure 3.2(i) in which $A = ab^{-1}ab^{-1}$, $B = bda^{-1}c^{-1}$, $C = c^3$ and $D = d^{-3}$. If $k = 4$ a spherical picture is given by Figure 3.2(ii) in which $D = d^{-4}$. If $k = 5$ there is a spherical picture consisting of 30 A regions, 60 B regions, 20 C regions and 12 $D = d^{-5}$ regions, half of which is shown in Figure 3.2(iii).

(iv) If $l = 4$ a spherical picture is given by Figure 3.3(i) in which $A = (ab^{-1})^3$, $B = bda^{-1}c^{-1}$, $C = c^2$ and $D = d^{-4}$. If $l = 5$ there is a spherical picture consisting of 20 A regions, 60 B regions, 30 C regions and 12 $D = d^{-5}$ regions, half of which is shown in Figure 3.3(ii).

(v) Spherical pictures for $k = 3$ and 4 are given in Figure 3.4(i) and (ii) in which $A = (ab^{-1})^3$ and $(ab^{-1})^4$ (respectively), $B = bda^{-1}c^{-1}$, $C = c^2$ and $D = d^{-3}$. If $k = 5$ there is a spherical picture consisting of 12 $A = (ab^{-1})^5$ regions, 60 B regions, 30 C regions and 20 D regions half of which is shown in Figure 3.4(iii).

Figure 3.3

(c) (i) If $|bda^{-1}c^{-1}| = n < \infty$ then there is a spherical picture over $\mathcal{P}$ consisting of two $4n$-gons each with label $(bda^{-1}c^{-1})^n$ together with n 2-gons with label c^2, n 2-gons with label d^{-2} and $2n$ 2-gons with label ab^{-1}. The sphere for $n = 3$ is shown in Figure 3.5(i).

(ii) If $|d| = n < \infty$ then there is a sphere consisting of two n-gons with label $D = d^{-n}$ and between them a layer consisting of n 8-gons with label $B = (bda^{-1}c^{-1})^2$, $2n$ 2-gons with label $A = ab^{-1}$ and n 2-gons with label $C = c^2$. The sphere for $n = 4$ is shown in Figure 3.5(ii).

Figure 3.4

(iii) A spherical picture for $l = 4$ is given by Figure 3.5(iii) in which $A = ab^{-1}$, $B = (bda^{-1}c^{-1})^3$, $C = c^2$ and $D = d^{-4}$. If $l = 5$ there is a sphere consisting of 60 A regions, 20 B regions, 30 C regions and 12 $D = d^{-5}$ regions half of which is given by Figure 3.5(iv).

(iv) Spherical pictures for $k = 3$ and 4 are given by Figure 3.6(i) and (ii) in which $A = ab^{-1}$, $B = (bda^{-1}c^{-1})^2$, $C = c^3$ and $D = d^{-3}$ and d^{-4} (respectively). If $k = 5$ there is a spherical picture consisting of 60 A regions, 30 B regions, 20 C regions and 12 $D = d^{-5}$ regions half of which is given by Figure 3.6(iii).

(v) Spherical pictures for $k = 3$ and 4 are given by Figure 3.7(i) and (ii) in which $A = ab^{-1}$; $B = (bda^{-1}c^{-1})^3$ and $(bda^{-1}c^{-1})^4$ (respectively), $C = c^2$ and $D = d^{-3}$. If $k = 5$ there is a spherical picture consisting of 60 A regions, 12 $B = (bda^{-1}c^{-1})^5$ regions, 30 C regions and 20 D regions half of which is given by Figure 3.7(iii). □

Figure 3.5

It follows from Theorem 1(2) in [1] that if $|t| < \infty$ in $G(\mathcal{P})$ then $\mathcal{P}$ fails to be aspherical. We apply this fact in the proof of the next lemma.

Lemma 3.2 *If any of the following conditions hold then $\mathcal{P}$ fails to be aspherical.*

(i) $|a^{-1}b| < \infty$, $|c| = |d| = 2$ *and* $a^{-1}cad = bdb^{-1}c = 1$.

(ii) $|d| < \infty$ *and* $c^{-1}ab^{-1} = db^{-1}a = 1$.

(iii) $|d| < \infty$ *and* $a^{-1}b = cada^{-1} = 1$.

(iv) $c^2 = cbda^{-1} = a^{-1}bd^{-2} = 1$.

(v) $c^2 = cbda^{-1} = (a^{-1}b)^2d^{-1} = 1$.

(vi) $a^{-1}b = c^2 = d^3 = 1$ *and* $cada^{-1}cad^{-1}a^{-1} = 1$.

(vii) $a^{-1}b = c^2ada^{-1} = 1$ *and* $4 \leq |c| \leq 5$.

(viii) $a^{-1}b = c^3ada^{-1} = 1$ *and* $|c| = 6$.

Proof

(i) It is enough to show that the group $G = \langle b, d, t \mid d^2 = b^k = 1, bd = db, t^2btdt^{-1}d = 1\rangle$ has order $2k(3^{2k} - 1)$. Now $G = \langle d, t \mid d^2, t^{-2}d^{-1}td^{-1}t^{-1}dtdt^{-1}dt^2d^{-1}, (t^3dt^{-1}d)^k\rangle$ and $G/G' = \langle d, t \mid d^2 = t^{2k} = [d, t] = 1\rangle$. Let $\mathcal{K}$ denote the covering 2-complex associated with G' [3]. Then $\mathcal{K}$ has edges $t_{0j}, t_{1j}, d_{j0}, d_{j1}$ $(1 \leq j \leq 2k)$ and 2-cells $d_{j0}d_{j1}$, $t_{0j}d_{1-j}t_{1j}^{-1}d_{2-j}$, $t_{i1}t_{i2}\dots t_{i2k}$ where $1 \leq i \leq 2$ and $1 \leq j \leq 2k$ and the d subscripts are mod $2k$. Collapsing the maximal subtree whose edges are d_{j0} $(1 \leq j \leq 2k)$, t_{0l} $(2 \leq l \leq 2k)$ and using the lifts of d^2 shows that $G' = \langle t_{01}, t_{1j}\ (1 \leq j \leq 2k)\rangle$. Using the lifts of the second relator it is easily shown that $G' = \langle t_{01}, t_{11}, t_{12}\rangle$ where $t_{11}t_{01}^{-1} = t_{12}^{-3^{2k-1}}$ and, finally, using the lift of the third relator $(t^3dt^{-1}d)^k$ one can show that $G' = \langle t_{12} \mid t_{12}^r\rangle$ where $r = \frac{1}{2}(3^{2k} - 1)$. We omit the details.

(ii) It is enough to show that $G = \langle d, x \mid d^k, t^2dtd^{-1}t^{-1}d\rangle$ has order $2k(1 + 4 + 4^2 + \ldots + 4^{k-1})$. Now $G = \langle u, t \mid (ut^{-2})^k, tut^{-1}u^{-2}\rangle$ and $G/G' = \langle u, t \mid u = t^{2k} = 1\rangle$. Let $\mathcal{L}$ denote the covering complex associated with G'. Then $\mathcal{L}$ has edges t_j, u_j $(1 \leq j \leq 2k)$ and 2-cells $t_1t_2\dots t_{2k}, u_j$ $(1 \leq j \leq 2k)$. Collapsing the maximal tree whose edges are $t_1, \ldots, t_{k-1}$ implies $G' = \langle t_{2k}, u_j\ (1 \leq j \leq 2k)\rangle$. The lifts of $tut^{-1}u^{-2}$ yield the relators $u_l = u_1^{2^{l-1}}$ for $2 \leq l \leq 2k$ and $t_{2k}u_1t_{2k}^{-1}u_1^{-4^k}$. The lifts of $(ut^{-2})^k$ yield the relators $t_{2k}^{-1} = \prod_{i=0}^{k-1} u_{2i+1} = \prod_{i=1}^{k} u_{2i}$. It follows that $G' = \langle u_1 \mid u_1^r\rangle$ where $r = 1 + 4 + 4^2 + \ldots + 4^{k-1}$.

(iii) Here $r = t^3dt^{-1}d^{-1}$ and $|d| < \infty$ implies $|t| < \infty$.

(iv) – (v) A spherical picture for (iv), (v) is shown in Figure 3.8(i), (ii) (respectively).

(vi) – (viii) For these cases we use GAP [8]. For (iv), $r = t^3ct^{-1}d$ together with the conditions yields $|t| \leq 12$; for (v) $r = t^3ct^{-1}c^{-2}$ and $|c| = 4, 5$ implies $|t| \leq 8, 10$ (respectively); and for (vi) $r = t^3ct^{-1}c^{-3}$ and $|c| = 6$ implies $|t| \leq 24$. □

Figure 3.6

Lemma 3.3 *If any of the conditions (i)–(iii) of Theorem 1.1 or (i)–(x) of Theorem 1.2 holds then* $\mathcal{P}$ *fails to be aspherical.*

Proof Consider Theorem 1.1. If (i) holds then $\mathcal{P}$ fails to be aspherical by Lemma 3.1(a)(i). If (ii) holds then $\mathcal{P}$ fails to be aspherical by Lemma 3.2(i). This leaves condition (iii). If

$a^{-1}b \neq 1$ and $bda^{-1}c^{-1} \neq 1$ then (iii) does not hold; and if $a^{-1}b = bda^{-1}c^{-1} = 1$ then H is cyclic. Let $a^{-1}b = 1$. Since $(|c|, |d|, |bda^{-1}c^{-1}|)$ is T-equivalent to $(|d|, |c|, |bda^{-1}c^{-1}|)$ it can be assumed without any loss that $|c| \leq |d|$. The resulting ten cases are dealt with by Lemma 3.1(c). Let $bda^{-1}c^{-1} = 1$. Since $(|a^{-1}b|, |c|, |d|)$ is T-equivalent to $(|a^{-1}b|, |d|, |c|)$ it can again be assumed without any loss that $|c| \leq |d|$. The resulting ten cases are dealt with by Lemma 3.1(b). Now consider Theorem 1.2. If (i) holds then $\mathcal{P}$ is aspherical by Lemma 3.1(a)(i); if (ii) holds then by Lemma 3.1(a)(ii); if (iii) holds then by Lemma 3.2(ii); if (iv) holds then by Lemma 3.2(i); if (v) holds then by Lemma 3.2(iv); if (vi) holds then by Lemma 3.2(v); if (vii) holds then by Lemmas 3.1(a)(iii) and 3.2(iii); if (viii) holds then by Lemma 3.2(vi); if (ix) holds then by Lemma 3.2(vii); and if (x) holds then by Lemma 3.2(viii). □

Figure 3.7

A *weight function* α on the star graph Γ of Figure 2.1(iii) is a real-valued function on the set of edges of Γ. Denote the edge labelled a, b, c, d by e_a, e_b, e_c, e_d (respectively). The function α is *weakly aspherical* if the following two conditions are satisfied:

(1) $\alpha(e_a) + \alpha(e_b) + \alpha(e_c) + \alpha(e_d) \leq 2$;

(2) each admissible cycle in Γ has weight at least 2.

If there is a weakly aspherical weight function on Γ then $\mathcal{P}$ is aspherical [2].

Lemma 3.4 *If any of the following conditions holds then $\mathcal{P}$ is aspherical.*

(i) $|c| = |d| = \infty$;

(ii) $1 < |b| < \infty$ *and* $|d| = \infty$;

(iii) $|c| < \infty$, $|d| < \infty$ *and* $|b| = \infty$.

Proof The following functions α are weakly aspherical.

(i) $\alpha(e_a) = \alpha(e_b) = 1$, $\alpha(e_c) = \alpha(e_d) = 0$.

(ii) $\alpha(e_a) = \alpha(e_b) = \frac{1}{2}$, $\alpha(e_c) = 1$, $\alpha(e_d) = 0$.

(iii) $\alpha(e_a) = \alpha(e_b) = 0$, $\alpha(e_c) = \alpha(e_d) = 1$. □

Figure 3.8

The following lemmas will be useful in later sections.

Lemma 3.5 *Let $d(\hat{\Delta}) = k$ where $\hat{\Delta}$ is a region of the spherical picture $\boldsymbol{P}$ over $\mathcal{P}$.*

(i) *If $\hat{\Delta}$ receives at most $\frac{\pi}{6}$ across each edge and $k \geq 6$ then $c^*(\hat{\Delta}) \leq 0$.*

(ii) *If $\hat{\Delta}$ receives at most $\frac{\pi}{6}$ across at most two-thirds of its edges, nothing across the remaining edges and $k \geq 5$ then $c^*(\hat{\Delta}) \leq 0$.*

(iii) *If $\hat{\Delta}$ receives at most $\frac{\pi}{4}$ across each edge and $k \geq 8$ then $c^*(\hat{\Delta}) \leq 0$.*

(iv) *If $\hat{\Delta}$ receives at most $\frac{\pi}{4}$ across at most two-thirds of its edges, nothing across the remaining edges and $k \geq 6$ then $c^*(\hat{\Delta}) \leq 0$.*

(v) *If $\hat{\Delta}$ receives at most $\frac{\pi}{4}$ across at most half of its edges, nothing across the remaining edges and $k \geq 5$ then $c^*(\hat{\Delta}) \leq 0$.*

(vi) *If $\hat{\Delta}$ receives at most $\frac{\pi}{2}$ across at most half of its edges, nothing across the remaining edges and $k \geq 7$ then $c^*(\hat{\Delta}) \leq 0$.*

(vii) *If $\hat{\Delta}$ receives at most $\frac{\pi}{2}$ across at most three-fifths of its edges, nothing across the remaining edges and $k \geq 8$ then $c^*(\hat{\Delta}) \leq 0$.*

Proof Recall that $c(\hat{\Delta}) = \pi\left(2 - \frac{k}{2}\right)$. (i) $c^*(\hat{\Delta}) \leq c(\hat{\Delta}) + k.\frac{\pi}{6} \leq 0$ for $k \geq 6$. (ii) $c^*(\hat{\Delta}) \leq c(\hat{\Delta}) + \frac{2k}{3}.\frac{\pi}{6} \leq 0$ for $k \geq 6$. If $k = 5$ then $\hat{\Delta}$ receives at most $\frac{\pi}{6}$ across at most three edges and so $c^*(\hat{\Delta}) \leq \pi\left(2 - \frac{5}{2}\right) + 3.\frac{\pi}{6} = 0$. (iii) $c^*(\hat{\Delta}) \leq c(\Delta) + k.\frac{\pi}{4} \leq 0$ for $k \geq 8$. (iv) $c^*(\hat{\Delta}) \leq c(\Delta) + \frac{2k}{3}.\frac{\pi}{4} \leq 0$ for $k \geq 6$. (v) $c^*(\hat{\Delta}) \leq c(\Delta) + \frac{k}{2}.\frac{\pi}{4} \leq 0$ for $k \geq 6$. If $k = 5$ then $\hat{\Delta}$ receives at most $\frac{\pi}{4}$ across at most two edges and $c^*(\hat{\Delta}) \leq \pi\left(2 - \frac{5}{2}\right) + 2.\frac{\pi}{4} = 0$. (vi) $c^*(\hat{\Delta}) \leq c(\Delta) + \frac{k}{2}.\frac{\pi}{2} \leq 0$ for $k \geq 8$. If $k = 7$ then $\hat{\Delta}$ receives at most $\frac{\pi}{2}$ across at most three edges and $c^*(\hat{\Delta}) \leq \pi\left(2 - \frac{7}{2}\right) + 3.\frac{\pi}{2} = 0$. (vii) $c^*(\hat{\Delta}) \leq c(\Delta) + \frac{3k}{5}.\frac{\pi}{2} \leq 0$ for $k \geq 10$. If $k = 9$ then $\hat{\Delta}$ receives at most $\frac{\pi}{2}$ across five edges and $c^*(\hat{\Delta}) \leq \pi(2 - \frac{9}{2}) + 5.\frac{\pi}{2} = 0$; if $k = 8$ then $\hat{\Delta}$ receives across at most four edges and $c^*(\hat{\Delta}) \leq \pi(2 - \frac{8}{2}) + 4.\frac{\pi}{2} = 0$. $\square$

Remark We will use the above lemmas as follows. Suppose that $\hat{\Delta}$ receives positive curvature across its edge e_i. If we know that it then never receives curvatures across e_{i-1} or across e_{i+1} then we can apply the "half" results; or if we know that it receives positive curvature across at most one of e_{i-1}, e_{i+1} then we can apply the "two-thirds" results.

Let $\hat{\Delta}$ be a region of $\boldsymbol{P}$ and let e be an edge of $\hat{\Delta}$. If $\hat{\Delta}$ receives no curvature across e then e is called a *gap*; if at most $\frac{\pi}{6}$ then e is called a *two-thirds gap*; and if at most $\frac{\pi}{4}$ then e is called a *half gap.*

Lemma 3.6 (The Four Gaps Lemma) *If $\hat{\Delta}$ has a total of at least four gaps (in particular, four edges across which $\hat{\Delta}$ does not receive any curvature) and the most curvature that crosses any edge is $\frac{\pi}{2}$ then $c^*(\hat{\Delta}) \leq 0$.*

Proof It follows that $c^*(\hat{\Delta}) \leq \pi\left(2-\frac{k}{2}\right)+(k-4)\frac{\pi}{2}=0$. $\square$

Checking the star graph shows that we will have the following LIST for the labels of regions of degree k where $k \in \{2,3,4,5,6,7\}$:

If $d(\Delta)=2$ then $l(\Delta) \in \{c^2, a^{-1}b, d^2\}$.

If $d(\Delta)=3$ then $l(\Delta) \in \{c^3, cab^{-1}, c^{-1}ab^{-1}, db^{-1}a, d^{-1}b^{-1}a, d^3\}$.

If $d(\Delta)=4$ then $l(\Delta) \in \{d^4, d^2a^{-1}b, d^2b^{-1}a, c^2ab^{-1}, c^2ba^{-1}, c^4, ab^{-1}ab^{-1}, d\{a^{-1},b^{-1}\}$ $\{c,c^{-1}\}\{a,b\}\}$.

If $d(\Delta)=5$ then $l(\Delta) \in \{d^5, d^3a^{-1}b, d^3b^{-1}a, c^3ab^{-1}, c^3ba^{-1}, c^5, cab^{-1}ab^{-1}, cba^{-1}ba^{-1},$ $da^{-1}ba^{-1}b, db^{-1}ab^{-1}a, d^2\{a^{-1},b^{-1}\}\{c,c^{-1}\}\{a,b\}, c^2\{a,b\}\{d,d^{-1}\}\{a^{-1},b^{-1}\}\}$.

If $d(\Delta)=6$ then $l(\Delta) \in \{d^6, d^4a^{-1}b, d^4b^{-1}a, c^4ab^{-1}, c^4ba^{-1}, c^6, ab^{-1}ab^{-1}ab^{-1}, d^2a^{-1}$ $ba^{-1}b, d^2b^{-1}ab^{-1}a, c^2ab^{-1}ab^{-1}, c^2ba^{-1}ba^{-1}, d^3\{a^{-1},b^{-1}\}\{c,c^{-1}\}\{a,b\}, d^2\{a^{-1},b^{-1}\}$ $\{c^2,c^{-2}\}\{a,b\}, d\{a^{-1},b^{-1}\}\{c^3,c^{-3}\}\{a,b\}, c\{ab^{-1},ba^{-1}\}\{c,c^{-1}\}\{ab^{-1},ba^{-1}\}, d\{a^{-1}$ $b, b^{-1}a\}\{d,d^{-1}\}\{a^{-1}b, b^{-1}a\}, c\{ab^{-1}a, ba^{-1}b\}\{d,d^{-1}\}\{a^{-1},b^{-1}\}, c\{a,b\}\{d,d^{-1}\}\{a^{-1}ba^{-1}, b^{-1}ab^{-1}\}\}$.

Where, for example $d^2\{a^{-1},b^{-1}\}\{c,c^{-1}\}\{a,b\}$ yields the eight labels $d^2a^{-1}c^{\pm 1}a$, $d^2a^{-1}c^{\pm 1}b$, $d^2b^{-1}c^{\pm 1}a$, $d^2b^{-1}c^{\pm 1}b$.

We will use the above LIST throughout the following sections often without explicit reference.

4 Proof of Case A

In this section we prove Theorems 1.1 and 1.2 for Case A, that is, we make the following assumptions:

$$|c|>2,\ |d|>2 \text{ and } a^{-1}b \neq 1.$$

This implies that $d(\Delta) \geq 3$ for each region Δ of the spherical diagram $\boldsymbol{P}$. If $d(\Delta)=3$ then we will fix the names of the fifteen neighbouring regions Δ_i $(1 \leq i \leq 15)$ of Δ as shown in Figure 4.1(i) and we use this notation throughout the section. We treat each of the cases (**A0**)–(**A10**) in turn.

(A0) $\boldsymbol{|c|>3}$, $\boldsymbol{|d|>3}$, $\boldsymbol{a^{-1}b \neq 1}$, $\boldsymbol{c^{\pm 1}ab^{-1} \neq 1}$, $\boldsymbol{d^{\pm 1}b^{-1}a \neq 1}$.
In this case $d(\Delta)>3$ for all regions Δ. Since the degree of each vertex equals 4 it follows that $c(\Delta)=(2-d(\Delta))\pi+d(\Delta)\frac{2\pi}{4} \leq 0$ and so $\mathcal{P}$ is aspherical.

Figure 4.1

(A1) $\boldsymbol{|c| = 3}$, $\boldsymbol{|d| > 3}$, $\boldsymbol{a^{-1}b \neq 1}$, $\boldsymbol{c^{\pm 1}ab^{-1} \neq 1}$, $\boldsymbol{d^{\pm 1}b^{-1}a \neq 1}$.
If $d(\Delta) = 3$ then $c(\Delta) = c(4,4,4) = \frac{\pi}{2}$, $l(\Delta) = c^3$ and Δ is given by Figure 4.1(ii). Observe that if $d(\Delta_i) = 4$ for $i \in \{1,3,5\}$ then $l(\Delta_i) = bd\omega$ and so $l(\Delta_i) \in \{bd^2a^{-1}, bda^{-1}c^{\pm 1}, bdb^{-1}c^{\pm 1}\}$. (See the LIST of Section 3.) But $bdb^{-1}c^{\pm 1}$ implies $|d| = |c|$, a contradiction. Observe further that at most one of bd^2a^{-1}, $bda^{-1}c$, $bda^{-1}c^{-1}$ equals 1 otherwise there is a contradiction to $|c| = 3$ or $|d| > 3$. This leaves the following cases: (i) $bd^2a^{-1} \neq 1$; $bda^{-1}c^{\pm 1} \neq 1$; (ii) $bd^2a^{-1} = 1$; (iii) $bda^{-1}c = 1$; (iv) $bda^{-1}c^{-1} = 1$.

Consider case (i). Here $d(\Delta_i) > 4$ for $i \in \{1,3,5\}$ of Figure 4.1(ii), so add $\frac{1}{3}c(\Delta) = \frac{\pi}{6}$ to each of $c(\Delta_i)$ across the bd edge as shown. Observe from Figure 4.1(ii) that Δ_1, say, does not receive any curvature from Δ_2 or Δ_6. Therefore if $\hat{\Delta}$ receives positive curvature, since $d(\hat{\Delta}) \geq 5$, it follows that $c^*(\hat{\Delta}) \leq 0$ by Lemma 3.5(ii).

Now consider case (ii), $bd^2a^{-1} = 1$. If $d(\Delta_i) > 4$ for at least two of the Δ_i where $i \in \{1,3,5\}$, say Δ_1 and Δ_3, then add $\frac{1}{2}c(\Delta) = \frac{\pi}{4}$ to each of $c(\Delta_1)$ and $c(\Delta_3)$ across the bd edge as in Figure 4.1(iii). Otherwise by symmetry it can be assumed without any loss of generality that $d(\Delta_1) = d(\Delta_3) = 4$. Then add $\frac{1}{2}c(\Delta) = \frac{\pi}{4}$ firstly to $c(\Delta_3)$ and then on to $c(\Delta_2)$ across the bd and ca edges as shown in Figure 4.1(iv). If $d(\Delta_5) > 4$ then add the remaining $\frac{1}{2}c(\Delta) = \frac{\pi}{4}$ to $c(\Delta_5)$ across the bd edge; otherwise $d(\Delta_5) = 4$ and similarly add the $\frac{\pi}{4}$ firstly to $c(\Delta_5)$ then on to $c(\Delta_4)$. Now observe that in Figure 4.1(iii) Δ_1 does not receive positive curvature from Δ_2 or Δ_6 and in Figure 4.1(iv) that Δ_2 does not receive positive curvature from Δ_1 or Δ_9. It follows that if $\hat{\Delta}$ receives positive curvature then $\hat{\Delta}$ receives curvature across at most half of its edges and so if $d(\hat{\Delta}) \geq 5$, it follows that $c^*(\hat{\Delta}) \leq 0$ by Lemma 3.5(v). This leaves the case when $d(\hat{\Delta}) = 4$ and $l(\hat{\Delta}) = cad^{-1}\omega$. Therefore $l(\hat{\Delta}) \in \{cad^{-1}b^{-1}, cad^{-1}a^{-1}\}$. In each case $l(\hat{\Delta})$ together with $bd^2a^{-1} = c^3 = 1$ implies $|d| = 3$, a contradiction.

Figure 4.2

Now consider (iii), $bda^{-1}c = 1$. As before, if $d(\Delta_1) > 4$ and $d(\Delta_3) > 4$ then add $\frac{1}{2}c(\Delta) = \frac{\pi}{4}$ to each of $c(\Delta_1)$ and $c(\Delta_3)$ across the bd edge as in Figure 4.1(iii). If $d(\Delta_1) = d(\Delta_3) = 4$ then add $\frac{1}{2}c(\Delta) = \frac{\pi}{4}$ firstly to $c(\Delta_1)$ and then on to $c(\Delta_2)$ across the bd and ad edges as shown in Figure 4.2(i). The remaining $\frac{1}{2}c(\Delta) = \frac{\pi}{4}$ is distributed either to Δ_5 when $d(\Delta_5) > 4$ or to Δ_6 via Δ_5 when $d(\Delta_5) = 4$. Now observe that in Figure 4.1(iii) Δ_1 does not receive positive curvature from Δ_2 or Δ_6; and in Figure 4.2(i) that Δ_2 does not receive positive curvature from Δ_3 or Δ_8. It follows that if the region $\hat{\Delta}$ receives positive curvature then it does so across at most half of its edges and therefore if $d(\hat{\Delta}) \geq 5$ then $c^*(\hat{\Delta}) \leq 0$ by Lemma 3.5(v). This leaves the case when $d(\hat{\Delta}) = 4$ and $l(\hat{\Delta}) = b^{-1}ad\omega$. Therefore $l(\hat{\Delta}) = b^{-1}ad^2$ and H is cyclic. If this occurs then add $\frac{1}{2}c(\Delta) = \frac{\pi}{4}$ to $c(\Delta_8)$ as shown in Figure 4.2(ii). If $d(\Delta_8) = 4$ then $l(\Delta_8) \in \{cad^{\pm 1}b^{-1}, cad^{\pm 1}a^{-1}, cab^{-1}c\}$ which, together with $b^{-1}add$, contradicts the (**A1**) assumptions. Also note that in Figure 4.2(ii), Δ_8 does not receive positive curvature from Δ_7 or from Δ_{16} and so if $\hat{\Delta}$ receives positive curvature it does so across at most half its edges and $c^*(\hat{\Delta}) \leq 0$.

Finally consider case (iv), $bda^{-1}c^{-1} = 1$. First assume that $b^2 \neq 1$. Again if $d(\Delta_1) > 4$ and $d(\Delta_3) > 4$ then add $\frac{1}{2}c(\Delta) = \frac{\pi}{4}$ to each of $c(\Delta_1)$ and $c(\Delta_3)$ across the bd edge as in Figure 4.1(iii). If $d(\Delta_1) = d(\Delta_3) = 4$ then add $\frac{1}{2}c(\Delta) = \frac{\pi}{4}$ firstly to $c(\Delta_1)$ and then on to $c(\Delta_2)$ across the bd and ab^{-1} edges as shown in Figure 4.2(iii) and add the remaining $\frac{1}{2}c(\Delta) = \frac{\pi}{4}$ to Δ_5 or Δ_6 as in the above. Now observe that in Figure 4.1(iii) Δ_1 does not receive positive curvature from Δ_2 or Δ_6 and in Figure 4.2(iii) that Δ_2 does not receive positive curvature from Δ_3 or Δ_8. Thus if $\hat{\Delta}$ receives positive curvature then it does so across at most half of its edges so $d(\hat{\Delta}) \geq 5$ implies $c^*(\hat{\Delta}) \leq 0$ by Lemma 3.5(v). This leaves the case when $d(\hat{\Delta}) = 4$ and $l(\hat{\Delta}) = b^{-1}ab^{-1}\omega$. Therefore $l(\hat{\Delta}) = b^{-1}ab^{-1}a$ and there is a contradiction to $b^2 \neq 1$. Now assume that $b^2 = 1$. The distribution in this case is different in that we add $\frac{1}{3}c(\Delta) = \frac{\pi}{6}$ to $c(\Delta_j)$ for $j \in \{1, 3, 5\}$ as in Figure 4.1(ii) across the bd edge. If $d(\Delta_1) = 4$, say, then distribute the $\frac{\pi}{6}$ further on to $c(\Delta_7)$ across the d^{-2} edge as shown in Figure 4.2(iv). Therefore if $\hat{\Delta}$ receives positive curvature and if $d(\hat{\Delta}) \geq 6$, it follows that $c^*(\hat{\Delta}) \leq 0$ by Lemma 3.5(i). This leaves the case when $4 \leq d(\hat{\Delta}) \leq 5$ and $l(\hat{\Delta}) \in \{bd\omega, d^{-2}\omega\}$. If $d(\hat{\Delta}) = 4$ then $l(\hat{\Delta}) = d^{-2}\omega$ and $l(\hat{\Delta}) \in \{d^{-4}, d^{-2}a^{-1}b, d^{-2}b^{-1}a\}$ therefore $d^4 = 1$ and there is a sphere by Lemma 3.1(b)(iii). So let $d(\hat{\Delta}) = 5$. If $\hat{\Delta}$ receives curvature across at most three edges then $c^*(\hat{\Delta}) \leq \pi(2 - \frac{5}{2}) + 3.\frac{\pi}{6} = 0$. Checking (the LIST) now shows that $c^*(\hat{\Delta}) \leq 0$ except when $l(\hat{\Delta}) = d^{-5}$ and in this case we obtain a sphere by Lemma 3.1(b)(iii).

In conclusion $\mathcal{P}$ fails to be aspherical in this case if and only if $b^2 = bda^{-1}c^{-1} = 1$ and $|d| \in \{4, 5\}$. Note that if these conditions hold then H is non-cyclic for otherwise we would obtain $d^6 = 1$, a contradiction.

Figure 4.3

(A2) $|c| = 3$, $|d| = 3$, $a^{-1}b \neq 1$, $c^{\pm 1}ab^{-1} \neq 1$, $d^{\pm 1}b^{-1}a \neq 1$.
If $d(\Delta) = 3$ then Δ is given by Figures 4.1(ii) and 4.3(i). If $d(\Delta) = 4$ and $l(\Delta) \in \{bd\omega, ca\omega\}$ then $l(\Delta) \in S = \{bda^{-1}c^{\pm 1}, bdb^{-1}c^{\pm 1}, cad^{\pm 1}a^{-1}, cad^{\pm 1}b^{-1}\}$ otherwise there is a contradiction to one of the assumptions. (Throughout this case unless otherwise stated this means one of the (**A2**) assumptions.)

The cases to be considered are (where for example case (ii) means $bda^{-1}c$ is the only member of S to equal 1): (i) $bda^{-1}c^{\pm 1} \neq 1$, $bdb^{-1}c^{\pm 1} \neq 1$, $cad^{\pm 1}a^{-1} \neq 1$, $cadb^{-1} \neq 1$; (ii) $bda^{-1}c = 1$; (iii) $bda^{-1}c^{-1} = 1$; (iv) $bdb^{-1}c = 1$; (v) $bdb^{-1}c^{-1} = 1$; (vi) $cada^{-1} = 1$; (vii) $cad^{-1}a^{-1} = 1$; (viii) $cadb^{-1} = 1$; (ix) $bdb^{-1}c = 1$, $cada^{-1} = 1$; (x) $bdb^{-1}c = 1$, $cad^{-1}a^{-1} = 1$; (xi) $bdb^{-1}c^{-1} = 1$, $cada^{-1} = 1$; (xii) $bdb^{-1}c^{-1} = 1$, $cad^{-1}a^{-1} = 1$.

Note that any other combination of these conditions gives a contradiction to one of the assumptions. Moreover, (ii) is T-equivalent to (viii); (iv) is T-equivalent to (vi); (v) is T-equivalent to (vii); and (x) is T-equivalent to (xi). So it remains to consider (i), (ii), (iii), (iv), (v), (ix), (x) and (xii).

Now let $c(\Delta) > 0$ and so $l(\Delta) \in \{c^3, d^3\}$. In cases (i), (ii), (iv) and (v) add $\frac{1}{3}c(\Delta) = \frac{\pi}{6}$ to $c(\Delta_i)$ for $i \in \{1, 3, 5\}$ as shown in Figures 4.1(ii) and 4.3(i). If $d(\Delta_i) > 4$ then no further distribution takes place. Suppose without any loss of generality that $d(\Delta_1) = 4$. This cannot happen in case (i); in case (ii) Δ_1 is given by Figure 4.3(ii) and so add the $\frac{\pi}{6}$ from $c(\Delta)$ to $c(\Delta_7)$ across the bd and bd^{-1} edges noting that $d(\Delta_7) > 4$ otherwise $l(\Delta_7) \in \{bd^{-2}a^{-1}, bd^{-1}a^{-1}c^{\pm 1}, bd^{-1}b^{-1}c^{\pm 1}\}$ which contradicts one of the assumptions; in case (iv) Δ_1 is given by Figure 4.3(iii) and so add the $\frac{\pi}{6}$ from $c(\Delta)$ to $c(\Delta_2)$ across the bd and ad edges noting that $d(\Delta_2) > 4$ otherwise $l(\Delta_2) \in \{ad^2b^{-1}, ada^{-1}c^{\pm 1}, ad^{-1}b^{-1}c^{\pm 1}\}$ which contradicts one of the assumptions or yields case (ix) or (x); and in case (v) Δ is given by Figure 4.3(iv) and so add the $\frac{\pi}{6}$ from $c(\Delta)$ to $c(\Delta_7)$ across the bd and $d^{-1}a^{-1}$ edges noting $d(\Delta_7) > 4$ otherwise $l(\Delta_7) \in \{d^{-2}a^{-1}b, d^{-1}a^{-1}c^{\pm 1}a, d^{-1}a^{-1}c^{\pm 1}b\}$ which contradicts one of the assumptions or yields case (xi) or (xii). Therefore if the region $\hat{\Delta}$ receives positive curvature then it receives $\frac{\pi}{6}$ across each edge and so if $d(\hat{\Delta}) \geq 6$ then $c^*(\hat{\Delta}) \leq 0$ by Lemma 3.5(i). This leaves the case when $d(\hat{\Delta}) = 5$. After checking for vertex labels that contain the sublabels (bd), (ca), (ad) and (bd^{-1}) corresponding to the edges crossed in Figures 4.1(ii) and 4.3(i)–(iv) we obtain $c^*(\hat{\Delta}) \leq \pi\left(2 - \frac{5}{2}\right) + 3.\frac{\pi}{6} = 0$. This completes cases (i), (ii), (iv) and (v).

Figure 4.4

Consider case (iii), $bda^{-1}c^{-1} = 1$. If $b^2 = 1$ then we obtain a sphere by Lemma 3.1(b)(iii). Note also that H is non-cyclic in this case otherwise we obtain $b = 1$, a contradiction. Suppose then that $b^2 \neq 1$. First let $l(\Delta) = c^3$. Add $\frac{1}{3}c(\Delta) = \frac{\pi}{6}$ to $c(\Delta_i)$ for $i \in \{1, 3, 5\}$ as in Figure 4.1(ii). If $d(\Delta_i) > 4$ then no further distribution takes place. Suppose that $d(\Delta_1) = 4$. Then add $\frac{\pi}{6}$ from $c(\Delta)$ to $c(\Delta_6)$ across the bd and ab^{-1} edges as shown in Figure 4.4(i) noting that $d(\Delta_6) > 4$, otherwise $l(\Delta_6) \in \{b^{-1}ab^{-1}a, b^{-1}ad^{\pm 2}\}$ which contradicts $b^2 \neq 1$ or one of the assumptions; and if $d(\Delta_3) = 4$ or $d(\Delta_5) = 4$ in Figure 4.1(ii) then similarly add $\frac{\pi}{6}$ to Δ_2 or Δ_4. Secondly, let $l(\Delta) = d^3$. Add $\frac{1}{3}c(\Delta) = \frac{\pi}{6}$ to $c(\Delta_i)$ for $i \in \{1, 3, 5\}$ as in Figure 4.3(i). If $d(\Delta_i) > 4$ then no further distribution takes place. Suppose without any loss of generality that $d(\Delta_1) = 4$. Then add $\frac{\pi}{6}$ from $c(\Delta)$ to $c(\Delta_2)$ across the ca and ba^{-1} edges as shown in Figure 4.4(ii), noting $d(\Delta_2) > 4$, otherwise $l(\Delta_2) \in \{ba^{-1}ba^{-1}, ba^{-1}c^{\pm 2}\}$ which contradicts $b^2 \neq 1$ or one of the assumptions. If $d(\Delta_3) = 4$ or $d(\Delta_5) = 4$ then similarly add $\frac{\pi}{6}$ to Δ_4 or Δ_6. If $\hat{\Delta}$ receives positive curvature and $d(\hat{\Delta}) \geq 6$,it follows by Lemma 3.5(i) that $c^*(\hat{\Delta}) \leq 0$. It remains to check $d(\hat{\Delta}) = 5$. After checking for vertex labels that contain the sublabels (bd),(ca),$(b^{-1}a)$ and (ba^{-1}) corresponding to the edges crossed in Figures 4.1(ii), 4.3(i) and 4.4(i)–(ii) we obtain $c^*(\hat{\Delta}) \leq \pi\left(2 - \frac{5}{2}\right) + 3.\frac{\pi}{6} = 0$ or $l(\hat{\Delta}) \in \{bda^{-1}ba^{-1}, cab^{-1}ab^{-1}\}$ and this contradicts one of the assumptions. Therefore $c^*(\hat{\Delta}) \leq 0$.

Figure 4.5

Consider case (ix), $bdb^{-1}c = 1$ and $cada^{-1} = 1$. If $d(\Delta_i) > 4$ for at least two of Δ_i where $i \in \{1, 3, 5\}$, say Δ_1 and Δ_3,then add $\frac{1}{2}c(\Delta) = \frac{\pi}{4}$ to each of $c(\Delta_1)$ and $c(\Delta_3)$

across the bd and ca edges as shown in Figures 4.1(iii) and 4.4(iii). By symmetry it can be assumed that $d(\Delta_1) = d(\Delta_3) = 4$. The two possibilities are given in Figures 4.4(iv) and 4.5(i) and in both cases add $\frac{1}{2}c(\Delta) = \frac{\pi}{4}$ to $c(\Delta_2)$ across the bd, ad or ca, bd^{-1} edges as shown. If $d(\Delta_5) > 4$ then add the remaining $\frac{1}{2}c(\Delta) = \frac{\pi}{4}$ to $c(\Delta_5)$; or if $d(\Delta_5) = 4$ then apply the above to Δ_1 and Δ_5 to distribute the remaining $\frac{1}{2}c(\Delta) = \frac{\pi}{4}$ similarly to $c(\Delta_6)$. Now observe that if Δ_1 receives positive curvature from Δ then it does not receive positive curvature from Δ_2; and if Δ_2 receives positive curvature from Δ (as in Figures 4.4(iv) and 4.5(i)) then it does not receive positive curvature from Δ_3. It follows that if the region $\hat{\Delta}$ receives positive curvature then it does so across at most two-thirds of its edges and therefore if $\hat{\Delta}$ receives positive curvature and if $d(\hat{\Delta}) \geq 6$ then $c^*(\hat{\Delta}) \leq 0$ by Lemma 3.5(iv). Note that $d(\Delta_2) > 4$ in Figures 4.4(iv) and 4.5(i) otherwise $l(\Delta_2) \in \{c^{-1}ada^{-1}, c^{-1}adb^{-1}, cbd^{-1}a^{-1}, cbd^{-1}b^{-1}\}$ which contradicts one of the assumptions. So there remains the case $d(\hat{\Delta}) = 5$ and $l(\hat{\Delta}) \in \{bd\omega, ca\omega, c^{-1}ad\omega, cbd^{-1}\omega\}$. Checking shows that in each case $c^*(\hat{\Delta}) \leq \pi(2 - \frac{5}{2}) + 2.\frac{\pi}{4} = 0$.

Figure 4.6

Consider (x), $bdb^{-1}c = 1$ and $cad^{-1}a^{-1} = 1$. First consider $l(\Delta) = d^3$. If at least two of the Δ_i where $i \in \{1, 3, 5\}$ have degree greater than four, say Δ_1 and Δ_3, then add $\frac{1}{2}c(\Delta) = \frac{\pi}{4}$ to $c(\Delta_1)$ and $c(\Delta_3)$ as shown in Figure 4.4(iii). So assume otherwise and without any loss of generality let $d(\Delta_1) = d(\Delta_3) = 4$ as shown in Figure 4.5(ii) where $d(\Delta_2) > 4$ otherwise $l(\Delta_2) = a^{-1}bd^{-2}$ which contradicts $d^{-1}b^{-1}a \neq 1$. So add $\frac{1}{2}c(\Delta) = \frac{\pi}{4}$ to $c(\Delta_2)$ as shown in Figure 4.5(ii). If $d(\Delta_5) > 4$ in Figure 4.5(ii) add the remaining $\frac{1}{2}c(\Delta) = \frac{\pi}{4}$ to $c(\Delta_5)$ otherwise use the same argument above for Δ_1 and Δ_5 and add $\frac{1}{2}c(\Delta) = \frac{\pi}{4}$ to $c(\Delta_6)$. Now consider $l(\Delta) = c^3$. If at least two of the Δ_i where $i \in \{1, 3, 5\}$ have degree > 4, say, Δ_1 and Δ_3, then add $\frac{1}{2}c(\Delta) = \frac{\pi}{4}$ to $c(\Delta_1)$ and $c(\Delta_3)$ as in Figure 4.1(iii). Suppose exactly two of the Δ_i have degree $= 4$, say, Δ_1 and Δ_3. Add $\frac{1}{2}c(\Delta) = \frac{\pi}{4}$ to $c(\Delta_5)$. If $d(\Delta_2) > 4$ then add the remaining $\frac{1}{2}c(\Delta) = \frac{\pi}{4}$ to $c(\Delta_2)$ as in Figure 4.4(iv). If $d(\Delta_2) = 4$ then add $\frac{1}{2}c(\Delta) = \frac{\pi}{4}$ to $c(\Delta_{10})$ as in Figure 4.5(iii). If now $d(\Delta_{10}) = 4$ then $l(\Delta_{10}) = ba^{-1}ba^{-1}$ and so add the $\frac{1}{2}c(\Delta) = \frac{\pi}{4}$ to $c(\Delta_9)$ as in Figure 4.6(i). Observe that $l(\Delta_9) = ad^{-1}d^{-1}w$ forces $d(\Delta_9) > 4$ otherwise there is a contradiction to $d^{-1}b^{-1}a \neq 1$. Finally suppose that $d(\Delta_i) = 4$ for $i \in \{1, 3, 5\}$. Then $\frac{1}{2}c(\Delta) = \frac{\pi}{4}$ is added to either Δ_2, Δ_{10} or Δ_9 exactly as above; similarly $\frac{1}{2}c(\Delta) = \frac{\pi}{4}$ is added to Δ_6, Δ_7 or Δ_{15} as shown in Figure 4.6(i).

Now observe that in Figures 4.4(iii) and 4.1(iii) Δ_1 does not receive positive curvature from Δ_2; in Figures 4.5(ii) and 4.4(iv) Δ_2 does not receive positive curvature from Δ_3; in Figure 4.5(iii) Δ_{10} does not receive positive curvature from Δ_{11}; and in Figure 4.6(i) Δ_9 does not receive positive curvature from Δ_2. Observe that if $\hat{\Delta}$ receives positive curvature then $d(\hat{\Delta}) \geq 5$. It follows from Lemma 3.5(iv) that if $d(\hat{\Delta}) \geq 6$ then $c^*(\hat{\Delta}) \leq 0$ so let $d(\hat{\Delta}) = 5$. If $\hat{\Delta}$ receives across at most two edges then $c^*(\hat{\Delta}) \leq 0$ so it remains to check if $\hat{\Delta}$ receives curvature more than two edges. From the above we see that positive curvature is transferred across $(ca), (bd), (bd^{-1}), (ad), (ba^{-1}), (ad^{-1})$-edges. The only two labels that contain more than two such sublabels and do not yield a contradiction are $a^{-1}ccad$ and

$a^{-1}cadd$ as shown in Figures 4.6(ii)–(iii). Let $l(\Delta) = a^{-1}ccad = 1$ as in Figure 4.6(ii). Here Δ receives nothing from $\hat{\Delta}_1$ or $\hat{\Delta}_5$. If $d(\hat{\Delta}_2) > 3$ then Δ receives nothing from $\hat{\Delta}_2$ and so $c^*(\hat{\Delta}) \leq 0$. If $d(\hat{\Delta}_2) = 3$ then $d(\hat{\Delta}_3) > 3$ and Δ receives nothing from $\hat{\Delta}_3$ via $\hat{\Delta}_4$ as in Figure 4.4(iv) and again $c^*(\hat{\Delta}) \leq 0$. Let $l(\Delta) = a^{-1}cadd = 1$ as in Figure 4.6(iii). Here Δ receives nothing from $\hat{\Delta}_4$ or $\hat{\Delta}_5$. If $d(\hat{\Delta}_1) > 3$ then Δ receives nothing from $\hat{\Delta}_1$ and so $c^*(\hat{\Delta}) \leq 0$. If $d(\hat{\Delta}_1) = 3$ then $d(\hat{\Delta}_2) > 3$ and Δ receives nothing from $\hat{\Delta}_2$ via $\hat{\Delta}_3$ and again $c^*(\hat{\Delta}) \leq 0$.

Figure 4.7

Finally consider case (xii), $bdb^{-1}c^{-1} = cad^{-1}a^{-1} = 1$. Then $c = d$ and so $bdb^{-1}d^2 = 1$. If now $b^2 = 1$ then we obtain $bdbd^2 = 1$ and $H = \langle bd \rangle$ is cyclic. Assume first that H is non-cyclic so, in particular, $|b| > 2$. Add $\frac{1}{3}c(\Delta) = \frac{\pi}{6}$ to $c(\Delta_i)$ for $i \in \{1, 3, 5\}$ as in Figures 4.1(ii) and 4.3(i) across the bd and ca edges. If, say, $d(\Delta_1) > 4$ then no further distribution takes place. If $d(\Delta_1) = 4$ then the $\frac{1}{3}c(\Delta) = \frac{\pi}{6}$ is added to $c(\Delta_2)$ if $l(\Delta) = c^3$ across the bd and ab^{-1} edges, or to $c(\Delta_6)$ if $l(\Delta) = d^3$ across the ca and $a^{-1}b$ edges as shown in Figures 4.7(i)–(ii). Observe that $d(\Delta_2) > 4$ and $d(\Delta_6) > 4$. If $\hat{\Delta}$ receives positive curvature and $d(\hat{\Delta}) \geq 6$, it follows by Lemma 3.5(i) that $c^*(\hat{\Delta}) \leq 0$. It remains to check $d(\hat{\Delta}) = 5$. After checking for vertex labels that contain the sublabels (bd),(ca),(ab^{-1}) and $(a^{-1}b)$ corresponding to the edges crossed in Figures 4.1(ii), 4.3(i) and 4.7(i)–(ii) it follows either that $\hat{\Delta}$ receives at most $3.\frac{\pi}{6}$ and so $c^*(\hat{\Delta}) \leq 0$ or $l(\hat{\Delta}) \in \{bda^{-1}ba^{-1}, cab^{-1}ab^{-1}\}$ which in each case yields a contradiction to H non-cyclic. Now assume that H is cyclic. If at least two of the Δ_i where $i \in \{1, 3, 5\}$ have degree greater than four, say Δ_1 and Δ_3, then add $\frac{1}{2}c(\Delta) = \frac{\pi}{4}$ to each of $c(\Delta_1)$ and $c(\Delta_3)$ as shown in Figures 4.1(iii) and 4.4(iii). By symmetry assume then that $d(\Delta_1) = d(\Delta_3) = 4$. The two possibilities are in Figures 4.7(iii) and 4.5(ii) and in each case add $\frac{1}{2}c(\Delta) = \frac{\pi}{4}$ to $c(\Delta_2)$ as shown. If $d(\Delta_5) > 4$ then add the remaining $\frac{1}{2}c(\Delta) = \frac{\pi}{4}$ to $c(\Delta_5)$; or if $d(\Delta_5) = 4$ then similarly distribute the remaining $\frac{1}{2}c(\Delta) = \frac{\pi}{4}$ via Δ_5 to Δ_4 or Δ_6. Now observe that if Δ_1 receives positive curvature from Δ it does not receive curvature from Δ_2; and if Δ_2 receives positive curvature from Δ it does not receive positive curvature from Δ_1 in Figure 4.7(iii) or from Δ_3 in Figure 4.5(ii). It follows that if $d(\hat{\Delta}) \geq 6$ then $c^*(\Delta) \leq 0$ by Lemma 3.5(iv). Now $d(\Delta_2) > 4$ in Figures 4.7(iii) and 4.5(ii) so there remains the case $d(\hat{\Delta}) = 5$ and $d(\hat{\Delta}) \in \{bdw, caw, c^{-1}aw, bd^{-1}w\}$. But checking shows that in all cases $c^*(\hat{\Delta}) \leq \pi\left(2 - \frac{5}{2}\right) + 2.\frac{\pi}{4} = 0$.

In conclusion $\mathcal{P}$ fails to be aspherical in this case if and only if $b^2 = bda^{-1}c^{-1} = 1$ (and H is non-cyclic).

(A3) $|c| > 3$, $|d| > 3$, $a^{-1}b \neq 1$, $cab^{-1} = 1$, $c^{-1}ab^{-1} \neq 1$, $d^{\pm 1}b^{-1}a \neq 1$.
If $d(\Delta) = 3$ then $l(\Delta) = cab^{-1}$ and Δ is given by Figure 4.7(iv). Add $c(\Delta) = \frac{\pi}{2}$ to $c(\Delta_1)$ across the d^2 edge. If Δ_5 receives no curvature across at least four edges then there are at least four *gaps* as defined in Section 3 and $c^*(\Delta_5) \leq 0$ by Lemma 3.6. This leaves the case when $l(\Delta_1) \in \{d^n, d^n a^{-1}b, d^n b^{-1}a\}$. If $l(\Delta_5) = d^n$ then there is a sphere by Lemma 3.1(a)(i); and if $d^n(a^{-1}b)^{\pm 1} = 1$ then H is cyclic. Therefore $\mathcal{P}$ is aspherical if and only if $|d| = \infty$.

(A4) $|c| > 3$, $|d| > 3$, $a^{-1}b \neq 1$, $cab^{-1} \neq 1$, $c^{-1}ab^{-1} = 1$, $d^{\pm 1}b^{-1}a \neq 1$.
If $d(\Delta) = 3$ then $l(\Delta) = c^{-1}ab^{-1}$ and Δ is given by Figure 4.8(i). First assume that H is non-cyclic. Add $c(\Delta) = \frac{\pi}{2}$ to $c(\Delta_1)$ across the db^{-1} edge and note that $l(\Delta_1) = db^{-1}\omega$ forces $d(\Delta_1) \geq 5$, otherwise $l(\Delta_1) \in \{db^{-1}ad, db^{-1}c^{\pm 1}a, db^{-1}c^{\pm 1}b\}$ which contradicts $|d| > 3$ or H non-cyclic. Observe also that Δ_1 does not receive curvature from Δ_2 or Δ_6 and so if $d(\Delta_1) \geq 7$, it follows that $c^*(\Delta_1) \leq 0$ by Lemma 3.5(vi); if $d(\Delta_1) = 5$ then $l(\Delta_1)$ contains at most one occurrence of $(db^{-1})^{\pm 1}$ and so $c^*(\Delta_1) \leq \pi\left(2 - \frac{5}{2}\right) + \frac{\pi}{2} = 0$; and if $d(\Delta_1) = 6$ then $l(\Delta_1)$ contains at most two occurrence of $(db^{-1})^{\pm 1}$ and so $c^*(\Delta_1) \leq \pi\left(2 - \frac{6}{2}\right) + 2.\frac{\pi}{2} = 0$. Now let H be cyclic. If $d(\Delta_1) > 4$ in Figure 4.8(i) then add $c(\Delta) = \frac{\pi}{2}$ to $c(\Delta_1)$. Assume otherwise and $d(\Delta_1) = 4$. Then $l(\Delta_1) \in \{db^{-1}ad, db^{-1}c^{\pm 1}a, db^{-1}c^{\pm 1}b\}$ which yields a contradiction to the (**A4**) assumptions except when $l(\Delta_1) \in \{db^{-1}ad, db^{-1}ca\}$. If $db^{-1}ca = 1$ then there is a sphere by Lemma 3.1(a)(ii) so assume otherwise. Thus if $d(\Delta_1) = 4$ then $l(\Delta_1) = db^{-1}ad$. In this case as shown in Figure 4.8(ii) add $c(\Delta) = \frac{\pi}{2}$ firstly to $c(\Delta_1)$ then on to $c(\Delta_6)$. Observe that $d(\Delta_6) > 4$ otherwise $l(\Delta_6) \in \{da^{-1}bd, da^{-1}c^{\pm 1}a, da^{-1}cb, da^{-1}c^{-1}b\}$ which either contradicts $|c| > 3$, $|d| > 3$ or $d^{\pm 1}b^{-1}a \neq 1$, or implies the exceptional case (**E1**) when $da^{-1}c^{-1}b = 1$. Now observe that if Δ_1 receives positive curvature from Δ then it does not receive positive curvature from Δ_2 or Δ_6 in Figure 4.8(i); and if Δ_6 receives positive curvature from Δ then it does not receive positive curvature from Δ_5 or Δ_{15} in Figure 4.8(ii). Therefore if $\hat{\Delta}$ receives positive curvature then it does so across at most half of its edges which implies that if $d(\hat{\Delta}) \geq 7$, then $c^*(\hat{\Delta}) \leq 0$ by Lemma 3.5(vi). If $d(\hat{\Delta}) = 5$ then checking shows that $l(\hat{\Delta})$ contains at most one occurrence of $(db^{-1})^{\pm 1}$ or $(da^{-1})^{\pm 1}$. It follows that $c^*(\hat{\Delta}) \leq 0$. If $d(\hat{\Delta}) = 6$ then checking shows that $l(\hat{\Delta})$ contains at most two occurrences of $(db^{-1})^{\pm 1}$ or $(da^{-1})^{\pm 1}$ and so $c^*(\hat{\Delta}) \leq 0$. In conclusion in this case $\mathcal{P}$ fails to be aspherical if and only if $db^{-1}ca = 1$ (and so H is cyclic).

(A5) $|c| = 3$, $|d| > 3$, $a^{-1}b \neq 1$, $cab^{-1} = 1$, $c^{-1}ab^{-1} \neq 1$, $d^{\pm 1}b^{-1}a \neq 1$.
If $|d| < \infty$ then there is a sphere by Lemma 3.1(a)(i) and if $|d| = \infty$ then $\mathcal{P}$ is aspherical by Lemma 3.4(ii).

Figure 4.8

(A6) $|c| = 3$, $|d| > 3$, $a^{-1}b \neq 1$, $cab^{-1} \neq 1$, $c^{-1}ab^{-1} = 1$, $d^{\pm 1}b^{-1}a \neq 1$.
If $d(\Delta) = 3$ then Δ is given by Figure 4.1(ii) and 4.8(iii). First assume that either H is non-cyclic or H is cyclic but $b \neq d^{\pm 2}$. If $l(\Delta) = c^3$ then add $\frac{1}{3}c(\Delta) = \frac{\pi}{6}$ to $c(\Delta_i)$ for $i \in \{1, 3, 5\}$ across the bd edge. Note that if $d(\Delta_i) = 4$ then $l(\Delta_i) \in \{bdda^{-1}, bda^{-1}c^{\pm 1}, bdb^{-1}c^{\pm 1}\}$ which contradicts $|d| > 3$ or $b \neq d^{\pm 2}$ and so $d(\Delta_i) \geq 5$. If $l(\Delta) = c^{-1}ab^{-1}$ then add $c(\Delta) = \frac{\pi}{2}$ to $c(\Delta_5)$ across the $d^{-1}a^{-1}$ edge and note that if $d(\Delta_5) = 4$ then $l(\Delta_5) \in \{d^{-1}a^{-1}bd^{-1}, d^{-1}a^{-1}c^{\pm 1}a, d^{-1}a^{-1}c^{\pm 1}b\}$ which contradicts $|d| > 3$ or $b \neq d^{\pm 2}$ and so $d(\Delta_5) \geq 5$. Observe now that Δ_1 does not receive curvature from Δ_2 or Δ_6 in Figure 4.1(ii); and Δ_5 does not receive curvature from Δ_4 or Δ_6 in Figure 4.8(iii). Therefore if $\hat{\Delta}$ receives positive curvature then it does so across at most half of its edges which implies that if $d(\hat{\Delta}) \geq 7$, then $c^*(\hat{\Delta}) \leq 0$ by Lemma 3.5(vi). It remains to consider $5 \leq d(\hat{\Delta}) \leq 6$. If $d(\hat{\Delta}) = 5$ then checking shows that $l(\hat{\Delta})$ contains at most one occurrence of $(bd)^{\pm 1}$ or $(d^{-1}a^{-1})^{\pm 1}$. It

follows that $c^*(\hat{\Delta}) \leq 0$. If $d(\hat{\Delta}) = 6$ then checking shows that $l(\hat{\Delta})$ contains at most two occurrences of $(bd)^{\pm 1}$ or $(d^{-1}a^{-1})^{\pm 1}$ and so $c^*(\hat{\Delta}) \leq 0$ and $\mathcal{P}$ is aspherical.

Now let $b = d^{-2}$. This implies $c = d^2$ and $d^6 = 1$. It follows that if $d(\Delta) \in \{4, 5\}$ then $l(\Delta) \in \{bdda^{-1}, ccab^{-1}, cab^{-1}ab^{-1}, dda^{-1}c^{-1}a, ddb^{-1}ca, ddb^{-1}c^{-1}b\}$. If $l(\Delta) = c^{-1}ab^{-1}$ then add $\frac{1}{2}c(\Delta) = \frac{\pi}{4}$ to $c(\Delta_1)$ and to $c(\Delta_5)$ as shown in Figure 4.8(iv). Let $l(\Delta) = c^3$. If at least two of $d(\Delta_i) > 4$, say $i = 1, 3$, then add $\frac{1}{2}c(\Delta) = \frac{\pi}{4}$ to $c(\Delta_1)$ and to $c(\Delta_3)$ as shown in Figure 4.1(iii); if say $d(\Delta_1) = d(\Delta_3) = 4$ and $d(\Delta_5) > 4$ then add $\frac{1}{2}c(\Delta) = \frac{\pi}{4}$ to $c(\Delta_5)$ and $\frac{1}{2}c(\Delta) = \frac{\pi}{4}$ to $c(\Delta_2)$ via Δ_3 as shown in Figure 4.1(iv); and if $d(\Delta_i) = 4$ for $i \in \{1, 3, 5\}$ then add $\frac{1}{3}c(\Delta) = \frac{\pi}{6}$ to each of $c(\Delta_j)$ for $j \in \{2, 4, 6\}$ similarly. Therefore if $\hat{\Delta}$ receives positive curvature then $d(\hat{\Delta}) \geq 5$. Observe that Δ_1 does not receive positive curvature from Δ_2 in Figure 4.1(iii); Δ_1, Δ_5 does not receive positive curvature from Δ_6, Δ_4 respectively in Figure 4.8(iv); and Δ_2 does not receive positive curvature from Δ_9 in Figure 4.1(iv). It follows that if $\hat{\Delta}$ receives positive curvature then it does so across at most two-thirds of its edges. Therefore $d(\hat{\Delta}) \geq 6$ implies $c^*(\hat{\Delta}) \leq 0$ by Lemma 3.5(iv). Since there are at most two occurrences of $(ad)^{\pm 1}$, $(db^{-1})^{\pm 1}$, $(bd)^{\pm 1}$, $(cad^{-1})^{\pm 1}$ in each of $cab^{-1}ab^{-1}$, $dda^{-1}c^{-1}a$, $ddb^{-1}ca$, $ddb^{-1}c^{-1}b$ it follows that if $d(\hat{\Delta}) = 5$ then again $c^*(\hat{\Delta}) \leq 0$.

Figure 4.9

Finally let $b = d^2$. This implies $c = d^{-2}$ and $d^6 = 1$. It follows that if $d(\Delta) \in \{4, 5\}$ then $l(\Delta) \in \{d^2b^{-1}a, c^2ab^{-1}, bd^2b^{-1}c, bd^2a^{-1}c^{-1}, d^2a^{-1}ca, cab^{-1}ab^{-1}\}$. Let $c(\Delta) = c^3$. If at least one of Δ_i where $i \in \{1, 3, 5\}$ has degree > 5, say $d(\Delta_1) > 5$, then add $c(\Delta) = \frac{\pi}{2}$ to $c(\Delta_1)$ as shown in Figure 4.9(i); or if $d(\Delta_i) = 5$ for $i \in \{1, 3, 5\}$ then add $\frac{1}{3}c(\Delta) = \frac{\pi}{6}$ to each $c(\Delta_i)$ as shown in Figure 4.9(ii) in which $l(\Delta_i) \in \{bd^2b^{-1}c, bd^2a^{-1}c^{-1}\}$. Let $c(\Delta) = c^{-1}ab^{-1}$. If $d(\Delta_3) > 4$ then add $c(\Delta) = \frac{\pi}{2}$ to $c(\Delta_3)$ as in Figure 4.8(v); if $d(\Delta_3) = 4$ and $d(\Delta_4) > 5$ then add $c(\Delta) = \frac{\pi}{2}$ to $c(\Delta_4)$ via Δ_3 as in Figure 4.9(iii); and if $d(\Delta_3) = 4$ and $d(\Delta_4) = 5$ then add $c(\Delta) = \frac{\pi}{2}$ to $c(\Delta_5)$ as in Figure 4.9(iv). Finally if $d(\hat{\Delta}) = 5$, $l(\hat{\Delta}) = bd^2b^{-1}c$ and $\hat{\Delta}$ receives positive curvature from two neighbouring regions then add $\frac{\pi}{2}$ from $c^*(\hat{\Delta})$ to $c(\hat{\Delta}_1)$ as shown in Figure 4.10(i). Now observe that Δ_1 does not receive positive curvature from Δ_6 in Figure 4.9(i); Δ_1, for example, does not receive from Δ_6 or Δ_2 in Figure 4.9(ii); Δ_3 does not receive from Δ_4 in Figure 4.8(v); Δ_4 does not receive from Δ_5 in Figure 4.9(iii); Δ_5 does not receive from Δ_4 or Δ_6 in Figure 4.9(iv); and $\hat{\Delta}_1$ does not receive from $\hat{\Delta}_2$ or $\hat{\Delta}_3$ in Figure 4.10(i). It follows that if $\hat{\Delta}$ receives positive curvature across two consecutive edges then $d(\hat{\Delta}) > 5$ and $\hat{\Delta}$ is given by Figure 4.10(ii) in which $\hat{\Delta}$ does not receive from $\hat{\Delta}_1$ or $\hat{\Delta}_2$. Since two sublabels of the form $(c^{-1}bd)^{\pm 1}$ must be separated by a sublabel of length at least 2 it follows that if $\hat{\Delta}$ receives positive curvature then it does so across at most $\frac{3}{5}$ of its edges. Therefore if $d(\hat{\Delta}) \geq 8$ then $c^*(\hat{\Delta}) \leq 0$ by Lemma 3.5(vii); and if $l(\hat{\Delta})$ does not involve $(c^{-1}bd)^{\pm 1}$ then Lemma 3.5(vi) applies and $c^*(\hat{\Delta}) \leq 0$ for $d(\hat{\Delta}) \geq 7$. If $d(\hat{\Delta}) = 5$ then either $c^*(\hat{\Delta}) \leq 0$ or $\hat{\Delta}$ is given by Figure 4.10(i) and $c^*(\hat{\Delta}) - \frac{\pi}{2} \leq 0$. If $l(\hat{\Delta})$ does not involve $(c^{-1}bd)^{\pm 1}$ it remains to consider $d(\hat{\Delta}) = 6$. But checking shows that each candidate for $l(\hat{\Delta})$ contains at most two occurrences of $(bd)^{\pm 1}$, $(b^{-1}c)^{\pm 1}$ or $(cad)^{\pm 1}$ so $c^*(\hat{\Delta}) \leq 0$.

Finally if $l(\hat{\Delta}) = c^{-1}bdw$ and $d(\hat{\Delta}) = 6$ then $l(\hat{\Delta}) = c^{-1}bd^2b^{-1}c^{-1}$ and $c^*(\hat{\Delta}) \leq 0$; or if $d(\hat{\Delta}) = 7$ then $l(\hat{\Delta}) \in \{c^{-1}bd^4b^{-1}, c^{-1}bda^{-1}bdb^{-1}, c^{-1}bda^{-1}bd^{-1}a^{-1}, c^{-1}bdb^{-1}ad^{-1}b^{-1}\}$ and $c^*(\hat{\Delta}) \leq c(\hat{\Delta}) + 3(\frac{\pi}{2}) = 0$. In conclusion $\mathcal{P}$ is aspherical for this case.

Figure 4.10

(A7) $|c| = 3$, $|d| > 3$, $a^{-1}b \neq 1$, $c^{\pm 1}ab^{-1} \neq 1$, $db^{-1}a = 1$, $d^{-1}b^{-1}a \neq 1$.
If $d(\Delta) = 3$ then $l(\Delta) \in \{c^3, db^{-1}a\}$ and Δ is given by Figures 4.1(ii) and 4.10(iii). Note that if $l(\Delta_i) = bdw$, $c^{-1}aw$ and $d(\Delta_i) = 4$ then $l(\Delta_i) = bda^{-1}c^{\pm 1}$, $c^{-1}ad^{-1}b^{-1}$ (respectively) otherwise there is a contradiction to $|d| > 3$. First assume that $bda^{-1}c^{\pm 1} \neq 1$. If $l(\Delta) = c^3$ then add $\frac{1}{3}c(\Delta) = \frac{\pi}{6}$ to $c(\Delta_i)$ for $i \in \{1, 3, 5\}$ across the bd edge as shown in Figure 4.1(ii); and if $l(\Delta) = db^{-1}a$ then add $c(\Delta) = \frac{\pi}{2}$ to $c(\Delta_5)$ as in Figure 4.10(iii). Observe then that Δ_1 does not receive positive curvature from Δ_2 or Δ_6 in Figure 4.1(ii); and Δ_5 does not receive from Δ_4 or Δ_6 in Figure 4.10(iii). It follows that $c^*(\hat{\Delta}) \leq 0$ for $d(\hat{\Delta}) \geq 7$ by Lemma 3.5(vi). If $d(\hat{\Delta}) = 5$ then checking shows that either $l(\hat{\Delta})$ contains at most one occurrence of $(bd)^{\pm 1}$ or $(c^{-1}a)^{\pm 1}$ and so $c^*(\hat{\Delta}) \leq 0$ or $l(\hat{\Delta}) = c^{-1}ad^{-2}b^{-1}$. If $d(\hat{\Delta}) = 6$ then checking shows that $l(\hat{\Delta})$ contains at most two occurrences of $(bd)^{\pm 1}$ or $(c^{-1}a)^{\pm 1}$ and so $c^*(\hat{\Delta}) \leq 0$. Thus if $c^*(\hat{\Delta}) > 0$ then $\hat{\Delta}$ is given by Figure 4.10(iv) and add $c^*(\hat{\Delta}) = \frac{\pi}{6}$ to $c(\hat{\Delta}_1)$ as shown. Now $\hat{\Delta}_1$ does not receive positive curvature from $\hat{\Delta}_2$ or $\hat{\Delta}_3$ in Figure 4.10(iv) and the above statements still hold for Δ_i of Figure 4.1(ii) and Δ_5 of Figure 4.10(iii). Therefore if $\hat{\Delta}$ receives positive curvature it does so across at most half of its edges and so $c^*(\hat{\Delta}) \leq 0$ for $d(\hat{\Delta}) \geq 7$. Let $d(\hat{\Delta}) < 7$. If $l(\hat{\Delta})$ does not involve $a^{-1}b$ then checking shows that $c^*(\hat{\Delta}) \leq 0$ so assume $l(\hat{\Delta}) = a^{-1}bw$. But if now $d(\hat{\Delta}) < 6$ then $l(\hat{\Delta})$ together with $c^{-1}ad^{-2}b^{-1}$ contradicts $|d| = 9$; and if $d(\hat{\Delta}) = 6$ then the only labels with more than two occurrences of $(a^{-1}b)^{\pm 1}$, $(bd)^{\pm 1}$ or $(c^{-1}a)^{\pm 1}$ that do not yield a contradiction are $a^{-1}ba^{-1}cbd$, $a^{-1}bda^{-1}cb$ and $a^{-1}bd^{-1}b^{-1}ad$ in which case $c^*(\hat{\Delta}) \leq c(\Delta) + 2(\frac{\pi}{6}) + \frac{\pi}{2} < 0$. If $bda^{-1}c = 1$ then, since $db^{-1}a \leftrightarrow c^{-1}ab^{-1}$ and $bda^{-1}c \leftrightarrow a^{-1}c^{-1}bd^{-1}$, there is a sphere by Lemma 3.1(a)(ii). This leaves the case $bda^{-1}c^{-1} = 1$ which implies $d^6 = 1$ and $c = d^2$. If $l(\Delta) = db^{-1}a$ then again add $c(\Delta) = \frac{\pi}{2}$ for $c(\Delta_5)$ as in Figure 4.10(iii). Let $l(\Delta) = c^3$. If at least two $d(\Delta_i) \geq 5$, say Δ_1 and Δ_3, then add $\frac{1}{2}c(\Delta) = \frac{\pi}{4}$ to $c(\Delta_1)$ and to $c(\Delta_3)$ as in Figure 4.1(iii). If, say, $d(\Delta_1) = d(\Delta_3) = 4$ then add $\frac{1}{2}c(\Delta) = \frac{\pi}{4}$ to $c(\Delta_2)$ via Δ_1 as in Figure 4.2(iii) and $\frac{1}{2}c(\Delta) = \frac{\pi}{4}$ to $c(\Delta_5)$ if $d(\Delta_5) \geq 5$ or to $c(\Delta_6)$ via Δ_5 if $d(\Delta_5) = 4$ also. Observe that Δ_5 does not receive positive curvature from Δ_4 or Δ_6 in Figure 4.10(iii); Δ_1 does not receive any from Δ_6 in Figure 4.1(iii); and Δ_2 does not receive any from Δ_3 in Figure 4.2(iii). It follows from Lemma 3.5(v) that if $d(\hat{\Delta}) \geq 7$ then $c^*(\hat{\Delta}) \leq 0$. If $d(\hat{\Delta}) = 6$ then checking the allowable labels shows that $l(\hat{\Delta})$ involves at most two occurrences of $(bd)^{\pm 1}$, $(c^{-1}a)^{\pm 1}$ or $(b^{-1}ab)^{\pm 1}$ and so $c^*(\hat{\Delta}) \leq 0$. If $d(\hat{\Delta}) = 5$ then $c^*(\hat{\Delta}) \leq 0$ except when $\hat{\Delta}$ is given by Figure 4.11(i) in which case add $c^*(\hat{\Delta}) = \frac{\pi}{4}$ to $c(\hat{\Delta}_1)$ across the bd edge as shown; and note that $\hat{\Delta}_1$ does not receive positive curvature from $\hat{\Delta}_2$ (and so none of the above is affected). If $c^*(\hat{\Delta}_1) \leq 0$ then we are done, otherwise $l(\hat{\Delta}_1) \in \{bda^{-1}c^{-1}, bda^{-1}c^2\}$. Let $l(\hat{\Delta}_1) = bda^{-1}c^{-1}$. Then add $c^*(\hat{\Delta}) = \frac{\pi}{4}$ to $\hat{\Delta}_2$ via $\hat{\Delta}_1$ as shown in Figure 4.11(ii) noting that $\hat{\Delta}_2$ does not receive positive curvature from the region Δ. It follows that if $c^*(\hat{\Delta}) > 0$ then $l(\hat{\Delta}) = b^{-1}ad^{-1}w$

and $d(\hat{\Delta}) \leq 6$. But this implies that $l(\hat{\Delta}) \in \{b^{-1}ad^{-1}a^{-1}bd, b^{-1}ad^{-1}a^{-1}ca\}$ and $c^*(\hat{\Delta}) \leq 0$. Finally if $\hat{\Delta}_1$ is given by Figure 4.11(iii) then add $c^*(\hat{\Delta}_1) = \frac{\pi}{4}$ to $\hat{\Delta}_3$ as shown and repeat the above argument. This procedure must terminate at a region $\hat{\Delta}_k$, say, where either $c^*(\hat{\Delta}_k) \leq 0$ or $d(\hat{\Delta}_k) = 4$ as in Figure 4.11(ii), in which case $c^*(\hat{\Delta}_k) = \frac{\pi}{4}$ is added to $c(\hat{\Delta}_{k+1})$ where $l(\hat{\Delta}_k) = b^{-1}ad^{-1}w$ and so $c^*(\hat{\Delta}_k) \leq 0$. In conclusion $\mathcal{P}$ is aspherical if and only if $bda^{-1}c \neq 1$. (Note that if $bda^{-1}c = 1$ then H is cyclic.)

Figure 4.11

(A8) $|c| = 3$, $|d| = 3$, $a^{-1}b \neq 1$, $cab^{-1} \neq 1$, $c^{-1}ab^{-1} = 1$, $d^{\pm 1}b^{-1}a \neq 1$.
If $d(\Delta) = 3$ then $l(\Delta) \in \{c^3, d^3, c^{-1}ab^{-1}\}$ and Δ is given by Figures 4.1(ii), 4.3(i) and 4.8(iii). If $l(\Delta) = c^3$ then add $\frac{1}{3}c(\Delta) = \frac{\pi}{6}$ to $c(\Delta_i)$ for $i \in \{1, 3, 5\}$ across the bd edge as shown in Figure 4.1(ii). Note that if $d(\Delta_i) = 4$ then $l(\Delta_i) \in \{bdda^{-1}, bda^{-1}c^{\pm 1}, bdb^{-1}c^{\pm 1}\}$ which contradicts $|d| = 3$ or $d^{\pm 1}b^{-1}a \neq 1$ and so $d(\Delta_i) \geq 5$. If $l(\Delta) = c^{-1}ab^{-1}$ then add $c(\Delta) = \frac{\pi}{2}$ to $c(\Delta_5)$ across the $d^{-1}a^{-1}$ edge as shown in Figure 4.8(iii) and note that if $d(\Delta_5) = 4$ then $l(\Delta_5) \in \{d^{-1}a^{-1}bd^{-1}, d^{-1}a^{-1}c^{\pm 1}a, d^{-1}a^{-1}c^{\pm 1}b\}$ which again contradicts $|d| = 3$ or $d^{\pm 1}b^{-1}a \neq 1$ and so $d(\Delta_5) \geq 5$. If $l(\Delta) = d^3$ then there are three subcases. First assume that $d(\Delta_i) \geq 5$ for $i \in \{1, 3, 5\}$. Then add $\frac{1}{3}c(\Delta) = \frac{\pi}{6}$ to each $c(\Delta_i)$ across the ca edge as shown in Figures 4.3(i). Now suppose that at least two of the Δ_i have degree four, without loss of generality Δ_1 and Δ_3. Then $l(\Delta_1) = l(\Delta_3) = cab^{-1}c$ (otherwise we will have a contradiction to $d^{\pm 1}b^{-1}a \neq 1$) which in turn forces $l(\Delta_2) = c^{-1}bd\omega$ and so $d(\Delta_2) > 4$ otherwise $l(\Delta_2) \in \{c^{-1}bda^{-1}, c^{-1}bdb^{-1}\}$ which contradicts $|d| = 3$ or $d^{\pm 1}b^{-1}a \neq 1$. In this case add $\frac{1}{2}c(\Delta) = \frac{\pi}{4}$ to $c(\Delta_2)$ and $c(\Delta_5)$ across the ca and bd edges as shown in Figure 4.11(iv) and if $d(\Delta_5) = 4$ then in the same way add $\frac{1}{2}c(\Delta) = \frac{\pi}{4}$ to $c(\Delta_6)$. This leaves the case where exactly one of Δ_i has degree four and without loss assume $d(\Delta_1) = 4$. Then add $\frac{1}{2}c(\Delta) = \frac{\pi}{4}$ to each of $c(\Delta_3)$ and $c(\Delta_5)$ as shown in Figure 4.11(v).

Observe that in Figure 4.1(ii) Δ_1 does not receive any positive curvature from Δ_2 or Δ_6; in Figure 4.3(i) Δ_1 does not receive positive curvature from Δ_2; in Figure 4.8(iii) Δ_5 does not receive any positive curvature from Δ_6; in Figure 4.11(iv) Δ_2 does not receive any positive curvature from Δ_3 or Δ_8 and Δ_5 does not receive any positive curvature from Δ_6; and in Figure 4.11(v) Δ_3, Δ_5 (respectively) does not receive any positive curvature from Δ_4, Δ_6 (respectively). Let $\hat{\Delta}$ receive positive curvature and first suppose that $l(\hat{\Delta})$ does not involve $(ad)^{\pm 1}$. It follows that $\hat{\Delta}$ receives positive curvature across at most half of its edges and so, since $d(\hat{\Delta}) \geq 5$, $c^*(\hat{\Delta}) \leq 0$ by Lemma 3.5(v). Now assume that $l(\hat{\Delta}) = ad\omega$ and put $d(\hat{\Delta}) = k$. Since positive curvature crosses (bd), (ca) and (ad) edges only it follows that if $l(\hat{\Delta})$ involves at least two occurrences of $(ad)^{\pm 1}$ then there are at least four gaps and $c^*(\hat{\Delta}) \leq 0$ by Lemma 3.6, so assume otherwise. Moreover, $l(\hat{\Delta}) = ad\omega$ forces $\hat{\Delta}$ to have at least two gaps and so $c^*(\hat{\Delta}) \leq \pi\left(2 - \frac{k}{2}\right) + \frac{\pi}{2} + (k-3).\frac{\pi}{4}$ which implies that if $k \geq 7$ then $c^*(\hat{\Delta}) \leq 0$. If $d(\hat{\Delta}) = 5$ then $l(\hat{\Delta}) \in \{ad^3b^{-1}, adb^{-1}ab^{-1}, ad^2a^{-1}c^{\pm 1}, ad^2b^{-1}c^{\pm 1}, ada^{-1}c^{\pm 2}, adb^{-1}c^{\pm 2}\}$ which contradicts $|b| = 3$ or $d^{\pm 1}b^{-1}a \neq 1$. If $d(\hat{\Delta}) = 6$ then checking shows that $\hat{\Delta}$ receives curvature across at most two edges and so $c^*(\hat{\Delta}) \leq 0$. In conclusion in this case $\mathcal{P}$ is aspherical.

(A9) $|c| = 3$, $|d| = 3$, $a^{-1}b \neq 1$, $cab^{-1} \neq 1$, $c^{-1}ab^{-1} = 1$, $db^{-1}a = 1$, $d^{-1}b^{-1}a \neq 1$.
Since H is cyclic $\mathcal{P}$ is aspherical by Lemma 3.2(ii).

(A10) $|c| > 3$, $|d| > 3$, $a^{-1}b \neq 1$, $cab^{-1} \neq 1$, $c^{-1}ab^{-1} = 1$, $db^{-1}a = 1$, $d^{-1}b^{-1}a \neq 1$.
Since H is cyclic, $|d| < \infty$ and so $\mathcal{P}$ is aspherical by Lemma 3.2(ii).

It follows from (**A0**)–(**A10**) above that either $\mathcal{P}$ is aspherical or modulo T-equivalence one of the conditions from Theorem 1.1(i), (iii) or Theorem 1.2(i), (ii), (iii) is satisfied and so Theorems 1.1 and 1.2 are proved for Case A.

5 Proof of Case B

In this section we prove Theorems 1.1 and 1.2 for Case B, that is, we make the following assumption: at least one of c^2, d^2, $a^{-1}b$ equals 1 in H.

If $d(\Delta) = 2$ then we will fix the names of the four neighbouring regions Δ_i $(1 \leq i \leq 4)$ of Δ as shown in Figure 5.1(i) and we use this notation throughout the section.

Remark Recall that if $c^2 = 1$ then $c^{\pm 2}$ cannot be a proper sublabel. This fact will be used often without explicit reference.

Figure 5.1

We treat each of the cases (**B1**) to (**B12**) in turn.

(B1) $|c| = 2$, $|d| > 3$, $a^{-1}b \neq 1$, $c^{\pm 1}ab^{-1} \neq 1$, $d^{\pm 1}b^{-1}a \neq 1$.
If $d(\Delta) = 2$ then Δ is given by Figure 5.1(ii). Observe that if $d(\Delta_i) = 4$ for $i \in \{1, 2\}$ then $l(\Delta_i) = \{bdda^{-1}, bda^{-1}c^{\pm 1}, bdb^{-1}c^{\pm 1}\}$. But $bdb^{-1}c^{\pm 1} = 1$ implies $|d| = |c|$, a contradiction. Observe further that at most one of $bd^2a^{-1}, bda^{-1}c^{\pm 1}$ equals 1 otherwise there is a contradiction to $|d| > 3$. This leaves the following cases:

(i) $bd^2a^{-1} \neq 1$, $bda^{-1}c^{\pm 1} \neq 1$;

(ii) $bd^2a^{-1} = 1$, $bda^{-1}c^{\pm 1} \neq 1$;

(iii) $bda^{-1}c^{\pm 1} = 1$, $bd^2a^{-1} \neq 1$.

Figure 5.2

(i) In this case $d(\Delta_i) > 4$ for Δ_i $(1 \leq i \leq 2)$ of Figure 5.1(ii) so add $\frac{1}{2}c(\Delta) = \frac{\pi}{2}$ to each of $c(\Delta_i)$ $(1 \leq i \leq 2)$. Observe from Figure 5.1(ii) that Δ_i does not receive positive curvature from Δ_j for $j \in \{3, 4\}$. It follows that if $\hat{\Delta}$ receives positive curvature then it does so across at most half of its edges and so $d(\hat{\Delta}) \geq 7$ implies that $c^*(\hat{\Delta}) \leq 0$ by Lemma 3.5(vi). Checking (the LIST) shows that if $d(\hat{\Delta}) = 5$ then $\hat{\Delta}$ receives positive curvature across at most one edge and so $c^*(\hat{\Delta}) \leq 0$. Also if $d(\hat{\Delta}) = 6$ then checking shows $\hat{\Delta}$ receives positive curvature across at most two edges and so $c^*(\hat{\Delta}) \leq 0$.

(ii) Suppose that $l(\Delta) = bd^2a^{-1} = 1$, $bda^{-1}c^{\pm 1} \neq 1$. If $d(\Delta_1) > 4$ and $d(\Delta_2) > 4$ then add $\frac{1}{2}c(\Delta) = \frac{\pi}{2}$ to $c(\Delta_1)$ and $c(\Delta_2)$ as in Figure 5.1(ii). If say $d(\Delta_2) = 4$ as in Figure 5.1(iii) then $l(\Delta_2) = bdda^{-1}$ which forces $l(\Delta_3) = ca\omega$. First assume that $cadb^{-1} \neq 1$. Then $d(\Delta_3) > 3$ and so add $\frac{\pi}{2}$ to $c(\Delta_3)$ via Δ_2. If $d(\Delta_1) = 4$ then add $\frac{\pi}{2}$ to $c(\Delta_4)$ via Δ_1 in a similar way. Observe that Δ_1 does not receive positive curvature from Δ_3 or Δ_4 in Figure 5.1(ii); and Δ_3 does not receive positive curvature from Δ_1 or Δ_5 in Figure 5.1(iii). It follows that if $\hat{\Delta}$ receives positive curvature then it does so across at most half of its edges and so $d(\hat{\Delta}) \geq 7$ implies that $c^*(\hat{\Delta}) \leq 0$ by Lemma 3.5(vi). It remains to study $5 \leq d(\hat{\Delta}) \leq 6$. Checking shows that if $d(\hat{\Delta}) = 5$ then either the label contradicts $|c| \neq 1$ or $\hat{\Delta}$ receives positive curvature across at most one edge and so $c^*(\hat{\Delta}) \leq 0$. Also if $d(\hat{\Delta}) = 6$ then checking shows that $\hat{\Delta}$ receives positive curvature across at most two edges and so $c^*(\hat{\Delta}) \leq 0$. Now assume that $cadb^{-1} = 1$, in which case $c = d^3$, $b = d^{-2}$ and $|d| = 6$. The distribution of curvature is exactly the same except when $d(\Delta_3) = 4$ in Figure 5.1(iii). In this case add $\frac{2}{3}c(\Delta) = \frac{2\pi}{3}$ to $c(\Delta_1)$ and $\frac{1}{3}c(\Delta) = \frac{\pi}{3}$ to $c(\Delta_5)$ via Δ_3 as shown in Figure 5.1(iv). Together with the observations above (which still hold) we also have that Δ_1 does not receive positive curvature from Δ_3, Δ_4 or Δ_7 and that Δ_5 does not receive positive curvature from Δ_9 or Δ_{10} in Figure 5.1(iv). An argument similar to those given in the proof of Lemma 3.5 now shows that if $d(\hat{\Delta}) \geq 8$ then $c^*(\hat{\Delta}) \leq 0$; and that if $l(\hat{\Delta})$ does not involve $(cbd)^{\pm 1}$ then $c^*(\hat{\Delta}) \leq 0$ for $d(\hat{\Delta}) \geq 7$ by Lemma 3.5(vi). The conditions on b, c and d imply that if $2 < d(\hat{\Delta}) < 6$ then $l(\hat{\Delta}) \in \{d^2a^{-1}b, db^{-1}c^{\pm 1}a\}$ so if $l(\hat{\Delta})$ does not involve $(cbd)^{\pm 1}$ it remains to consider $d(\hat{\Delta}) = 6$. But checking shows that $l(\hat{\Delta})$ will then either involve at most two non-adjacent occurrences of $(bd)^{\pm 1}$, $(ca)^{\pm 1}$ or $(ba^{-1})^{\pm 1}$ and so $c^*(\hat{\Delta}) \leq 0$ or $l(\hat{\Delta}) = (ab^{-1})^3$ in which case $c^*(\hat{\Delta}) \leq c(\hat{\Delta}) + 3(\frac{\pi}{3}) = 0$. Finally if $l(\hat{\Delta}) = cbd\omega$ and $d(\hat{\Delta}) \leq 7$ then $l(\hat{\Delta}) \in \{cbda^{-1}ba^{-1}, cbdb^{-1}ab^{-1}, cbd^3b^{-1}\}$ and $c^*(\hat{\Delta}) \leq c(\hat{\Delta}) + \frac{2\pi}{3} + \frac{\pi}{3} = 0$.

Figure 5.3

(iii) Now suppose that $l(\Delta) = bda^{-1}c^{\pm 1} = 1$, $bd^2a^{-1} \neq 1$. First assume that $|b| \geq 3$. Add $\frac{1}{2}c(\Delta) = \frac{\pi}{2}$ to $c(\Delta_1)$ and $c(\Delta_2)$ as in Figure 5.1(ii). If say $d(\Delta_2) = 4$ then $l(\Delta_2) = bda^{-1}c^{\pm 1}$. First let $l(\Delta_2) = bda^{-1}c$ as in Figure 5.2(i). This forces $l(\Delta_4) = ad\omega$ and so $d(\Delta_4) = 4$ forces $l(\Delta_4) = addb^{-1}$. But if $b = d^2$ then $c = d^3$ and there is a sphere by Lemma 3.2(iv), so it can be assumed that $d(\Delta_4) > 4$. So add $\frac{\pi}{2}$ to $c(\Delta_4)$ via Δ_2 as shown. Suppose now that $l(\Delta_2) = bda^{-1}c^{-1}$ as in Figure 5.2(ii). This forces $l(\Delta_4) = ab^{-1}\omega$ and so $d(\Delta_4) > 4$, otherwise there is a contradiction to $|b| \geq 3$. So add $\frac{\pi}{2}$ to $c(\Delta_4)$ via Δ_2 as shown. Similarly add $\frac{1}{2}c(\Delta) = \frac{\pi}{2}$ to $c(\Delta_3)$ if $d(\Delta_1) = 4$. Observe that Δ_1 does not receive positive curvature from Δ_3 or Δ_4 in Figure 5.1(ii); Δ_2 does not receive positive curvature from Δ_3 or Δ_4 in Figure 5.1(ii); and Δ_4 does not receive positive curvature from Δ_1 or Δ_8 in Figures 5.2(i), (ii). It follows that if $\hat{\Delta}$ receives positive curvature then it does so across at most half of its edges and so $d(\hat{\Delta}) \geq 7$ implies that $c^*(\hat{\Delta}) \leq 0$ by Lemma 3.5(vi). It remains to study $5 \leq d(\hat{\Delta}) \leq 6$.

If $|b| > 3$ then checking shows that if $d(\hat{\Delta}) = 5$ then either the label contradicts one of the (**B2**) assumptions or $\hat{\Delta}$ receives positive curvature across at most one edge and so $c^*(\hat{\Delta}) \leq 0$. Checking shows that if $d(\hat{\Delta}) = 6$ then $\hat{\Delta}$ receives positive curvature across at most two edges or $l(\hat{\Delta}) = (ab^{-1})^3$ contradicting $|b| > 3$, and so $c^*(\hat{\Delta}) \leq 0$.

Let $|b| = 3$. If H is cyclic then $bdc = 1$ implies $b = d^2$ and we obtain a sphere as before, so assume otherwise. If $|b| = 3$ and $|d| \in \{4, 5\}$ then we obtain a sphere by Lemma 3.1(b)(iv). So let $|b| = 3$, $|d| \geq 6$. Distribute curvature from Δ as above and as shown in Figure 5.2. Checking shows that if $d(\hat{\Delta}) = 5$ then either the label contradicts $|b| = 3$ or $|d| \geq 6$ or $\hat{\Delta}$ receives positive curvature across at most one edge and so $c^*(\hat{\Delta}) \leq 0$. Checking shows that if $d(\hat{\Delta}) = 6$ then $\hat{\Delta}$ receives positive curvature across at most two edges and so $c^*(\hat{\Delta}) \leq 0$ except when $l(\hat{\Delta}) = ba^{-1}ba^{-1}ba^{-1}$. This case is shown in Figure 5.3(i) where $c^*(\hat{\Delta}) = \frac{\pi}{2}$ and so add $\frac{1}{3}c^*(\hat{\Delta}) = \frac{\pi}{6}$ to $c(\hat{\Delta}_i)$ for $i \in \{1, 2, 3\}$ across the edge ad^{-1}. If $d(\hat{\Delta}_i) = 4$ then $l(\hat{\Delta}_i) \in \{ad^{-1}b^{-1}c^{-1}, ad^{-1}b^{-1}c\}$. Suppose that $l(\hat{\Delta}_1) = ad^{-1}b^{-1}c^{-1}$ as in Figure 5.3(ii). Then $l(\hat{\Delta}_4) = d^2b^{-1}\omega$ and $d(\hat{\Delta}_4) > 4$ otherwise there is a contradiction to H non-cyclic. So add $\frac{\pi}{6}$ to $c(\hat{\Delta}_4)$ across the edge db^{-1}. If $l(\hat{\Delta}_1) = ad^{-1}b^{-1}c$ as in Figure 5.3(iii) then $l(\hat{\Delta}_4) = d^3\omega$ and $d(\hat{\Delta}_4) > 4$ otherwise there is a contradiction to $|d| \geq 6$. So add $\frac{\pi}{6}$ to $c(\hat{\Delta}_4)$ across the edge d^2. Observe that if $\hat{\Delta}$ receives positive curvature then it receives $\frac{\pi}{2}$ across the edges ab^{-1}, ad or bd; and receives $\frac{\pi}{6}$ across the edges ad^{-1}, db^{-1} or dd. Thus there is a gap preceding $c^{\pm 1}$, b, a and a gap after $c^{\pm 1}$, b^{-1}, a^{-1} and there is a two-thirds gap across the edges ad^{-1}, db^{-1} and dd. Also there is always a gap between two d's (other than when the subword is $d^{\pm 2}$). Now since c^2 cannot be a proper sublabel it follows that if there are at least two occurrences of $c^{\pm 1}$ then we obtain four gaps. Suppose now that there is at most one occurrence of $c^{\pm 1}$. If there is exactly one occurrence of c and either no occurrences of b or no occurrences of d then H is cyclic, a contradiction; and if there are no occurrences of c and exactly one occurrence of d or of b then again H is cyclic, a contradiction. So assume otherwise. It follows that $l(\hat{\Delta})$ contains at least four gaps or $l(\hat{\Delta}) \in \{c^{\pm 1}ad^{\pm 1}a^{-1}ba^{-1}, c^{\pm 1}bd^{\pm 1}a^{-1}ba^{-1}, c^{\pm 1}ad^{\pm 1}b^{-1}ab^{-1}, c^{\pm 1}bd^{\pm 1}b^{-1}ab^{-1},$ $d(a^{-1}b)^{\pm 1}d^{\pm 1}(a^{-1}b)^{\pm 1}\}$. But since $c = bd$ each of these labels contradicts H non-cyclic, $|d| \geq 6$ or $|b| = 3$ except when $l(\hat{\Delta}) = da^{-1}bda^{-1}b$. In this case if $c^*(\hat{\Delta}) > 0$ then it can be assumed without loss of generality that $\hat{\Delta}$ is given by Figure 5.3(iv) and $\hat{\Delta}$ receives $\frac{1}{3}c^*(\hat{\Delta}_1) = \frac{\pi}{6}$ from $c(\hat{\Delta}_1)$. This implies that $l(\hat{\Delta}_3) = a^{-1}c^{-1}bd$ and $l(\hat{\Delta}_4) = ad^{-1}d^{-1}\omega$. So add $\frac{\pi}{6}$ from $c(\hat{\Delta})$ to $c(\hat{\Delta}_4)$. Since this $\frac{\pi}{6}$ is across an ad^{-1} edge and since $l(\hat{\Delta}_4) = ad^{-2}\omega$ it follows from the above that $c^*(\hat{\Delta}_4) \leq 0$. If $l(\hat{\Delta}_2) = b^{-1}ab^{-1}ab^{-1}$ in Figure 5.3(iv) then a similar argument applies to $c(\hat{\Delta}_5)$.

Finally let $|b| = 2$. In particular, H is non-cyclic for otherwise $d^2 = 1$, a contradiction. If $|d| < \infty$ then we obtain a sphere by Lemma 3.1(b)(i) and if $|d| = \infty$ then $\mathcal{P}$ is aspherical by Lemma 3.4(ii).

In conclusion $\mathcal{P}$ is aspherical in this case except when H is non-cyclic, $bda^{-1}c^{\pm 1} = 1$ and either $|b| = 3$, $|d| \in \{4, 5\}$ or $|b| = 2$, $|d| < \infty$; or when H is cyclic, $b = d^2$, $c = d^3$ and $|d| = 6$.

Figure 5.4

(B2) $|c|=2$, $|d|=2$, $a^{-1}b\neq 1$, $c^{\pm 1}ab^{-1}\neq 1$, $d^{\pm 1}b^{-1}a\neq 1$.
If H is cyclic then $c=d$. Therefore $bd^{-1}b^{-1}c=cada^{-1}=1$ and there is a sphere by Lemma 3.2(i). So suppose from now on that H is non-cyclic. If $d(\Delta)=2$ then Δ is given by Figures 5.1(ii) and 5.4(i). Moreover, if $d(\Delta_i)=4$ for $i\in\{1,2\}$ then $l(\Delta_i)\in\{bd^2a^{-1}, bda^{-1}c^{\pm 1}, bdb^{-1}c^{\pm 1}, cad^{\pm 1}a^{-1}, cad^{\pm 1}b^{-1}\}$. But $bdda^{-1}=1$ implies that $b=1$ which is a contradiction. This leaves the following cases:

(i) $bda^{-1}c^{\pm 1}\neq 1$, $bdb^{-1}c^{\pm 1}\neq 1$, $cad^{\pm 1}a^{-1}\neq 1$;

(ii) $bda^{-1}c^{\pm 1}=1$, $bdb^{-1}c^{\pm 1}\neq 1$, $cad^{\pm 1}a^{-1}\neq 1$;

(iii) $bdb^{-1}c^{\pm 1}=1$, $bda^{-1}c^{\pm 1}\neq 1$, $cad^{\pm 1}a^{-1}\neq 1$;

(iv) $cad^{\pm 1}a^{-1}=1$, $bda^{-1}c^{\pm 1}\neq 1$, $bdb^{-1}c^{\pm 1}\neq 1$;

(v) $bdb^{-1}c^{\pm 1}=1$, $cad^{\pm 1}a^{-1}=1$, $bda^{-1}c^{\pm 1}\neq 1$.

(Note that any other combination implies $b=1$.)

(i) In this case $d(\Delta_1)>4$ and $d(\Delta_2)>4$ in Figures 5.1(ii) and 5.4(i) so add $\frac{1}{2}c(\Delta)=\frac{\pi}{2}$ to each of $c(\Delta_1)$ and $c(\Delta_2)$. Observe that Δ_1 and Δ_2 do not receive positive curvature from Δ_3 or Δ_4 in Figures 5.1(ii) and 5.4(i). It follows that if $\hat{\Delta}$ receives positive curvature then it does so across at most half of its edges and so $d(\hat{\Delta})\geq 7$ implies that $c^*(\hat{\Delta})\leq 0$ by Lemma 3.5(vi). It remains to study $5\leq d(\hat{\Delta})\leq 6$. Checking shows that if $d(\hat{\Delta})=5$ then either the label contradicts $c^{\pm 1}ab^{-1}\neq 1$ or $\hat{\Delta}$ receives positive curvature across at most one edge and so $c^*(\hat{\Delta})\leq 0$. Also if $d(\hat{\Delta})=6$ then $\hat{\Delta}$ receives positive curvature across at most two edges and so $c^*(\hat{\Delta})\leq 0$.

(ii) In this case the labels $bda^{-1}c^{\pm 1}$ can occur. If $|b|<\infty$ then we obtain spheres by Lemma 3.1(b)(ii) and if $|b|=\infty$ then $\mathcal{P}$ is aspherical by Lemma 3.4(iii).

(iii) In this case the labels $bdb^{-1}c^{\pm 1}$ can occur and H is non-Abelian. If $d(\Delta_1)>4$ and $d(\Delta_2)>4$ as in Figures 5.1(ii) and 5.4(i) then add $\frac{1}{2}c(\Delta)=\frac{\pi}{2}$ to $c(\Delta_1)$ and $c(\Delta_2)$. If say $d(\Delta_2)=4$ as in Figure 5.4(ii) then $l(\Delta_3)=c^{-1}a\omega$ and so $d(\Delta_3)>4$, otherwise there is a contradiction to $c\neq d$ or $b\neq 1$. So add $\frac{\pi}{2}$ to $c(\Delta_3)$ via Δ_2 as shown. If $d(\Delta_1)=4$ then similarly add $\frac{1}{2}c(\Delta)=\frac{\pi}{2}$ to $c(\Delta_4)$ via Δ_1. We see from Figures 5.1(ii) and 5.4(i)–(ii) that if $\hat{\Delta}$ receives positive curvature then it does so across the edges bd, ca or $c^{-1}a$. It follows that the only word of length 3 that contains no gaps is $a^{-1}c^{\pm 1}a$. Suppose that $l(\hat{\Delta})=\omega_1 a^{-1}c^{\pm 1}a\omega_2\omega$ where ω_1 and ω_2 have length 2 and ω has length at least 0. Then there is a gap preceding a^{-1} in $\omega_1 a^{-1}$ and after a in $a\omega_2$. Moreover, if ω_1 does not contain a gap then $\omega_1=bd$ and if ω_2 does not contain a gap then $\omega_2=d^{-1}b^{-1}$. It follows that if $l(\omega)>0$ then $l(\hat{\Delta})$

contain at least 4 gaps and $c^*(\hat{\Delta}) \leq 0$. If $l(\omega) = 0$ and $l(\hat{\Delta}) = \omega_1 a^{-1}c^{\pm 1}ad^{-1}b^{-1}$ then $l(\hat{\Delta}) \in \{c^{\pm 1}ba^{-1}c^{\pm 1}ad^{-1}b^{-1}, ad^{\pm 1}a^{-1}c^{\pm 1}ad^{-1}b^{-1}\}$ and there are 4 gaps; if $l(\omega) = 0$ and $l(\hat{\Delta}) = bda^{-1}c^{\pm 1}a\omega_2$ then $l(\hat{\Delta}) \in \{bda^{-1}c^{\pm 1}ad^{\pm 1}a^{-1}, bda^{-1}c^{\pm 1}ab^{-1}c^{\pm 1}\}$ and again there are 4 gaps. If now $l(\hat{\Delta}) = a^{-1}c^{\pm 1}aw$ and $d(\hat{\Delta}) \leq 6$ then either $l(\hat{\Delta}) = a^{-1}c^{\pm 1}ad^{\pm 2}$, a contradiction, or there are 4 gaps. Now suppose that $l(\hat{\Delta})$ does not contain the subword $a^{-1}c^{\pm 1}a$. Then $\hat{\Delta}$ receives positive curvature across at most half of its edges and so $d(\hat{\Delta}) \geq 7$ implies $c^*(\hat{\Delta}) \leq 0$ by Lemma 3.5(vi). So let $d(\hat{\Delta}) \leq 6$ and $l(\hat{\Delta}) \in \{ca\omega, bd\omega, c^{-1}a\omega\}$. If $d(\hat{\Delta}) < 6$ then checking shows that there is a contradiction to H non-Abelian; and if $d(\hat{\Delta}) = 6$ then checking shows that $\hat{\Delta}$ receives positive curvature across at most two edges and so $c^*(\hat{\Delta}) \leq 0$.

Figure 5.5

(iv) In this case the labels $cad^{\pm 1}a^{-1}$ can occur and H is non-Abelian. If $d(\Delta_1) > 4$ and $d(\Delta_2) > 4$ as in Figures 5.1(ii) and 5.4(i) then add $\frac{1}{2}c(\Delta) = \frac{\pi}{2}$ to $c(\Delta_1)$ and $c(\Delta_2)$. If say $d(\Delta_2) = 4$ then $l(\Delta_2) = cad^{\pm 1}a^{-1}$ which forces $l(\Delta_4) = bd^{-1}\omega$ and so $d(\Delta_3) > 4$, otherwise there is a contradiction to $b \neq 1$ or $bd^{-1}b^{-1}c^{\pm 1} \neq 1$. So add $\frac{\pi}{2}$ to $c(\Delta_4)$ via Δ_2 as shown in Figure 5.4(iii). If $d(\Delta_1) = 4$ then similarly add $\frac{1}{2}c(\Delta) = \frac{\pi}{2}$ to $c(\Delta_3)$ via Δ_1. We see from Figures 5.1(ii), 5.4(i) and 5.4(iii) that if $\hat{\Delta}$ receives positive curvature then it does so across the edges bd, bd^{-1} or ca. It follows that the only word of length 3 that contains no gaps is $bd^{\pm 1}b^{-1}$. Suppose that $l(\hat{\Delta}) = \omega_1 bd^{\pm 1}b^{-1}\omega_2\omega$ where ω_1 and ω_2 have length 2 and ω has length at least 0. Then there is a gap preceding b in $\omega_1 b$ and after b^{-1} in $b^{-1}\omega_2$. Moreover, if ω_1 does not contain a gap then $\omega_1 = a^{-1}c^{-1}$ and if ω_2 does not contain a gap then $\omega_2 = ca$. It follows that if $l(\omega) > 0$ then $l(\hat{\Delta})$ contain at least 4 gaps and $c^*(\hat{\Delta}) \leq 0$. If $l(\omega) = 0$ and $l(\hat{\Delta}) = \omega_1 bd^{\pm 1}b^{-1}ca$ then $l(\hat{\Delta}) \in \{d^{\pm 1}a^{-1}bdb^{-1}ca, bc^{\pm 1}bdb^{-1}ca\}$ and there are 4 gaps; if $l(\omega) = 0$ and $l(\hat{\Delta}) = a^{-1}c^{-1}bd^{\pm 1}b^{-1}\omega_2$ then $l(\hat{\Delta}) \in \{a^{-1}c^{-1}bdb^{-1}ad^{\pm 1}, a^{-1}c^{-1}bdb^{-1}c^{\pm 1}b\}$ and again there are 4 gaps. If $l(\hat{\Delta}) = bd^{\pm 1}b^{-1}\omega$ and $d(\hat{\Delta}) \leq 6$ then there are 4 gaps. Now suppose that $l(\hat{\Delta})$ does not contain the subword $bd^{\pm 1}b^{-1}$. Then $\hat{\Delta}$ receives positive curvature across at most half of its edges and so $d(\hat{\Delta}) \geq 7$ implies $c^*(\hat{\Delta}) \leq 0$ by Lemma 3.5(vi). So let $d(\hat{\Delta}) \leq 6$ and $l(\hat{\Delta}) \in \{bd\omega, bd^{-1}\omega, ca\omega\}$. If $d(\hat{\Delta}) < 6$ then checking shows that there is a contradiction to H non-Abelian; and if $d(\hat{\Delta}) = 6$ then checking shows that $\hat{\Delta}$ receives positive curvature across at most two edges and so $c^*(\hat{\Delta}) \leq 0$.

(v) In this case the labels $bdb^{-1}c^{\pm 1}$ and $cad^{\pm 1}a^{-1}$ can occur. If $|b| = \infty$ then $\mathcal{P}$ is aspherical by Lemma 3.4(iii); and if $|b| < \infty$ then there is a sphere by Lemma 3.2(i).

In conclusion $\mathcal{P}$ fails to be aspherical in this case either when H is cyclic or when H is non-cyclic and when $bda^{-1}c^{-1} = 1$, $|b| < \infty$ or when $bdb^{-1}c^{-1} = a^{-1}cad = 1$, $|b| < \infty$.

(B3) $|c| = 2$, $|d| = 3$, $a^{-1}b \neq 1$, $c^{\pm 1}ab^{-1} \neq 1$, $d^{\pm 1}b^{-1}a \neq 1$.
If $d(\Delta) = 2$ then Δ is given by Figure 5.1(ii). If $d(\Delta) = 3$ then Δ is given by Figure 4.3(i).

Moreover, if $d(\Delta_i) = 4$ and $l(\Delta_i) = bdw$ or caw then $l(\Delta_i) \in \{bd^2a^{-1}, bda^{-1}c^{\pm1}, bdb^{-1}c^{\pm1}, cad^{\pm1}a^{-1}, cad^{\pm1}b^{-1}\}$. But each of $bd^2a^{-1} = 1$, $bdb^{-1}c^{\pm1} = 1$ and $cad^{\pm1}a^{-1} = 1$ implies a contradiction to one of the (**B3**) assumptions. Thus we have the following cases:

(i) $bda^{-1}c^{\pm1} \neq 1$, $cadb^{-1} \neq 1$;

(ii) $bda^{-1}c^{\pm1} = 1$, $cadb^{-1} \neq 1$;

(iii) $cadb^{-1} = 1$, $bda^{-1}c^{\pm1} \neq 1$.

(i) In this case $d(\Delta_1) > 4$ and $d(\Delta_2) > 4$ in Figure 5.1(ii), so add $\frac{1}{2}c(\Delta) = \frac{\pi}{2}$ to each of $c(\Delta_1)$ and $c(\Delta_2)$. In Figure 4.3(i) $d(\Delta_1) > 4$, $d(\Delta_3) > 4$ and $d(\Delta_5) > 4$ so add $\frac{1}{3}c(\Delta) = \frac{\pi}{6}$ to each of $c(\Delta_1)$, $c(\Delta_3)$ and $c(\Delta_5)$. Observe that Δ_1 and Δ_2 do not receive positive curvature from Δ_3 or Δ_4 in Figure 5.1(ii). Also Δ_1, Δ_3 and Δ_5 do not receive positive curvature from Δ_m for $m \in \{2,4,6\}$ in Figure 4.3(i). It follows that if $\hat{\Delta}$ receives positive curvature then it does so across at most half of its edges and so $d(\hat{\Delta}) \geq 7$ implies that $c^*(\hat{\Delta}) \leq 0$ by Lemma 3.5(vi). It remains to study $5 \leq d(\hat{\Delta}) \leq 6$. Checking shows that if $d(\hat{\Delta}) = 5$ then either the label contradicts $cab^{-1} \neq 1$ or $\hat{\Delta}$ receives positive curvature across at most one edge and so $c^*(\hat{\Delta}) \leq 0$. Also if $d(\hat{\Delta}) = 6$ then $\hat{\Delta}$ receives positive curvature across at most two edges and so $c^*(\hat{\Delta}) \leq 0$.

Figure 5.6

(ii) In this case the labels $bda^{-1}c^{\pm1}$ can occur. If H is cyclic then $d = b^2$, $c = b^3$ and there is a sphere by Lemma 3.2(v), so assume that H is non-cyclic. If $|b| \in \{2,3,4,5\}$ then we obtain spheres by Lemma 3.1(b)(i), (v), so assume that $|b| \geq 6$. In Figure 5.1(ii) if $d(\Delta_1) > 4$ and $d(\Delta_2) > 4$ then add $\frac{1}{2}c(\Delta) = \frac{\pi}{2}$ to $c(\Delta_1)$ and $c(\Delta_2)$ as shown. If say $d(\Delta_1) > 4$ and $d(\Delta_2) = 4$ as in Figure 5.5(i) then add $\frac{\pi}{2}$ to $c(\Delta_1)$. This implies that $l(\Delta_2) = bda^{-1}c^{\pm1}$ as shown. This forces $l(\Delta_3) = b^{-1}a\omega$ and so $d(\Delta_3) > 4$, otherwise there is a contradiction to $|b| \geq 6$ so add $\frac{\pi}{2}$ to $c(\Delta_3)$ via Δ_2 as shown. If $d(\Delta_1) = d(\Delta_2) = 4$ then add $\frac{\pi}{2}$ to $c(\Delta_j)$ for $j \in \{3,4\}$ as shown in Figure 5.5(ii). The one exception to the above is when $l(\Delta_1) = bda^{-1}\omega$ and $d(\Delta_1) > 4$. Then $d(\Delta_4) > 4$ and in this situation add the $\frac{\pi}{2}$ from $c(\Delta)$ to $c(\Delta_4)$ via Δ_1 as shown in Figure 5.5(iii). The same applies to Δ_2. In Figure 4.3(i) if $d(\Delta_1) > 4$, $d(\Delta_3) > 4$ and $d(\Delta_5) > 4$ then add $\frac{1}{3}c(\Delta) = \frac{\pi}{6}$ to each of $c(\Delta_1)$, $c(\Delta_2)$ and $c(\Delta_5)$. If say $d(\Delta_1) = 4$, $d(\Delta_3) > 4$ and $d(\Delta_5) > 4$ then $l(\Delta_2) = ba^{-1}\omega$ and $d(\Delta_2) > 4$ otherwise there is a contradiction to $|b| \geq 6$ so add $\frac{\pi}{6}$ to $c(\Delta_2)$, $c(\Delta_3)$ and $c(\Delta_5)$ as shown in Figure 5.6(i). Now suppose that $d(\Delta_1) = 4$ and $d(\Delta_3) = 4$. This implies that $l(\Delta_2) = l(\Delta_4) = ba^{-1}\omega$ as shown in Figure 5.6(ii). So add $\frac{\pi}{6}$ to $c(\Delta_2)$, $c(\Delta_4)$ and $c(\Delta_5)$. If $d(\Delta_1) = d(\Delta_3) = d(\Delta_5) = 4$ then similarly add $\frac{\pi}{6}$ to $c(\Delta_m)$ for $m \in \{2,4,6\}$.

We now see that if $\hat{\Delta}$ receives positive curvature then it receives at most $\frac{\pi}{2}$ across $(bd)^{\pm 1}$ and $(b^{-1}a)^{\pm 1}$; and it receives at most $\frac{\pi}{6}$ across $(ca)^{\pm 1}$ and $(ab^{-1})^{\pm 1}$. Thus there is always a gap immediately preceding c and d^{-1}; and there is a gap immediately after c^{-1} and d. This implies that if there are at least four occurrences of $c^{\pm 1}$ or $d^{\pm 1}$ then $l(\hat{\Delta})$ contains at least four gaps and so $c^*(\hat{\Delta}) \leq 0$ by Lemma 3.6. Suppose that there are at most three occurrences of $c^{\pm 1}$ or $d^{\pm 1}$ in $l(\hat{\Delta})$. Observe that in addition to the four gaps mentioned above the following sublabels yield gaps: $(cb)^{\pm 1}$ and $(ad)^{\pm 1}$ each yields a gap; $(bda^{-1})^{\pm 1}$ yields two gaps (see Figure 5.5(iii)); and $(ca)^{\pm 1}$ and $(ab^{-1})^{\pm 1}$ each yields the equivalent of a two-thirds gap. If $l(\hat{\Delta}) = (b^{-1}a)^{\pm n}$ where $n \geq 1$ then $l(\hat{\Delta})$ obtains at least four gaps since $|b| \geq 6$. If $l(\hat{\Delta}) \in \{d^{\pm 1}(b^{-1}a)^{\pm n}, (ab^{-1})^{\pm n}c^{\pm 1}\}$ then H is cyclic so it can be assumed that $l(\hat{\Delta})$ involves either two or three occurrences of $c^{\pm 1}$ or $d^{\pm 1}$. It follows that if there are three occurrences then $c^*(\hat{\Delta}) \leq 0$; or if exactly two occurrences then either $c^*(\hat{\Delta}) \leq 0$ or
$l(\hat{\Delta}) \in \{d^{\pm 1}b^{-1}ada^{-1}b, d^{\pm 1}b^{-1}ada^{-1}ba^{-1}b, d^{\pm 1}b^{-1}ab^{-1}ada^{-1}b, cba^{-1}bd^{\pm 1}b^{-1}, cbd^{\pm 1}b^{-1}ab^{-1}\}$.
But each of these labels forces H cyclic or $|b| < 6$ or a (**B3**) contradiction, therefore $c^*(\hat{\Delta}) \leq 0$ by Lemma 3.6.

Figure 5.7

(iii) In this case the label $cadb^{-1}$ can occur. First assume that H is non-cyclic. In Figure 5.1(ii) $d(\Delta_1) > 4$ and $d(\Delta_2) > 4$ otherwise there is a contradiction to $|c| = 2$ or $|d| = 3$, so add $\frac{1}{2}c(\Delta) = \frac{\pi}{2}$ to $c(\Delta_1)$ and $c(\Delta_2)$. In Figure 4.3(i) if $d(\Delta_1) > 4$, $d(\Delta_3) > 4$ and $d(\Delta_5) > 4$ add $\frac{1}{3}c(\Delta) = \frac{\pi}{6}$ to each of $c(\Delta_1)$, $c(\Delta_2)$ and $c(\Delta_5)$. If say $d(\Delta_1) = 4$ only then add $\frac{\pi}{4}$ to $c(\Delta_3)$ and $c(\Delta_5)$. Now suppose that $d(\Delta_1) = d(\Delta_3) = 4$. This implies that their label is $cadb^{-1}$ which forces $l(\Delta_2) = cba^{-1}\omega$ as shown in Figure 5.6(iii). So add $\frac{\pi}{4}$ to $c(\Delta_2)$ via Δ_1 and to $c(\Delta_5)$. Finally if $d(\Delta_1) = d(\Delta_3) = d(\Delta_5) = 4$ then in a similar way add $\frac{\pi}{6}$ to $c(\Delta_m)$ for $m \in \{2, 4, 6\}$. Observe that Δ_1 does not receive positive curvature from Δ_3 and Δ_2 does not receive positive curvature from Δ_4 in Figure 5.1(ii). In Figure 4.3(i) Δ_1 does not receive positive curvature from Δ_2. In Figure 5.6(iii) Δ_2 does not receive positive curvature from Δ_3. Since $\hat{\Delta}$ receives $\frac{\pi}{2}$ across the bd edge and $\frac{\pi}{4}$ across the ca, ba^{-1} edges it follows that $\hat{\Delta}$ receives an average of $\frac{\pi}{4}$ across each of its edges, so $d(\hat{\Delta}) \geq 8$ implies that $c^*(\hat{\Delta}) \leq 0$. It remains to study $5 \leq d(\hat{\Delta}) \leq 7$. Checking shows that if $d(\hat{\Delta}) = 5$ then either the label contradicts $|d| = 3$ or H non-cyclic or $\hat{\Delta}$ receives positive curvature across at most one edge and so $c^*(\hat{\Delta}) \leq 0$, except when $l(\hat{\Delta}) = bdda^{-1}c^{-1}$ as in Figure 5.7(i). In this case $\hat{\Delta}$ receives $\frac{\pi}{2}$ from $c(\hat{\Delta}_1)$ and $\frac{\pi}{4}$ from $c(\hat{\Delta}_2)$. If $|b| > 2$ then this implies that $d(\hat{\Delta}_3) > 4$ and so add $\frac{\pi}{4}$ to $c(\hat{\Delta}_3)$ noting that this is a similar edge to the one crossed in Figure 5.6(iii) so there is no change to the above argument and $c^*(\hat{\Delta}) \leq 0$ in this case. Suppose now that $|b| = 2$ and $l(\hat{\Delta}_3) = ab^{-1}ab^{-1}$ as in Figure 5.7(ii). If $d(\hat{\Delta}_4) > 5$ then add $\frac{\pi}{4}$ to $c(\hat{\Delta}_4)$ across the da^{-1} edge. If $d(\hat{\Delta}_4) = 5$ then $l(\hat{\Delta}_4) = da^{-1}c^{-1}bd$ which implies that $l(\hat{\Delta}_5) = ca\omega$ and so if $d(\hat{\Delta}_5) > 5$ then add $\frac{\pi}{4}$ to $c(\hat{\Delta}_5)$ as in Figure 5.7(iii). If $d(\hat{\Delta}_5) \in \{4, 5\}$ then $l(\hat{\Delta}_5) \in \{cadb^{-1}, cad^{-2}b^{-1}\}$ and this forces $l(\hat{\Delta}_6) = c^{-1}ba^{-1}\omega$ and so $d(\hat{\Delta}_6) > 5$ otherwise there is a contradiction to $|c| \neq 1$. So add $\frac{\pi}{4}$ to $c(\hat{\Delta}_6)$ again as shown in Figure 5.7(iii).

Observe that $\hat{\Delta}_4$ in Figure 5.7(ii) can now receive $\frac{\pi}{4}$ from $c(\hat{\Delta})$, however it receives no positive curvature from $\hat{\Delta}_3$ or any other region across the da^{-1} edge. Moreover, it is clear from Figure 5.7(iii) that $\hat{\Delta}_5$ receives only the $\frac{\pi}{4}$ from $\hat{\Delta}_4$ across its ca edge; and $\hat{\Delta}_6$ receives only the $\frac{\pi}{4}$ from $\hat{\Delta}_5$ across its ba^{-1} edge. Finally observe that Figures 5.7(ii)–(iii) do not alter the fact that Δ_1 does not receive positive curvature from Δ_3 and Δ_2 does not receive positive curvature from Δ_4 in Figure 5.1(ii). Therefore the average positive curvature that $\hat{\Delta}$ receives across each edge is still $\frac{\pi}{4}$ and so if $d(\hat{\Delta}) \geq 8$ then $c^*(\hat{\Delta}) \leq 0$. It remains to check $6 \leq d(\hat{\Delta}) \leq 7$ for the sublabels $(bd)^{\pm 1}(\pi/2)$ and $(ca)^{\pm 1}, (ab^{-1})^{\pm 1}, (da^{-1}c^{-1})^{\pm 1}(\pi/4)$. Checking shows that if $d(\hat{\Delta}) = 6$ then the most curvature that $\hat{\Delta}$ can receive is either $2(\frac{\pi}{2})$ or $\frac{\pi}{2} + 2(\frac{\pi}{4})$ or $4(\frac{\pi}{4})$ and so $c^*(\hat{\Delta}) \leq 0$. If $d(\hat{\Delta}) = 7$ then the most curvature received is $3(\frac{\pi}{2})$ or $2(\frac{\pi}{2}) + 2(\frac{\pi}{4})$ or $\frac{\pi}{2} + 4(\frac{\pi}{4})$ or $6(\frac{\pi}{4})$ and $c^*(\hat{\Delta}) \leq 0$ except for $l(\hat{\Delta}) = da^{-1}c^{-1}bda^{-1}b$; but this implies $cd = 1$, a contradiction.

Now let H be cyclic. Then $d = b^4$ and $c = b^3$. Again add $\frac{1}{2}c(\Delta) = \frac{\pi}{2}$ to each of $c(\Delta_1)$, $c(\Delta_2)$ as in Figure 5.1(ii). In Figure 4.3(i) if say $d(\Delta_1) > 5$ then add $c(\Delta) = \frac{\pi}{2}$ to $c(\Delta_1)$ and so it can be assumed that $d(\Delta_i) \leq 5$ for $i \in \{1, 3, 5\}$ in which case $l(\Delta_i) \in \{cadb^{-1}, cad^{-2}b^{-1}\}$. If say $d(\Delta_1) = 4$ then add $c(\Delta) = \frac{\pi}{2}$ to $c(\Delta_6)$ via Δ_1 as shown in Figure 5.7(iv). It can be assumed then that $d(\Delta_i) = 5$ for $i \in \{1, 3, 5\}$ in which case add $\frac{1}{3}c(\Delta) = \frac{\pi}{6}$ to each $c(\hat{\Delta})$ via Δ_i where $i \in \{1, 3, 5\}$ as shown in Figure 5.8(i). If say $\hat{\Delta} = \hat{\Delta}_1$ and $d(\hat{\Delta}_1) = 5$ then repeat the above, that is, add the $\frac{\pi}{6}$ from $c(\Delta)$ across another ca edge and continue in this way until $\frac{\pi}{6}$ is eventually added to a region $\hat{\Delta}_k$ where either $d(\hat{\Delta}_k) > 5$ (and so the process terminates) or $d(\hat{\Delta}_k) = 4$ in which case the $\frac{\pi}{6}$ from $c(\Delta)$ is added to $c(\hat{\Delta}_{k+1})$ as shown in Figure 5.8(ii), where $k = 3$. If $d(\hat{\Delta}_{k+1}) > 5$ then the process terminates (and note that $l(\hat{\Delta}_{k+1}) = d^{-1}b^{-1}c^{-1}w$); otherwise $l(\hat{\Delta}_{k+1}) = d^{-1}b^{-1}c^{-1}ad^{-1}$ and the $\frac{\pi}{6}$ from $c(\Delta)$ is added to $c(\hat{\Delta}_{k+2})$ where $\hat{\Delta}_{k+2}$ is the region shown in Figure 5.8(ii) with $k = 3$. Observe that $l(\hat{\Delta}_{k+2}) = ba^{-1}c^{-1}w$ so $d(\hat{\Delta}_{k+2}) > 5$ and the process terminates. This completes the distribution of curvature that occurs. It follows that if $\hat{\Delta}$ receives positive curvature across an edge e_i say then $\hat{\Delta}$ does not receive any curvature across the adjacent edges e_{i-1}, e_{i+1} except when $\hat{\Delta}$ is given by $\hat{\Delta}_{k+1} = \hat{\Delta}_4$ in Figure 5.8(ii). Therefore if $l(\hat{\Delta})$ does not involve $(cbd)^{\pm 1}$ then Lemma 3.5(vi) applies and $c^*(\hat{\Delta}) \leq 0$ for $d(\hat{\Delta}) \geq 7$; and if $d(\hat{\Delta}) = 6$ then checking for $(bd)^{\pm 1}$, $(ca)^{\pm 1}$ and $(cba^{-1})^{\pm 1}$ shows that $\hat{\Delta}$ receives positive curvature across at most two edges and $c^*(\hat{\Delta}) \leq 0$. Finally if $l(\hat{\Delta}) = cbdw$ then we see from Figure 5.8(ii) that the maximum amount $\hat{\Delta}$ receives is on average $\frac{\pi}{3}$ across $\frac{2}{3}$ of its edges and so if $d(\hat{\Delta}) \geq 8$ then $c^*(\hat{\Delta}) \leq 0$ by Lemma 3.5(vii). Checking shows that if $6 \leq d(\hat{\Delta}) \leq 7$ then $l(\hat{\Delta}) \in \{cbdb^{-1}ab^{-1}, cbda^{-1}bdb^{-1}, cbda^{-1}c^{-1}ba^{-1}, cbda^{-1}cba^{-1}\}$ and so if $d(\hat{\Delta}) = 6, 7$ then $\hat{\Delta}$ receives curvature across at most $2, 3$ edges (respectively) and $c^*(\hat{\Delta}) \leq 0$.

In conclusion $\mathcal{P}$ fails to be aspherical in this case when H is non-cyclic, $bda^{-1}c^{-1} = 1$ and $|b| \in \{2, 3, 4, 5\}$; or when H is cyclic and $bda^{-1}c^{\pm 1} = 1$.

Figure 5.8

(B4) $|c| = 2$, $|d| > 3$, $a^{-1}b \neq 1$, $c^{\pm 1}ab^{-1} = 1$, $d^{\pm 1}b^{-1}a \neq 1$.
If $|d| < \infty$ then there is a sphere by Lemma 3.1(a)(i) and if $|d| = \infty$ then $\mathcal{P}$ is aspherical by Lemma 3.4(ii).

(B5) $|c| = 2$, $|d| > 3$, $a^{-1}b \neq 1$, $c^{\pm 1}ab^{-1} \neq 1$, $db^{-1}a = 1$, $d^{-1}b^{-1}a \neq 1$.
If $d(\Delta) = 2$ then Δ is given by Figure 5.1(ii) in which case add $\frac{1}{2}c(\Delta) = \frac{\pi}{2}$ to each of $c(\Delta_1)$ and $c(\Delta_2)$ as shown; and if $d(\Delta) = 3$ then Δ is given by Figure 4.10(iii) in which case add $c(\Delta) = \frac{\pi}{2}$ to $c(\Delta_5)$ as shown. Assume that H is non-cyclic. If say $d(\Delta_1) = 4$ in Figure 5.1(ii) then $l(\Delta_1) \in \{bd^2a^{-1}, bda^{-1}c^{\pm 1}, bdb^{-1}c^{\pm 1}\}$ which contradicts $|d| > 3$ or H non-cyclic; and if $d(\Delta_5) = 4$ in Figure 4.10(iii) then $l(\Delta_5) \in \{c^{-1}ad^{\pm 1}a^{-1}, c^{-1}ad^{\pm 1}b^{-1}\}$ which contradicts $|d| > 3$ or H non-cyclic. Observe that in Figure 5.1(ii) Δ_1 say does not receive positive curvature from Δ_3 or Δ_4; and in Figure 4.10(iii) Δ_5 does not receive any from Δ_4 or Δ_6. Therefore $c^*(\hat{\Delta}) \leq 0$ for $d(\hat{\Delta}) \geq 7$ by Lemma 3.5(vi) and so it remains to consider $5 \leq d(\hat{\Delta}) \leq 6$. But checking for subwords $(bd)^{\pm 1}, (c^{-1}a)^{\pm 1}$ shows that if $d(\hat{\Delta}) = 5, 6$ then $\hat{\Delta}$ receives positive curvature across $1, 2$ edges respectively except when $l(\hat{\Delta}) = d^2a^{-1}cb$. But H would then be cyclic, so $c^*(\hat{\Delta}) \leq 0$.

Now assume that H is cyclic. If $bda^{-1}c = 1$ then the conditions are T-equivalent to those of Lemma 3.1(a)(ii) and there is a sphere, so assume otherwise. Then $d(\Delta_1) > 4$ in Figure 5.1(ii) and $d(\Delta_5) > 4$ in Figure 4.10(iii). The argument above now applies except when $l(\Delta_1) = bd^2a^{-1}c$ in which case add $\frac{1}{2}c(\Delta) = \frac{\pi}{2}$ to $c(\Delta_3)$ via Δ_1 as shown in Figure 5.8(iii). We see from Figures 4.10(iii), 5.1(ii) and 5.8(iii) that if $\hat{\Delta}$ receives positive curvature then it does so across at most half of its edges and so $c^*(\hat{\Delta}) \leq 0$ for $d(\hat{\Delta}) \geq 7$ by Lemma 3.5(vi), so let $4 \leq d(\hat{\Delta}) \leq 6$. Then either $l(\hat{\Delta}) = bd^2a^{-1}c^{-1}$ and $c^*(\hat{\Delta}) \leq c(\Delta) + \frac{\pi}{2} = 0$; or $d(\hat{\Delta}) = 6$, $\hat{\Delta}$ receives across at most two edges (checking for $(bd)^{\pm 1}, (c^{-1}a)^{\pm 1}, (ad)^{\pm 1}$) and again $c^*(\hat{\Delta}) \leq 0$.

In conclusion $\mathcal{P}$ is aspherical if and only if $bda^{-1}c \neq 1$.

(B6) $|c| = 2$, $|d| = 3$, $a^{-1}b \neq 1$, $c^{\pm 1}ab^{-1} \neq 1$, $db^{-1}a = 1$, $d^{-1}b^{-1}a \neq 1$.
If $d(\Delta) = 2$ then Δ is given by Figure 5.1(ii). If $d(\Delta) = 3$ then Δ is given by Figures 4.3(i) and 4.10(iii). In Figure 5.1(ii) if $d(\Delta_1) > 4$ and $d(\Delta_2) > 4$ then add $\frac{1}{2}c(\Delta) = \frac{\pi}{2}$ to $c(\Delta_1)$ and $c(\Delta_2)$. If $d(\Delta_1) > 4$ and $d(\Delta_2) = 4$ then $l(\Delta_2) \in \{bd^2a^{-1}, bda^{-1}c^{\pm 1}, bdb^{-1}c^{\pm 1}\}$ and so $l(\Delta_2) = bd^2a^{-1}$ otherwise there is a contradiction to $|d| = 3$. Therefore Δ_2 is given by Figure 5.1(iii) forcing $l(\Delta_3) = ca\omega$ and so $d(\Delta_3) > 4$, otherwise there is a contradiction to $|c| = 2$, so add $\frac{\pi}{2}$ to $c(\Delta_1)$ and to $c(\Delta_3)$ via Δ_2. If $d(\Delta_1) = 4$ then add $\frac{1}{2}c(\Delta) = \frac{\pi}{2}$ to $c(\Delta_4)$ via Δ_1. Observe in Figure 4.10(iii) that if $d(\Delta_5) = 4$ then $l(\Delta_5) \in \{c^{-1}ad^{\pm 1}a^{-1}, c^{-1}ad^{\pm 1}b^{-1}\}$ which contradicts $|c| = 2$ so add $c(\Delta) = \frac{\pi}{2}$ to $c(\Delta_5)$ as shown. In Figure 4.3(i) if $d(\Delta_i) = 4$ where $i \in \{1, 3, 5\}$ then $l(\Delta_i) \in \{cad^{\pm 1}a^{-1}, cad^{\pm 1}b^{-1}\}$ which contradicts $|c| = 2$ so add $c(\Delta) = \frac{\pi}{6}$ to $c(\Delta_i)$ as shown. We see from Figures 4.3(i), 4.10(iii) and 5.1(ii)–(iii) that if $\hat{\Delta}$ receives positive curvature then it does so across the edges bd, ca or $c^{-1}a$. It follows that the only word of length 3 that contains no gaps is $a^{-1}c^{\pm 1}a$. Suppose that $l(\hat{\Delta}) = \omega_1 a^{-1}c^{\pm 1}a\omega_2\omega$

where ω_1 and ω_2 have length 2 and ω has length at least 0. Then there is a gap preceding a^{-1} in $\omega_1 a^{-1}$ and after a in $a\omega_2$. Moreover, if ω_1 does not contain a gap then $\omega_1 = bd$ and if ω_2 does not contain a gap then $\omega_2 = d^{-1}b^{-1}$. It follows that if $l(\omega) > 0$ then $l(\hat{\Delta})$ contain at least 4 gaps and $c^*(\hat{\Delta}) \leq 0$. If $l(\omega) = 0$ and $l(\hat{\Delta}) = \omega_1 a^{-1}c^{\pm 1}ad^{-1}b^{-1}$ then $l(\hat{\Delta}) \in \{c^{\pm 1}ba^{-1}c^{\pm 1}ad^{-1}b^{-1}, ad^{\pm 1}a^{-1}c^{\pm 1}ad^{-1}b^{-1}\}$ and there are 4 gaps; if $l(\omega) = 0$ and $l(\hat{\Delta}) = bda^{-1}c^{\pm 1}a\omega_2$ then $l(\hat{\Delta}) \in \{bda^{-1}c^{\pm 1}ad^{\pm 1}a^{-1}, bda^{-1}c^{\pm 1}ab^{-1}c^{\pm 1}\}$ and again there are 4 gaps. If $l(\hat{\Delta}) = a^{-1}c^{\pm 1}a\omega$ and $d(\hat{\Delta}) \leq 6$ then either we obtain a contradiction to $|c| = 2$ or $l(\hat{\Delta}) = a^{-1}c^{\pm 1}ab^{-1}c^{\pm 1}b$ and there are 4 gaps. Now suppose that $l(\hat{\Delta})$ does not contain the subword $a^{-1}c^{\pm 1}a$. Then $\hat{\Delta}$ receives positive curvature across at most half of its edges and so $d(\hat{\Delta}) \geq 7$ implies $c^*(\hat{\Delta}) \leq 0$ by Lemma 3.5(vi). So let $d(\hat{\Delta}) \leq 6$ and $l(\hat{\Delta}) \in \{ca\omega, bd\omega, c^{-1}a\omega\}$. If $d(\hat{\Delta}) < 6$ then checking shows that there is a contradiction to a (**B6**) assumption or $l(\hat{\Delta}) = bda^{-1}ba^{-1}$ and there are four gaps; and if $d(\hat{\Delta}) = 6$ then checking for $c^{-1}a\omega$ shows that either there is a contradiction to a (**B6**) assumption or $\hat{\Delta}$ receives positive curvature across at most two edges and so $c^*(\hat{\Delta}) \leq 0$. Therefore $\mathcal{P}$ is aspherical.

Figure 5.9

(B7) $|c| > 3$, $|d| > 3$, $a^{-1}b = 1$, $c^{\pm 1}ab^{-1} \neq 1$, $d^{\pm 1}b^{-1}a \neq 1$.

If $d(\Delta) = 2$ then Δ is given by Figure 5.9(i). Here $l(\Delta_1) = b^{-1}c\omega$ and $l(\Delta_2) = ad^{-1}\omega$. First assume that H is non-cyclic. This implies that $d(\Delta_1) > 4$, $d(\Delta_2) > 4$, otherwise there is a contradiction to H non-cyclic, $|c| > 3$, or $|d| > 3$. So add $\frac{1}{2}c(\Delta) = \frac{\pi}{2}$ to $c(\Delta_1)$ and $c(\Delta_2)$. Observe that each of Δ_1 and Δ_2 does not receive positive curvature from Δ_j for $j \in \{3, 4\}$. It follows that if $\hat{\Delta}$ receives positive curvature then it does so across at most half of its edges and so $d(\hat{\Delta}) \geq 7$ implies that $c^*(\hat{\Delta}) \leq 0$ by Lemma 3.5(vi). Checking shows that if $d(\hat{\Delta}) = 5$ then $l(\hat{\Delta})$ contradicts $|c| > 3$, $|d| > 3$ or H non-cyclic. Also if $d(\hat{\Delta}) = 6$ then $\hat{\Delta}$ receives positive curvature across at most two edges and so $c^*(\hat{\Delta}) \leq 0$.

Now assume that H is cyclic. If $c = d^{\pm 1}$ there is a sphere by Lemma 3.1(a)(iii) and Lemma 3.2(iii) so assume from now on that $c \neq d^{\pm 1}$. If $c \neq d^2$ and $d \neq c^2$ then $\mathcal{P}$ is aspherical by the above argument. If $c = d^2$ and $d = c^2$ then $c^3 = 1$, a contradiction, so assume that $c = d^2$ and $d \neq c^2$. Then $c^*(\hat{\Delta}) \leq 0$ as above except when $l(\hat{\Delta}) = b^{-1}cad^{-2}$ and $\hat{\Delta}$ is given by Figure 5.9(ii). In this case add $c^*(\hat{\Delta}) = \frac{\pi}{2}$ to $c(\hat{\Delta}_1)$ as shown noting that $\hat{\Delta}_1$ does not receive any curvature from Δ. Therefore if a region receives positive curvature across an edge it is across a $b^{-1}c$ or ad^{-1} or $a^{-1}c^{-1}$ edge; and if across consecutive edges then the sublabel is $(da^{-1}c^{-1})^{\pm 1}$. It follows that if $l(\hat{\Delta})$ does not involve $(da^{-1}c^{-1})^{\pm 1}$ and $d(\hat{\Delta}) \geq 7$ then $c^*(\hat{\Delta}) \leq 0$. Observe that each occurrence of $da^{-1}c^{-1}$ contributes two gaps. It follows that if $l(\hat{\Delta})$ contains at least two occurrences of $(da^{-1}c^{-1})^{\pm 1}$ or if $l(\hat{\Delta})$ contains exactly one occurrence and $d(\hat{\Delta}) \geq 8$ then there are four gaps and $c^*(\hat{\Delta}) \leq 0$. But if $l(\hat{\Delta}) = da^{-1}c^{-1}w$ and $d(\hat{\Delta}) = 7$ then checking possible $l(\hat{\Delta})$ shows that again there are four gaps and $c^*(\hat{\Delta}) \leq 0$. If $5 \leq d(\hat{\Delta}_1) \leq 6$ then either there is a contradiction to one of our assumptions or $c^*(\hat{\Delta}_1) \leq 0$ except when $l(\hat{\Delta}_1) \in \{a^{-1}c^{-2}ad^{-1}, a^{-1}c^{-2}bd^{-1}, a^{-1}c^{-1}ad^2, a^{-1}c^{-1}bd^2, a^{-1}c^{-3}bd\}$. If $d = c^{-2}$ or $d = c^3$ then $|c| = 5$ and there is a sphere by Lemma 3.2(vii). This leaves

$l(\hat{\Delta}_1) \in \{a^{-1}c^{-1}ad^2, a^{-1}c^{-1}bd^2\}$ and these are given by Figure 5.9(iii), (iv). Add $c^*(\hat{\Delta}_1) = \frac{\pi}{2}$ to $c(\hat{\Delta}_2)$ as shown. Repeat the argument for $\hat{\Delta}_2$ noting that as before $\hat{\Delta}_2$ does not receive positive curvature from the corresponding region Δ. This procedure will terminate at a region $\hat{\Delta}_k$, say, such that $\hat{\Delta}_i \neq \hat{\Delta}_j$ for $i \neq j$ where $1 \leq i, j \leq k$ and $c^*(\hat{\Delta}_k) \leq 0$. Repeat this for each copy of the region $\hat{\Delta}_1$ to conclude that $\mathcal{P}$ is aspherical. If $d = c^2$ and $c \neq d^2$ then the argument is the same by symmetry.

In conclusion $\mathcal{P}$ is aspherical in this case except when either $c = d^{\pm 1}$ or $c^5 = 1$ and $c = d^2$ or $d^5 = 1$ and $d = c^2$.

(B8) $|c| = 2$, $|d| > 3$, $a^{-1}b = 1$, $c^{\pm 1}ab^{-1} \neq 1$, $d^{\pm 1}b^{-1}a \neq 1$.
If $d(\Delta) = 2$ then Δ is given by Figures 5.1(ii) and 5.9(i). In Figure 5.9(i) $l(\Delta_1) = b^{-1}c\omega$ and $l(\Delta_2) = ad^{-1}\omega$. This implies that $d(\Delta_1) > 4$,$d(\Delta_2) > 4$, otherwise there is a contradiction to $|d| > 3$ so add $\frac{1}{2}c(\Delta) = \frac{\pi}{2}$ to $c(\Delta_1)$ and $c(\Delta_2)$. In Figure 5.1(ii) $l(\Delta_1) = l(\Delta_2) = bd\omega$. This similarly implies that $d(\Delta_1) > 4$ and $d(\Delta_2) > 4$, so add $\frac{1}{2}c(\Delta) = \frac{\pi}{2}$ to $c(\Delta_1)$ and $c(\Delta_2)$. Observe that in Figure 5.9(i) Δ_1 does not receive positive curvature from Δ_4; and Δ_2 does not receive positive curvature from Δ_4. Observe also that if $\hat{\Delta}$ receives positive curvature then it does so across the edges $b^{-1}c$, ad^{-1} or bd. Thus there is always a gap immediately preceding c^{-1} and a; and there is a gap after c and a^{-1}. This implies that if there are at least four occurrences of $c^{\pm 1}$ then $l(\hat{\Delta})$ contains at least four gaps and so $c^*(\hat{\Delta}) \leq 0$ by Lemma 3.6. We will proceed according to the number of occurrences of $c^{\pm 1}$ in $l(\hat{\Delta})$. If there are no occurrences of $c^{\pm 1}$ then either $l(\hat{\Delta}) = (ab^{-1})^k$ where $|k| \geq 2$ and there are four gaps, or $l(\hat{\Delta}) = d(ab^{-1})^{k_1} \ldots d(ab^{-1})^{k_m}$ where $k_i \in \mathbb{Z}$ $(1 \leq i \leq m)$. But since there is always at least one gap between any two occurrences of $d^{\pm 1}$ it follows that again there are four gaps or $|d| \leq 3$, a contradiction. So $c^*(\hat{\Delta}) \leq 0$ in this case.

Assume first that H is non-cyclic. If there is exactly one occurrence of $c^{\pm 1}$ in $l(\hat{\Delta})$ then H is cyclic so suppose that there are either two or three occurrences of $c^{\pm 1}$. Then either the label contains at least four gaps or it contradicts one of the (**B8**) assumptions or one of the following cases $\hat{\Delta}_i$ $(1 \leq i \leq 9)$ occurs:

(1) $cad^{-1}b^{-1}cad^{-1}b^{-1}$;

(2) $cad^{-1}b^{-1}cbd^{-1}b^{-1}$;

(3) $cad^{-1}b^{-1}c^{-1}ad^{-1}b^{-1}$;

(4) $cad^{-1}b^{-1}c^{-1}bd^{-1}b^{-1}$;

(5) $cad^{-1}b^{-1}cbda^{-1}$;

(6) $cad^{-1}b^{-1}cbdb^{-1}$;

(7) $cad^{-1}b^{-1}c^{-1}bda^{-1}$;

(8) $cad^{-1}b^{-1}c^{-1}bdb^{-1}$;

(9) $(bda^{-1}c^{-1})^3$.

Figure 5.10

If any of (1)–(4) occurs with any of (5)–(8) or with (9) then $|d| = 2$, a contradiction. Also if any of (5)–(8) occurs with (9) then $c = d^3$ and H is cyclic, so assume otherwise.

Consider (1)–(4). These yield the relator $(cd)^2$ and it follows that $cd^k = d^{-k}c$ for $k \in \mathbb{Z}$. Moreover if $|d| < \infty$ then there is a sphere by Lemma 3.1(c)(ii) so it can be assumed that $|d| = \infty$. In case (1) $\hat{\Delta}_1$ is given by Figure 5.10(i) where, given that $c^*(\hat{\Delta}_1) > 0$, it can be assumed that $d(\Delta_1) = d(\Delta_2) = 2$ and at least one of $d(\Delta_3)$, $d(\Delta_4)$ equals 2. Add $\frac{\pi}{2}$ from $c(\hat{\Delta}_1)$ to $c(\hat{\Delta}_{10})$ as shown in Figure 5.10(i); and if $d(\Delta_3) = d(\Delta_4) = 2$ add a further $\frac{\pi}{2}$ of $c(\hat{\Delta}_1)$ to $c(\hat{\Delta}_{12})$ as shown. In cases (2)–(4) $c^*(\hat{\Delta}) \leq \frac{\pi}{2}$ where $\hat{\Delta} \in \{\hat{\Delta}_2, \hat{\Delta}_3, \hat{\Delta}_4\}$ and $\frac{\pi}{2}$ is added from $c(\hat{\Delta})$ to $c(\hat{\Delta}_{10})$ as shown in Figure 5.10(ii). Observe that $x \neq b$ in Figure 5.10(i), (ii) for otherwise c^2 would be a proper sublabel, and so $x \in \{a, d\}$. If $x = a$ then the sublabel ad yields a gap so let $x = d$. Then either dd yields a gap or $\hat{\Delta}_{11} \in \{\hat{\Delta}_i \colon 1 \leq i \leq 4\}$ and $\frac{\pi}{2}$ is added to $c(\hat{\Delta}_{10})$ from $c(\hat{\Delta}_{11})$. Continuing this way, since $|d| = \infty$, eventually we get a sublabel ad or dd which contributes a gap. Consider $l(\hat{\Delta}_{10})$. If it contains an odd number of occurrences of c then $cd^k = d^{-k}c$ implies that $c \in \langle d \rangle$ and H is cyclic. This leaves the case when there are exactly two occurrences of c and $cd^{\alpha_1}cd^{\alpha_2} = 1$ for $\alpha_1, \alpha_2 \in \mathbb{Z}\backslash\{0\}$. If $|\alpha_1|, |\alpha_2| > 1$ then there are four gaps and $c^*(\hat{\Delta}_{10}) \leq 0$; and if $|\alpha_1| > 1$, $|\alpha_2| = 1$ this implies $|d| < \infty$, a contradiction.

Consider (5)–(8). These yield the relator $cdcd^{-1}$ and H is Abelian. Observe that $|d| = 4$ yields (**E**), so assume otherwise. In each case add $c^*(\hat{\Delta}) = \frac{\pi}{2}$ to $c(\hat{\Delta}_{13})$ as shown in Figure 5.10(iii)–(vi). Observe that $\hat{\Delta}_{13}$ receives no curvature from Δ_1 or Δ_2; that $l(\hat{\Delta}_{13}) = adw$ implies $d(\hat{\Delta}_{13}) > 4$ otherwise there is a contradiction to $|d| > 3$; and there is still a gap between each pair of occurrences of d. If $l(\hat{\Delta}_{13})$ contains an odd number of occurrences of c then H is cyclic so it can be assumed that $l(\hat{\Delta}_{13})$ yields the relator $cd^{\beta_1}cd^{\beta_2}$. If $|\beta_1| > 1$ and $|\beta_2| > 1$ then there are four gaps and if $(\beta_1, \beta_2) \in \{(2, 1), (2, -1), (1, 1)\}$ then $|d| \leq 3$, so this leaves the case $\beta_1 = 1$, $\beta_2 = -1$. Again there are four gaps except when $l(\hat{\Delta}_{13}) = adb^{-1}cad^{-1}b^{-1}c$ and this is shown in Figure 5.10(vii): add $c^*(\hat{\Delta}_{13}) = \frac{\pi}{2}$ to $c(\hat{\Delta}_{14})$ and observe that $\hat{\Delta}_{14}$ does not receive positive curvature from Δ. Consider $l(\hat{\Delta}_{14}) = bddw$. If there are at least four occurrences of c then $c^*(\hat{\Delta}_4) \leq 0$; and if there is an odd number of occurrences then H is cyclic. Suppose firstly that there are no occurrences of c in $l(\hat{\Delta}_{14})$. Since $|d| \geq 5$, if there is one occurrence of b then $l(\hat{\Delta}_4) = a^{-1}bd^k$ $(k \geq 5)$ and there are four gaps; and since each $(a^{-1}b)^{\pm 1}$ yields a gap and each $(bd^l)^{\pm 1}$ $(l \geq 2)$ yields a gap it follows that if there are at least two occurrences of b then again $c^*(\hat{\Delta}_{14}) \leq 0$. Suppose finally that there are two occurrences of c and so $cd^{\beta_1}cd^{\beta_2} = 1$ where $\beta_1 \geq 2$ and $|\beta_2| \geq 0$. If $|\beta_2| > 1$ then there are four gaps; and if $|\beta_2| = 1$ then $\beta_1 \geq 4$, otherwise there is a contradiction to $|d| > 4$, and again there are four gaps, so $c^*(\hat{\Delta}_{14}) \leq 0$.

Finally consider case (9). In this case $\hat{\Delta}_9$ is given by Figure 5.10(viii). Suppose that $c^*(\hat{\Delta}_9) > 0$. Then it can be assumed that $d(\Delta_i) = 2$ for $1 \leq i \leq 6$ and $c^*(\hat{\Delta}) = \frac{\pi}{2}$ so add

$\frac{1}{3}c^*(\hat{\Delta}) = \frac{\pi}{6}$ to $c(\hat{\Delta}_l)$ for $l \in \{15, 16, 17\}$. In this case if $|d| \in \{4, 5\}$ then we obtain a sphere by Lemma 3.1(c)(iii). Now if $|d| \geq 6$ then as shown in Figure 5.10(viii) $l(\Delta_l) = d^{-2}\omega$ and d^{-2} will contribute two-thirds of a gap. If there are now at least two occurrences of c then either $|d| < 6$ or H is cyclic, a contradiction, or there are four gaps; if there is exactly one occurrence of c then this contradicts H non-cyclic; and if there are no occurrences of c then $l(\Delta_l) = d^{k_1}(b^{-1}a)^{m_1} \ldots d^{k_n}(b^{-1}a)^{m_n}$ where $m_i \in \mathbb{Z}, k_i \geq 1$. Since $k_1 + \ldots + k_n \geq 6$ it follows that there are at least four gaps and $c^*(\hat{\Delta}_l) \leq 0$.

Now let H be cyclic. If $c = d^2$ or $c = d^3$ then there is a sphere by T-equivalence and Lemma 3.2(vii), (viii); and $c = d^4$ is (**E4**), so assume otherwise. In particular, $|d| > 4$. We follow the same argument as above and so if $l(\hat{\Delta})$ contains no occurrences or at least four occurrences of c then, as before, $c^*(\hat{\Delta}) \leq 0$; and if $l(\hat{\Delta})$ contains an odd number of occurrences of c then $c = d^k$ for some $k \geq 4$ which implies there are at least four gaps and $c^*(\hat{\Delta}) \leq 0$. Suppose then that $l(\hat{\Delta})$ involves c exactly twice. Subcases (1)–(4) imply $d^2 = 1$ and (9) implies $c = d^3$, a contradiction. This leaves subcases (5)–(8).

Add $c^*(\hat{\Delta}) = \frac{\pi}{2}$ to $c(\hat{\Delta}_{13})$ as in Figure 5.10(iii)-(vi). Since there is still a gap between each pair of occurrences of d it follows from the above paragraph and the previous argument that $c^*(\hat{\Delta}_{13}) \leq 0$ except when $l(\hat{\Delta}_{13}) = adb^{-1}cad^{-1}b^{-1}c$. Again add $c^*(\hat{\Delta}_{13}) = \frac{\pi}{2}$ to $c(\hat{\Delta}_{14})$ as shown in Figure 5.10(vii). If $l(\hat{\Delta}_{14}) = bddw$ involves at least three occurrences of c then, since $\hat{\Delta}_4$ does not receive positive curvature from Δ in Figure 5.10(vii), there are at least four gaps and $c^*(\hat{\Delta}_4) \leq 0$. Otherwise checking the possible labels for $l(\hat{\Delta}_4) = bddw$ shows that there are four gaps or a contradiction to $|d| > 4$ or $c \notin \{d^3, d^4\}$.

In conclusion $\mathcal{P}$ is aspherical except when $|cd| = 2$, $|d| < \infty$ or when $|cd| = 3$, $|d| \in \{4, 5\}$ or when H is cyclic and $c = d^2$ or d^3.

Figure 5.11

(B9) $|c| = 3$, $|d| > 3$, $a^{-1}b = 1$, $c^{\pm 1}ab^{-1} \neq 1$, $d^{\pm 1}b^{-1}a \neq 1$.
If $d(\Delta) = 2$ then Δ is given by Figure 5.9(i). If $d(\Delta) = 3$ then Δ is given by Figure 4.1(ii). In Figure 5.9(i) $l(\Delta_1) = b^{-1}c\omega$ and $l(\Delta_2) = ad^{-1}\omega$ which implies that $d(\Delta_1) > 4$ and $d(\Delta_2) > 4$ otherwise there is a contradiction to $|d| > 3$ or $|c| = 3$, so add $\frac{1}{2}c(\Delta) = \frac{\pi}{2}$ to $c(\Delta_1)$ and $c(\Delta_2)$. In Figure 4.1(ii) $l(\Delta_1) = l(\Delta_3) = l(\Delta_5) = bd\omega$ which implies that $d(\Delta_i) > 4$ where $i \in \{1, 3, 5\}$ otherwise there is a contradiction to $|d| > 3$ so add $\frac{1}{3}c(\Delta) = \frac{\pi}{6}$ to $c(\Delta_i)$. Observe that if $\hat{\Delta}$ receives positive curvature then it receives $\frac{\pi}{2}$ across the edges $b^{-1}c$, ad^{-1} or receives $\frac{\pi}{6}$ across the edge bd. Thus there is always a gap immediately preceding c^{-1} and a; and there is a gap after c and a^{-1}. Also there is a two thirds of a gap across the edge bd. This implies that if there are at least four occurrences of $c^{\pm 1}$ then $l(\hat{\Delta})$ contains at least four gaps and so $c^*(\hat{\Delta}) \leq 0$ by Lemma 3.6. We will proceed according to the number of occurrences of $c^{\pm 1}$ in $l(\hat{\Delta})$. If there are no occurrences of $c^{\pm 1}$ then either $l(\hat{\Delta}) = (ab^{-1})^k$ where $|k| \geq 3$ and there are four gaps, or $l(\hat{\Delta}) = d(ab^{-1})^{k_1} \ldots d(ab^{-1})^{k_m}$ where $k_i \in \mathbb{Z}$ $(1 \leq i \leq m)$. But since there is always at least one gap between any two

occurrences of $d^{\pm 1}$ it follows that again there are four gaps or $|d| \leq 3$, a contradiction. So $c^*(\hat{\Delta}) \leq 0$ in this case. On the other hand if there are between one and three occurrences of $c^{\pm 1}$ inclusive and $c^*(\hat{\Delta}) > 0$ then either there is a contradiction to one of the (**B9**) assumptions or $c \in \{d^{\pm 2}, d^3\}$ or $l(\hat{\Delta}) = cad^{-1}b^{-1}cad^{-1}b^{-1}$. If $c = d^{\pm 2}$ then (**E2**) or (**E3**) occurs, so assume otherwise.

First suppose that $l(\hat{\Delta}) = cad^{-1}b^{-1}cad^{-1}b^{-1}$ (in particular, H is non-cyclic) as in Figure 5.11(i). If $c^*(\hat{\Delta}) > 0$ then it can be assumed that $d(\Delta_i) = 2$ for $1 \leq i \leq 4$ and $c^*(\hat{\Delta}) = \frac{\pi}{3}$, so add $\frac{1}{2}c^*(\hat{\Delta}) = \frac{\pi}{6}$ to $c(\hat{\Delta}_j)$ for $j \in \{1, 2\}$. Therefore $d^{\pm 2}$ contributes a two-thirds gap. If $|d| \in \{4, 5\}$ then we obtain a sphere by Lemma 3.1(c)(iv). Assume now that $|d| \geq 6$ and without any loss of generality assume that $\hat{\Delta}_l \neq \hat{\Delta}$ of Figure 5.11(i). If $l(\hat{\Delta}_l)$ has at least two occurrences of c then either there is a contradiction to H non-cyclic or one of the (**B9**) assumptions or there are four gaps and $c^*(\hat{\Delta}_l) \leq 0$; if there is exactly one occurrence of c then this contradicts H non-cyclic; and if there are no occurrences of c then $l(\hat{\Delta}_l) = d^{k_1}(b^{-1}a)^{m_1}...d^{k_n}(b^{-1}a)^{m_n}$ where $m_i \in \mathbb{Z}$, $k_i \geq 1$. Since $k_1 + \ldots + k_n \geq 6$ it follows that there are at least four gaps and $c^*(\hat{\Delta}_l) \leq 0$.

Now let $c = d^3$. It follows from the above argument and checking possible vertex labels that if $\hat{\Delta}_1$ receives positive curvature then $c^*(\hat{\Delta}_1) \leq 0$ except when $l(\hat{\Delta}_1) = c^{-1}bd^3a^{-1}$ and $c^*(\hat{\Delta}) = \frac{\pi}{6}$. Add $c^*(\hat{\Delta}_1)$ to $c(\hat{\Delta}_2)$ as shown in Figure 5.11(ii) (where we note that $\hat{\Delta}_2$ receives no curvature from Δ). Checking $\{d^{-1}, a, b\}d^{-2}w$ with the understanding that there is a gap preceding the d^{-2} and each d^{-2} contributes a two-thirds gap it follows that either there is a contradiction or $c^*(\hat{\Delta}_2) \leq 0$ except when $l(\hat{\Delta}_2) = l(\hat{\Delta}_1)^{-1} = cad^{-3}b^{-1}$. If $\hat{\Delta}_2$ is given by Figure 5.11(iii) then $c^*(\hat{\Delta}_2) \leq c(\hat{\Delta}_2) + \frac{\pi}{2} + 2(\frac{\pi}{6}) < 0$, so let $\hat{\Delta}_2$ be given by Figure 5.11(iv). Observe that the neighbour Δ_1 has label $c^{-2}bw$ and no longer has degree 3 and that if either $d(\Delta_2) > 2$ or $d(\Delta_3) > 2$ then $c^*(\hat{\Delta}) < 0$ and so Figure 5.11(iv) shows the only possibility for $c^*(\hat{\Delta}_2) > 0$. Now repeat the argument for $\hat{\Delta}_2$ and $\hat{\Delta}_3$. The conditions on the neighbouring regions Δ_1, Δ_2 and Δ_3 ensure that this procedure will terminate at a region $\hat{\Delta}_k$, say, such that $\hat{\Delta}_i \neq \hat{\Delta}_j$ for $i \neq j$ where $1 \leq i, j \leq k$ and $c^*(\hat{\Delta}_k) \leq 0$. Repeat this for each copy of the region $\hat{\Delta}_1$ to conclude $\mathcal{P}$ is aspherical.

In conclusion $\mathcal{P}$ is aspherical except when H is non-cyclic, $|c^{-1}d| = 2$ and $|d| \in \{4, 5\}$.

(B10) $|c| = 2$, $|d| = 2$, $a^{-1}b = 1$, $c^{\pm 1}ab^{-1} \neq 1$, $d^{\pm 1}b^{-1}a \neq 1$.
If $|cd| < \infty$ then we obtain a sphere by Lemma 3.1(c)(i), so assume otherwise (in particular, H is non-cyclic). Observe that if a label involves c and is not $c^{\pm 2}$ then it yields the relation $(cd)^k = 1$ and so it follows that if $c(\Delta) > 0$ then Δ is given by Figure 5.1(ii) in which $l(\Delta_1) = l(\Delta_2) = bdw$. Add $\frac{1}{2}c(\Delta) = \frac{\pi}{2}$ to $c(\Delta_1)$ and $c(\Delta_2)$ as shown. This completes the distribution of curvature. Consider Δ_1 of Figure 5.1(ii). Since positive curvature only crosses the edge $(bd)^{\pm 1}$ it follows that if $l(\Delta_1)$ involves at least four occurrences of d then $c^*(\Delta_1) \leq 0$. This leaves the case when $l(\Delta_1)$ involves exactly two occurrences of d. But d^2 cannot be a sublabel since there would then be a region with label caw, and so $l(\Delta_1) = bd\{a^{-1}b, b^{-1}a\}da^{-1}$ has four gaps and $c^*(\Delta_1) \leq 0$. In conclusion $\mathcal{P}$ is aspherical in this case if and only if $|cd| = \infty$.

Figure 5.12

(B11) $|c| = 2$, $|d| = 3$, $a^{-1}b = 1$, $c^{\pm 1}ab^{-1} \neq 1$, $d^{\pm 1}b^{-1}a \neq 1$.
If H is cyclic then there is a sphere by Lemma 3.2(vi), so assume otherwise. If $d(\Delta) = 2$ then Δ is given by Figures 5.1(ii) and 5.9(i); and if $d(\Delta) = 3$ then Δ is given by Figure 4.3(i). In Figure 5.9(i) $l(\Delta_1) = b^{-1}c\omega$ and $l(\Delta_2) = ad^{-1}\omega$. This implies that $d(\Delta_1) > 4$ and $d(\Delta_2) > 4$ otherwise there is a contradiction to $|d| = 3$ so add $\frac{1}{2}c(\Delta) = \frac{\pi}{2}$ to $c(\Delta_1)$ and $c(\Delta_2)$. In Figure 5.1(ii) $l(\Delta_1) = l(\Delta_2) = bd\omega$. This implies that $d(\Delta_1) > 4$ and $d(\Delta_2) > 4$ otherwise there is a contradiction to $|d| = 3$ so add $\frac{1}{2}c(\Delta) = \frac{\pi}{2}$ to $c(\Delta_1)$ and $c(\Delta_2)$. In Figure 4.3(i) $l(\Delta_1) = l(\Delta_3) = l(\Delta_5) = ca\omega$. This implies that $d(\Delta_1) > 4$, $d(\Delta_3) > 4$ and $d(\Delta_5) > 4$ otherwise there is a contradiction to $|d| = 3$ so add $\frac{1}{3}c(\Delta) = \frac{\pi}{6}$ to $c(\Delta_1)$, $c(\Delta_3)$ and $c(\Delta_5)$.

Observe that if $\hat{\Delta}$ receives positive curvature then it does so across the edges $b^{-1}c$, ad^{-1}, bd, ca and there is a two-thirds gap in ca. If there is only one occurrence of c or d then this contradicts H non-cyclic. Also if $|cd| \in \{2, 3, 4, 5\}$ then we obtain a sphere by Lemma 3.1(c)(ii),(v). So assume that $|cd| > 5$. If there are between two and five occurrences of c inclusive then the resulting relator contradicts H non-cyclic or $|cd| > 5$ except for $(cd)^2(cd^{-1})^2$ and $(cdcd^{-1})^2$. But each of these yields four gaps. Since at least six occurrences of c yields four gaps, this leaves the case when c does not occur and this implies that there is at least one gap between any two occurrences of d. It follows that there can only be exactly three occurrences of d. Checking shows there are four gaps except when the following subcases occur:

(1) $l(\hat{\Delta}) = bddda^{-1}$;

(2) $l(\hat{\Delta}) = bd^2a^{-1}bda^{-1}$;

(3) $l(\hat{\Delta}) = bda^{-1}bda^{-1}bda^{-1}$.

Consider (1), this case is given by Figure 5.12(i). If $d(\Delta_2) > 2$ then $c^*(\hat{\Delta}) \leq 0$. Assume otherwise and add $c^*(\hat{\Delta}) = \frac{\pi}{2}$ to $c(\hat{\Delta}_1)$ as shown. Observe that $l(\hat{\Delta}_1) = cax\omega$ and $x \in \{b^{-1}, d, d^{-1}\}$. If $x \in \{b^{-1}, d\}$ then $\hat{\Delta}_1$ cannot receive positive curvature across the edges ab^{-1} or ad. Suppose that $x = d^{-1}$. Then $\hat{\Delta}_1$ cannot receive positive curvature across the edge ad^{-1} otherwise $l(\Delta_1) \neq c^2$, a contradiction. So there will be a gap between c and the next occurrence of d and the previous argument does not change. It follows that $c^*(\hat{\Delta}_1) \leq 0$.

Consider (2), this case is given by Figure 5.12(ii). If $d(\Delta_2) > 2$ or $d(\Delta_4) > 2$ then $c^*(\hat{\Delta}) \leq 0$. Assume otherwise and add $c^*(\hat{\Delta}) = \frac{\pi}{2}$ to $c(\hat{\Delta}_2)$ as shown. Observe that $l(\hat{\Delta}_2) = cax\omega$ and the same argument as for (1) can be applied to deduce that $c^*(\hat{\Delta}_2) \leq 0$.

Consider (3), this case is given by Figure 5.12(iii). If $d(\Delta_2) > 2$ or $d(\Delta_4) > 2$ or $d(\Delta_6) > 2$ then $c^*(\hat{\Delta}) \leq 0$. Assume otherwise and add $\frac{1}{3}c^*(\hat{\Delta}) = \frac{\pi}{6}$ to $c(\hat{\Delta}_i)$ for $i \in \{3, 4, 5\}$ as shown.

Consider $l(\hat{\Delta}_3) = ad^{-1}x\omega$ where $x \in \{b^{-1}, a^{-1}, d^{-1}\}$. If $x = b^{-1}$ then $l(\hat{\Delta}_6) = c^{-2}bd\omega$, a contradiction. If $x = a^{-1}$ then exactly as above $l(\hat{\Delta}_3)$ either contains four gaps or there is a contradiction to either H non-cyclic, $|cd| > 5$ or one of the (**B11**) assumptions and so $c^*(\hat{\Delta}_3) \leq 0$. Finally let $x = d^{-1}$. If $l(\hat{\Delta}_3) \notin \{ad^{-3}b^{-1}, ad^{-2}b^{-1}ad^{-1}b^{-1}\}$ then again as above $c^*(\hat{\Delta}_3) \leq 0$ so assume otherwise. But if $l(\hat{\Delta}_3) = ad^{-3}b^{-1}$ then we are back in subcase (1) and if $l(\hat{\Delta}_3) = ad^{-2}b^{-1}ad^{-1}b^{-1}$ then we are back in subcase (2), the only difference being that $\hat{\Delta}$ in Figures 5.12(i), (ii) this time receives $\pi/6$ from Δ_2 rather than $\pi/2$.

In conclusion $\mathcal{P}$ is aspherical except when H is non-cyclic and $|cd| \in \{2, 3, 4, 5\}$ or when H is cyclic.

(B12) $|c| = 3$, $|d| = 3$, $a^{-1}b = 1$, $c^{\pm 1}ab^{-1} \neq 1$, $d^{\pm 1}b^{-1}a \neq 1$.
If H is cyclic then $c = d^{\pm 1}$ and so there is a sphere by Lemma 3.1(a)(iii) or Lemma 3.2(iii), so assume otherwise. If $d(\Delta) = 2$ then Δ is given by Figure 5.9(i); and if $d(\Delta) = 3$ then Δ is given by Figures 4.1(ii) and 4.3(i). In Figure 5.9(i) $l(\Delta_1) = b^{-1}c\omega$ and $l(\Delta_2) = ad^{-1}\omega$. This implies that $d(\Delta_1) > 4$ $d(\Delta_2) > 4$, otherwise there is a contradiction to H non-cyclic or $|c| = 3$ or $|d| = 3$ so add $\frac{1}{2}c(\Delta) = \frac{\pi}{2}$ to $c(\Delta_1)$ and $c(\Delta_2)$. In Figure 4.1(ii) $l(\Delta_1) = l(\Delta_3) = l(\Delta_5) = bd\omega$. This implies that $d(\Delta_1) > 4$, $d(\Delta_3) > 4$ and $d(\Delta_5) > 4$ otherwise there is a contradiction to H non-cyclic or $|c| = 3$ or $|d| = 3$ so add $\frac{1}{3}c(\Delta) = \frac{\pi}{6}$ to $c(\Delta_1)$, $c(\Delta_3)$ and $c(\Delta_5)$. In Figure 4.3(i) $l(\Delta_1) = l(\Delta_3) = l(\Delta_5) = ca\omega$. This implies that $d(\Delta_1) > 4$, $d(\Delta_3) > 4$ and $d(\Delta_5) > 4$ otherwise there is a contradiction to H non-cyclic or $|c| = 3$ or $|d| = 3$ so add $\frac{1}{3}c(\Delta) = \frac{\pi}{6}$ to $c(\Delta_1)$, $c(\Delta_3)$ and $c(\Delta_5)$.

Observe that if $\hat{\Delta}$ receives positive curvature then it does so across the edges $b^{-1}c$, ad^{-1}, bd, ca and there is a two-thirds gap in bd and ca. If there is only one occurrence of c or d in $l(\hat{\Delta})$ then this contradicts H non-cyclic so assume otherwise. Also if $|c^{-1}d| = 2$ then we obtain a sphere as by Lemma 3.1(c)(iv). So assume that $|c^{-1}d| > 2$. Checking now shows that there is at least one gap between any two occurrences of c or between any two occurrences of d; and there is at least a two-thirds gap between any two occurrences of c and d. It follows that if there are at least four occurrences of c or of d then $c^*(\hat{\Delta}) \leq 0$. Suppose there are three occurrences of c. If there are three occurrences of d then there are at least $6.\frac{2}{3} = 4$ gaps; and if there are two occurrences of d then there are four gaps or $l(\hat{\Delta})$ yields the relator $c^2d^{-1}cd^{-1}$ which forces H to be cyclic. Suppose there are two occurrences of c. If there are three occurrences of d then there are four gaps or $l(\hat{\Delta})$ yields the relator $c^{-1}dc^{-1}d^2$ which forces H to be cyclic; and if there are two occurrences of d then $l(\hat{\Delta})$ forces $|c^{-1}d| = 2$ or H cyclic or is one of $c\{a,b\}d\{a^{-1},b\}c\{a,b\}d\{a^{-1},b^{-1}\}$, $c\{a,b\}d\{a^{-1},b^{-1}\}c^{-1}\{a,b\}d^{-1}\{a^{-1},b^{-1}\}$ and there are four gaps. This leaves the case when c does not occur. Checking $l(\hat{\Delta})$ shows that there is a contradiction to $|d| = 3$ or there are at least four gaps except when $l(\hat{\Delta}) = bddda^{-1}$ as shown in Figure 5.12(iv). In this case add $c^*(\hat{\Delta}) = \frac{\pi}{6}$ to $c(\hat{\Delta}_1)$ as shown. Since $\frac{\pi}{6}$ is distributed across ca it follows that $c^*(\hat{\Delta}_1) \leq 0$ as above.

In conclusion $\mathcal{P}$ is aspherical except when H is cyclic or when H is non-cyclic and $|cd| = 2$.

It follows from (**B1**)–(**B12**) that either $\mathcal{P}$ is aspherical or modulo T-equivalence one of the conditions of Theorem 1.1 (i)–(iii) or Theorem 1.2 (i), (ii), (iv)–(x) is satisfied and so Theorems 1.1 and 1.2 are proved for Case B.

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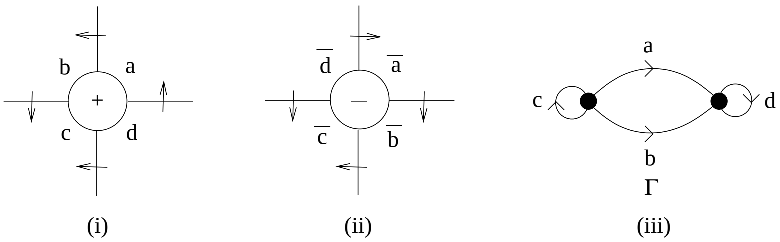

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Figure 2.1

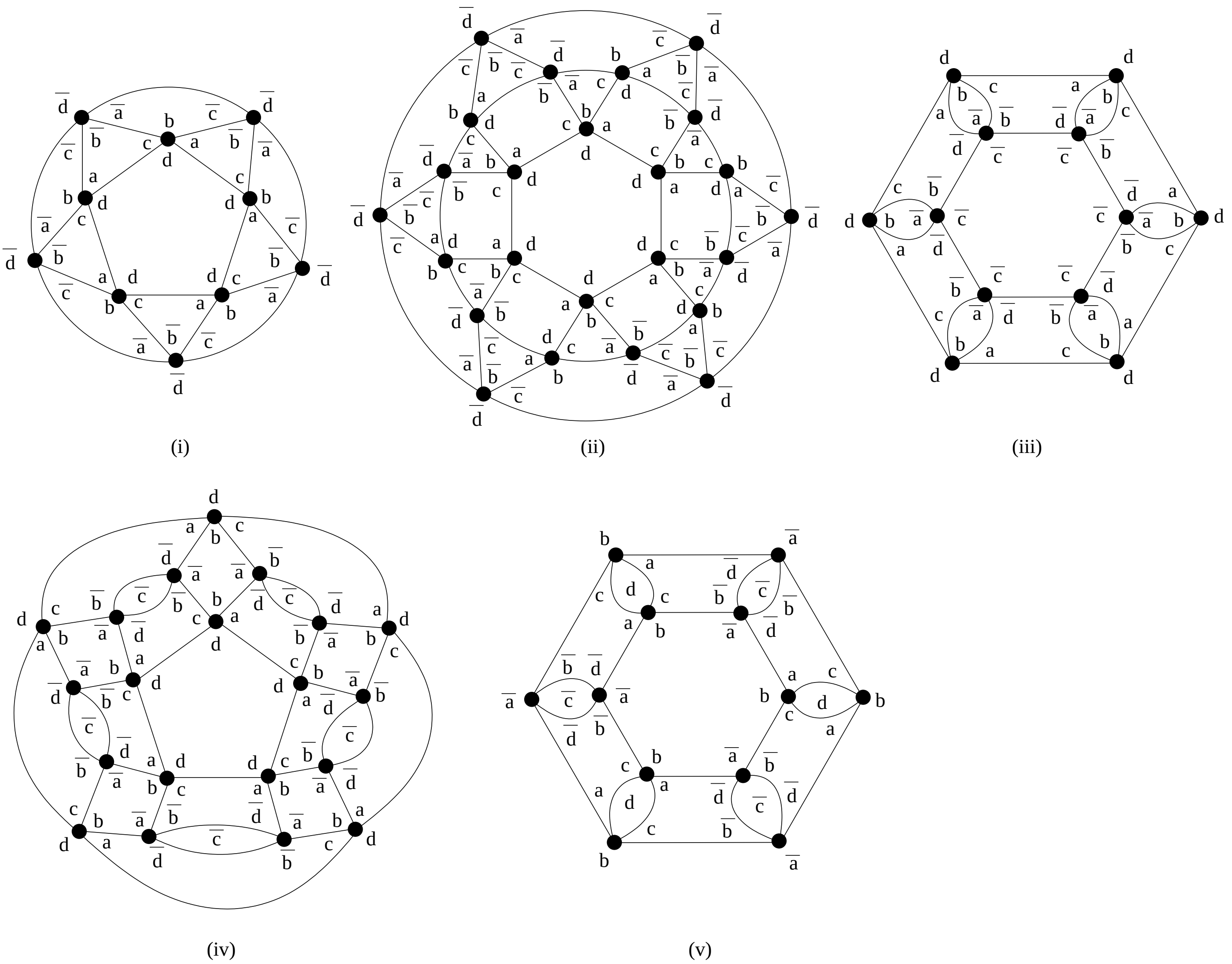
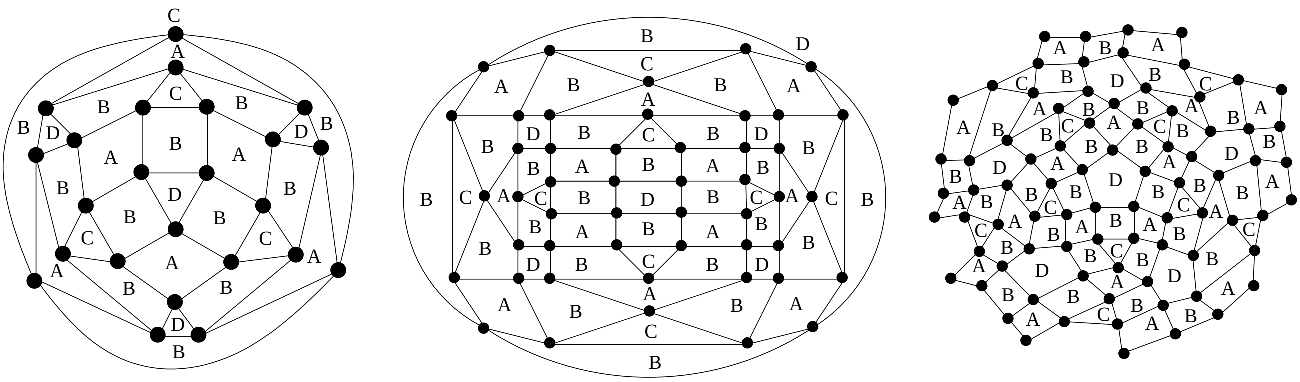
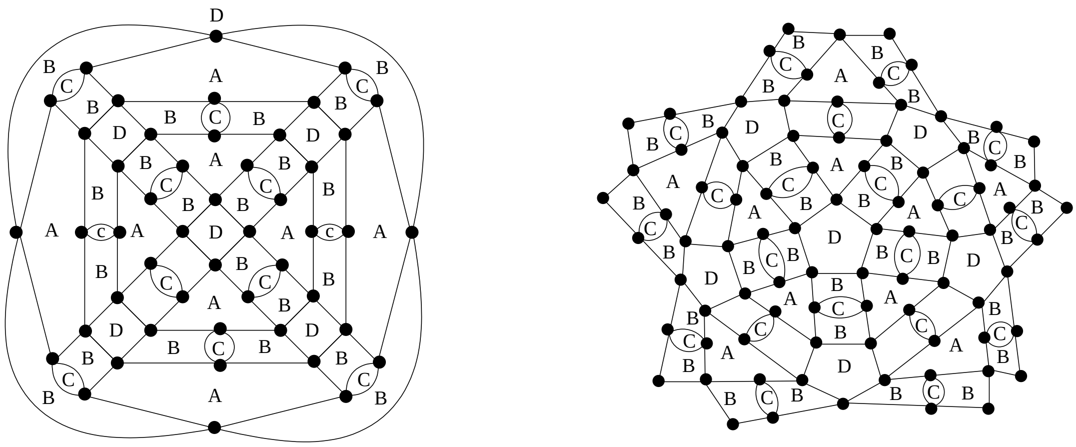


Figure 3.1



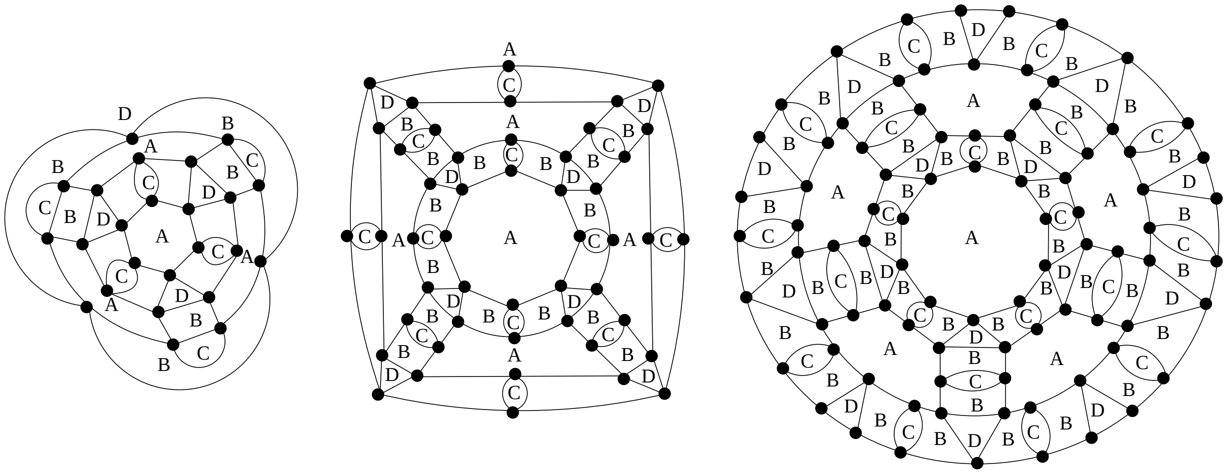
(i) (ii) (iii)

Figure 3.2



(i) (ii)

Figure 3.3



(i) (ii) (iii)

Figure 3.4

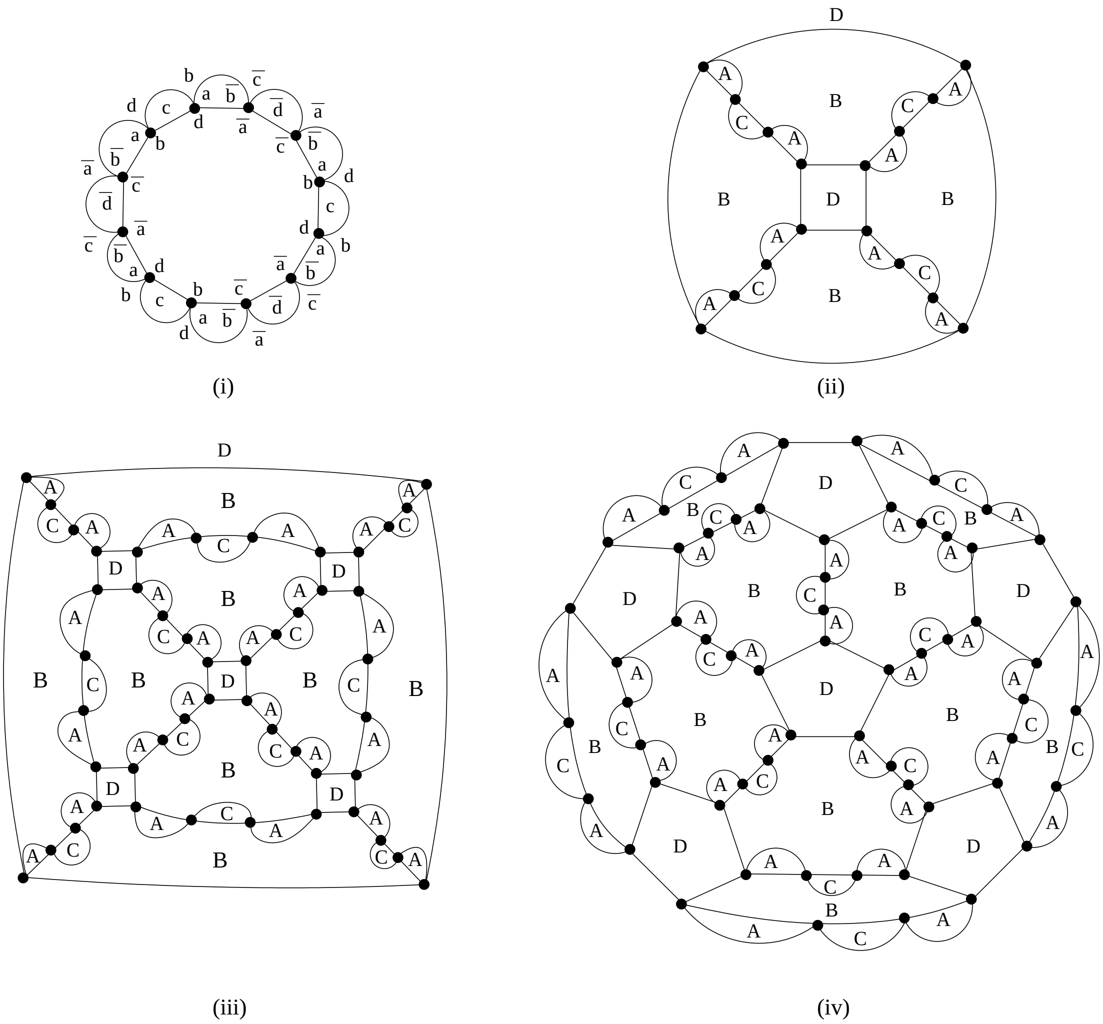
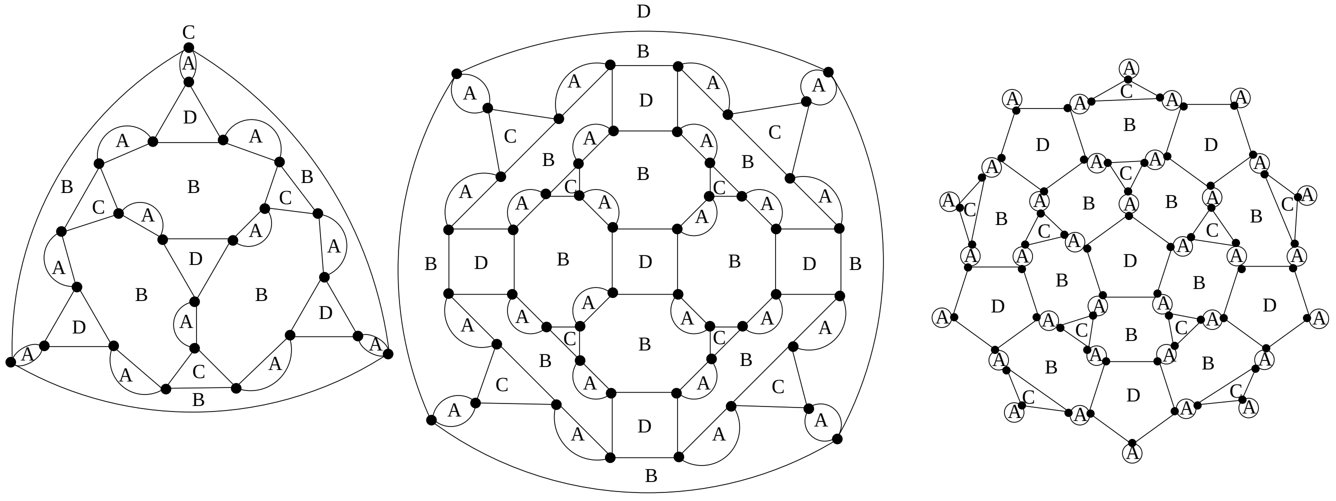


Figure 3.5



(i) (ii) (iii)

Figure 3.6

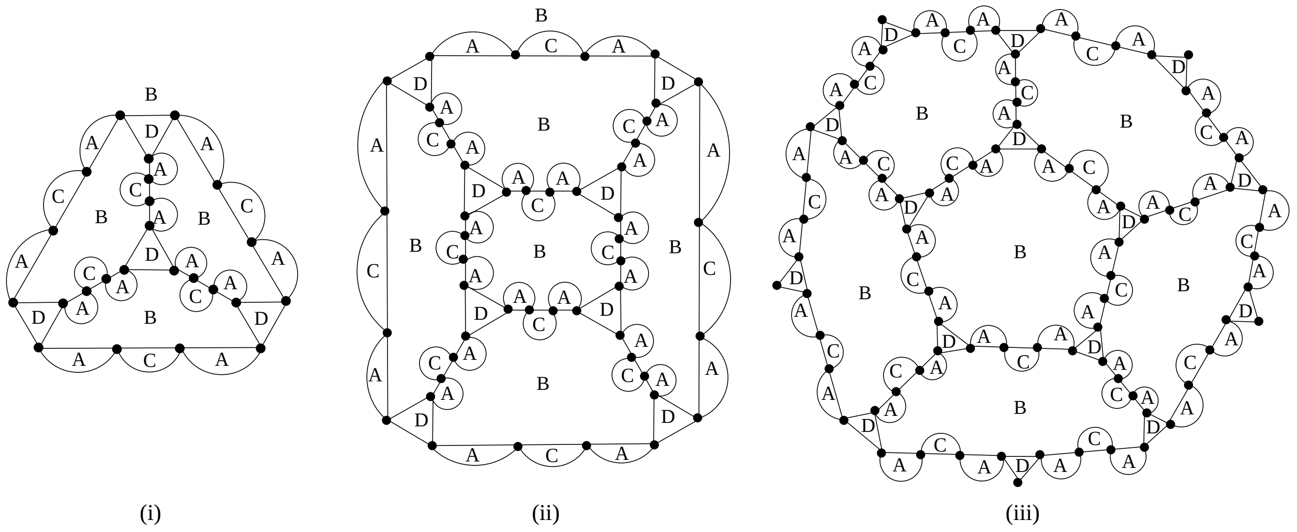
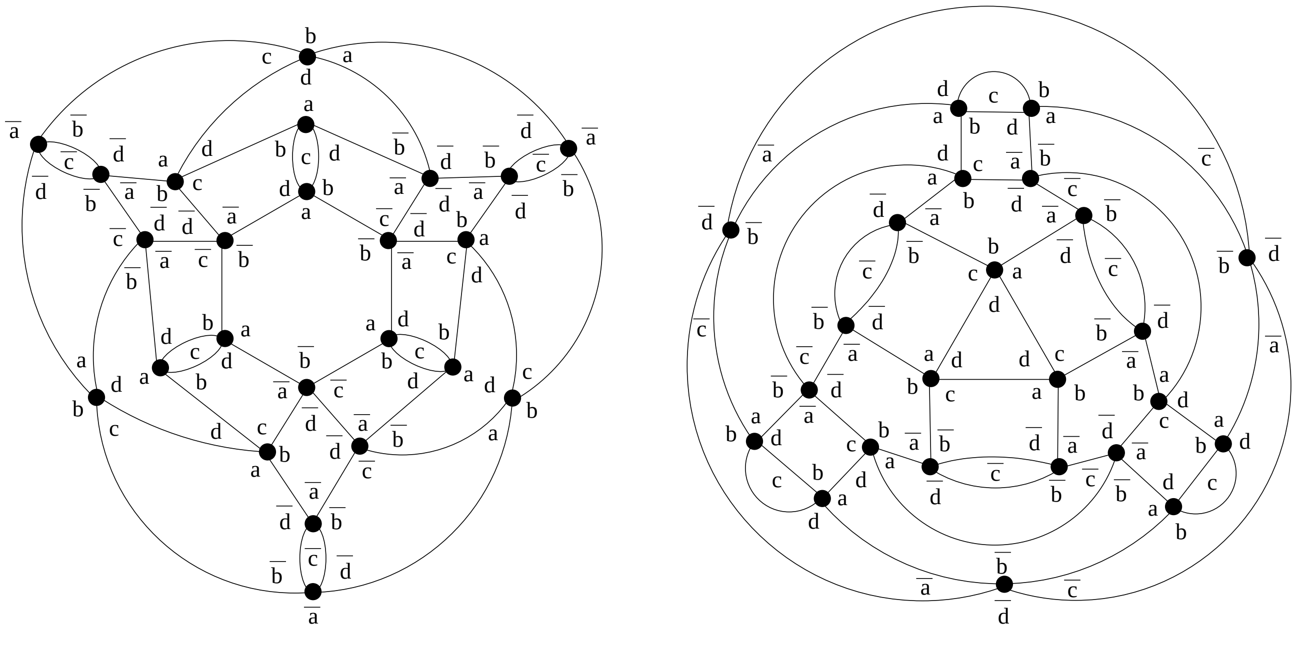
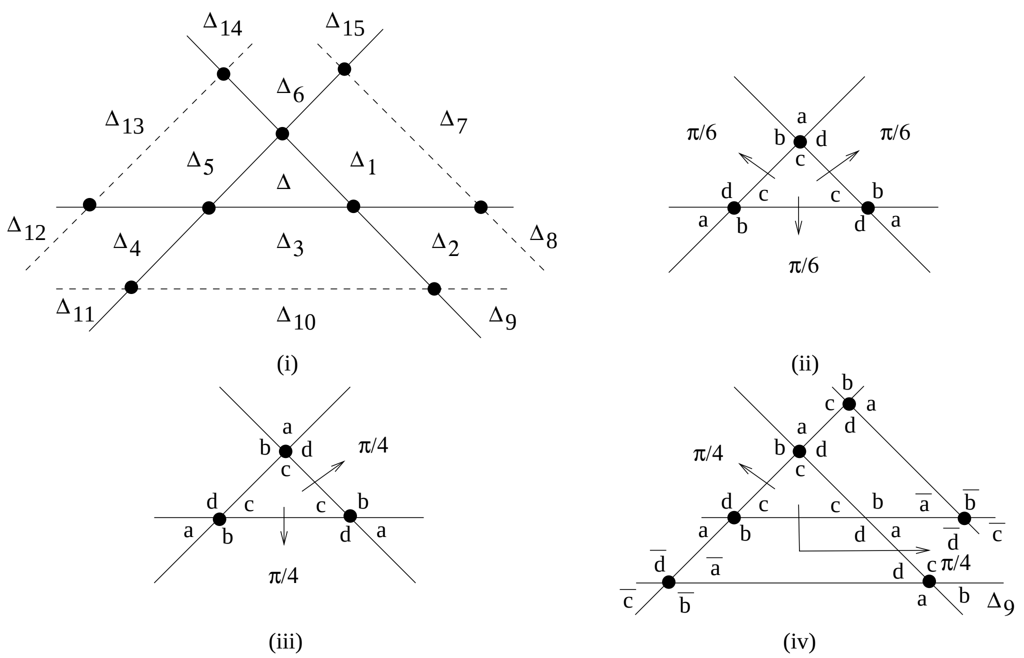


Figure 3.7



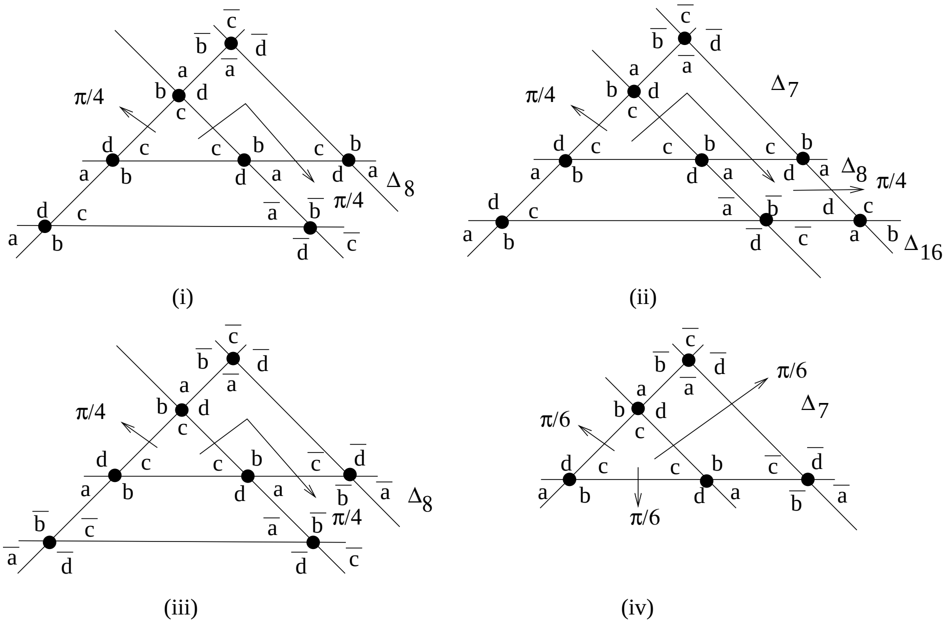
(i) (ii)

Figure 3.8



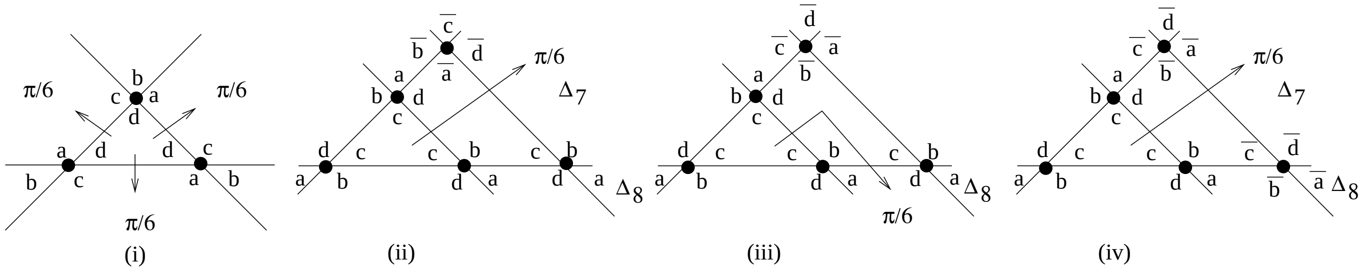
Δ14 Δ15 Δ6 Δ13 Δ7 Δ5 Δ1 Δ Δ12 Δ4 Δ3 Δ2 Δ8 Δ11 Δ10 Δ9
(i)
π/6 π/6 π/6
(ii)
π/4 π/4
(iii)
π/4 π/4 Δ9
(iv)

Figure 4.1



π/4
Δ_8
(i)
π/4
Δ_7
Δ_8
Δ_{16}
(ii)
π/4
Δ_8
(iii)
π/6
Δ_7
(iv)

Figure 4.2



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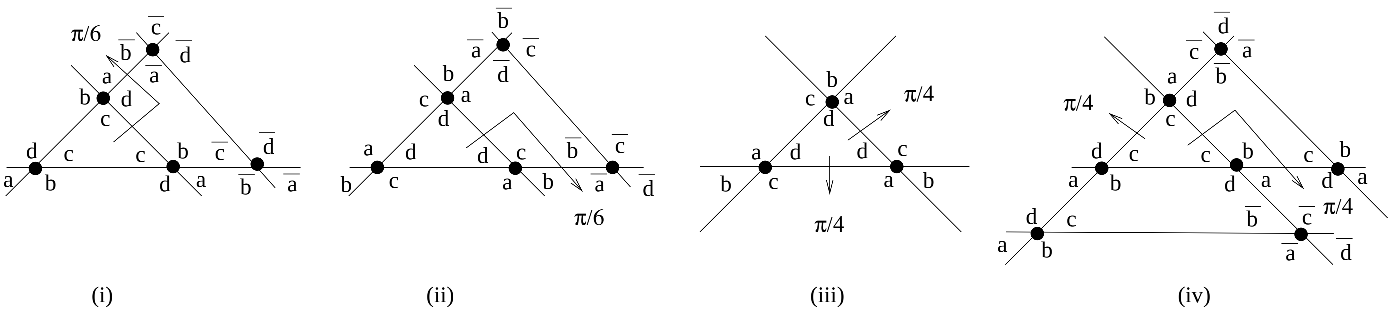


Figure 4.4

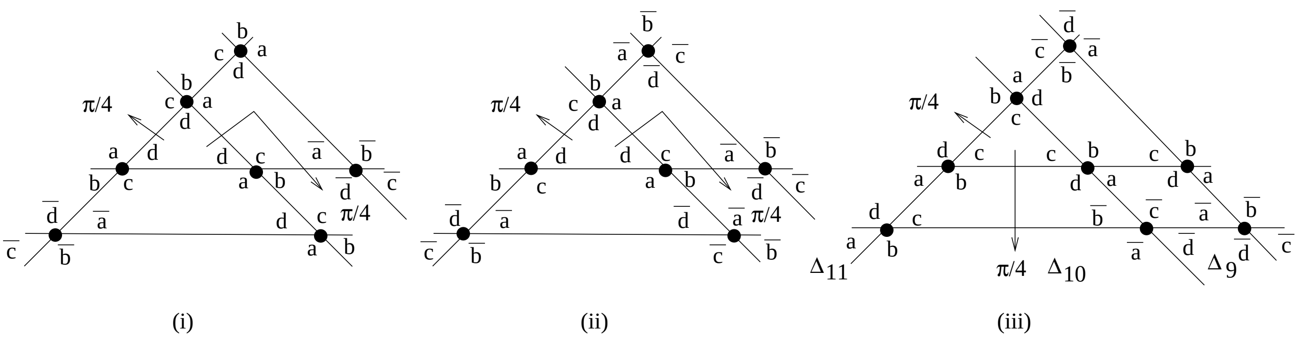


Figure 4.5

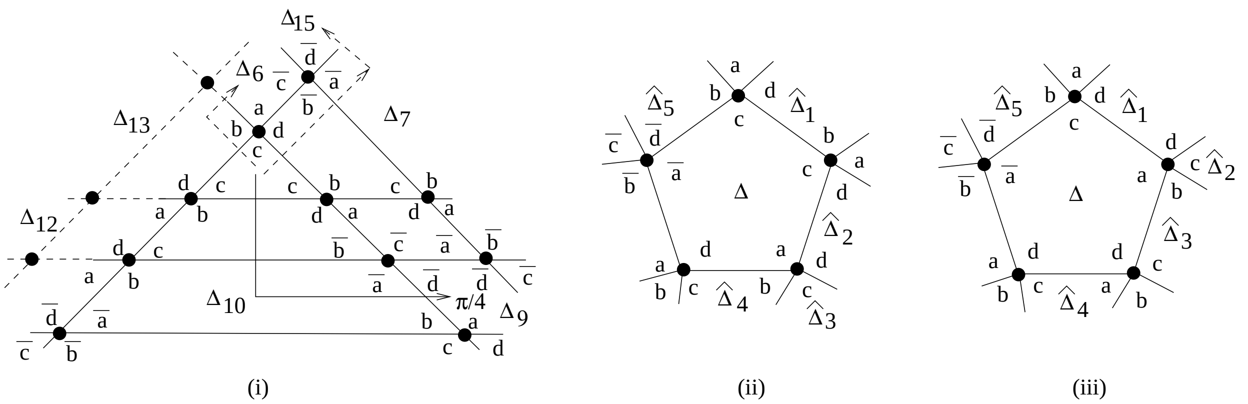


Figure 4.6

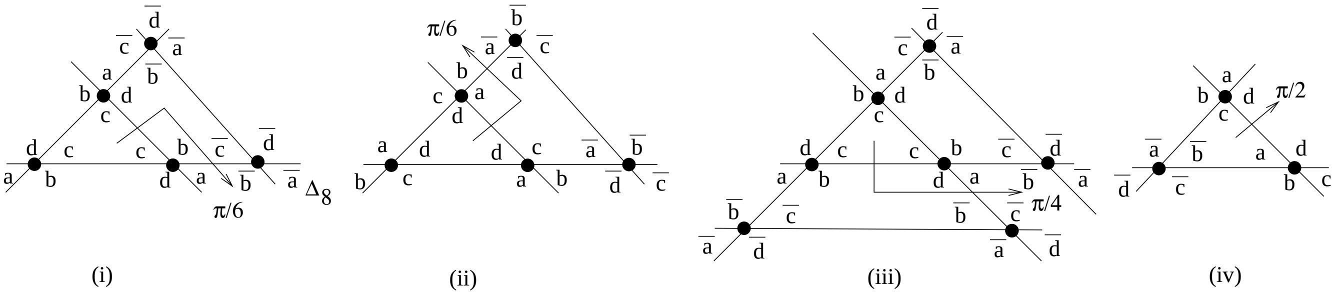


Figure 4.7

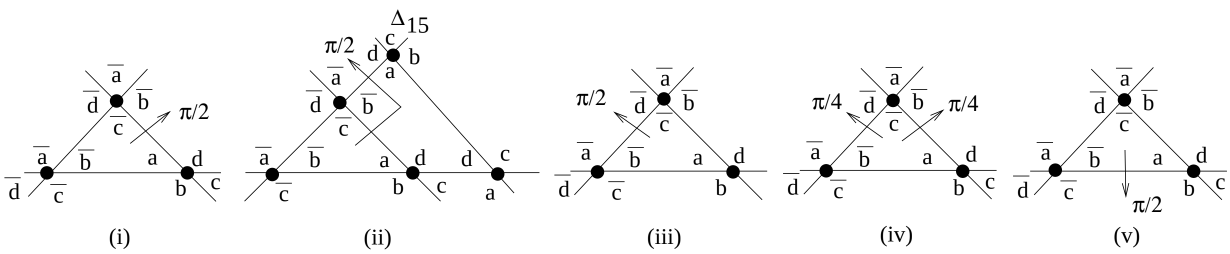


Figure 4.8

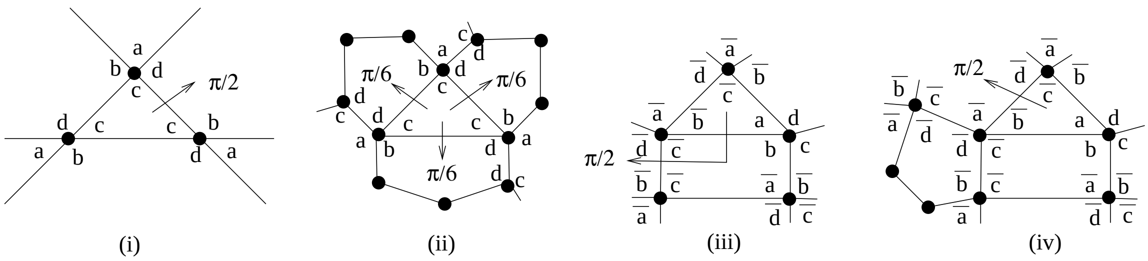


Figure 4.9

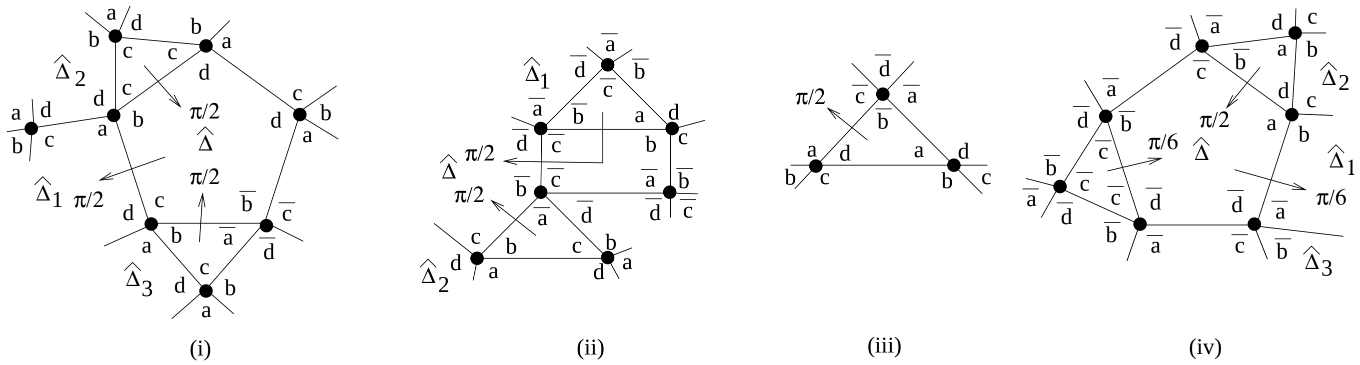


Figure 4.10

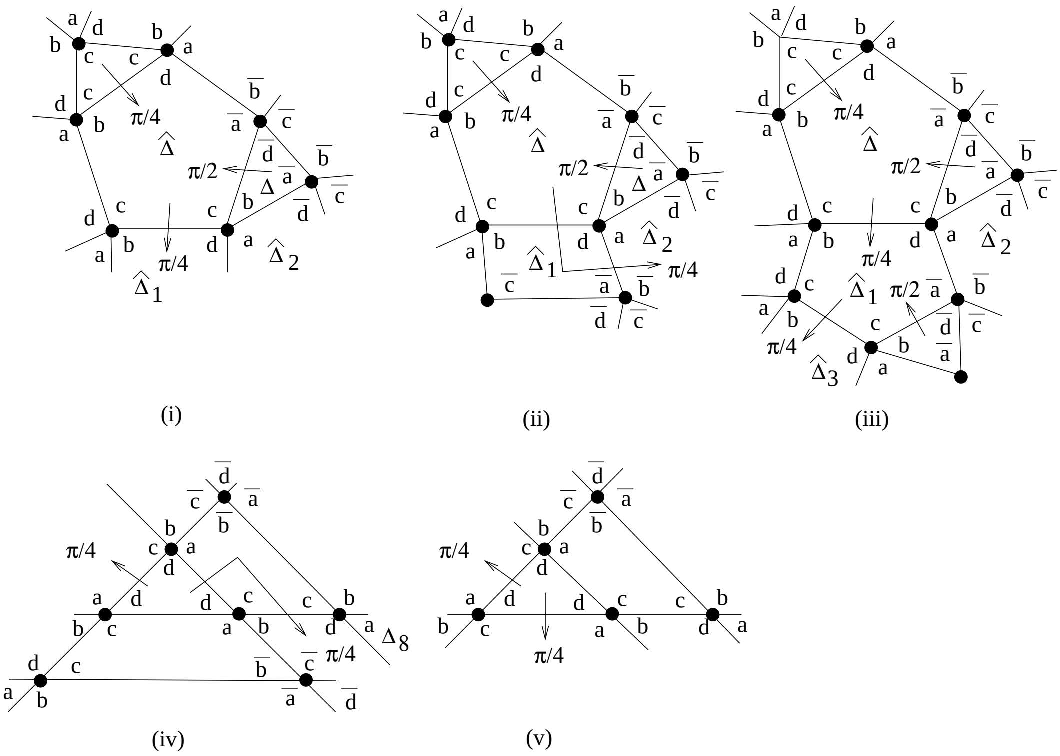


Figure 4.11

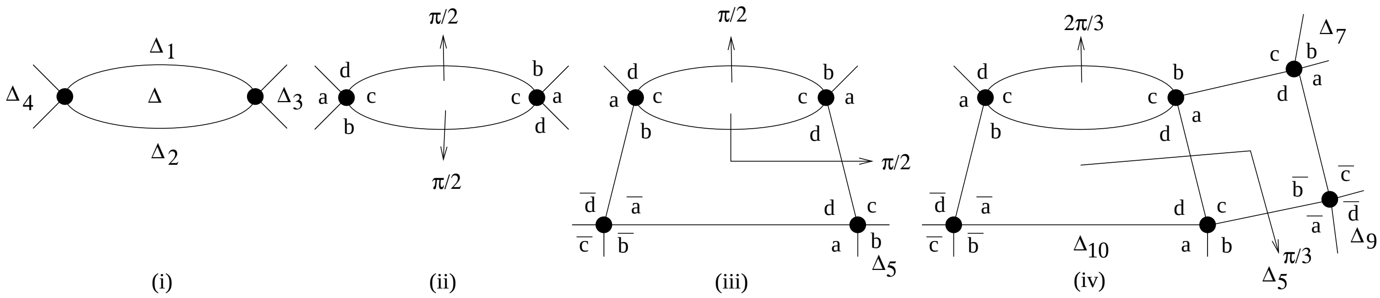


Figure 5.1

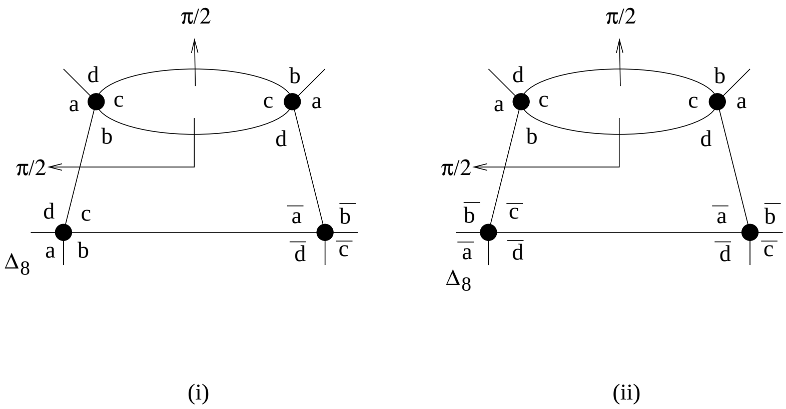

π/2
d c a b
b c a d
π/2
d c a b
$\bar{a}$ $\bar{b}$ $\bar{d}$ $\bar{c}$
Δ_8
$\bar{b}$ $\bar{c}$ $\bar{a}$ $\bar{d}$
$\bar{a}$ $\bar{b}$ $\bar{d}$ $\bar{c}$
Δ_8
(i)
(ii)

Figure 5.2

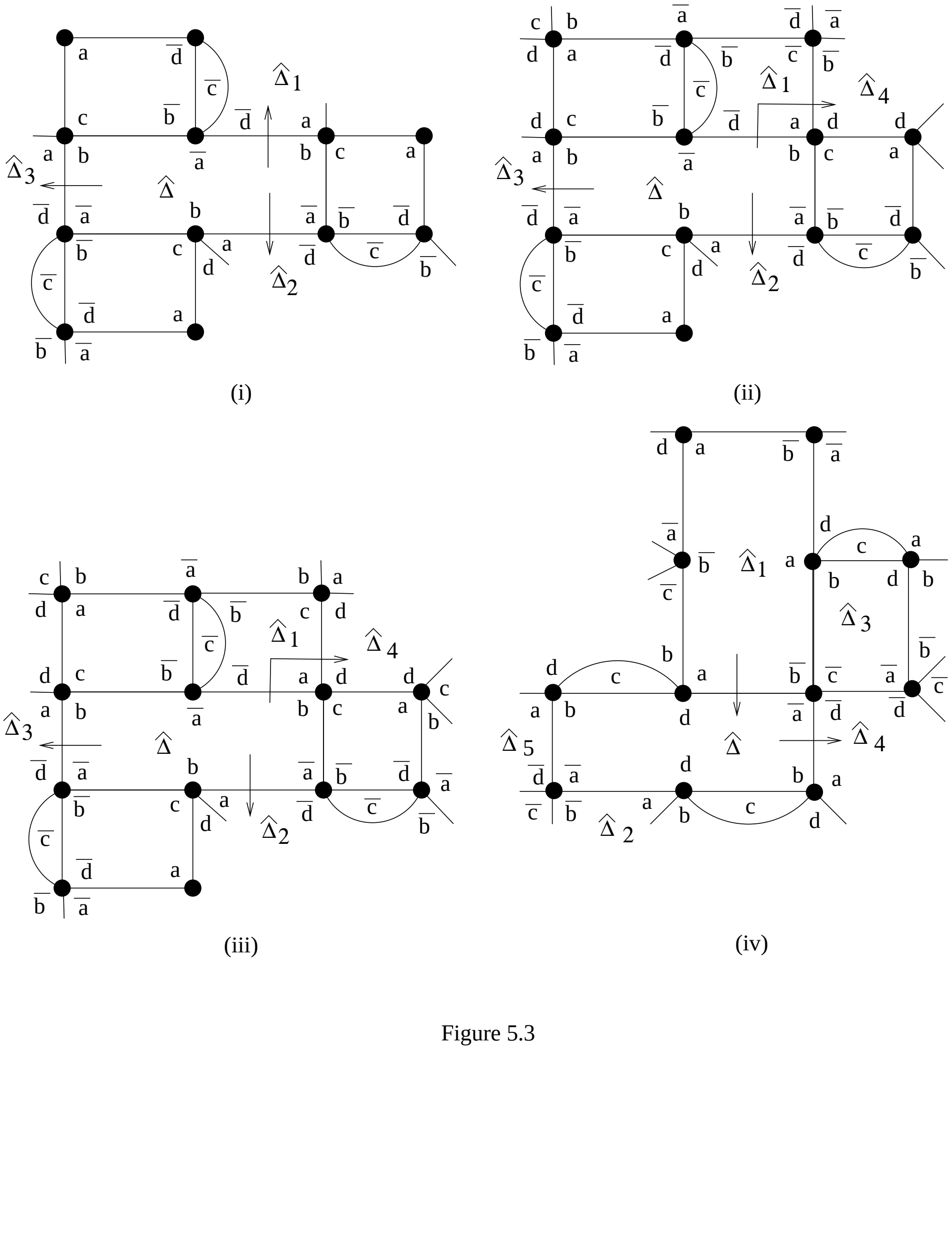
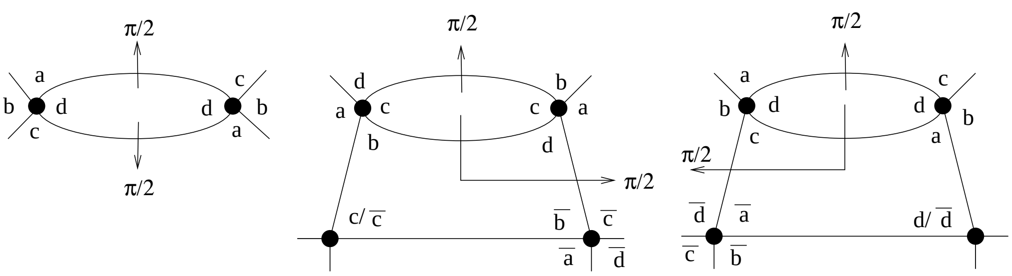
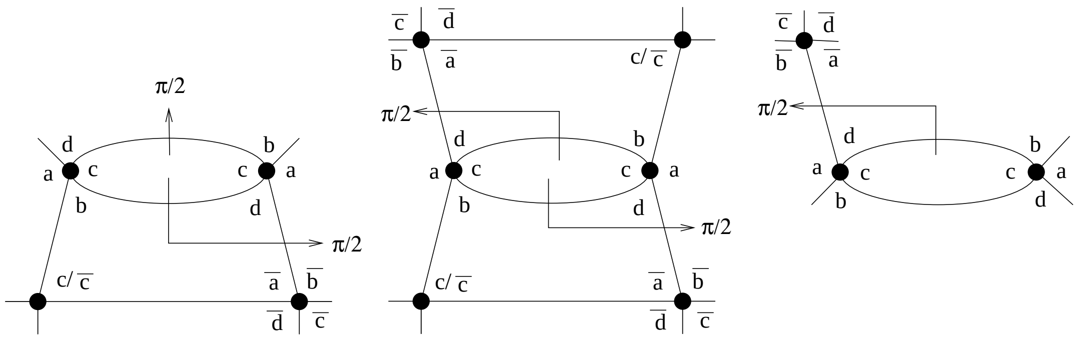


Figure 5.3



(i) (ii) (iii)

Figure 5.4



(i) (ii) (iii)

Figure 5.5

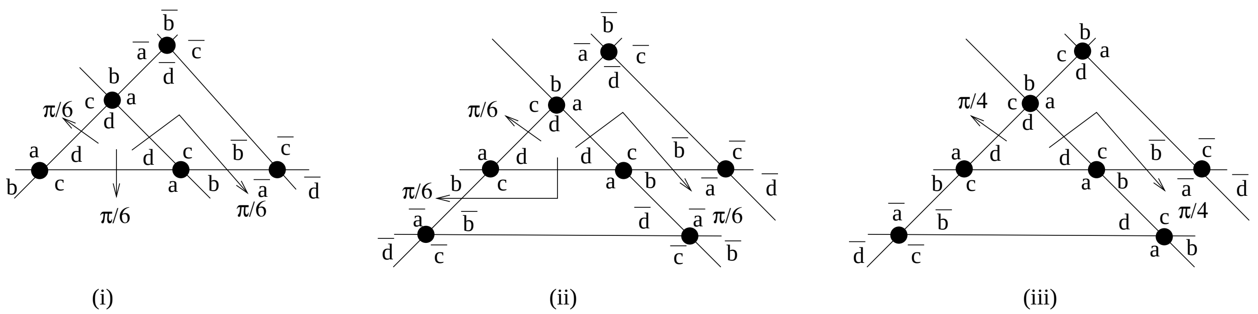


Figure 5.6

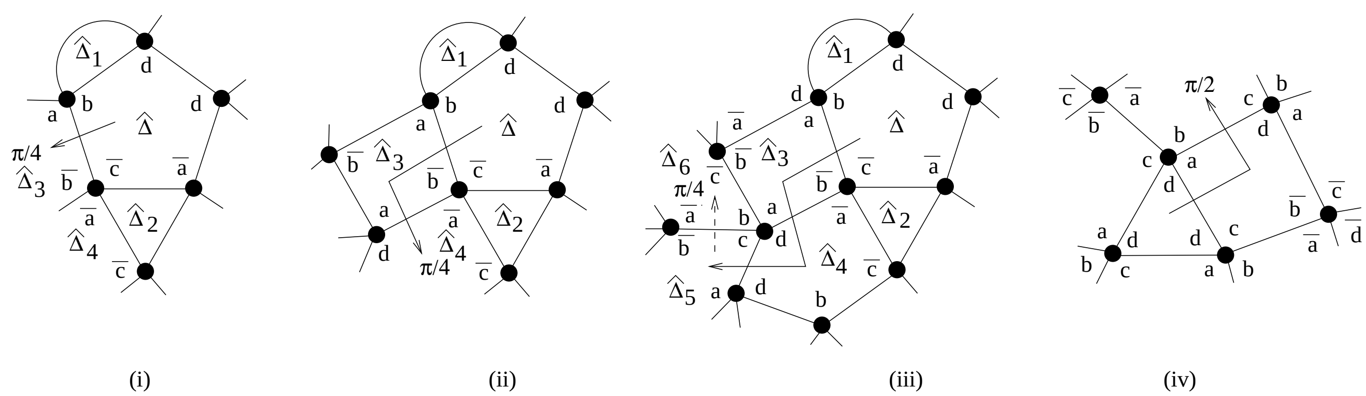
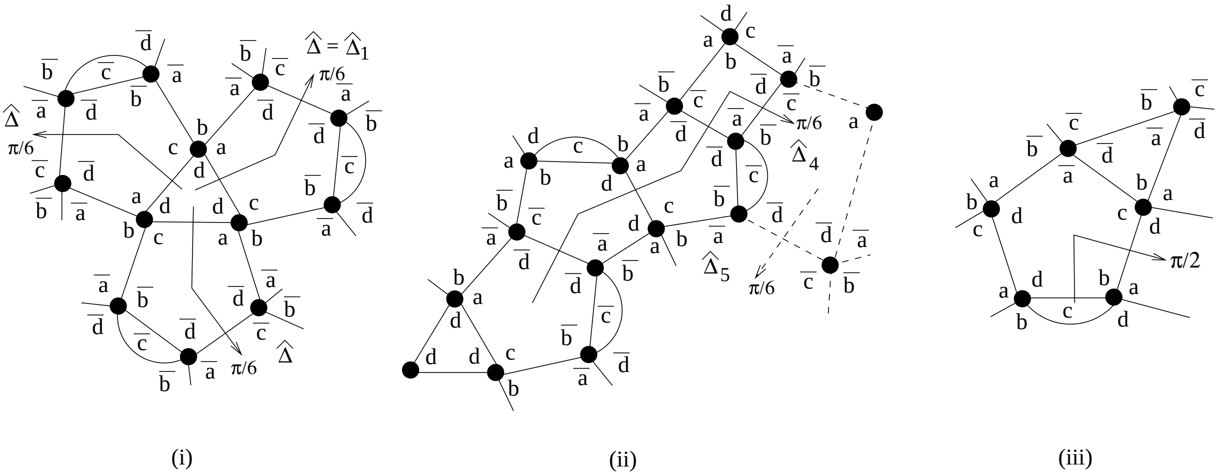
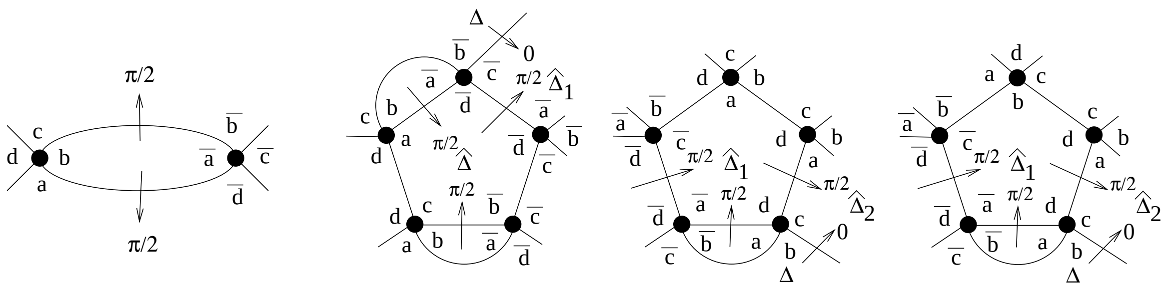


Figure 5.7



$\hat{\Delta} = \hat{\Delta}_1$
$\pi/6$
$\hat{\Delta}$
$\hat{\Delta}_4$
$\hat{\Delta}_5$
$\pi/2$
(i)
(ii)
(iii)

Figure 5.8



(i) (ii) (iii) (iv)

Figure 5.9

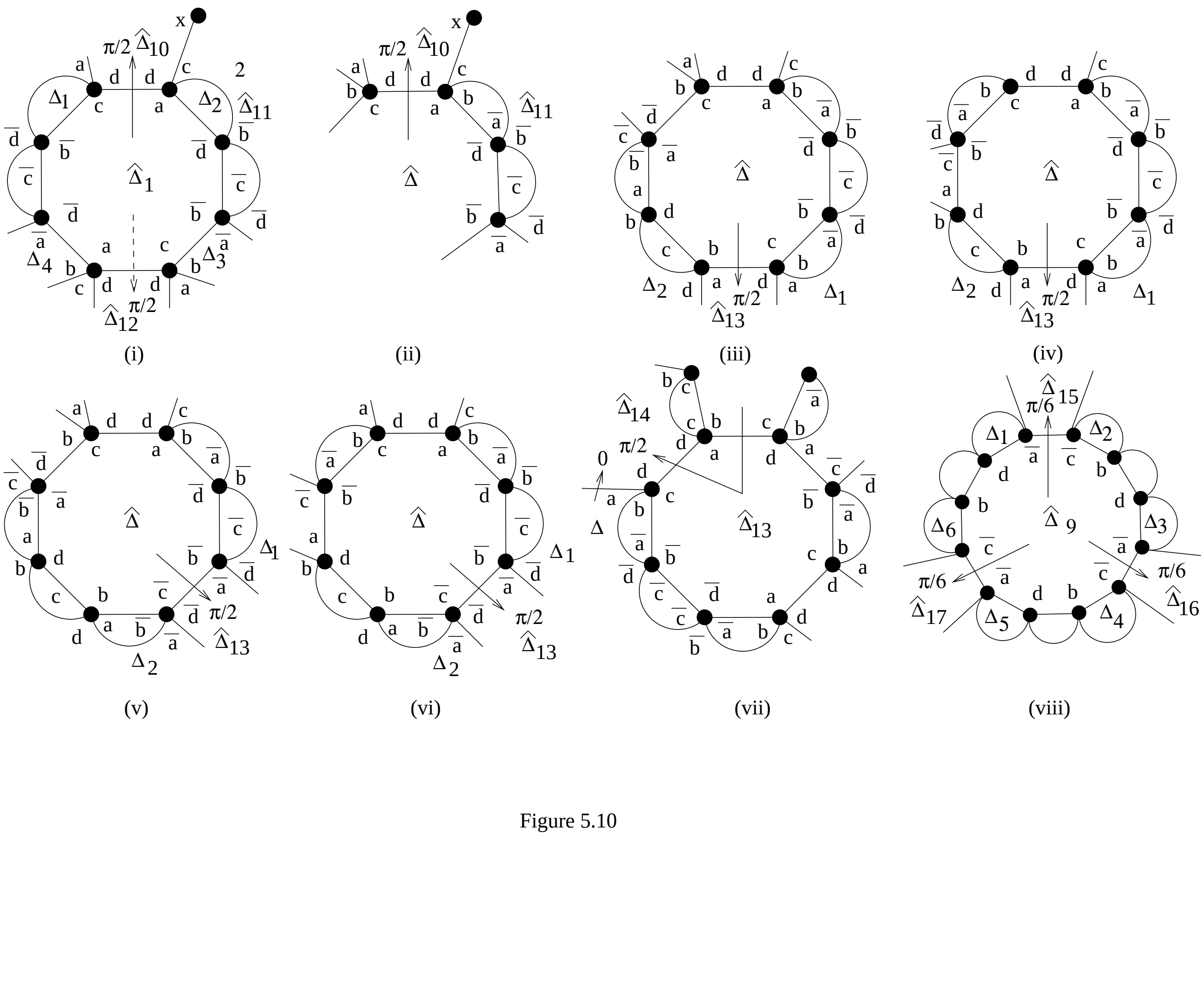
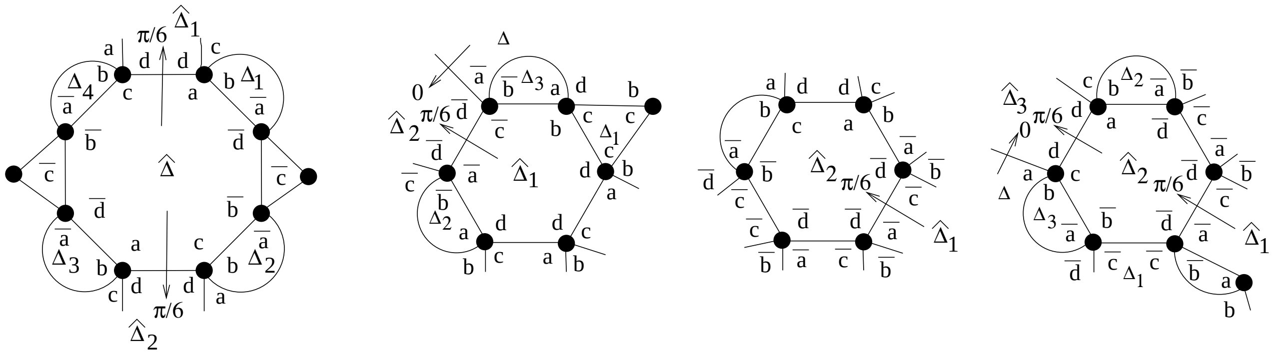


Figure 5.10



(i) (ii) (iii) (iv)

Figure 5.11

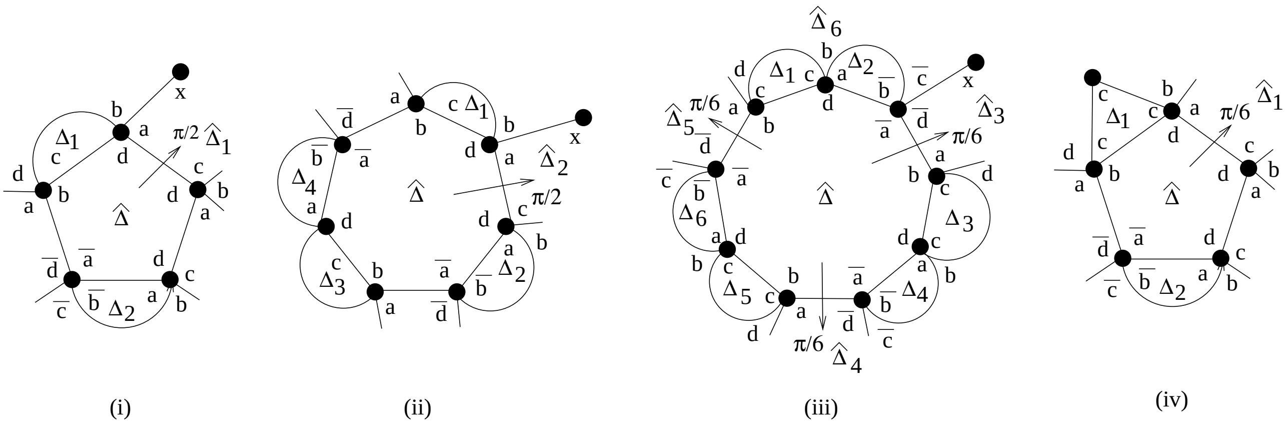


Figure 5.12