The infinite Fibonacci groups and relative asphericity – Technical Report

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Dedicated to David L. Johnson

1 Introduction

The generalised Fibonacci group $F(r,n)$ is the group defined by the cyclic presentation

$$\langle x_1,\ldots,x_n \mid x_1x_2\ldots x_rx_{r+1}^{-1}, x_2x_3\ldots x_{r+1}x_{r+2}^{-1},\ldots,x_{n-1}x_nx_1\ldots x_{r-2}x_{r-1}^{-1}, x_nx_1x_2\ldots x_{r-1}x_r^{-1}\rangle,$$

where $r > 1$, $n > 1$ and all the subscripts are assumed to be reduced modulo n. There has been a great deal of interest in the study of these groups since the question in [5] by Conway about the order of $F(2,5)$. Up to now the order of $F(r,n)$ was known except for the two infinite families $F\{7,5\}$ and $F\{8,5\}$ where $F\{r,n\} := \{F(r+kn,n)\colon k \geq 0\}$. The reader is referred to [15] and the references therein together with [4] and [14] for further details. In this paper we will show that each group in $F\{7,5\}$ or $F\{8,5\}$ is infinite. This together with previous results yields the following theorem.

Theorem 1.1 *The generalised Fibonacci group $F(r,n)$ is finite if and only if one of the following conditions is satisfied:*

(i) $r = 2$ *and* $n \in \{2,3,4,5,7\}$: *indeed* $F(2,2)$ *is trivial*; $F(2,3) \cong Q_8$, the quaternion group of order 8; $F(2,4) \cong \mathbb{Z}_5$; $F(2,5) \cong \mathbb{Z}_{11}$; and $F(2,7) \cong \mathbb{Z}_{29}$;

(ii) $r = 3$ *and* $n \in \{2,3,5,6\}$: *indeed* $F(3,2) \cong Q_8$; $F(3,3) \cong \mathbb{Z}_2$; $F(3,5) \cong \mathbb{Z}_{22}$; *and* $F(3,6)$ *is non-metacyclic, soluble of order* 1512;

(iii) $r \geq 4$ *and* $r \equiv 0 \pmod n$, *in which case* $F(r,n) \cong \mathbb{Z}_{r-1}$;

(iv) $r \geq 4$ *and* $r \equiv 1 \pmod n$, *in which case* $F(r,n)$ *is metacyclic of order* $r^n - 1$;

(v) $r \geq 4$, $n = 4$ *and* $r \equiv 2 \pmod n$, *in which case* $F(r,n) = F(4k+2,4)$ $(k \geq 1)$ *is metacyclic of order* $(4k+1)(2(4)^{2k} + 2(-4)^k + 1)$.

A *relative group presentation* is a presentation of the form $\mathcal{P} = \langle G, \boldsymbol{x} | \boldsymbol{r} \rangle$ where G is a group, $\boldsymbol{x}$ a set disjoint from G, and $\boldsymbol{r}$ a set of cyclically reduced words in the free product $G * \langle \boldsymbol{x} \rangle$ where $\langle \boldsymbol{x} \rangle$ denotes the free group on $\boldsymbol{x}$ [2]. If $G(\mathcal{P})$ denotes the group defined by $\mathcal{P}$ then $G(\mathcal{P})$ is the quotient group $G * \langle \boldsymbol{x} \rangle / N$ where N denotes the normal closure in $G * \langle \boldsymbol{x} \rangle$ of $\boldsymbol{r}$. A relative group presentation is defined in [2] to be *aspherical* if every spherical picture over it contains a dipole, that is, fails to be reduced. There is interest in when a relative presentation is aspherical, see, for example, [1], [2], [6], [7], [9] and [13]. In this paper we consider the situation when $G = \langle t | t^5 \rangle$, $\boldsymbol{x} = \{u\}$ and $\boldsymbol{r} = \{t^2 u t u^{-n}\}$ and prove the following theorem.

Theorem 1.2 *The relative presentation $\mathcal{P}_n = \langle t, u | t^5, t^2 u t u^{-n} \rangle$ is aspherical for $n \geq 7$.*

Applying, for example, statement **(0.4)** in the introduction of [2] and the fact that the group defined by $\mathcal{P}_n$ is neither trivial nor cyclic of order 5 we immediately obtain

Corollary 1.3 *If $G(\mathcal{P}_n)$ is the group defined by $\mathcal{P}_n$ then $G(\mathcal{P}_n)$ is infinite for $n \geq 7$, indeed u has infinite order in $G(\mathcal{P}_n)$ for $n \geq 7$.*

We shall show in Section 2 that Corollary 1.3 implies that each group in $F\{7,5\}, F\{8,5\}$ is infinite. The remaining Sections 3–8 of the paper will be devoted entirely to proving Theorem 1.2.

2 Fibonacci groups

Consider the generalised Fibonacci group $F(r,n)$ of the introduction. If $r = 2$ or $2 \leq n \leq 4$ or $(r,n) \in \{(3,5),(3,6)\}$ or n divides r or $r \equiv 1 \pmod{n}$ then Theorem 1.1 applies and these cases are discussed fully with relevant references in [15]. Assume then that none of these conditions holds. In particular $r \geq 3$ and $n \geq 5$. In [14] it is shown that if n does not divide any of $r \pm 1$, $r+2$, $2r$, $2r+1$ or $3r$ then $F(r,n)$ is infinite. If n divides $r+1$ then $F(r,n)$ is infinite for $r \geq 3$ [11] so assume otherwise. We are left therefore to consider the families $F\{r, r+2\}$; $F\{r, 2r\}$; $F\{r, 2r+1\}$ and $F\{r, 3r\}$. In [4] it is shown that if $r \geq 4$ then each group in $F\{r, r+2\}$ and $F\{r, 2r\}$ is infinite; and if $r \geq 3$ then each group in $F\{r, 2r+1\}$ is infinite. This leaves $F\{8,5\}$, $F\{9,6\}$, $F\{7,5\}$ and $F\{r,3r\}$. In [14] it is also shown that if n does not divide any of $r \pm 1$, $r \pm 2$, $r+3$, $2r$, $2r+1$ then $F(r,n)$ is infinite. If n divides $3r$ and $r+2$ we obtain the family $F\{4,6\}$ which is $F\{r,r+2\}$ for $r=4$; if n divides $3r$ and $r-2$ we obtain $F\{8,6\}$ and each group in this family is infinite [3]; and if n divides $3r$ and $r+3$ we obtain $F\{6,9\}$. By our assumptions n does not divide $3r$ together with any of $r \pm 1$, $2r$ or $2r+1$. It is also shown in [4] that each group in $F\{9,6\}$ or $F\{6,9\}$ is infinite, all of which leaves $F\{7,5\}$ and $F\{8,5\}$. These families are

$$\{F(7+5k,5) \colon k \geq 0\} \quad \text{and} \quad \{F(8+5k,5) \colon k \geq 0\},$$

where $F(7+5k,5)$ and $F(8+5k,5)$ are defined respectively by the presentations

$$\langle x_1, x_2, x_3, x_4, x_5 \mid (x_1x_2x_3x_4x_5)^{k+1}x_1x_2x_3^{-1}, \ldots, (x_5x_1x_2x_3x_4)^{k+1}x_5x_1x_2^{-1} \rangle,$$
$$\langle x_1, x_2, x_3, x_4, x_5 \mid (x_1x_2x_3x_4x_5)^{k+1}x_1x_2x_3x_4^{-1}, \ldots, (x_5x_1x_2x_3x_4)^{k+1}x_5x_1x_2x_3^{-1} \rangle.$$

We show how Corollary 1.3 can be used to prove Theorem 1.1. Forming a semi-direct product with the cyclic group of order 5 in the standard way (see, for example, Chapter 10 of [10]) yields the groups $E(7+5k,5)$ and $E(8+5k,5)$ defined respectively by the presentations

$$\langle x,t \mid t^5, (xt^{-1})^{7+5k}x^{-1}t^2\rangle,$$
$$\langle x,t \mid t^5, (xt^{-1})^{8+5k}x^{-1}t^3\rangle.$$

Now

$$\begin{aligned}\langle x,t \mid t^5, (xt^{-1})^{7+5k}x^{-1}t^2\rangle &= \langle x,t,y \mid t^5, (xt^{-1})^{7+5k}x^{-1}t^2, y^{-1}xt^{-1}\rangle\\ &= \langle y,t \mid t^5, y^{7+5k}t^{-1}y^{-1}t^2\rangle\\ &= \langle y,t \mid t^5, y^{7+5k}ty^{-1}t^3\rangle \quad (\text{replacing } t \text{ by } t^{-1})\\ &= \langle y,t,s \mid t^5, y^{7+5k}ty^{-1}t^3, st^{-3}\rangle \quad (s^2=t^6=t)\\ &= \langle y,s \mid s^5, y^{7+5k}s^2y^{-1}s\rangle \quad (s=t^3)\\ &= \langle u,t \mid t^5, t^2utu^{-(7+5k)}\rangle \quad (s\leftrightarrow t,\ y=u^{-1}) \quad (\text{cyclic conjugate})\end{aligned}$$

and

$$\begin{aligned}\langle x,t \mid t^5, (xt^{-1})^{8+5k}x^{-1}t^3\rangle &= \langle x,t,y \mid t^5, (xt^{-1})^{8+5k}x^{-1}t^3, y^{-1}xt^{-1}\rangle\\ &= \langle t,y \mid t^5, y^{8+5k}t^{-1}y^{-1}t^3\rangle \quad (\text{inverse},\ t^{-3}=t^2)\\ &= \langle u,t \mid t^5, t^2utu^{-(8+5k)}\rangle \quad (y=u).\end{aligned}$$

Therefore Corollary 1.3 implies that each group in $\{E(7+5k,5) \text{ and } E(8+5k,5)\colon k\geq 0\}$ is infinite and, given this, Theorem 1.1 now follows.

3 The amended picture and curvature

The reader is referred to [2] and [12] for definitions of many of the basic terms used in this and subsequent sections.

Suppose by way of contradiction that the relative presentation

$$\mathcal{P}_n = \langle t,u \mid t^5, t^2utu^{-n}\rangle \quad (n\geq 7)$$

is not aspherical, that is, there exists a *reduced spherical picture* $\boldsymbol{P}$ over $\mathcal{P}_n$. Then each *arc* of $\boldsymbol{P}$ is equipped with a normal orientation and labelled by an element of $\{u,u^{-1}\}$; each *corner* of $\boldsymbol{P}$ is labelled by an element of $\{t^i\colon -2\leq i\leq 2\}$; reading the labels clockwise on the corners and arcs at a given vertex yields t^2utu^{-n} (up to cyclic permutation and inversion); and the product of the sequence of corner labels encountered in an anti-clockwise traversal of any given *region* of $\boldsymbol{P}$ yields the identity in $G=\langle t \mid t^5\rangle$.

Now let $\boldsymbol{D}$ be the dual of the picture $\boldsymbol{P}$ with the labelling of $\boldsymbol{D}$ inherited from that of $\boldsymbol{P}$. Then $\boldsymbol{D}$ is a tessellation of the 2-sphere S^2 such that: each corner label of $\boldsymbol{D}$ is t^i where $-2 \leq i \leq 2$; each edge is oriented and labelled u or u^{-1}; and each region Δ of $\boldsymbol{D}$ is given (up to cyclic permutation and inversion) by Figure 3.1(i). (In all subsequent figures for ease of presentation we will not show the orientation of the edges or the edge labels u, u^{-1}.) For convenience we will use the following notation for corner labels:

a for t^1; b for t^2; λ for t^0;

and in figures we denote the inverse θ^{-1} of a corner label θ by $\overline{\theta}$. Note that the sum of the powers of t read around any given vertex of $\boldsymbol{D}$ is congruent to 0 modulo 5.

The *star graph* Γ for $\boldsymbol{D}$ is given by Figure 3.1(ii) with the following convention: we use μ for λ^{-1} with the understanding that the edges labelled λ and μ in Γ are traversed only in the direction indicated.

We can make the following assumptions without any loss of generality:

A1. $\boldsymbol{D}$ is minimal with respect to number of regions.

A2. Subject to assumption 1, $\boldsymbol{D}$ is maximal with respect to number of vertices of degree 2.

We introduce some further notation. If v is a vertex of $\boldsymbol{D}$ then $l(v)$, the *label* of v, is the cyclic word obtained from the corner labels of v in a *clockwise* direction; and $d(v)$ denotes the *degree* of v. If Δ is a region of $\boldsymbol{D}$ then $d(\Delta)$ denotes the degree of Δ. A (v_1, v_2)-*edge* is an edge with endpoints v_1 and v_2; and an edge is a (θ_1, θ_2)-*edge relative to the region* Δ if its corner labels in Δ are θ_1 and θ_2. (When there is no ambiguity we will simply talk of a (θ_1, θ_2)-edge.)

Lemma 3.1 *If v is a vertex of $\boldsymbol{D}$ then $l(v) \neq (\lambda\mu)^{\pm k}$ for $k \geq 2$.*

Proof. The proof is by induction on k. Consider the vertex of Figure 3.2(i) having label $(\lambda\mu)^2$. Apply $m = \min\{l_1, l_2\}$ bridge moves of the type shown in Figure 3.2(ii). Then each of the first $m-1$ bridge moves will create and destroy two vertices of degree 2, leaving the total number unchanged. The mth bridge move however will create two vertices of degree 2 but destroy at most one. Since bridge moves leave the total number of regions unchanged we obtain a contradiction to assumption **A2**. Now consider the vertex of Figure 3.2(iii) having label $(\lambda\mu)^k$ where $k \geq 3$. Again apply $m = \min\{l_1, l_2\}$ bridge moves of the type shown in Figure 3.2(iv). The first such bridge move may decrease the total number of vertices of degree 2 by one, each subsequent bridge move creates two and destroys two until the mth bridge which increases the number by at least one. This produces a new diagram with at most the same number of vertices of degree 2 as $\boldsymbol{D}$. But applying an induction argument to the vertex v of Figure 3.2(iv) where $l(v) = (\lambda\mu)^{k-1}$ will yield a contradiction to **A2** as before. □

Lemma 3.2 *Let $v \in \boldsymbol{D}$. (i) If $d(v) = 2$ then $l(v) = (\lambda\mu)^{\pm 1}$ and (ii) if $d(v) > 2$ then $l(v)$ contains at least three occurrences of $a^{\pm 1}, b^{\pm 1}$.*

Proof. Both statements follow from the fact that the sum of the corner labels is congruent to $0 \bmod 5$ together with Lemma 3.1 for (ii) and the fact that no adjacent corner labels are inverse to each other. □

We amend $\boldsymbol{D}$ as follows. Delete all vertices of degree 2 and all edges that are not (b, a)-edges (relative to any region), and relabel the corners accordingly to obtain $\boldsymbol{K}$. Then Lemmas 3.1 and 3.2 ensure the existence of a connected, simply connected, component $\boldsymbol{K}_0$ of $\boldsymbol{K}$ such that $d(v) \geq 3$ for each $v \in \boldsymbol{K}_0$ and that $\boldsymbol{K}_0$ is a *map* in the sense of [12]. Indeed let $\boldsymbol{K}_0^{(1)}$ be the 1-skeleton of $\boldsymbol{K}_0$. Since $\boldsymbol{K}_0$ is connected, $\boldsymbol{K} \backslash \boldsymbol{K}_0^{(1)}$ is the disjoint union of connected and simply connected submaps of $\boldsymbol{K}$. These submaps $\Delta_{\boldsymbol{K}}$ are the *regions* of $\boldsymbol{K}_0$ in the sense of [12] and $c := \partial\Delta_{\boldsymbol{K}}$ is a minimal closed curve. We claim that *each 2-segment of $\Delta_{\boldsymbol{K}}$ in $\boldsymbol{K}$ has its endpoints in c.* Now, $\Delta_{\boldsymbol{K}}$ cannot contain a vertex $v \in \boldsymbol{K}_0 \backslash c$ for otherwise minimality would force the intersection of $\Delta_{\boldsymbol{K}}$ and the connected component containing v in $\boldsymbol{K}_0$ to be a tree, contradicting each vertex in $\boldsymbol{K}_0$ has degree ≥ 3 by Lemma 3.2. Consequently it follows from Lemma 3.1 that all inner vertices of $\Delta_{\boldsymbol{K}}$ in $\boldsymbol{K}$ have degree 2 and then each 2-segment of $\Delta_{\boldsymbol{K}}$ in $\boldsymbol{K}$ has its endpoints in c.

The corner labels of $\boldsymbol{K}_0$ are:

$$\begin{aligned}
\tilde{a} &= a(\lambda\mu)^{k_1} && \text{(odd length)}\\
\tilde{b} &= (\mu\lambda)^{k_2} b && \text{(odd length)}\\
\tilde{\lambda} &= (\lambda\mu)^{k_3}\lambda && \text{(odd length)}\\
x &= \tilde{a}\lambda && \text{(even length)}\\
y &= \lambda\tilde{b} && \text{(even length)}\\
z &= \tilde{a}\lambda\tilde{b} && \text{(odd length)}
\end{aligned} \tag{3.1}$$

where $k_i \geq 0$ $(1 \leq i \leq 3)$. The star graph Γ_0 for $\boldsymbol{K}_0$ is given by Figure 3.3(i), and the table in Figure 3.3(ii) gives the power of t each corner label represents. Observe that $(\lambda\mu)^k$ for $k \geq 1$ cannot be a corner label in $\boldsymbol{K}_0$ for otherwise $\boldsymbol{K}_0$ would contain a subdiagram of the form shown in Figure 3.3(iii) and this contradicts **A1** since after *bridge moves* and cancellation it would be possible to reduce the number of regions of $\boldsymbol{D}$ by at least two.

Lemma 3.3 *Let $v \in \boldsymbol{K}_0$. If $d(v) \leq 6$ then $l(v)$ is one of the following:*

(i) $d(v) = 3 : \quad \tilde{a}xy^{-1}$

$\tilde{b}\tilde{\mu}z$

(ii) $d(v) = 4 : \quad \tilde{a}\tilde{a}z\tilde{\mu}$

$\tilde{b}\tilde{b}x^{-1}y$

(iii) $d(v) = 5 : \quad \tilde{a}\tilde{a}\tilde{a}\tilde{a}\tilde{a}$

$\tilde{b}\tilde{b}\tilde{b}\tilde{b}\tilde{b}$

$\tilde{a}zx^{-1}y\tilde{\mu}$

$\tilde{b}x^{-1}\tilde{\lambda}z^{-1}y$

Proof. This follows from checking all reduced closed paths in Γ_0 whose exponent sum is 0 modulo 5 together with the fact that equations (3.1) can be used to show that the following paths of length 2 together with their inverses do not occur as sublabels: $\tilde{a}\tilde{\lambda}$; $\tilde{a}y$; $\tilde{a}^{-1}x$; $\tilde{a}^{-1}z$; $\tilde{b}y^{-1}$; $\tilde{b}z^{-1}$; $\tilde{b}^{-1}x^{-1}$; $\tilde{b}^{-1}\tilde{\mu}$; $\tilde{\lambda}x^{-1}$; $\tilde{\lambda}\mu$; $\tilde{\mu}y$; $\tilde{\mu}\tilde{\lambda}$; $x^{-1}z$; yz^{-1}. For example $\tilde{a}\tilde{\lambda} = a(\lambda\mu)^{k_1}(\lambda\mu)^{k_2}\lambda = a(\lambda\mu)^{k_1+k_2}\lambda = \tilde{a}\lambda = x$ after rewriting using equations (3.1). □

Convention: We will usually write a, b, λ, μ for $\tilde{a}, \tilde{b}, \tilde{\lambda}, \tilde{\mu}$ simply for ease of presentation. For example if $v \in \boldsymbol{D}$ has label $l(v) = a\lambda\mu a\lambda\mu\lambda b^{-1}\mu\lambda\mu$ then in $\boldsymbol{K}_0$ this transforms uniquely to $(a\lambda\mu)(a\lambda\mu\lambda)(b^{-1}\mu\lambda\mu) = \tilde{a}xy^{-1}$ which we write as axy^{-1} or as $ax\bar{y}$ in the figures. This is illustrated in Figure 3.3(iv).

We turn now to the regions of $\boldsymbol{K}_0$. The edges or *2-segments* deleted in forming $\boldsymbol{K}$ from $\boldsymbol{D}$ will be referred to as *shadow edges* and will usually be denoted by dotted edges in our figures. The number of edges in a 2-segment will be called its *length*. Much use will be made of the fact that *the number of edges in a region of $\boldsymbol{D}$ is $n+1$*. By *length contradiction* we mean either a contradiction to this fact or to the fact that $n \geq 7$.

We will also use the fact that no region of $\boldsymbol{K}_0$ can contain the configuration of edges and shadow edges shown in Figure 3.4. To see this observe in Figure 3.4(i) that $\{\phi_1, \phi_2\} \subseteq \{\lambda, \mu\}$ forcing each $\theta_i \in \{a^{\pm 1}, b^{\pm 1}\}$ and any attempt at labelling forces $\theta_2\theta_3 = aa^{-1}$ or bb^{-1}, a contradiction to $\boldsymbol{D}$ being reduced. In Figure 3.4(ii) each $\phi_i \in \{\lambda, \mu\}$ and this produces a region in $\boldsymbol{D}$ without corner label $a^{\pm 1}$ or $d^{\pm 1}$. We refer to the existence of each of these situations as a *basic labelling contradiction.*

For example suppose that $\Delta \in \boldsymbol{K}_0$ and $d(\Delta) = 6$. If Δ contains no shadow edges as in Figure 3.5(i) then we obtain the length contradiction $n + 1 = 6$. Let (pq) denote the shadow edge with endpoints p and q. If Δ contains exactly one shadow edge e then, working modulo cyclic permutation and inversion, $e \in \{(13), (14)\}$. But $e = (13)$ yields the length contradiction $n + 1 = n + 3$ as shown in Figure 3.5(ii) since the length of (13) must be $n - 1$. If Δ contains exactly two shadow edges e_1 and e_2 then $(e_1, e_2) \in$

$\{((13),(14)),((13),(15)),((13),(46))\}$. But $(e_1,e_2) = ((13),(14))$ yields the length contradiction $n+1 = 4$; and $(e_1,e_2) = ((13),(15))$ or $((13),(46))$ implies $n+1 = 2n$ (see Figure 3.5(iii)-(v)). Finally if Δ contains three shadow edges e_1, e_2 and e_3 then $(e_1,e_2,e_3) = ((13),(14),(15))$ or $((13),(15),(35))$ yielding a basic labelling contradiction (see Figure 3.5(vi)-(vii)); or $(e_1,e_2,e_3) = ((13),(14),(46))$.

Similar elementary but somewhat lengthy arguments are given in the Appendix to prove the following.

Lemma 3.4 *Let Δ be a region of $\boldsymbol{K}_0$. If $d(\Delta) \leq 9$ then $d(\Delta) \in \{4,6,8,9\}$ and Δ is given by Figure 3.6.*

For example it follows from Lemma 3.4 that if $d(\Delta) = 6$ then up to cyclic permutation and inversion Δ is given by Figure 3.7. In particular, if Δ contains a (a,b)-edge or (x,y)-edge then $d(\Delta) \geq 8$.

We will use similar curvature arguments to those used in [8]. Thus, if Δ is an m-gon of $\boldsymbol{K}_0$ and the degrees of the vertices of Δ are d_i $(1 \leq i \leq m)$, then the *curvature* of Δ is given by

$$c(\Delta) = c(d_1,\ldots,d_m) = (2-m)\pi + 2\pi\sum_{i=1}^{m}\frac{1}{d_i}. \tag{3.2}$$

(Observe that if ρ is any permutation of $\{1,\ldots,m\}$ then $c(\Delta) = c(d_{\rho(1)},\ldots,d_{\rho(m)})$. This fact will be used throughout without explicit reference.) A list of $c(d_1,\ldots,d_m)$ used in the paper is given in the tables below for the reader's benefit.

$c(3,3,3,3) = \frac{2\pi}{3}$	$c(3,3,5,5) = \frac{2\pi}{15}$	$c(3,4,5,5) = -\frac{\pi}{30}$
$c(3,3,3,4) = \frac{\pi}{2}$	$c(3,3,5,6) = \frac{\pi}{15}$	$c(3,4,5,6) = -\frac{\pi}{10}$
$c(3,3,3,5) = \frac{2\pi}{5}$	$c(3,3,5,7) = \frac{2\pi}{105}$	$c(3,4,5,7) = -\frac{31\pi}{210}$
$c(3,3,3,6) = \frac{\pi}{3}$	$c(3,3,6,6) = 0$	$c(3,4,6,6) = \frac{\pi}{6}$
$c(3,3,4,4) = \frac{\pi}{3}$	$c(3,4,4,5) = \frac{\pi}{15}$	$c(3,5,5,5) = -\frac{2\pi}{15}$
$c(3,3,4,5) = \frac{7\pi}{30}$	$c(3,4,4,6) = 0$	$c(4,4,4,6) = -\frac{\pi}{6}$
$c(3,3,4,6) = \frac{\pi}{6}$	$c(3,4,4,7) = -\frac{\pi}{21}$	$c(4,4,6,6) = -\frac{\pi}{3}$

$c(3,3,3,3,3,4) = -\frac{\pi}{6}$	$c(3,3,3,4,4,4) = -\frac{\pi}{2}$
$c(3,3,3,3,4,4) = -\frac{\pi}{3}$	$c(3,3,3,4,4,5) = -\frac{3\pi}{5}$
$c(3,3,3,3,4,5) = -\frac{13\pi}{30}$	$c(3,3,3,4,4,6) = -\frac{2\pi}{3}$
$c(3,3,3,3,4,6) = -\frac{\pi}{2}$	$c(3,3,4,4,4,4) = -\frac{2\pi}{3}$

We now give a brief outline of the strategy for the proof of Theorem 1.2.

Suppose first that $\boldsymbol{K}_0 = \boldsymbol{K}$. Our aim is to show that $c(\boldsymbol{K}_0) := \sum_{\Delta \in \boldsymbol{K}_0} c(\Delta) \leq 0$. From this the result follows by essentially known methods (see Section 8). By Lemma 3.4, $\boldsymbol{K}_0$ has no regions of degree 5 and since $c(\Delta) \leq 0$ for $d(\Delta) \geq 6$ it follows that

$$c(\boldsymbol{K}_0) = \sum_{d(\Delta)=4} c(\Delta) + \sum_{d(\Delta)=6} c(\Delta) + \sum_{d(\Delta)\geq 8} c(\Delta) := \Sigma_4 + \Sigma_6 + \Sigma_{8+}.$$

Clearly $\Sigma_6 \leq 0$ and $\Sigma_{8+} < 0$ while Σ_4 may be positive. The idea is to compensate the positive curvature obtained from Σ_4 by the non-positive and negative curvature obtained from Σ_6 and Σ_{8+} in a uniform and systematic way by *distribution of positive curvature rules*, depending on configuration.

Now suppose that $\boldsymbol{K}_0 \neq \boldsymbol{K}$. In this case we delete all vertices and edges in $\boldsymbol{K} \backslash \boldsymbol{K}_0$ to produce a tessellation $\boldsymbol{K}_1$ of S^2 consisting of the union of $\boldsymbol{K}_0$ and a single region Δ_0. Note that Lemma 3.3 holds for $\boldsymbol{K}_1$. Therefore

$$c(\boldsymbol{K}_1) \leq \Sigma_4 + \Sigma_6 + \Sigma_{8+} + c(\Delta_0).$$

The first step of the proof, given in Section 4, will be to locate regions $\Delta \neq \Delta_0$ for which $c(\Delta) > 0$, and so $d(\Delta) = 4$, and distribute $c(\Delta)$ (in a way to be made precise later) to near regions $\hat{\Delta}$ of Δ.

(**Remark**. Throughout the paper Δ or Δ_i will generally be used to denote regions from which positive curvature is transferred, and $\hat{\Delta}$, $\hat{\Delta}_j$ regions that receive positive curvature.)

For the region $\hat{\Delta}$ define $c^*(\hat{\Delta})$ to equal $c(\hat{\Delta})$ plus all the positive curvature $\hat{\Delta}$ receives minus all the curvature $\hat{\Delta}$ distributes as a result of the distribution of positive curvature that has been defined. The main result of Section 4 is (for the cases $\boldsymbol{K}_0 = \boldsymbol{K}$ and $\boldsymbol{K}_0 \neq \boldsymbol{K}$, respectively)

$$c(\boldsymbol{K}_0) \leq \sum_{d(\hat{\Delta})\geq 6} c^*(\hat{\Delta}) \quad \text{or} \quad c(\boldsymbol{K}_1) \leq \sum_{\substack{d(\hat{\Delta})\geq 6 \\ \hat{\Delta}\neq\Delta_0}} c^*(\hat{\Delta}) + c^*(\Delta_0).$$

The second step of the proof, given in Section 5, will be to define a positive curvature distribution scheme for regions of degree 6, that is, we locate regions $\Delta \neq \Delta_0$ of degree 6 for which $c^*(\Delta) > 0$ and distribute $c^*(\Delta)$ to near regions. The main result of Section 5 is

$$c(\boldsymbol{K}_0) \leq \sum_{d(\hat{\Delta})\geq 8} c^*(\hat{\Delta}) \quad \text{or} \quad c(\boldsymbol{K}_1) \leq \sum_{\substack{d(\hat{\Delta})\geq 8 \\ \hat{\Delta}\neq\Delta_0}} c^*(\hat{\Delta}) + c^*(\Delta_0).$$

In Sections 6 and 7 we prove that if $d(\hat{\Delta}) \geq 8$ and $\hat{\Delta} \neq \Delta_0$ then $c^*(\hat{\Delta}) \leq 0$, that is, $c(\boldsymbol{K}_0) \leq 0$ or $c(\boldsymbol{K}_1) \leq c^*(\Delta_0)$.

Finally in Section 8 we complete the proof of Theorem 1.2.

4 Distribution of positive curvature from 4-gons

In the section we will describe the distribution of positive curvature from regions Δ of the diagram $\boldsymbol{K}_0$ such that $c(\Delta) > 0$. It follows from Lemma 3.4 that $d(\Delta) = 4$ and Δ is given by Figure 4.1(i) with neighbouring regions $\hat{\Delta}_i$ $(1 \leq i \leq 4)$ and vertices v_i $(1 \leq i \leq 4)$ which we fix for the remainder of this section. There are 15 cases to consider according to which vertices of Δ have degree 3. Our approach will be to consider neighbouring regions of Δ, the valency and labels of their vertices and, if necessary, the neighbours of these also.

There will be exactly fourteen exceptions to the distribution of positive curvature rules given for the 15 cases. These are contained within six exceptional Configurations A-F and will be fully described later in this section.

Note. In the figures the upper bound of the amount of curvature transferred will generally be indicated.

Note. It should be emphasised that whenever we identify regions, we do so modulo cyclic permutation and inversion. For example in what follows we will identify Δ of Figure 4.15(v) with Δ_1 of Figure 4.27(i).

$\boldsymbol{d(v_i) = 3\ (1 \leq i \leq 4)}$: (here and in what follows we use Lemma 3.3 to classify the possible labellings of the vertices) add $\frac{1}{4}c(\Delta) = \frac{1}{4}c(3,3,3,3) = \frac{\pi}{6}$ to each of $c(\hat{\Delta}_i)$ $(1 \leq i \leq 4)$ as shown in Figure 4.1(ii).

$\boldsymbol{d(v_i) = 3\ (1 \leq i \leq 3)}$ (Figures 4.1(iii)-(vii)): if $d(v_4) > 5$ then add $\frac{1}{2}c(\Delta) \leq \frac{1}{2}c(3,3,3,6) = \frac{\pi}{6}$ to each of $c(\hat{\Delta}_1)$ and $c(\hat{\Delta}_2)$ as in Figure 4.1(iii); if $d(v_4) = 5$ then $c(\Delta) = c(3,3,3,5) = \frac{2\pi}{5}$ in which case add $\frac{\pi}{6}$ to $c(\hat{\Delta}_1), c(\hat{\Delta}_2)$ and $\frac{\pi}{30}$ to $c(\hat{\Delta}_3), c(\hat{\Delta}_4)$ when v_4 is given by Figure 4.1(iv), or add $\frac{\pi}{5}$ to $c(\hat{\Delta}_1), c(\hat{\Delta}_2)$ when v_4 is given by Figure 4.1(v); if $d(v_4) = 4$ then either add $\frac{1}{3}c(\Delta) = \frac{1}{3}c(3,3,3,4) = \frac{\pi}{6}$ to $c(\hat{\Delta}_1)$, $c(\hat{\Delta}_2)$ and $c(\hat{\Delta}_4)$ when v_4 is given by Figure 4.1(vi), or add $\frac{1}{3}c(\Delta) = \frac{\pi}{6}$ to $c(\hat{\Delta}_i)$ $(1 \leq i \leq 3)$ when v_4 is given by Figure 4.1(vii).

$\boldsymbol{d(v_1) = d(v_2) = d(v_4) = 3}$ (Figures 4.2(i)-(iv)): if $d(v_3) > 5$ then add $\frac{1}{2}c(\Delta) \leq \frac{1}{2}c(3,3,3,6) = \frac{\pi}{6}$ to each of $c(\hat{\Delta}_1)$ and $c(\hat{\Delta}_4)$ as shown in Figure 4.2(i); if $d(v_3) = 5$ then $c(\Delta) = c(3,3,3,5) = \frac{2\pi}{5}$ so add $\frac{\pi}{6}$ to each of $c(\hat{\Delta}_1)$ and $c(\hat{\Delta}_4)$, and add the remaining $\frac{\pi}{15}$ to $c(\hat{\Delta}_2)$ as shown in the two possibilities for v_5, Figure 4.2(ii) and (iii); and if $d(v_3) = 4$ then $c(\Delta) = c(3,3,3,4) = \frac{\pi}{2}$ so add $\frac{\pi}{5}$ to each of $c(\hat{\Delta}_1)$ and $c(\hat{\Delta}_4)$, and add the remaining $\frac{\pi}{10}$ to $c(\hat{\Delta}_2)$ as shown in Figure 4.2(iv).

$\boldsymbol{d(v_1) = d(v_3) = d(v_4) = 3}$ (Figures 4.3(i)-(v) and 4.30(ii), (iv)) (**Configurations E, F**): if $d(v_2) > 4$ then add $\frac{1}{2}c(\Delta) \leq \frac{\pi}{5}$ to each of $c(\hat{\Delta}_3)$ and $c(\hat{\Delta}_4)$ as in Figure 4.3(i); and if $d(v_2) = 4$ then either Δ is given by Figure 4.3(ii) in which case add $\frac{1}{3}c(\Delta) = \frac{\pi}{6}$ to each of $c(\hat{\Delta}_2)$, $c(\hat{\Delta}_3)$ and $c(\hat{\Delta}_4)$, or Δ is given by Figure 4.3(iii) in which case add $\frac{1}{3}c(\Delta) = \frac{\pi}{6}$ to each of $c(\hat{\Delta}_1)$, $c(\hat{\Delta}_3)$ and $c(\hat{\Delta}_4)$. There are exactly two exceptions to the rules given here, namely when Δ occurs in Configurations E and F, that is, when Δ (and its neighbourhood) is given by Figures 4.3(iv) and (v). If Δ is given by Figure 4.3(iv) then Δ occurs in Configuration F as Δ_1 in Figure 4.30(iv); and if Δ is given by Figure 4.3(v) then Δ occurs in Configuration E as (the inverse of) Δ_1 in 4.30(ii). In Figure 4.3(iv), (v) the exceptional rules that are applied in Configuration F, E (respectively) are shown in dotted lines.

$\boldsymbol{d(v_i) = 3\ (2 \leq i \leq 4)}$ (Figures 4.3(vi)-(vii)): if $d(v_1) > 4$ then add $\frac{1}{2}c(\Delta) \leq \frac{\pi}{5}$ to each of $c(\hat{\Delta}_2)$ and $c(\hat{\Delta}_3)$ as in Figure 4.3(vi); and if $d(v_1) = 4$ then $c(\Delta) = \frac{\pi}{2}$ so add $\frac{\pi}{5}$ to each of $c(\hat{\Delta}_2)$ and $c(\hat{\Delta}_3)$, and add the remaining $\frac{\pi}{10}$ to $c(\hat{\Delta}_1)$ as in Figure 4.3(vii).

$\boldsymbol{d(v_1) = d(v_2) = 3}$ (Figures 4.4(i)-(viii)): $((d(v_3), d(v_4)) \in \{(4,4), (4,5), (4, \geq 6), (5,4), (\geq 6, 4), (\geq 5, \geq 5)\})$ if $d(v_3) = 4$ and $d(v_4) \geq 6$ or $d(v_3) \geq 5$ and $d(v_4) \geq 5$ or $d(v_3) \geq 6$ and $d(v_4) = 4$ then $c(\Delta) \leq c(3,3,4,6) = \frac{\pi}{6}$ so add $\frac{2\pi}{15}$ to $c(\hat{\Delta}_1)$ and the remaining (at most) $\frac{\pi}{30}$ to $c(\hat{\Delta}_2)$ as in Figure 4.4(i); if $d(v_3) = 5$ and $d(v_4) = 4$ then $c(\Delta) = c(3,3,4,5) = \frac{7\pi}{30}$ so add $\frac{2\pi}{15}$ to $c(\hat{\Delta}_1)$, $\frac{\pi}{15}$ to $c(\hat{\Delta}_2)$ and $\frac{\pi}{30}$ to $c(\hat{\Delta}_3)$ in each of the four possibilities shown in Figure 4.4(ii)–(v); if $d(v_3) = 4$ and $d(v_4) = 5$ then $c(\Delta) = \frac{7\pi}{30}$ so add $\frac{2\pi}{15}$ to $c(\hat{\Delta}_1)$ and the remaining $\frac{\pi}{10}$ to $c(\hat{\Delta}_2)$ as in Figure 4.4(vi); and if $d(v_3) = d(v_4) = 4$ then $c(\Delta) = \frac{\pi}{3}$ so add $\frac{2\pi}{15}$ to $c(\hat{\Delta}_1)$, $\frac{\pi}{10}$ to $c(\hat{\Delta}_2)$ and $\frac{\pi}{10}$ to $c(\hat{\Delta}_3)$ as in Figure 4.4(vii)–(viii) where the two possibilities for Δ are shown.

$\boldsymbol{d(v_2) = d(v_3) = 3}$ (Figures 4.5(i)-(viii)): if $d(v_1) = 4$ and $d(v_4) \geq 6$ or $d(v_1) \geq 5$ and $d(v_4) \geq 5$ or $d(v_1) \geq 6$ and $d(v_4) = 4$ then $c(\Delta) \leq c(3,3,4,6) = \frac{\pi}{6}$ so add $\frac{\pi}{30}$ to $c(\hat{\Delta}_1)$ and the remaining $\frac{2\pi}{15}$ to $c(\hat{\Delta}_2)$ as in Figure 4.5(i); if $d(v_1) = 5$ and $d(v_4) = 4$ then $c(\Delta) = \frac{7\pi}{30}$ so add $\frac{2\pi}{15}$ to $c(\hat{\Delta}_2)$, $\frac{\pi}{15}$ to $c(\hat{\Delta}_4)$ and $\frac{\pi}{30}$ either to $c(\hat{\Delta}_3)$ when Δ is given by Figure 4.5(ii)–(iii), or to $c(\hat{\Delta}_1)$ when Δ is given by Figure 4.5(iv)–(v); if $d(v_1) = 4$ and $d(v_4) = 5$ then $c(\Delta) = \frac{7\pi}{30}$ so add $\frac{\pi}{10}$ to $c(\hat{\Delta}_1)$ and $\frac{2\pi}{15}$ to $c(\hat{\Delta}_2)$ as in Figure 4.5(vi); and if $d(v_1) = d(v_4) = 4$ then $c(\Delta) = \frac{\pi}{3}$ so add $\frac{\pi}{10}$ to each of $c(\hat{\Delta}_1)$ and $c(\hat{\Delta}_4)$ and add $\frac{2\pi}{15}$ to $c(\hat{\Delta}_2)$ as in Figure 4.5(vii)–(viii) where the two possibilities for Δ are shown.

$\boldsymbol{d(v_3) = d(v_4) = 3}$ (Figures 4.6(i), (iii), (iv), 4.27(ii)-(iv), and 4.30(i)) (**Configurations A, C and E**): add $c(\Delta) \leq c(3,3,4,4) = \frac{\pi}{3}$ to $c(\hat{\Delta}_3)$ as in Figure 4.6(i). There are exactly five exceptions to the rules given here. The first three exceptions are when Δ occurs in Configuration A where it is given by Δ_3 of Figure 4.27(ii)-(iv); the fourth exception is when Δ occurs in Configuration C and is given by Figure 4.29(i), and the fifth exception is when Δ occurs in Configuration E and is given by Δ_1 of Figure 4.30(i) (the case $d(v) > 3$). In Figures 4.6(iii) and (iv) the exceptional rules that are applied in Configurations A and C are shown in dotted lines.

$\boldsymbol{d(v_4) = d(v_1) = 3}$ (Figures 4.6(ii), (v), (vi), 4.28(ii)-(iv), and 4.30(iii)) (**Configurations B, D and F**): add $c(\Delta) \leq c(3,3,4,4) = \frac{\pi}{3}$ to $c(\hat{\Delta}_4)$ as in Figure 4.6(ii). There are exactly five exceptions to the rules given here. The first three exceptions are when Δ occurs in Configuration B where it is given by Δ_3 of Figure 4.28(ii)-(iv); the fourth exception is when Δ occurs in Configuration D and is given by Figure 4.29(ii); and the fifth exception is when Δ occurs in Configuration F and is given by Δ_1 of Figure 4.30(iii) (the case $d(v) > 3$). In Figures 4.6(v) and (vi) the exceptional rules that are applied in Configurations B and D are shown in dotted lines.

$\boldsymbol{d(v_2) = d(v_4) = 3}$ (Figures 4.7-4.14): if $d(v_3) = 4$ and $d(v_1) > 5$ then $c(\Delta) \leq c(3,3,4,6) = \frac{\pi}{6}$ so add $\frac{\pi}{15}$ to $c(\hat{\Delta}_1)$ and $\frac{\pi}{10}$ to $c(\hat{\Delta}_2)$ as in Figure 4.7(i); if $d(v_3) = 4$ and $d(v_1) = 5$ then $c(\Delta) = c(3,3,4,5) = \frac{7\pi}{30}$ so add $\frac{\pi}{10}$ to $c(\hat{\Delta}_2)$ and either $\frac{\pi}{15}$ to each of $c(\hat{\Delta}_1)$ and $c(\hat{\Delta}_4)$ when Δ is given by Figure 4.7(ii), or $\frac{2\pi}{15}$ to $c(\hat{\Delta}_1)$ if Δ is given by Figure 4.7(iii); if $d(v_1) = 4$ and $d(v_3) > 5$ then $c(\Delta) \leq \frac{\pi}{6}$ so add $\frac{\pi}{10}$ to $c(\hat{\Delta}_1)$ and $\frac{\pi}{15}$ to $c(\hat{\Delta}_2)$ as in Figure 4.7(iv); if $d(v_1) = 4$ and $d(v_3) = 5$ then $c(\Delta) = \frac{7\pi}{30}$ so add $\frac{\pi}{10}$ to $c(\hat{\Delta}_1)$ and either $\frac{\pi}{15}$ to each of $c(\hat{\Delta}_2)$ and $c(\hat{\Delta}_3)$ if Δ is given by Figure 4.7(v), or add $\frac{2\pi}{15}$ to $c(\hat{\Delta}_2)$ if Δ is given by Figure 4.7(vi); if $d(v_1) \geq 5$ and $d(v_3) \geq 5$ then $c(\Delta) \leq c(3,3,5,5) = \frac{2\pi}{15}$ or $c(\Delta) \leq c(3,3,5,6) = \frac{\pi}{15}$ so add $\frac{\pi}{15}$ to $c(\hat{\Delta}_1)$ if $d(v_1) = 5$ as shown in Figure 4.7(vii)–(viii) and if $d(v_3) = 5$ add $\frac{\pi}{15}$ either to $c(\hat{\Delta}_3)$ when Δ is given by Figure 4.7(ix), or add $\frac{\pi}{15}$ to $c(\hat{\Delta}_2)$ when Δ is given by Figure 4.7(x).

This leaves the case $d(v_1) = d(v_3) = 4$ and $c(\Delta) = c(3,3,4,4) = \frac{\pi}{3}$. Add $\frac{\pi}{10}$ to each of $c(\hat{\Delta}_1)$ and $c(\hat{\Delta}_2)$ as in Figure 4.7(xi) leaving a further $\frac{2\pi}{15}$ to be distributed from $c(\Delta)$. If $d(\hat{\Delta}_3) \geq 6$ and $d(\hat{\Delta}_4) \geq 6$ then add $\frac{\pi}{15}$ to each of $c(\hat{\Delta}_3)$ and $c(\hat{\Delta}_4)$ as shown in Figure 4.7(xi); if $d(\hat{\Delta}_3) = 4$ and $d(\hat{\Delta}_4) > 6$ then add $\frac{2\pi}{15}$ to $c(\hat{\Delta}_4)$ as in Figure 4.7(xii) and if $d(\hat{\Delta}_3) > 6$ and $d(\hat{\Delta}_4) = 4$ then add $\frac{2\pi}{15}$ to $c(\hat{\Delta}_3)$ as in Figure 4.7(xiii). It can be assumed from now on that $(d(\hat{\Delta}_3), d(\hat{\Delta}_4)) \in \{(4,6),(6,4),(4,4)\}$ in which case add $\frac{\pi}{15}$ to each of $c(\hat{\Delta}_3)$ and $c(\hat{\Delta}_4)$ *except in two specific cases* which occur when $d(\hat{\Delta}_3) = d(\hat{\Delta}_4) = 4$ and $d(u_2) \geq 6$ and will be made explicit in what follows (**ex4.1**). (For the reader's benefit we have included a few signposts ex*.* for "exit" and en*.* for "entrance" within the text.) It remains to describe the further transfer of positive curvature (if any) from $c(\hat{\Delta}_3)$ and $c(\hat{\Delta}_4)$.

Let $d(\hat{\Delta}_3) = 4$ and $d(\hat{\Delta}_4) = 6$. This is shown in Figure 4.8(i) in which $d(u_1) \geq 3$ and $d(u_2) \geq 4$. If $c(\hat{\Delta}_3) \leq -\frac{\pi}{15}$ then the $\frac{\pi}{15}$ from $c(\Delta)$ remains with $c(\hat{\Delta}_3)$ as in Figure 4.8(i); and if $-\frac{\pi}{15} < c(\hat{\Delta}_3) \leq 0$ then $\frac{\pi}{15} + c(\hat{\Delta}_3) \leq \frac{\pi}{15}$ is added to $c(\hat{\Delta}_4)$ as in Figure 4.8(ii). We now proceed according to the values of $d(u_1)$ and $d(u_2)$. If $d(u_1) = 4$ and $d(u_2) = 5$ then $(c(\hat{\Delta}_3) = c(3,4,4,5) = \frac{\pi}{15}$ and) $\frac{\pi}{15} + c(\hat{\Delta}_3) = \frac{2\pi}{15}$ so add $\frac{\pi}{15}$ to each of $c(\hat{\Delta}_4)$ and $c(\hat{\Delta}_6)$ as in Figure 4.8(iii); if $d(u_1) = 4 = d(u_2)$ then $\frac{\pi}{15} + c(\hat{\Delta}_3) = \frac{7\pi}{30}$ so add $\frac{\pi}{15}$ to $c(\hat{\Delta}_4)$ and $\frac{\pi}{6}$ to $c(\hat{\Delta}_6)$ as in (iv); if $d(u_1) = 5$ and $d(u_2) = 4$ then add $\frac{\pi}{15} + c(\hat{\Delta}_3) = \frac{2\pi}{15}$ to $c(\hat{\Delta}_4)$ as in (v); if $d(u_1) = 3$ and $d(u_2) \geq 6$ then $\frac{\pi}{15} + c(\hat{\Delta}_3) \leq \frac{7\pi}{30}$ so add $\frac{\pi}{15}$ to $c(\hat{\Delta}_4)$ and $\frac{\pi}{6}$ to $c(\hat{\Delta}_5)$ as in (vi); if $d(u_1) = 3$ and $d(u_2) = 5$ then $\frac{\pi}{15} + c(\hat{\Delta}_3) = \frac{9\pi}{30}$ so add $\frac{\pi}{15}$ to $c(\hat{\Delta}_4)$ and $c(\hat{\Delta}_6)$, and add $\frac{\pi}{6}$ to $c(\hat{\Delta}_5)$ as in (vii); and if $d(u_1) = 3$ and $d(u_2) = 4$ then $\frac{\pi}{15} + c(\hat{\Delta}_3) = \frac{6\pi}{15}$ so add $\frac{2\pi}{15}$ to $c(\hat{\Delta}_4)$ and $\frac{4\pi}{15}$ to $c(\hat{\Delta}_5)$ as in (viii).

Now let $d(\hat{\Delta}_3) = 6$ and $d(\hat{\Delta}_4) = 4$. This is shown in Figure 4.9(i) where $d(u_3) \geq 3$ and $d(u_2) \geq 4$. If $c(\hat{\Delta}_4) \leq -\frac{\pi}{15}$ then the $\frac{\pi}{15}$ from $c(\Delta)$ remains with $c(\hat{\Delta}_4)$ as in Figure 4.9(i); and if $-\frac{\pi}{15} < c(\hat{\Delta}_4) \leq 0$ then $\frac{\pi}{15} + c(\hat{\Delta}_4) \leq \frac{\pi}{15}$ is added to $c(\hat{\Delta}_3)$ as in Figure 4.9(ii). We proceed according to the values of $d(u_2)$ and $d(u_3)$. If $d(u_3) = 4$ and $d(u_2) = 5$ then $\frac{\pi}{15} + c(\hat{\Delta}_4) = \frac{2\pi}{15}$ so add $\frac{\pi}{15}$ to each of $c(\hat{\Delta}_3)$ and $c(\hat{\Delta}_7)$ as in Figure 4.9(iii); if $d(u_3) = 4 = d(u_2)$ then

$\frac{\pi}{15} + c(\hat{\Delta}_4) = \frac{7\pi}{30}$ so add $\frac{\pi}{15}$ to $c(\hat{\Delta}_3)$ and $\frac{\pi}{6}$ to $c(\hat{\Delta}_7)$ as in (iv); if $d(u_3) = 5$ and $d(u_2) = 4$ then add $\frac{\pi}{15} + c(\hat{\Delta}_4) = \frac{2\pi}{15}$ to $c(\hat{\Delta}_3)$ as in (v); if $d(u_3) = 3$ and $d(u_2) \geq 6$ then add $\frac{\pi}{15} + c(\hat{\Delta}_4) \leq \frac{7\pi}{30}$ to $c(\hat{\Delta}_8)$ as in (vi); if $d(u_3) = 3$ and $d(u_2) = 5$ then $\frac{\pi}{15} + c(\hat{\Delta}_4) = \frac{3\pi}{10}$ so add $\frac{\pi}{15}$ to $c(\hat{\Delta}_3)$ and $\frac{7\pi}{30}$ to $c(\hat{\Delta}_8)$ as in (vii); and if $d(u_3) = 3$ and $d(u_2) = 4$ then $\frac{\pi}{15} + c(\hat{\Delta}_4) = \frac{6\pi}{15}$ so add $\frac{2\pi}{15}$ to $c(\hat{\Delta}_3)$ and $\frac{4\pi}{15}$ to $c(\hat{\Delta}_8)$ as in (viii).

Finally let $d(\hat{\Delta}_3) = d(\hat{\Delta}_4) = 4$ as shown in Figure 4.10(i) in which $d(u_1) \geq 3$, $d(u_2) \geq 4$ and $d(u_3) \geq 3$. If $c(\hat{\Delta}_3) \leq -\frac{\pi}{15}$ then the $\frac{\pi}{15}$ from $c(\Delta)$ remains with $c(\hat{\Delta}_3)$ and similarly for $c(\hat{\Delta}_4)$ as shown in Figure 4.10(i). Assume from now on that $c(\hat{\Delta}_3) > -\frac{\pi}{15}$ and $c(\hat{\Delta}_4) > -\frac{\pi}{15}$.

Let $d(u_2) = 4$. If $d(u_1) \geq 6$ then add $\frac{\pi}{15} + c(\hat{\Delta}_3) \leq \frac{\pi}{15}$ to $c(\hat{\Delta}_6)$ as in Figure 4.10(ii); if $d(u_1) = 5$ then add $\frac{\pi}{15} + c(\hat{\Delta}_3) = \frac{2\pi}{15}$ to $c(\hat{\Delta}_6)$ if $l(u_1)$ is given by (iii), or add $\frac{\pi}{15}$ to each of $c(\hat{\Delta}_5)$ and $c(\hat{\Delta}_6)$ if $l(u_1)$ is given by (iv); if $d(u_1) = 4$ then add $\frac{\pi}{15} + c(\hat{\Delta}_3) = \frac{7\pi}{30}$ to $c(\hat{\Delta}_6)$ as in (v); if $d(u_1) = 3$ then $\frac{\pi}{15} + c(\hat{\Delta}_3) = \frac{6\pi}{15}$ so add $\frac{4\pi}{15}$ to $c(\hat{\Delta}_5)$ and $\frac{2\pi}{15}$ to $c(\hat{\Delta}_6)$ as in (vi); if $d(u_3) \geq 6$ then add $\frac{\pi}{15} + c(\hat{\Delta}_4) \leq \frac{\pi}{15}$ to $c(\hat{\Delta}_7)$ as in (ii); if $d(u_3) = 5$ then add $\frac{\pi}{15} + c(\hat{\Delta}_4) = \frac{2\pi}{15}$ to $c(\hat{\Delta}_7)$ if $l(u_3)$ is given by (vii), or add $\frac{\pi}{15}$ to each of $c(\hat{\Delta}_7)$ and $c(\hat{\Delta}_8)$ if $l(u_3)$ is given by (viii); if $d(u_3) = 4$ then add $\frac{\pi}{15} + c(\hat{\Delta}_4) = \frac{7\pi}{30}$ to $c(\hat{\Delta}_7)$ as in (ix); and if $d(u_3) = 3$ then $\frac{\pi}{15} + c(\hat{\Delta}_4) = \frac{6\pi}{15}$ so add $\frac{2\pi}{15}$ to $c(\hat{\Delta}_7)$ and $\frac{4\pi}{15}$ to $c(\hat{\Delta}_8)$ as in (x).

Let $d(u_2) = 5$ and so $l(u_2) = a^5$. If $d(u_1) = 5$ then add $\frac{\pi}{30}$ from $c(\Delta)$ to $c(\hat{\Delta}_3) = c(3,4,5,5) = -\frac{\pi}{30}$ and $\frac{\pi}{30}$ from $c(\Delta)$ to $c(\hat{\Delta}_6)$ as in Figure 4.11(i) and (ii); if $d(u_1) = 4$ then add $\frac{\pi}{15} + c(\hat{\Delta}_3) = \frac{2\pi}{15}$ to $c(\hat{\Delta}_6)$ as in (iii); if $d(u_1) = 3$ then add $\frac{\pi}{15} + c(\hat{\Delta}_3) = \frac{9\pi}{30}$ to $c(\hat{\Delta}_5)$ as in (iv); if $d(u_3) = 5$ then add $\frac{\pi}{30}$ from $c(\Delta)$ to $c(\hat{\Delta}_4)$ and $\frac{\pi}{30}$ from $c(\Delta)$ to $c(\hat{\Delta}_7)$ as in (v) and (vi); if $d(u_3) = 4$ then add $\frac{\pi}{15} + c(\hat{\Delta}_4) = \frac{2\pi}{15}$ to $c(\hat{\Delta}_7)$ as in (vii); and if $d(u_3) = 3$ then add $\frac{\pi}{15} + c(\hat{\Delta}_4) = \frac{3\pi}{10}$ to $c(\hat{\Delta}_8)$ as in (viii).

Let $d(u_2) \geq 6$ so that by assumption $3 \leq d(u_1), d(u_3) \leq 4$ (since $c(3,4,5,6) = -\frac{\pi}{10}$). If $d(u_1) = 3$ then add $\frac{\pi}{15} + c(\hat{\Delta}_3) \leq \frac{7\pi}{30}$ to $c(\hat{\Delta}_5)$ as in Figure 4.12(i); and if $d(u_3) = 3$ then add $\frac{\pi}{15} + c(\hat{\Delta}_4) \leq \frac{7\pi}{30}$ to $c(\hat{\Delta}_8)$ as in (ii). If $d(u_1) = 4$ and $d(\hat{\Delta}_6) > 4$ then add *all* of the $\frac{2\pi}{15}$ from $c(\Delta)$ to $c(\hat{\Delta}_6)$ as in Figure 4.12(iii); otherwise if $d(u_3) = 4$ and $d(\hat{\Delta}_7) > 4$ then add *all* of the $\frac{2\pi}{15}$ to $c(\hat{\Delta}_7)$ as in (iii) (**en4.1**). (*These are the two specific cases mentioned above.*)

This leaves $d(u_1) = d(u_3) = d(\hat{\Delta}_6) = d(\hat{\Delta}_7) = 4$. First consider $\hat{\Delta}_6$ as shown in Figure 4.13. If $d(u_4) > 3$ and $d(u_5) > 3$ then add $\frac{\pi}{15} + c(\hat{\Delta}_3) \leq \frac{\pi}{15}$ to $c(\hat{\Delta}_6) \leq c(4,4,4,6) = -\frac{\pi}{6}$ as in (i); if $d(u_4) = 3$ and $d(u_5) > 3$ then add $\frac{\pi}{15} + c(\hat{\Delta}_3) + c(\hat{\Delta}_6) \leq \frac{\pi}{15}$ to $c(\hat{\Delta}_9)$ as in (ii); if $d(u_4) = 3 = d(u_5)$ then $\frac{\pi}{15} + c(\hat{\Delta}_3) + c(\hat{\Delta}_6) \leq \frac{7\pi}{30}$ so add $\frac{\pi}{10}$ to $c(\hat{\Delta}_9)$ and $\frac{2\pi}{15}$ to $c(\hat{\Delta}_{10})$ as in (iii); if $d(u_4) = 4$, $d(u_5) = 3$ and $d(u_2) > 6$ then $c(\hat{\Delta}_3) \leq -\frac{\pi}{21}$ so add $\frac{\pi}{21}$ from $c(\Delta)$ to $c(\hat{\Delta}_3)$ and the remaining $\frac{\pi}{15} - \frac{\pi}{21} = \frac{2\pi}{105}$ to $c(\hat{\Delta}_6) \leq -\frac{\pi}{21}$ as in (iv); if $d(u_4) = 4$, $d(u_5) = 3$ and $d(u_2) = 6$ then (checking the star graph Γ_0 for possible labels shows that) u_2 is given by (v) in which case add $\frac{\pi}{15} + c(\hat{\Delta}_3) + c(\hat{\Delta}_6) = \frac{\pi}{15}$ to $c(\hat{\Delta}_{11})$ as in (v); and if $d(u_4) > 4$ and $d(u_5) = 3$ then add $\frac{\pi}{15} + c(\hat{\Delta}_3) \leq \frac{\pi}{15}$ to $c(\hat{\Delta}_6) \leq -\frac{\pi}{10}$ as in (vi).

Now consider $\hat{\Delta}_7$ as in Figure 4.14(i). If $d(u_6) > 3$ and $d(u_7) > 3$ then add $\frac{\pi}{15} + c(\hat{\Delta}_4) \leq \frac{\pi}{15}$ to $c(\hat{\Delta}_7) \leq -\frac{\pi}{6}$ as in (i); if $d(u_7) = 3$ and $d(u_6) > 3$ then add $\frac{\pi}{15} + c(\hat{\Delta}_6) \leq \frac{\pi}{15}$ to $c(\hat{\Delta}_{14})$ as in (ii); if $d(u_7) = 3 = d(u_6)$ then $\frac{\pi}{15} + c(\hat{\Delta}_4) + c(\hat{\Delta}_7) = \frac{7\pi}{30}$ so add $\frac{2\pi}{15}$ to $c(\hat{\Delta}_{13})$ and $\frac{\pi}{10}$ to $c(\hat{\Delta}_{14})$ as in (iii); if $d(u_7) = 4$, $d(u_6) = 3$ and $d(u_2) > 6$ then $c(\hat{\Delta}_4) \leq -\frac{\pi}{21}$ so add $\frac{\pi}{21}$ from $c(\Delta)$ to $c(\hat{\Delta}_4)$ and the remaining $\frac{\pi}{15} - \frac{\pi}{21} = \frac{2\pi}{105}$ to $c(\hat{\Delta}_7) \leq -\frac{\pi}{21}$ as in (iv); if $d(u_7) = 4$, $d(u_6) = 3$ and $d(u_2) = 6$ then u_2 is given by (v) in which case add $\frac{\pi}{15} + c(\hat{\Delta}_4) + c(\hat{\Delta}_7) = \frac{\pi}{15}$ to $c(\hat{\Delta}_{12})$ as in (v); and if $d(u_7) > 4$ and $d(u_6) = 3$ then add $\frac{\pi}{15} + c(\hat{\Delta}_4) \leq \frac{\pi}{15}$ to $c(\hat{\Delta}_7) \leq -\frac{\pi}{10}$ as in (vi).

$\boldsymbol{d(v_1) = d(v_3) = 3}$ (Figures 4.15-4.19, 4.27(i) and 4.28(i)) (**Configurations A and B**): if $d(v_2) = d(v_4) = 4$ then add $\frac{1}{2}c(\Delta) = \frac{\pi}{6}$ to each of $c(\hat{\Delta}_2)$ and $c(\hat{\Delta}_4)$ if Δ is given by Figure 4.15(i), or to $c(\hat{\Delta}_1)$ and $c(\hat{\Delta}_4)$ if by (ii), or to $c(\hat{\Delta}_2)$ and $c(\hat{\Delta}_3)$ if by (iii), or to $c(\hat{\Delta}_1)$ and $c(\hat{\Delta}_3)$ if by (iv); if $d(v_2) = 4$ and $d(v_4) = 5$ and v_2 and v_4 are given by (v) then add $c(\Delta) = \frac{7\pi}{30}$ to $c(\hat{\Delta}_2)$ as shown apart from the one exception when Δ occurs in Configuration B in which case Δ is given by Δ_1 of Figure 4.28(i) (and in Figure 4.15(v) the exceptional rule applied in Configuration B is shown in dotted lines); if $d(v_2) = 4$ and $d(v_4) = 5$ and v_2 and v_4 are given by (vi) then add $c(\Delta) = \frac{7\pi}{30}$ to $c(\hat{\Delta}_1)$ as shown apart from the one exception when Δ occurs in Configuration A in which case Δ is given by Δ_1 of Figure 4.27(i) (and in Figure 4.15(vi) the exceptional rule applied in Configuration A is shown in dotted lines); if $d(v_2) = 4$ and $d(v_4) = 5$ and v_2 and v_4 are given by (vii) then add $\frac{2\pi}{15}$ to $c(\hat{\Delta}_2)$, $\frac{\pi}{15}$ to $c(\hat{\Delta}_1)$ and $\frac{\pi}{30}$ to $c(\hat{\Delta}_4)$ as shown; if $d(v_2) = 4$ and $d(v_4) = 5$ and v_2 and v_4 are given by (viii) then add $\frac{2\pi}{15}$ to $c(\hat{\Delta}_1)$, $\frac{\pi}{15}$ to $c(\hat{\Delta}_3)$ and $\frac{\pi}{30}$ to $c(\hat{\Delta}_4)$ as shown; if $d(v_2) = 5$ and $d(v_4) = 4$ then add $c(\Delta) = \frac{7\pi}{30}$ to $c(\hat{\Delta}_4)$ if Δ is given by (ix), or to $c(\hat{\Delta}_3)$ if by (x); if $d(v_2) = 4$ and $d(v_4) \geq 6$ then add $c(\Delta) \leq \frac{\pi}{6}$ to $c(\hat{\Delta}_2)$ if Δ is given by (xi), or to $c(\hat{\Delta}_1)$ if Δ is given by (xii); if $d(v_2) \geq 6$ and $d(v_4) = 4$ then add $c(\Delta) \leq \frac{\pi}{6}$ to $c(\hat{\Delta}_4)$ if Δ is given by (xiii), or to $c(\hat{\Delta}_3)$ if Δ is given by (xiv). This leaves the two cases $d(v_2) \geq 6$, $d(v_4) = 5$ and $d(v_2) = 5$, $d(v_4) \geq 5$ (**ex4.2**).

First let $d(v_2) \geq 6$ and $d(v_4) = 5$. If Δ is given by Figure 4.15(xv) then add $\frac{1}{2}c(\Delta) \leq \frac{\pi}{30}$ to each of $c(\hat{\Delta}_3)$ and $c(\hat{\Delta}_4)$ as shown. Otherwise $l(v_4) = a^5$ and this subcase is now considered using Figures 4.16 and 4.17.

Let $d(v_2) \geq 6$ and $l(v_4) = a^5$. Then $c(\Delta) \leq \frac{\pi}{15}$, half of which ($\leq \frac{\pi}{30}$) is distributed to $c(\hat{\Delta}_1)$ and $c(\hat{\Delta}_4)$ whilst the other half is distributed to $c(\hat{\Delta}_2)$ and $c(\hat{\Delta}_3)$ (**ex4.3**). The distribution of $\frac{1}{2}c(\Delta)$ to $c(\hat{\Delta}_1)$ and $c(\hat{\Delta}_4)$ is as follows. If $d(\hat{\Delta}_1) > 4$ then add $\frac{1}{2}c(\Delta) \leq \frac{\pi}{30}$ to $c(\hat{\Delta}_1)$ as in Figure 4.16(i), or if $d(\hat{\Delta}_1) = 4$ and $d(\hat{\Delta}_4) > 4$ then add $\frac{1}{2}c(\Delta) \leq \frac{\pi}{30}$ to $c(\hat{\Delta}_4)$ again as in (i). It can be assumed therefore that Δ, $\hat{\Delta}_1$ and $\hat{\Delta}_j$ ($4 \leq j \leq 8$) are given by Figure 4.16(ii). We proceed according to $d(u_4) \geq 3$, $d(u_5) \geq 4$, $d(u_6) \geq 3$ of Figure 4.16(ii). If $d(u_6) = 3$, $d(u_5) = 4$ and $d(u_4) \geq 5$ then add $\frac{1}{2}c(\Delta) \leq \frac{\pi}{30}$ to $c(\hat{\Delta}_4) \leq -\frac{\pi}{30}$ as in Figure 4.16(iii); if $d(u_6) = 3$, $d(u_5) = 4$ and $d(u_4) = 4$ then add $\frac{1}{2}c(\Delta) \leq \frac{\pi}{30}$ to $c(\hat{\Delta}_4)$ and then add $\frac{\pi}{30} + c(\hat{\Delta}_4) \leq \frac{\pi}{10}$ to $c(\hat{\Delta}_5)$ as in (iv); if $d(u_6) = 3$, $d(u_5) = 4$ and $d(u_4) = 3$ then add $\frac{1}{2}c(\Delta) \leq \frac{\pi}{30}$ to $c(\hat{\Delta}_4)$ and then add $\frac{\pi}{30} + c(\hat{\Delta}_4) \leq \frac{4\pi}{15}$ to $c(\hat{\Delta}_6)$ as in (v); if $d(u_6) = 3$ and

$d(u_5) = 5$ then add $\frac{1}{2}c(\Delta) \leq \frac{\pi}{30}$ to $c(\hat{\Delta}_1) \leq \frac{\pi}{15}$ and then add $\frac{\pi}{15}$ to $c(\hat{\Delta}_7)$ and add $\frac{\pi}{30}$ to $c(\hat{\Delta}_8)$ as in (vi); if $d(u_6) = 3$ and $d(u_5) \geq 6$ then add $\frac{1}{2}c(\Delta) \leq \frac{\pi}{30}$ to $c(\hat{\Delta}_1) \leq 0$ and then add $\frac{\pi}{30}$ to $c(\hat{\Delta}_8)$ as in (vii); if $d(u_6) = 4$, $d(u_5) = 4$ and $d(v_2) = 7$ (note that $c(3,3,5,8) < 0$) then add $\frac{1}{2}c(\Delta) \leq \frac{\pi}{105}$ to $c(\hat{\Delta}_1) \leq -\frac{\pi}{21}$ as in (viii). Let $d(u_6) = 4$, $d(u_5) = 4$ and $d(v_2) = 6$ so, in particular, $c(\hat{\Delta}_1) = 0$. If u_6 is given by Figure 4.16(ix) then add $\frac{1}{2}c(\Delta) = \frac{\pi}{30}$ to $c(\hat{\Delta}_7)$, so from now on suppose that u_6 is given by Figure 4.16(x). If $d(\hat{\Delta}_8) > 4$ then add $\frac{1}{2}c(\Delta) = \frac{\pi}{30}$ to $\hat{\Delta}_8$ as shown in Figure 4.16(x), so suppose from now on that $d(\hat{\Delta}_8) = 4$. *Suppose that* $\hat{\Delta}_8$ *is given by Figure 4.16(xi).* If $d(u_4) \geq 5$ then add $\frac{1}{2}c(\Delta) \leq \frac{\pi}{30}$ to $c(\hat{\Delta}_4) \leq -\frac{\pi}{30}$ as in Figure 4.16(xii); if $d(u_4) = 4$ then add $\frac{1}{2}c(\Delta) \leq \frac{\pi}{30}$ to $c(\hat{\Delta}_4)$ and then add $\frac{\pi}{30} + c(\hat{\Delta}_4) \leq \frac{\pi}{10}$ to $c(\hat{\Delta}_5)$ as in (xiii); and if $d(u_4) = 3$ then add $\frac{1}{2}c(\Delta) \leq \frac{\pi}{30}$ to $c(\hat{\Delta}_4)$ and then add $\frac{\pi}{30} + c(\hat{\Delta}_4) \leq \frac{4\pi}{15}$ to $c(\hat{\Delta}_6)$ as in (xiv). *Suppose now that* $\hat{\Delta}_8$ *is not given by Figure 4.16(xi).* Then again add $\frac{1}{2}c(\Delta) = \frac{\pi}{30}$ to $\hat{\Delta}_8$ as in Figure 4.16(x). We proceed according to the degrees of the vertices w_1 and w_2 of Figure 4.16(x). If $d(w_1) = d(w_2) = 3$ then $\frac{1}{2}c(\Delta) + c(\hat{\Delta}_8) = \frac{\pi}{5}$ so add $\frac{\pi}{10}$ to $c(\hat{\Delta}_9)$ and $\frac{\pi}{10}$ to $c(\hat{\Delta}_{10})$ as shown in Figure 4.16(xv); if $d(w_1) = 3$ and $d(w_2) > 3$ then $\frac{1}{2}c(\Delta) + c(\hat{\Delta}_8) = \frac{\pi}{30}$ so add $\frac{\pi}{30}$ to $c(\hat{\Delta}_9)$ as shown in (xvi); if $d(w_1) = 4$ and $d(w_2) = 3$ then by assumption $\hat{\Delta}_8$ is given by (xvii) and $c(\hat{\Delta}_8) = 0$ so add $\frac{1}{2}c(\Delta) + c(\hat{\Delta}_8) = \frac{\pi}{30}$ to $c(\hat{\Delta}_{10})$ as shown; and if either $d(w_1) \geq 5$ and $d(w_2) = 3$ or $d(w_1) \geq 4$ and $d(w_2) \geq 4$ then add $\frac{1}{2}c(\Delta) \leq \frac{\pi}{30}$ to $c(\hat{\Delta}_8) \leq -\frac{\pi}{10}$ as shown in (xviii). This completes the subcase $d(u_6) = 4$, $d(u_5) = 4$ and $d(v_2) = 6$. Finally if $d(u_5) \geq 5$ and $d(u_6) \geq 4$ then add $\frac{1}{2}c(\Delta) = \frac{\pi}{30}$ to $c(\hat{\Delta}_1) \leq c(3,4,5,6) = -\frac{\pi}{10}$ as shown in Figure 4.16(xix).

The remaining $\frac{1}{2}c(\Delta) \leq \frac{\pi}{30}$ is distributed to $c(\hat{\Delta}_2)$ and $c(\hat{\Delta}_3)$ as follows (**en4.3**). If $d(\hat{\Delta}_2) > 4$ then add $\frac{1}{2}c(\Delta) \leq \frac{\pi}{30}$ to $c(\hat{\Delta}_2)$ as in Figure 4.16(i), or if $d(\hat{\Delta}_2) = 4$ and $d(\hat{\Delta}_3) > 4$ then add $\frac{1}{2}c(\Delta) \leq \frac{\pi}{30}$ to $c(\hat{\Delta}_3)$ again as in (i). It can be assumed therefore that Δ, $\hat{\Delta}_2$, $\hat{\Delta}_3$ and $\hat{\Delta}_j$ ($5 \leq j \leq 8$) are now given by Figure 4.17(i). We proceed according to $d(u_1) \geq 3$, $d(u_2) \geq 4$, $d(u_3) \geq 3$ of Figure 4.17(i). If $d(u_1) = 3$, $d(u_2) = 4$ and $d(u_3) \geq 5$ then add $\frac{1}{2}c(\Delta) \leq \frac{\pi}{30}$ to $c(\hat{\Delta}_3) \leq -\frac{\pi}{30}$ as in Figure 4.17(ii); if $d(u_1) = 3$, $d(u_2) = 4$ and $d(u_3) = 4$ then add $\frac{1}{2}c(\Delta) \leq \frac{\pi}{30}$ to $c(\hat{\Delta}_3)$ and then add $\frac{\pi}{30} + c(\hat{\Delta}_3) \leq \frac{\pi}{10}$ to $c(\hat{\Delta}_5)$ as in (iii); if $d(u_1) = 3$, $d(u_2) = 4$ and $d(u_3) = 3$ then add $\frac{1}{2}c(\Delta) \leq \frac{\pi}{30}$ to $c(\hat{\Delta}_3)$ and then add $\frac{\pi}{30} + c(\hat{\Delta}_3) \leq \frac{4\pi}{15}$ to $c(\hat{\Delta}_6)$ as in (iv); if $d(u_1) = 3$ and $d(u_2) = 5$ then add $\frac{1}{2}c(\Delta) \leq \frac{\pi}{30}$ to $c(\hat{\Delta}_2) \leq \frac{\pi}{15}$ and add $\frac{\pi}{15}$ to $c(\hat{\Delta}_7)$ and $\frac{\pi}{30}$ to $c(\hat{\Delta}_8)$ as in (v); if $d(u_1) = 3$ and $d(u_2) \geq 6$ then add $\frac{1}{2}c(\Delta) \leq \frac{\pi}{30}$ to $c(\hat{\Delta}_2) \leq 0$ and then add $\frac{\pi}{30}$ to $c(\hat{\Delta}_8)$ as in (vi); if $d(u_1) = 4$, $d(u_2) = 4$ and $d(v_2) = 7$ then add $\frac{1}{2}c(\Delta) \leq \frac{\pi}{105}$ to $c(\hat{\Delta}_2) \leq -\frac{\pi}{21}$ as in (vii). Let $d(u_1) = 4$, $d(u_2) = 4$ and $d(v_2) = 6$ so, in particular, $c(\hat{\Delta}_2) = 0$. If u_2 is given by Figure 4.17(viii) then add $\frac{1}{2}c(\Delta) = \frac{\pi}{30}$ to $c(\hat{\Delta}_7)$, so from now on suppose that u_2 is given by Figure 4.17(ix). If $d(\hat{\Delta}_8) > 4$ then add $\frac{1}{2}c(\Delta) = \frac{\pi}{30}$ to $\hat{\Delta}_8$ as shown in Figure 4.17(ix), so suppose from now on that $d(\hat{\Delta}_8) = 4$. *Suppose that* $\hat{\Delta}_8$ *is given by Figure 4.17(x).* If $d(u_3) \geq 5$ then add $\frac{1}{2}c(\Delta) \leq \frac{\pi}{30}$ to $c(\hat{\Delta}_3) \leq -\frac{\pi}{30}$ as in Figure 4.17(xi); if $d(u_3) = 4$ then add $\frac{1}{2}c(\Delta) \leq \frac{\pi}{30}$ to $c(\hat{\Delta}_3)$ and then add $\frac{\pi}{30} + c(\hat{\Delta}_3) \leq \frac{\pi}{10}$ to $c(\hat{\Delta}_5)$ as in (xii); and if $d(u_3) = 3$ then add $\frac{1}{2}c(\Delta) \leq \frac{\pi}{30}$ to $c(\hat{\Delta}_3)$ and then add $\frac{\pi}{30} + c(\hat{\Delta}_3) \leq \frac{4\pi}{15}$

to $c(\hat{\Delta}_6)$ as in (xiii). *Suppose now that $\hat{\Delta}_8$ is not given by Figure 4.17(x).* Then again add $\frac{1}{2}c(\Delta) = \frac{\pi}{30}$ to $\hat{\Delta}_8$ as in Figure 4.17(ix). We proceed according to the degrees of the vertices w_3 and w_4 of Figure 4.17(ix). If $d(w_3) = d(w_4) = 3$ then $\frac{1}{2}c(\Delta) + c(\hat{\Delta}_8) = \frac{\pi}{30} + \frac{\pi}{6} = \frac{\pi}{5}$ so add $\frac{\pi}{10}$ to $c(\hat{\Delta}_9)$ and $\frac{\pi}{10}$ to $c(\hat{\Delta}_{10})$ as shown in Figure 4.17(xiv); if $d(w_3) = 3$ and $d(w_4) > 3$ then $\frac{1}{2}c(\Delta) + c(\hat{\Delta}_8) = \frac{\pi}{30}$ so add $\frac{\pi}{30}$ to $c(\hat{\Delta}_9)$ as shown in (xv); if $d(w_3) = 4$ and $d(w_4) = 3$ then by assumption $\hat{\Delta}_8$ is given by (xvi) and $c(\hat{\Delta}_8) = 0$ so add $\frac{1}{2}c(\Delta) = \frac{\pi}{30}$ to $c(\hat{\Delta}_{10})$ as shown; and if either $d(w_3) \geq 5$ and $d(w_4) = 3$ or $d(w_3) \geq 4$ and $d(w_4) \geq 4$ then add $\frac{1}{2}c(\Delta) \leq \frac{\pi}{30}$ to $c(\hat{\Delta}_8) \leq -\frac{\pi}{10}$ as shown in (xvii). This completes the subcase $d(u_1) = 4$, $d(u_2) = 4$ and $d(v_2) = 6$. Finally if $d(u_1) \geq 4$ and $d(u_2) \geq 5$ then add $\frac{1}{2}c(\Delta) = \frac{\pi}{30}$ to $c(\hat{\Delta}_2) \leq c(3,4,5,6) = -\frac{\pi}{10}$ as shown in Figure 4.17(xviii).

Finally let $d(v_2) = 5$ and $d(v_4) \geq 5$ (**en4.2**). If Δ is given by Figure 4.15(xvi) then add $\frac{1}{2}c(\Delta) \leq \frac{\pi}{15}$ to each of $c(\hat{\Delta}_1)$ and $c(\hat{\Delta}_2)$. Otherwise $l(v_2) = bx^{-1}\lambda z^{-1}y$ and Δ is given by Figure 4.15(xvii). Here add $\frac{1}{2}c(\Delta) \leq \frac{\pi}{15}$ to $c(\hat{\Delta}_4)$ if $d(\hat{\Delta}_4) > 4$, otherwise add $\frac{1}{2}c(\Delta) \leq \frac{\pi}{15}$ to $c(\hat{\Delta}_1)$; and add $\frac{1}{2}c(\Delta) \leq \frac{\pi}{15}$ to $c(\hat{\Delta}_3)$ if $d(\hat{\Delta}_3) > 4$, otherwise add $\frac{1}{2}c(\Delta) \leq \frac{\pi}{15}$ to $c(\hat{\Delta}_2)$. If $\frac{1}{2}c(\Delta) \leq \frac{\pi}{15}$ is added to $c(\hat{\Delta}_1)$ and $d(\hat{\Delta}_1) > 4$ there is no further distribution of curvature from $\hat{\Delta}_1$ and the same statement holds for $\hat{\Delta}_2$. This leaves the subcases $d(\hat{\Delta}_1) = d(\hat{\Delta}_4) = 4$ and $d(\hat{\Delta}_2) = d(\hat{\Delta}_3) = 4$.

Assume first that $d(\hat{\Delta}_1) = d(\hat{\Delta}_4) = 4$ in Figure 4.15(vii). Then Δ is given by Figure 4.18(i) where $d(u_5) \geq 4$. If $d(u_5) = 4$ and $d(u_6) = 3$ then $\frac{\pi}{15} + c(\hat{\Delta}_1) \leq \frac{3\pi}{10}$ so add $\frac{\pi}{10}$ to $c(\hat{\Delta}_7)$ and $\frac{\pi}{5}$ to $c(\hat{\Delta}_8)$ as in Figure 4.18(ii); if $d(u_5) = 4$ and $d(u_6) = 4$ then add $\frac{1}{2}(\frac{\pi}{15} + c(\hat{\Delta}_1)) \leq \frac{\pi}{15}$ to each of $c(\hat{\Delta}_7)$ and $c(\hat{\Delta}_8)$ if u_6 is given by (iii), or add $\frac{\pi}{15} + c(\hat{\Delta}_1) \leq \frac{2\pi}{15}$ to $c(\hat{\Delta}_8)$ if u_6 is given by (iv); if $d(u_5) = 4$ and $d(u_6) = 5$ then $c(\hat{\Delta}_1) = -\frac{\pi}{30}$ so add $\frac{\pi}{15} + c(\hat{\Delta}_1) \leq \frac{\pi}{30}$ to $c(\hat{\Delta}_8)$ as in (v); if $d(u_5) = 4$ and $d(u_6) \geq 6$ then add $\frac{1}{2}c(\Delta) \leq \frac{\pi}{15}$ to $c(\hat{\Delta}_1) \leq -\frac{\pi}{10}$ as in (vi); if $d(u_5) = 5$ and $d(u_6) = 3$ then $\frac{\pi}{15} + c(\hat{\Delta}_1) \leq \frac{\pi}{5}$ so add $\frac{\pi}{15}$ to $c(\hat{\Delta}_7)$ and $\frac{2\pi}{15}$ to $c(\hat{\Delta}_8)$ as in (vii); if $d(u_5) = 5$ and $d(u_6) = 4$ then $c(\hat{\Delta}_1) \leq -\frac{\pi}{30}$ so add $\frac{\pi}{15} + c(\hat{\Delta}_1) \leq \frac{\pi}{30}$ to $c(\hat{\Delta}_8)$ as shown in the two possibilities for u_6, namely (viii) and (ix); if $d(u_5) = 5$ and $d(u_6) \geq 5$ then add $\frac{1}{2}c(\Delta) \leq \frac{\pi}{15}$ to $c(\hat{\Delta}_1) \leq -\frac{2\pi}{15}$ as in (x); if $d(u_5) > 5$ and $d(u_6) = 3$ then add $\frac{\pi}{15} + c(\hat{\Delta}_1) \leq \frac{2\pi}{15}$ to $c(\hat{\Delta}_8)$ as in (xi); and if $d(u_5) > 5$ and $d(u_6) > 3$ then add $\frac{1}{2}c(\Delta) \leq \frac{\pi}{15}$ to $c(\hat{\Delta}_1) \leq -\frac{\pi}{10}$ as in (xii).

Now assume that $d(\hat{\Delta}_2) = d(\hat{\Delta}_3) = 4$ in Figure 4.15(vii). Then Δ is given by Figure 4.19(i). We proceed according to $d(u_1) \geq 3$ and $d(u_2) \geq 4$. If $d(u_2) = 4$ and $d(u_1) = 3$ then $\frac{\pi}{15} + c(\hat{\Delta}_2) \leq \frac{3\pi}{10}$ so add $\frac{\pi}{5}$ to $c(\hat{\Delta}_9)$ and $\frac{\pi}{10}$ to $c(\hat{\Delta}_{10})$ as in Figure 4.19(ii); if $d(u_2) = 4$ and $d(u_1) = 4$ then add $\frac{1}{2}(\frac{\pi}{15} + c(\hat{\Delta})) \leq \frac{\pi}{15}$ to each of $c(\hat{\Delta}_9)$ and $c(\hat{\Delta}_{10})$ if u_1 is given by (iii), or $\frac{\pi}{15} + c(\hat{\Delta}_2) \leq \frac{2\pi}{15}$ to $c(\hat{\Delta}_9)$ if u_1 is given by (iv); if $d(u_2) = 4$ and $d(u_1) = 5$ then $c(\hat{\Delta}_2) = -\frac{\pi}{30}$ so $\frac{\pi}{15} + c(\hat{\Delta}_2) \leq \frac{\pi}{30}$ is added to $c(\hat{\Delta}_9)$ as shown in (v); if $d(u_2) = 4$ and $d(u_1) \geq 6$ then add $\frac{1}{2}c(\Delta) \leq \frac{\pi}{15}$ to $c(\hat{\Delta}_2) \leq -\frac{\pi}{10}$ as in (vi); if $d(u_2) = 5$ and $d(u_1) = 3$ then $\frac{\pi}{15} + c(\hat{\Delta}_2) \leq \frac{\pi}{5}$ so add $\frac{2\pi}{15}$ to $c(\hat{\Delta}_9)$ and $\frac{\pi}{15}$ to $c(\hat{\Delta}_{10})$ as in (vii); if $d(u_2) = 5$ and $d(u_1) = 4$ then $c(\hat{\Delta}_2) = -\frac{\pi}{30}$ so add $\frac{\pi}{15} + c(\hat{\Delta}_2) = \frac{\pi}{30}$ to $c(\hat{\Delta}_9)$ as shown in the two possibilities (viii) and (ix); if $d(u_2) = 5$

and $d(u_1) > 4$ then add $\frac{1}{2}c(\Delta) \leq \frac{\pi}{15}$ to $c(\hat{\Delta}_2) \leq -\frac{2\pi}{15}$ as in (x); if $d(u_2) > 5$ and $d(u_1) = 3$ then add $\frac{\pi}{15} + c(\hat{\Delta}_2) \leq \frac{2\pi}{15}$ to $c(\hat{\Delta}_9)$ as in (xi); and if $d(u_2) > 5$ and $d(u_1) > 3$ then add $\frac{1}{2}c(\Delta) \leq \frac{\pi}{15}$ to $c(\hat{\Delta}_2) \leq -\frac{\pi}{10}$ as in (xii).

$\boldsymbol{d(v_1) = 3}$ (Figure 4.20): if $d(v_3) = 5$ then add $c(\Delta) = \frac{\pi}{15}$ to $c(\hat{\Delta}_3)$ if Δ is given by Figure 4.20 (i) or (ii), or add $c(\Delta) = \frac{\pi}{15}$ to $c(\hat{\Delta}_4)$ if Δ is given by (iii); if $d(v_4) = 5$ then add $c(\Delta) = \frac{\pi}{15}$ to $c(\hat{\Delta}_3)$ if Δ is given by (iv), or add $c(\Delta) = \frac{\pi}{15}$ to $c(\hat{\Delta}_4)$ if Δ is given by (v). Let $d(v_3) = d(v_4) = 4$. If (reading clockwise from Δ) $l(v_4) = a^{-2}\lambda z^{-1}$ and $d(v_2) = 4$ then add $\frac{\pi}{15}$ to $c(\hat{\Delta}_3)$ and $c(\Delta) - \frac{\pi}{15} = \frac{\pi}{10}$ to $c(\hat{\Delta}_4)$ as in Figure 4.20(vi); if $l(v_4) = a^{-2}\lambda z^{-1}$ and $d(v_2) = 5$ then add $c(\Delta) = \frac{\pi}{15}$ to $c(\hat{\Delta}_4)$ as in (vii); if $l(v_4) = a^{-1}\lambda z^{-1}a^{-1}$ and $d(v_2) = 4$ then add $c(\Delta) = \frac{\pi}{6}$ to $c(\hat{\Delta}_3)$ as in (viii); and if $l(v_4) = a^{-1}\lambda z^{-1}a^{-1}$ and $d(v_2) = 5$ then add $c(\Delta) = \frac{\pi}{15}$ to $c(\hat{\Delta}_3)$ as in (ix).

$\boldsymbol{d(v_2) = 3}$ (Figure 4.21): if $d(v_1) = 5$ then add $c(\Delta) = \frac{\pi}{15}$ to $c(\hat{\Delta}_2)$ as in Figure 4.21(i); if $d(v_3) = 5$ then add $c(\Delta) = \frac{\pi}{15}$ to $c(\hat{\Delta}_1)$ as in (ii); and if $d(v_1) = d(v_3) = 4$ then add $\frac{1}{2}c(\Delta) \leq \frac{\pi}{12}$ to each of $c(\hat{\Delta}_1)$ and $c(\hat{\Delta}_2)$ as in (iii).

$\boldsymbol{d(v_3) = 3}$ (Figure 4.22): if $d(v_1) = 5$ then add $c(\Delta) = \frac{\pi}{15}$ to $c(\hat{\Delta}_3)$ if Δ is given by Figure 4.22(i), or add $c(\Delta) = \frac{\pi}{15}$ to $c(\hat{\Delta}_4)$ if Δ is given by (ii) or (iii); if $d(v_4) = 5$ then add $c(\Delta) = \frac{\pi}{15}$ to $c(\hat{\Delta}_4)$ as shown in (iv) and (v). Let $d(v_1) = d(v_4) = 4$. If (reading clockwise from Δ) $l(v_4) = a^{-2}\lambda z^{-1}$ and $d(v_2) = 4$ then add $c(\Delta) = \frac{\pi}{6}$ to $c(\hat{\Delta}_4)$ as in Figure 4.22(vi); if $l(v_4) = a^{-2}\lambda z^{-1}$ and $d(u_2) = 5$ then add $c(\Delta) = \frac{\pi}{15}$ to $c(\hat{\Delta}_4)$ as in (vii); if $l(v_4) = a^{-1}\lambda z^{-1}a^{-1}$ and $d(v_2) = 4$ then add $\frac{\pi}{10}$ to $c(\hat{\Delta}_3)$ and $c(\Delta) - \frac{\pi}{10} = \frac{\pi}{15}$ to $c(\hat{\Delta}_4)$ as in (viii); and if $l(v_4) = a^{-1}\lambda z^{-1}a^{-1}$ and $d(v_2) = 5$ then add $c(\Delta) = \frac{\pi}{15}$ to $c(\hat{\Delta}_3)$ as in (ix).

$\boldsymbol{d(v_4) = 3}$ (Figures 4.23-4.26): if $d(v_1) = 5$ then add $c(\Delta) = \frac{\pi}{15}$ to $c(\hat{\Delta}_4)$ if Δ is given by Figure 4.23(i), or to $c(\hat{\Delta}_1)$ if Δ is given by (ii) or (iii); if $d(v_3) = 5$ then add $c(\Delta) = \frac{\pi}{15}$ to $c(\hat{\Delta}_3)$ if Δ is given by (iv), or to $c(\hat{\Delta}_2)$ if Δ is given by (v) or (vi); if $d(v_1) = d(v_3) = 4$ and $d(v_2) = 4$ then add $c(\Delta) = \frac{\pi}{6}$ to $c(\hat{\Delta}_1)$ if Δ is given by (vii), or to $c(\hat{\Delta}_2)$ if Δ is given by (viii); and if $d(v_1) = d(v_3) = 4$ and $l(v_2) = bx^{-1}\lambda z^{-1}y$ then add $c(\Delta) = \frac{\pi}{15}$ to $c(\hat{\Delta}_1)$ as in (ix).

This leaves the case when $d(v_1) = d(v_3) = 4$ and $l(v_2) = b^5$. If $d(\hat{\Delta}_3) \geq 6$ and $d(\hat{\Delta}_4) \geq 6$ then add $\frac{1}{2}c(\Delta) = \frac{\pi}{30}$ to each of $c(\hat{\Delta}_3)$ and $c(\hat{\Delta}_4)$ as shown in Figure 4.23(x); if $d(\hat{\Delta}_3) \geq 6$ and $d(\hat{\Delta}_4) = 4$ then add $\frac{\pi}{15}$ to $c(\hat{\Delta}_3)$ as in (xi); and if $d(\hat{\Delta}_3) = 4$ and $d(\hat{\Delta}_4) \geq 6$ then add $\frac{\pi}{15}$ to $c(\hat{\Delta}_4)$ as in (xii). Assume from now on that $d(\hat{\Delta}_3) = d(\hat{\Delta}_4) = 4$ as shown in Figure 4.24. If $c(\hat{\Delta}_3) \leq -\frac{\pi}{15}$ then add $c(\Delta) = \frac{\pi}{15}$ to $c(\hat{\Delta}_3)$, or if $c(\hat{\Delta}_4) \leq -\frac{\pi}{15}$ then add $c(\Delta) = \frac{\pi}{15}$ to $c(\hat{\Delta}_4)$ as shown in Figure 4.24(i). Assume from now on that $c(\hat{\Delta}_3) > -\frac{\pi}{15}$ and $c(\hat{\Delta}_4) > -\frac{\pi}{15}$. We proceed according to $d(u_2) \geq 4$, $d(u_1) \geq 3$ and $d(u_3) \geq 3$.

Let $d(u_2) = 4$. If $d(u_1) \geq 6$ then add $c(\Delta) + c(\hat{\Delta}_3) \leq \frac{\pi}{15}$ to $c(\hat{\Delta}_6)$ as in Figure 4.24(ii); if $d(u_1) = 5$ then add $c(\Delta) + c(\hat{\Delta}_3) = \frac{2\pi}{15}$ to $c(\hat{\Delta}_6)$ if $\hat{\Delta}_3$ is given by (iii), or add $\frac{1}{2}(c(\Delta) +$

$c(\hat{\Delta}_3)) = \frac{\pi}{15}$ to each of $c(\hat{\Delta}_5)$ and $c(\hat{\Delta}_6)$ if $\hat{\Delta}_3$ is given by (iv); if $d(u_1) = 4$ then add $c(\Delta) + c(\hat{\Delta}_3) = \frac{7\pi}{30}$ to $c(\hat{\Delta}_6)$ as in (v); and if $d(u_1) = 3$ then $c(\Delta) + c(\hat{\Delta}_3) = \frac{6\pi}{15}$ so add $\frac{4\pi}{15}$ to $c(\hat{\Delta}_5)$ and $\frac{2\pi}{15}$ to $c(\hat{\Delta}_6)$ as in (vi).

Let $d(u_2) = 5$ in which case $l(u_2) = a^5$. In this case add $\frac{1}{2}c(\Delta) = \frac{\pi}{30}$ to each of $c(\hat{\Delta}_3)$ and $c(\hat{\Delta}_4)$. If $d(u_1) \geq 5$ then $c(\hat{\Delta}_3) \leq c(3,4,5,5) = -\frac{\pi}{30}$ and the $\frac{\pi}{30}$ from $c(\Delta)$ remains with $c(\hat{\Delta}_3)$ as shown in Figure 4.25(i); if $d(u_1) = 4$ then add $\frac{\pi}{30} + c(\hat{\Delta}_3) = \frac{\pi}{10}$ to $c(\hat{\Delta}_6)$ as in (ii); and if $d(u_1) = 3$ then add $\frac{\pi}{30} + c(\hat{\Delta}_3) = \frac{4\pi}{15}$ to $c(\hat{\Delta}_5)$ as in (iii). If $d(u_3) \geq 5$ then $c(\hat{\Delta}_4) \leq -\frac{\pi}{30}$ and the $\frac{\pi}{30}$ from $c(\Delta)$ remains with $c(\hat{\Delta}_4)$ as shown in Figure 4.25(i); if $d(u_3) = 4$ then add $\frac{\pi}{30} + c(\hat{\Delta}_4) = \frac{\pi}{10}$ to $c(\hat{\Delta}_7)$ as in (iv); and if $d(u_3) = 3$ then add $\frac{\pi}{30} + c(\hat{\Delta}_4) = \frac{4\pi}{15}$ to $c(\hat{\Delta}_8)$ as in (v).

Finally let $d(u_2) \geq 6$. If $d(u_1) > 4$ then add $c(\Delta) = \frac{\pi}{15}$ to $c(\hat{\Delta}_3) \leq -\frac{\pi}{10}$; or if $d(u_3) > 4$ then add $c(\Delta) = \frac{\pi}{15}$ to $c(\hat{\Delta}_4) \leq -\frac{\pi}{10}$ as in Figure 4.25(vi). If $d(u_1) = 3$ then add $c(\Delta) + c(\hat{\Delta}_3) \leq \frac{\pi}{15} + \frac{\pi}{6} = \frac{7\pi}{30}$ to $c(\hat{\Delta}_5)$; or if $d(u_3) = 3$ then add $c(\Delta) + c(\hat{\Delta}_4) \leq \frac{7\pi}{30}$ to $c(\hat{\Delta}_8)$ as shown in Figure 4.25(vii). This leaves $d(u_1) = d(u_3) = 4$. If $d(u_2) \geq 7$ then add $\frac{1}{2}c(\Delta) = \frac{\pi}{30}$ to each of $c(\hat{\Delta}_3) \leq -\frac{\pi}{21}$ and $c(\hat{\Delta}_4) \leq -\frac{\pi}{21}$ as in Figure 4.25(viii), so assume that $d(u_2) = 6$. If $d(\hat{\Delta}_6) > 4$ then add $c(\Delta) = \frac{\pi}{15}$ to $c(\hat{\Delta}_6)$, or if $d(\hat{\Delta}_7) > 4$ then add $c(\Delta) = \frac{\pi}{15}$ to $c(\hat{\Delta}_7)$ as in Figure 4.25(ix). It can be assumed that $d(\hat{\Delta}_6) = d(\hat{\Delta}_7) = 4$ which forces $l(u_2) = aaxy^{-1}xy^{-1}$ as shown in Figure 4.26(i). If $d(u_4) > 3$ and $d(u_5) > 3$ in Figure 4.26(i) then add $c(\Delta) + c(\hat{\Delta}_3) = \frac{\pi}{15}$ to $c(\hat{\Delta}_6) \leq -\frac{\pi}{6}$ as shown; if $d(u_4) = 3$ and $d(u_5) > 3$ then add $c(\Delta) + c(\hat{\Delta}_3) + c(\hat{\Delta}_6) \leq \frac{\pi}{15}$ to $c(\hat{\Delta}_9)$ as in (ii); if $d(u_4) = d(u_5) = 3$ then $c(\Delta) + c(\hat{\Delta}_3) + c(\hat{\Delta}_6) = \frac{7\pi}{30}$ and so add $\frac{\pi}{10}$ to $c(\hat{\Delta}_9)$, $\frac{\pi}{15}$ to $c(\hat{\Delta}_{10})$ and $\frac{\pi}{15}$ to $c(\hat{\Delta}_{11})$ as in (iii); and if $d(u_4) > 3$ and $d(u_5) = 3$ then add $c(\Delta) + c(\hat{\Delta}_3) + c(\hat{\Delta}_6) = \frac{\pi}{15}$ to $c(\hat{\Delta}_{11})$ as in (iv).

This completes the description of distribution of curvature from Δ when $d(\Delta) = 4$ except for six exceptional configurations which we now describe and for which there is an amendment to the rules given above.

Configurations A and B: These are shown in Figures 4.27(i) and 4.28(i) where $c(\Delta_1) = \frac{7\pi}{30}$ and $c(\Delta_3) = \frac{\pi}{3}$. The region Δ_1 of Figure 4.27(i) corresponds to (the inverse of) the region Δ of Figure 4.15(vi); Δ_1 of Figure 4.28(i) to the region Δ of Figure 4.15(v); Δ_3 of Figure 4.27(i) to the region Δ of Figure 4.6(i) and (iii); and Δ_3 of Figure 4.28(i) to the region Δ of Figure 4.6(ii) and (v). The new rule is: add $\frac{\pi}{5}$ from $c(\Delta_1)$ to $c(\hat{\Delta})$ and add $\frac{\pi}{30}$ from $c(\Delta_1)$ to $c(\hat{\Delta}_1)$ as shown except when the neighbouring regions of Δ_3 are given by Figure 4.27(ii)-(iv) and Figure 4.28(ii)-(iv). There it is assumed that $\hat{\Delta}_2$ receives $\frac{\pi}{5}$ from Δ_4. In these cases add all of $c(\Delta_1) = \frac{7\pi}{30}$ to $c(\hat{\Delta})$ as usual, and the new rule is: add $\frac{3\pi}{10}$ from $c(\Delta_3)$ to $c(\hat{\Delta})$ and add $\frac{\pi}{30}$ from $c(\Delta_3)$ to $c(\hat{\Delta}_3)$ as shown in Figure 4.27(ii)-(iv) and Figure 4.28(ii)-(iv). If $d(\hat{\Delta}_3) = 4$ then add $\frac{\pi}{30} + c(\hat{\Delta}_3) = \frac{\pi}{10}$ to $c(\hat{\Delta}_4)$ as shown in Figure 4.27(ii) and Figure 4.28(ii). Note that Δ_4 of Figure 4.27(ii) and (iii) and Δ_4 of Figure 4.28(ii) and (iii) is given by Δ of Figure 4.1(v); Δ_4 of Figure 4.27(iv) is given by Figures 4.3(vi), (vii); and Δ_4 of Figure 4.28(iv) is given by Figure 4.2(iv). Note further that it is assumed that

$d(\hat{\Delta}_3) \neq 4$ in Figure 4.27(iii) and Figure 4.28(iii), in which case $\hat{\Delta}_3$ is not given by Figure 3.6(ii) or (iii) and so $d(\hat{\Delta}_3) \geq 8$ which also holds for Figures 4.27(iv) and 4.28(iv).

Configurations C and D: These are shown in Figure 4.29 where $c(\Delta) = \frac{\pi}{3}$. The region Δ of Figure 4.29(i) corresponds to the region Δ of Figure 4.6(i) and (iv); and Δ of Figure 4.29(ii) to the region Δ of Figure 4.6(ii) and (vi). In both cases the new rule is: add $\frac{3\pi}{10}$ from $c(\Delta)$ to $c(\hat{\Delta})$ and add $\frac{\pi}{30}$ from $c(\Delta)$ to $c(\hat{\Delta}_1)$.

Configurations E and F: These are shown in Figure 4.30. For each configuration there are two cases, namely when $d(v) \geq 4$ and when $d(v) = 3$ for the vertex v indicated. If $d(v) = 3$ then Δ_1 of Configuration E, Figure 4.30(ii) corresponds to the region Δ of Figure 4.3(v); and Δ_1 of Configuration F, Figure 4.30(iv) corresponds to the region Δ of Figure 4.3(iv). If $d(v) > 3$ then either we are in the case $d(v_3) = d(v_4) = 3$ only and Δ_1 of Configuration E, Figure 4.30(i) corresponds to the region Δ of Figure 4.6(i); or we are in the case $d(v_1) = d(v_4) = 3$ only and Δ_1 of Configuration F, Figure 4.30(iii) corresponds to Δ of Figure 4.6(ii). In Figure 4.30(i) and (iii) $c(\hat{\Delta})$ receives at most $\frac{\pi}{3}$ from $c(\Delta_1)$; and in (ii) and (iv) the rules above (Figure 4.3(ii)-(iii)) state that $c(\hat{\Delta})$ receives $\frac{\pi}{6}$ from $c(\Delta_1)$. The new rules are: add $\min\{c(\Delta_1), \frac{\pi}{5}\}$ from $c(\Delta_1)$ to $c(\hat{\Delta}_1)$ via $\hat{\Delta}$ across the edge shown in Figure 4.30(i) and (iii); add $\frac{\pi}{5}$ from $c(\Delta_1)$ to $c(\hat{\Delta}_1)$ across the edge shown in Figure 4.30(ii) and (iv); and add (at most) $\frac{2\pi}{15}$ from $c(\Delta_1)$ to $c(\hat{\Delta})$ across the edge shown in Figure 4.30(i)–(iv). Observe that $d(\hat{\Delta}_1) \geq 8$ in Figure 4.30.

Lemma 4.1 *Let $\hat{\Delta}$ be a region of degree 4 that receives positive curvature across at least one edge. Then one of the following occurs.*

(i) $c^*(\hat{\Delta}) \leq 0$;

(ii) $c^*(\hat{\Delta}) > 0$ *is distributed to a region of degree* > 4;

(iii) $c^*(\hat{\Delta}) > 0$ *is distributed to a region* Δ' *of degree 4 and either* $c^*(\Delta') \leq 0$ *or* $c^*(\Delta') > 0$ *is distributed to a region of degree* > 4.

Proof. Let $d(\hat{\Delta}) = 4$. If $\hat{\Delta}$ receives positive curvature across at least one edge then inspection of Figures 4.1–4.30 shows that either $d(v_2) = d(v_4) = 3$ in Δ and $\hat{\Delta}$ occurs in Figures 4.8–4.14 in which case we say that $\hat{\Delta}$ is a *T24 region*; or $d(v_1) = d(v_3) = 3$ in Δ and $\hat{\Delta}$ occurs in Figures 4.16–4.19 in which case $\hat{\Delta}$ is a *T13 region*; or $\hat{\Delta}$ occurs in Figures 4.24–4.26 and $\hat{\Delta}$ is a *T4 region*; or $\hat{\Delta}$ is the region $\hat{\Delta}_3$ of Figure 4.27(ii) or 4.28(ii). In all other cases, that is, Figures 4.1–4.7, 4.15, 4.20–4.23, 4.27(i), (iii) and (iv), 4.28(i), (iii) and (iv), 4.29 and 4.30 there is no region $\hat{\Delta}$ of degree 4 that receives positive curvature. For these cases the statements of the Lemma trivially hold and so they will not be considered for the rest of the proof. Thus we have: every region that contributes positive curvature to $\hat{\Delta}$ is exactly one of the types T24, T13, T4 or $\hat{\Delta}_3$ of Figures 4.27(ii), 4.28(ii).

We divide the proof of the lemma into two parts. The first (easy and short) deals with the cases when $\hat{\Delta}$ receives positive curvature across exactly one edge and the second part deals with the cases in which $\hat{\Delta}$ receives positive curvature across at least two edges.

If $\hat{\Delta}$ receives positive curvature across exactly one edge then we see by inspection of Figures 4.8-14, 4.16-19, 4.24-26, 4.27(ii) and 4.28(ii) that in all cases either $c^*(\hat{\Delta}) \leq 0$ or $c^*(\hat{\Delta})$ is distributed from $\hat{\Delta}$ to a neighbouring region of degree > 4 except when $\hat{\Delta}$ is given by Figures 4.13, 4.14, 4.16, 4.17 and 4.26 where $c^*(\hat{\Delta})$ is initially distributed further to a region Δ' of degree 4. But in each of these cases either $c^*(\Delta') \leq 0$ (under the assumption that Δ' receives positive curvature across exactly one edge – the case when Δ' may receive across more than one edge is considered below) or $c^*(\Delta')$ is again distributed to a region of degree > 4.

Now suppose that $\hat{\Delta}$ receives positive curvature across at least two edges. If $\hat{\Delta}$ receives from a T24 then an inspection of Figures 4.8-14 shows that there are six cases for $\hat{\Delta}$, namely $\hat{\Delta}_3$ of Figure 4.8(i); $\hat{\Delta}_4$ of Figure 4.9(i); $\hat{\Delta}_3$ or $\hat{\Delta}_4$ of Figure 4.10(i); $\hat{\Delta}_6$ of Figure 4.13(i) where we no longer assume, however, that $d(u_4) > 3$ and $d(u_5) > 3$; and $\hat{\Delta}_7$ of Figure 4.14(i) where we no longer assume, however, that $d(u_6) > 3$ and $d(u_7) > 3$. If $\hat{\Delta}$ receives from a T13 region then an inspection of Figures 4.16-19 shows that there are six cases for $\hat{\Delta}$, namely $\hat{\Delta}_1$ of Figure 4.16(ii) but with $d(v_2) \geq 5$ to take Figure 4.18 into account; $\hat{\Delta}_4$ of Figure 4.16(ii); $\hat{\Delta}_8$ of Figure 4.16(x) *under the assumption* that $d(\hat{\Delta}_8) = 4$ and $\hat{\Delta}_8$ is not given by Figure 4.16(xi); $\hat{\Delta}_2$ of Figure 4.17(i) but with $d(v_2) \geq 5$ to take Figure 4.19 into account; $\hat{\Delta}_3$ of Figure 4.17(i); and $\hat{\Delta}_8$ of Figure 4.17(ix) *under the assumption* that $d(\hat{\Delta}_8) = 4$ and $\hat{\Delta}_8$ is not given by Figure 4.17(x). If $\hat{\Delta}$ receives from a T4 region then an inspection of Figures 4.24-26 shows that there are three cases, namely $\hat{\Delta}_3$ or $\hat{\Delta}_4$ of Figure 4.24(i); and $\hat{\Delta}_6$ of Figure 4.26(i) where we no longer assume, however, that $d(u_4) > 3$ and $d(u_5) > 3$.

An inspection of the labelling and degrees of the vertices in each of these 17 figures immediately rules out the following combinations: a T24 region with a T24; a T24 with a T4; a T4 with a T4; and either Figure 4.27(ii) or 4.28(ii) with any of the other 16 possibilities.

Suppose that $\hat{\Delta}$ receives positive curvature from at least two T13 regions. An inspection of the six T13 regions shows that all combinations are immediately ruled out by the labelling and degree of vertices except for two cases. The first case is when $\hat{\Delta}$ coincides with $\hat{\Delta}_8$ of Figure 4.16(x) and $\hat{\Delta}_8$ of Figure 4.17(ix). But this forces $l(v_2) = bx^{-1}a^{-1}ybw$ in Figure 4.16(x), and the fact that $d(v_2) = 6$ then forces $l(v_2) = bx^{-1}a^{-1}ybb$, a label whose t-exponent sum is equal to 6, a contradiction. The second case is when $\hat{\Delta}$ coincides with $\hat{\Delta}_1$ of Figure 4.16(ii) and with $\hat{\Delta}_2$ of Figure 4.17(i). This is shown in Figure 4.31, and it follows that a combination of more than two T13 regions cannot occur.

Consider Figure 4.31(i) in which $\hat{\Delta}$ receives positive curvature from the T13 regions each contributing at most $\frac{\pi}{15}$ to $c(\hat{\Delta})$. (Note that we use Δ_1, Δ_2 and not Δ as before to denote regions from which positive curvature is distributed.) Let $d(w_1) > 5$ and $d(w_2) > 5$. If $d(u) > 3$ then $c^*(\hat{\Delta}) \leq c(3,4,6,6) + 2\left(\frac{\pi}{30}\right) < 0$; and if $d(u) = 3$ then $c(\hat{\Delta}) + 2\left(\frac{\pi}{30}\right) \leq$

$c(3,3,6,6)+2\left(\frac{\pi}{30}\right)=\frac{\pi}{15}$ so add $\frac{\pi}{30}$ to each of $c(\hat{\hat{\Delta}}_1), c(\hat{\hat{\Delta}}_2)$ as shown in Figure 4.31(ii). Let $d(w_1)>5$ and $d(w_2)=5$. If $d(u)>3$ then $c^*(\hat{\Delta})\leq c(3,4,5,6)+\frac{\pi}{30}+\frac{\pi}{15}=0$; and if $d(u)=3$ then $c(\hat{\Delta})+\frac{\pi}{30}+\frac{\pi}{15}\leq c(3,3,5,6)+\frac{\pi}{30}+\frac{\pi}{15}=\frac{\pi}{6}$ so add $\frac{2\pi}{15}$ to $c(\hat{\hat{\Delta}}_1)$ and $\frac{\pi}{30}$ to $c(\hat{\hat{\Delta}}_2)$ as shown in Figure 4.31(iii). Let $d(w_1)=5$ and $d(w_2)>5$. If $d(u)>3$ then $c^*(\hat{\Delta})\leq c(3,4,5,6)+\frac{\pi}{15}+\frac{\pi}{30}=0$; and if $d(u)=3$ then $c(\hat{\Delta})+\frac{\pi}{15}+\frac{\pi}{30}\leq c(3,3,5,6)+\frac{\pi}{15}+\frac{\pi}{30}=\frac{\pi}{6}$ so add $\frac{\pi}{30}$ to $c(\hat{\hat{\Delta}}_1)$ and $\frac{2\pi}{15}$ to $c(\hat{\hat{\Delta}}_2)$ as shown in Figure 4.31(iv). This leaves $d(w_1)=d(w_2)=5$. If $d(u)>4$ then $c^*(\hat{\Delta})\leq c(3,5,5,5)+2\left(\frac{\pi}{15}\right)=0$; if $d(u)=4$ then $c(\hat{\Delta})+2\left(\frac{\pi}{15}\right)=c(3,4,5,5)+2\left(\frac{\pi}{15}\right)=\frac{\pi}{10}$ so add $\frac{\pi}{15}$ to $c(\hat{\hat{\Delta}}_1)$ and $\frac{\pi}{30}$ to $c(\hat{\hat{\Delta}}_2)$ as shown in Figures 4.31(v), (vi); and if $d(u)=3$ then $c(\hat{\Delta})+2\left(\frac{\pi}{15}\right)\leq c(3,3,5,5)+\frac{2\pi}{15}=\frac{4\pi}{15}$ so add $\frac{2\pi}{15}$ to $c(\hat{\hat{\Delta}}_1)$ and $\frac{2\pi}{15}$ to $c(\hat{\hat{\Delta}}_2)$ as shown in Figure 4.31(vii).

Now suppose that $\hat{\Delta}$ receives positive curvature from a T4 region and a T13 region. Again an inspection of the labelling and degrees of the vertices involved immediately rules out all combinations except for three cases. The first case is $\hat{\Delta}_3$ of Figure 4.24(i) with $\hat{\Delta}_8$ of Figure 4.16(x), but this forces $\hat{\Delta}_8$ to be given by Figure 4.16(xi), a contradiction; and the second case is $\hat{\Delta}_4$ of Figure 4.24(i) with $\hat{\Delta}_8$ of Figure 4.17(ix), but this forces $\hat{\Delta}_8$ to be given by Figure 4.17(x), a contradiction. The third case is when $\hat{\Delta}=\hat{\Delta}_6$ of Figure 4.26(i) and $\hat{\Delta}=\hat{\Delta}_8$ of Figure 4.17(ix). But then $c^*(\hat{\Delta})\leq c(4,4,6,6)+\frac{\pi}{15}+\frac{\pi}{30}<0$.

Finally suppose that $\hat{\Delta}$ receives positive curvature from a T24 region and a T13 region. An inspection of the 36 possible combinations immediately rules out all but the following 12 cases. If $\hat{\Delta}=\hat{\Delta}_3$ of Figure 4.8(i) or 4.10(i) and $\hat{\Delta}=\hat{\Delta}_8$ of Figure 4.16(x) then this forces $\hat{\Delta}_8$ to be given by Figure 4.16(xi), a contradiction; or if $\hat{\Delta}=\hat{\Delta}_4$ of Figure 4.9(i) or 4.10(i) and $\hat{\Delta}=\hat{\Delta}_8$ of Figure 4.17(ix) then this forces $\hat{\Delta}_8$ to be given by Figure 4.17(x), a contradiction. If $\hat{\Delta}=\hat{\Delta}_6$ of Figure 4.13(i) and $\hat{\Delta}=\hat{\Delta}_8$ of Figure 4.17(ix), or if $\hat{\Delta}=\hat{\Delta}_7$ of Figure 4.14(i) and $\hat{\Delta}=\hat{\Delta}_8$ of Figure 4.16(x) then $c^*(\hat{\Delta})\leq c(4,4,6,6)+\frac{\pi}{15}+\frac{\pi}{30}<0$.

This leaves $\hat{\Delta}=\hat{\Delta}_6$ of Figure 4.13(i) and either $\hat{\Delta}=\hat{\Delta}_1$ of Figure 4.16(ii) or $\hat{\Delta}=\hat{\Delta}_4$ of Figure 4.16(ii) or $\hat{\Delta}=\hat{\Delta}_2$ of Figure 4.17(i) (see Figure 4.32(i)); or $\hat{\Delta}=\hat{\Delta}_7$ of Figure 4.14(i) and either $\hat{\Delta}=\hat{\Delta}_1$ of Figure 4.16(ii) or $\hat{\Delta}=\hat{\Delta}_2$ of Figure 4.17(ii) or $\hat{\Delta}=\hat{\Delta}_3$ of Figure 4.17(i) (see Figure 4.32(ii)). Consider Figure 4.32. Since $l(v)\in\{b^{-1}y^{-1}a^2xw, y^{-1}a^2xb^{-1}w\}$ forces $d(v)\geq 7$ it follows that $c(\hat{\Delta}_1)\leq c(3,4,4,7)=-\frac{\pi}{21}$ and $c(\Delta_2)\leq c(3,3,5,7)=\frac{2\pi}{105}$. In both configurations $\frac{\pi}{21}$ is added from $c(\Delta_1)=\frac{\pi}{15}$ to $c(\hat{\Delta}_1)$ and the remaining $\frac{\pi}{15}-\frac{\pi}{21}=\frac{2\pi}{105}$ to $\hat{\Delta}$ as shown. If $\hat{\Delta}$ does not receive positive curvature from Δ_3 then $c^*(\hat{\Delta})\leq c(3,4,4,7)+2\left(\frac{2\pi}{105}\right)<0$ so it can be assumed without any loss that $\hat{\Delta}$ receives from Δ_1 (via $\hat{\Delta}_1$), Δ_2 and Δ_3. But then $\hat{\Delta}=\hat{\Delta}_1$ of Figure 4.16(ii) or $\hat{\Delta}=\hat{\Delta}_2$ of Figure 4.17(i) forces $d(u)\geq 5$ and $c^*(\hat{\Delta}\leq c(3,4,5,7)+\frac{\pi}{15}+2\left(\frac{2\pi}{105}\right)<0$. □

An immediate consequence of Lemma 4.1 is the following (where $\boldsymbol{K}_0$ and $\boldsymbol{K}_1$ have already been defined in Section 3).

Proposition 4.2 If $\boldsymbol{K}=\boldsymbol{K}_0$ then $c(\boldsymbol{K})\leq\sum_{d(\hat{\Delta})\geq 6}c^*(\hat{\Delta})$; or if $\boldsymbol{K}=\boldsymbol{K}_1$ then $c(\boldsymbol{K})\leq$

$\sum_{\substack{d(\hat{\Delta})\geq 6 \\ \hat{\Delta}\neq\Delta_0}} c^*(\hat{\Delta}) + c^*(\Delta_0)$.

Note. In Figure 4.33(i) the maximum amount of curvature, denoted $c(u,v)$, distributed across an edge e_i with endpoints u,v according to the description of curvature given in Figures 4.1-4.32 above is shown for each choice of corner labels (the list excludes (b,a)-edges and the (x,y)-edges of Figure 4.6); and in Figure 4.33(ii) $c(u,v)$ is shown when at least one of $d(u),d(v)$ is greater than 4. The integers shown are multiples of $\frac{\pi}{30}$ with 7(5), 4(2) meaning that if $c(u,v) < \frac{7\pi}{30}, \frac{2\pi}{15}$ then $c(u,v) = \frac{\pi}{6}, \frac{\pi}{15}$ respectively. This will be used throughout what follows often without explicit reference.

5 Regions of degree 6

We now study $c^*(\hat{\Delta})$ for $d(\hat{\Delta}) = 6$ and so $\hat{\Delta}$ is given by Figure 3.7. In Figures 5.1 and 5.2 we fix the labelling of the six neighbours $\hat{\Delta}_i$ $(1 \leq i \leq 6)$ of $\hat{\Delta}$ as shown. *First assume that $\hat{\Delta}$ is not $\hat{\Delta}_1$ of Configuration A–D in Figures 4.27-4.29* (**ex5.1**). Checking the distribution of curvature described in Figures 4.1-4.32 yields the following table in which vertex subscripts are modulo 6; the entries under $c(u_i,u_{i+1})$ are multiples of $\frac{\pi}{30}$ and denote the maximum amount of curvature that $\hat{\Delta}$ can receive across the edge with endpoints u_i, u_{i+1} according to Figure 4.33; and 5^+, 6^+ means ≥ 5, ≥ 6. Moreover the table applies to $\hat{\Delta}$ both of Figure 3.7(i) and of Figure 3.7(ii).

$d(u_i)$	$d(u_{i+1})$	$c(u_1,u_2)$	$c(u_2,u_3)$	$c(u_3,u_4)$	$c(u_4,u_5)$	$c(u_5,u_6)$	$c(u_6,u_1)$
3	3	0	0	0	6	0	0
3	4	0	3	0	0	0	2
4	3	0	0	0	0	7	0
3	5	0	2	2	0	0	0
5	3	0	2	2	2	2	0
3	6^+	0	2	2	2	0	0
6^+	3	0	2	2	2	2	0
4	4	7	0	0	0	0	0
4	5	2	0	0	2	0	0
5	4	2	2	0	0	4	0
4	6^+	4	0	1	0	2	0
6^+	4	2	0	0	0	1	4
5^+	5^+	1	0	0	1	1	0

Notes.

1. (See Figures 5.1 and 5.2.) $d(u_1) = 3$ ($\Rightarrow d(\hat{\Delta}_1) > 4$, $d(\hat{\Delta}_2) > 4$) $\Rightarrow c(u_1, u_2) = c(u_6, u_1) = 0$; $d(u_2) = 3 \Rightarrow c(u_1, u_2) = 0$; $d(u_2) = 4 \Rightarrow c(u_2, u_3) = 0$; $d(u_5) = 3 \Rightarrow c(u_5, u_6) = 0$; and $d(u_5) = 4 \Rightarrow c(u_4, u_5) = 0$.

2. $c(u_1, u_2) > 0$ and $c(u_2, u_3) > 0 \Rightarrow$ (see above table) $c(u_1, u_2) + c(u_2, u_3) \leq \frac{2\pi}{15} + \frac{\pi}{15}$ and since $c(u_1, u_2) \leq \frac{7\pi}{30}$, $c(u_2, u_3) \leq \frac{\pi}{10}$ we have $c(u_1, u_2) + c(u_2, u_3) \leq \frac{7\pi}{30}$.

3. $c(u_4, u_5) > 0$ and $c(u_5, u_6) > 0 \Rightarrow c(u_4, u_5) + c(u_5, u_6) \leq \frac{\pi}{15} + \frac{2\pi}{15}$ and since $c(u_4, u_5) \leq \frac{\pi}{5}$, $c(u_5, u_6) \leq \frac{7\pi}{30}$ we have $c(u_4, u_5) + c(u_5, u_6) \leq \frac{7\pi}{30}$.

4. Let $d(u_5) = 5$, $d(u_6) = 4$. If $c(u_5, u_6) = \frac{2\pi}{15}$ then checking $l(u_5), l(u_6)$ shows that $c(u_4, u_5) = 0$ (see Figures 4.18(iv) and 4.19(iv)); moreover if $c(u_5, u_6) \neq \frac{2\pi}{15}$ then $c(u_5, u_6) = \frac{\pi}{15}$.

In what follows much use will be made of Lemma 3.3 when determining the vertex labels and the above table when determining $c(u, v)$.

Lemma 5.1 *If $\hat{\Delta}$ is given by Figure 3.7 and $\hat{\Delta}$ receives positive curvature across at least one edge then $c^*(\hat{\Delta}) \leq \frac{2\pi}{15}$ and if $c^*(\hat{\Delta}) > 0$ then $\hat{\Delta}$ is given by one of the regions of Figure 5.1 and Figure 5.2.*

Proof. It follows from the table and notes above that $c^*(\hat{\Delta}) \leq c(\hat{\Delta}) + (c(u_1, u_2) + c(u_2, u_3)) + c(u_3, u_4) + (c(u_4, u_5) + c(u_5, u_6)) + c(u_6, u_1) \leq c(\hat{\Delta}) + \frac{7\pi}{30} + \frac{\pi}{15} + \frac{7\pi}{30} + \frac{2\pi}{15} = c(\hat{\Delta}) + \frac{2\pi}{3}$. Therefore if $\hat{\Delta}$ has at most two vertices of degree 3 then $c^*(\hat{\Delta}) \leq c(3, 3, 4, 4, 4, 4) + \frac{2\pi}{3} = 0$.

Let $\hat{\Delta}$ have exactly three vertices of degree 3 so that $c(\hat{\Delta}) \leq -\frac{\pi}{2}$. If $d(u_1) = 3$ then $c(u_1, u_2) = c(u_6, u_1) = 0$ and $c^*(\hat{\Delta}) \leq -\frac{\pi}{2} + \frac{\pi}{10} + \frac{\pi}{15} + \frac{7\pi}{30} < 0$, so assume $d(u_1) \geq 4$. If $d(u_2) = 3$ then $c(u_1, u_2) = 0$ so if $d(u_6) \geq 6$ then $c^*(\hat{\Delta}) \leq c(3, 3, 3, 4, 4, 6) + \frac{\pi}{10} + \frac{\pi}{15} + \frac{7\pi}{30} + \frac{2\pi}{15} < 0$; otherwise $c(u_6, u_1) = \frac{\pi}{15}$ and $c^*(\hat{\Delta}) \leq -\frac{\pi}{2} + \frac{\pi}{10} + \frac{\pi}{15} + \frac{7\pi}{30} + \frac{\pi}{15} < 0$; so assume $d(u_2) \geq 4$. This leaves four subcases. First let $d(u_3) = d(u_4) = d(u_5) = 3$. Then $c(u_3, u_4) = c(u_5, u_6) = 0$. Moreover if $d(u_6) < 6$ then $c(u_6, u_1) = 0$ and $c^*(\hat{\Delta}) \leq -\frac{\pi}{2} + \frac{7\pi}{30} + \frac{\pi}{5} = 0$; and if $d(u_6) \geq 6$ then $c^*(\hat{\Delta}) \leq c(3, 3, 3, 4, 4, 6) + \frac{7\pi}{30} + \frac{\pi}{5} + \frac{2\pi}{15} < 0$. Let $d(u_3) = d(u_4) = d(u_6) = 3$. Then $c(u_2, u_3) = \frac{\pi}{15}$, $c(u_3, u_4) = 0$, $c(u_4, u_5) + c(u_5, u_6) \leq \frac{7\pi}{30}$ and $c(u_6, u_1) = \frac{\pi}{15}$. If either $d(u_1) > 4$ or $d(u_2) > 4$ then $c(u_1, u_2) \leq \frac{2\pi}{15}$ and $c^*(\hat{\Delta}) \leq c(3, 3, 3, 4, 4, 5) + \frac{\pi}{2} < 0$; otherwise $d(u_1) = d(u_2) = 4$ which implies $c(u_2, u_3) = 0$ and the labelling (of u_1 and u_2) either forces $c(u_1, u_2) = 0$ and $c^*(\hat{\Delta}) \leq -\frac{\pi}{2} + \frac{3\pi}{10} < 0$ or forces $c(u_6, u_1) = 0$ and $c^*(\hat{\Delta}) \leq -\frac{\pi}{2} + \frac{7\pi}{30} + \frac{7\pi}{30} < 0$. Let $d(u_3) = d(u_5) = d(u_6) = 3$. Then $c(u_4, u_5) = \frac{\pi}{15}$, $c(u_5, u_6) = 0$ and $c(u_6, u_1) = \frac{\pi}{15}$. Therefore $c^*(\hat{\Delta}) \leq -\frac{\pi}{2} + \frac{7\pi}{30} + \frac{\pi}{15} + \frac{\pi}{15} + \frac{\pi}{15} < 0$. Finally let $d(u_4) = d(u_5) = d(u_6) = 3$. Then $c(u_5, u_6) = 0$ and $c(u_6, u_1) = \frac{\pi}{15}$. If $d(u_1) > 4$ or $d(u_2) > 4$ then $c(u_1, u_2) = \frac{2\pi}{15}$ and $c^*(\hat{\Delta}) \leq -\frac{3\pi}{5} + \frac{8\pi}{15} < 0$; otherwise $d(u_1) = d(u_2) = 4$ so $c(u_2, u_3) = 0$ and the labelling either forces $c(u_1, u_2) = 0$ and $c^*(\hat{\Delta}) \leq -\frac{\pi}{2} + \frac{\pi}{3} < 0$ or forces $c(u_6, u_1) = 0$ and $c^*(\hat{\Delta}) \leq -\frac{\pi}{2} + \frac{7\pi}{30} + \frac{\pi}{15} + \frac{\pi}{5} = 0$.

Now let $\hat{\Delta}$ have exactly four vertices of degree 3 so that $c(\hat{\Delta}) \leq -\frac{\pi}{3}$. There are fifteen cases to consider. In fact if $(d(u_1), d(u_2), d(u_3), d(u_4), d(u_5), d(u_6)) \in \{(3,3,3,3,*,*), (3,3,3,*,3,*), (3,3,3,*,*,3), (3,3,*,*,3,3), (3,*,3,3,3,*), (3,*,3,3,*,3), (3,*,3,*,3,3), (3,*,*,3,3,3), (*,3,3,3,3,*), (*,3,3,3,*,3), (*,3,3,*,3,3)\}$ then a straightforward check using the above table and notes shows that $c^*(\hat{\Delta}) \leq -\frac{\pi}{3} + \frac{\pi}{3} = 0$. Let $d(u_1) = d(u_2) = d(u_4) = d(u_5) = 3$. Then $c(u_1, u_2) = c(u_5, u_6) = c(u_6, u_1) = 0$. If $d(u_3) > 4$ then $c^*(\hat{\Delta}) \leq -\frac{13\pi}{30} + \frac{11\pi}{30} < 0$; otherwise $d(u_3) = 4$ forcing $c(u_3, u_4) = 0$ and $c^*(\hat{\Delta}) \leq -\frac{\pi}{3} + \frac{3\pi}{10} < 0$. Let $d(u_1) = d(u_2) = d(u_4) = d(u_6) = 3$. Then $c(u_1, u_2) = c(u_6, u_1) = 0$. If $d(u_3) > 4$ then $c^*(\hat{\Delta}) \leq -\frac{13\pi}{30} + \frac{2\pi}{5} < 0$; otherwise $d(u_3) = 4$ forcing $c(u_3, u_4) = 0$ and $c^*(\hat{\Delta}) \leq -\frac{\pi}{3} + \frac{\pi}{3} = 0$. Let $d(u_2) = d(u_4) = d(u_5) = d(u_6) = 3$. Then $c(u_1, u_2) = c(u_5, u_6) = 0$ and $c(u_6, u_1) = \frac{\pi}{15}$. If $d(u_1) > 4$ or $d(u_3) > 4$ then $c^*(\hat{\Delta}) \leq -\frac{13\pi}{30} + \frac{13\pi}{30} = 0$, so assume $d(u_1) = d(u_3) = 4$. Then $c(u_3, u_4) = 0$ and $l(u_1)$ either forces $c(u_6, u_1) = 0$ and $c^*(\hat{\Delta}) \leq -\frac{\pi}{3} + \frac{3\pi}{10} < 0$ or $\hat{\Delta}$ is given by Figure 5.1(i) or 5.2(i) in which case $c^*(\hat{\Delta}) \leq -\frac{\pi}{3} + \frac{11\pi}{30} = \frac{\pi}{30}$. (Note that if $c^*(\hat{\Delta}) > 0$ then $\hat{\Delta}$ must receive $\frac{\pi}{15}$ from $\hat{\Delta}_6$ and this forces $\hat{\Delta}_6 = \Delta$ where Δ is given by Figure 4.7(xi)) This leaves the case $d(u_j) = 3$ $(3 \leq j \leq 6)$. Then $c(u_3, u_4) = c(u_5, u_6) = 0$ and $c(u_6, u_1) = \frac{\pi}{15}$. If $d(u_1) \geq 5$ and $d(u_2) \geq 5$ then $c^*(\hat{\Delta}) \leq -\frac{8\pi}{15} + \frac{\pi}{2} < 0$. If $d(u_1) = 4$ and $d(u_2) = 5$ or $d(u_1) \geq 5$ and $d(u_2) = 4$ then $c(u_1, u_2) = \frac{\pi}{15}$ and $c^*(\hat{\Delta}) \leq c(3,3,3,3,4,5) + \frac{\pi}{15} + \frac{\pi}{10} + \frac{\pi}{5} + \frac{\pi}{15} = 0$; and if $d(u_1) = 4$ and $d(u_2) \geq 6$ then $c^*(\hat{\Delta}) \leq c(3,3,3,3,4,6) + \frac{2\pi}{15} + \frac{\pi}{10} + \frac{\pi}{5} + \frac{\pi}{15} = 0$. Let $d(u_1) = d(u_2) = 4$ so $c(u_2, u_3) = 0$. Then $l(u_1)$ either forces $c(u_1, u_2) = 0$ and $c^*(\hat{\Delta}) \leq -\frac{\pi}{3} + \frac{\pi}{5} + \frac{\pi}{15} < 0$ or $\hat{\Delta}$ is given by Figure 5.1(ii) or 5.2(ii) where $c^*(\hat{\Delta}) \leq -\frac{\pi}{3} + \frac{7\pi}{30} + \frac{\pi}{5} = \frac{\pi}{10}$.

Now suppose that $\hat{\Delta}$ has exactly five vertices of degree 3 so that $c(\hat{\Delta}) \leq -\frac{\pi}{6}$. If $d(u_6) > 3$ then $c(u_1, u_2) = c(u_2, u_3) = c(u_3, u_4) = c(u_5, u_6) = c(u_6, u_1) = 0$, $c^*(\hat{\Delta}) \leq -\frac{\pi}{6} + \frac{\pi}{5} = \frac{\pi}{30}$ and $\hat{\Delta}$ is given by Figure 5.1(iii) or 5.2(iii). If $d(u_5) > 3$ then $c(u_i, u_{i+1}) = 0$ except for $c(u_4, u_5)$ and $c(u_5, u_6)$ so $c^*(\hat{\Delta}) \leq -\frac{\pi}{6} + \frac{7\pi}{30} = \frac{\pi}{15}$ and $\hat{\Delta}$ is given by Figure 5.1(iv) or 5.2(iv). If $d(u_4) > 3$ then $c(u_i, u_{i+1}) = 0$ except for $c(u_3, u_4) = c(u_4, u_5) = \frac{\pi}{15}$ and $c^*(\hat{\Delta}) \leq -\frac{\pi}{6} + \frac{2\pi}{15} < 0$. Let $d(u_3) > 3$. Then $c(u_1, u_2) = c(u_5, u_6) = c(u_6, u_1) = 0$. If $d(u_3) \geq 6$ then $c^*(\hat{\Delta}) \leq -\frac{\pi}{3} + 2\left(\frac{\pi}{15}\right) + \frac{\pi}{5} = 0$; if $d(u_3) = 5$ then $l(u_3)$ forces either $c(u_2, u_3) = 0$ or $c(u_3, u_4) = 0$ so $c^*(\hat{\Delta}) \leq -\frac{4\pi}{15} + \frac{\pi}{15} + \frac{\pi}{5} = 0$; and if $d(u_3) = 4$ then $c(u_3, u_4) = 0$, $c^*(\hat{\Delta}) \leq \frac{\pi}{6} + \frac{\pi}{10} + \frac{\pi}{5} = \frac{2\pi}{15}$ and $\hat{\Delta}$ is given by Figure 5.1(v) or 5.2(v). If $d(u_2) > 3$ then $c(u_1, u_2) = c(u_3, u_4) = c(u_5, u_6) = c(u_6, u_1) = 0$. If $d(u_2) \geq 5$ then $c^*(\hat{\Delta}) \leq -\frac{4\pi}{15} + \frac{\pi}{15} + \frac{\pi}{5} = 0$; and if $d(u_2) = 4$ then $c(u_2, u_3) = 0$, $c^*(\hat{\Delta}) \leq -\frac{\pi}{6} + \frac{\pi}{5} = \frac{\pi}{30}$ and $\hat{\Delta}$ is given by Figure 5.1(vi) or 5.2(vi). Finally if $d(u_1) > 3$ then $c(u_i, u_{i+1}) = 0$ except for $c(u_4, u_5) = \frac{\pi}{5}$ and $c(u_6, u_1) = \frac{\pi}{15}$. So if $d(u_1) \geq 5$ then $c^*(\hat{\Delta}) \leq -\frac{4\pi}{15} + \frac{4\pi}{15} = 0$; and if $d(u_1) = 4$ then $c(u_6, u_1) = \frac{\pi}{15}$ or $c(u_6, u_1) = 0$, $c^*(\hat{\Delta}) \leq -\frac{\pi}{6} + \frac{4\pi}{15} = \frac{\pi}{10}$ and the two cases for $\hat{\Delta}$ are shown in Figure 5.1(vii), (viii) or 5.2(vii), (viii).

This leaves the case $d(u_i) = 3$ $(1 \leq i \leq 6)$. Then $c(u_i, u_{i+1}) = 0$ except for $c(u_4, u_5) = \frac{\pi}{5}$, $c^*(\hat{\Delta}) \leq 0 + \frac{\pi}{5} = \frac{\pi}{5}$ and $\hat{\Delta}$ is given by Figure 5.1(ix) or 5.2(ix). $\square$

We now describe the distribution of curvature from each of the 18 regions $\hat{\Delta}$ of Figures 5.1 and 5.2.

Figure 5.1(i) and 5.2(i): $c^*(\hat{\Delta}) \leq -\frac{\pi}{3} + \frac{11\pi}{30}$; distribute $\frac{\pi}{30}$ from $\hat{\Delta}$ to $\hat{\Delta}_1$ in each case.

Figure 5.1(ii) and 5.2(ii): $c^*(\hat{\Delta}) \leq -\frac{\pi}{3} + \frac{13\pi}{30}$; distribute $\frac{\pi}{10}$ from $\hat{\Delta}$ to $\hat{\Delta}_6$ in each case.

Figure 5.1(iii) and 5.2(iii): $c^*(\hat{\Delta}) \leq -\frac{\pi}{6} + \frac{\pi}{5}$; distribute $\frac{\pi}{30}$ from $\hat{\Delta}$ to $\hat{\Delta}_2$ in each case.

Figure 5.1(iv) and 5.2(iv): $c^*(\hat{\Delta}) \leq -\frac{\pi}{6} + \frac{7\pi}{30}$; distribute $\frac{\pi}{15}$ from $\hat{\Delta}$ to $\hat{\Delta}_2$ in each case.

Figure 5.1(v) and 5.2(v): $c^*(\hat{\Delta}) \leq -\frac{\pi}{6} + \frac{3\pi}{10}$; distribute $\frac{2\pi}{15}$ from $\hat{\Delta}$ to $\hat{\Delta}_1$ in each case. (5.1)

Figure 5.1(vi) and 5.2(vi): $c^*(\hat{\Delta}) \leq -\frac{\pi}{6} + \frac{\pi}{5}$; distribute $\frac{\pi}{30}$ from $\hat{\Delta}$ to $\hat{\Delta}_3$ in each case.

Figure 5.1(vii) and 5.2(vii): $c^*(\hat{\Delta}) \leq -\frac{\pi}{6} + \frac{\pi}{5}$; distribute $\frac{\pi}{30}$ from $\hat{\Delta}$ to $\hat{\Delta}_2$ in each case.

Figure 5.1(viii) and 5.2(viii): $c^*(\hat{\Delta}) \leq -\frac{\pi}{6} + \frac{4\pi}{15}$; distribute $\frac{\pi}{15}$ from $\hat{\Delta}$ to $\hat{\Delta}_2$ and $\frac{\pi}{30}$ from $\hat{\Delta}$ to $\hat{\Delta}_3$ in each case.

Figure 5.1(ix) and 5.2(ix): $c^*(\hat{\Delta}) \leq 0 + \frac{\pi}{5}$; distribute $\frac{\pi}{10}$ from $\hat{\Delta}$ to $\hat{\Delta}_1$, $\frac{\pi}{15}$ from $\hat{\Delta}$ to $\hat{\Delta}_2$ and $\frac{\pi}{30}$ from $\hat{\Delta}$ to $\hat{\Delta}_3$ in each case.

Note: in all of the above cases $d(\hat{\Delta}_i) > 6$ for each region $\hat{\Delta}_i$ that receives positive curvature from $\hat{\Delta}$ except possibly for $\hat{\Delta}_1$ in Figures 5.1(i) and 5.2(i). Moreover the upper bounds $c(u, v)$ of Figure 4.33 remain unchanged.

Now consider Configurations A and B of Figure 4.27(i), 4.28(i) and assume that $d(\hat{\Delta}_1) = 6$ (**en5.1a**). Then $\hat{\Delta}_1$ is given by Figure 5.3(i), 5.4(i). A ***key observation*** (**ex5.2**) is the following. Since, by definition of distribution in Configuration A, $\hat{\Delta}_1$ receives $\frac{\pi}{30}$ from Δ_1 it follows that $\hat{\Delta}_2$ of Figure 5.3(i), 5.4(i) cannot be the region $\hat{\Delta}$ of Figure 5.2(i), 5.1(i) respectively. Otherwise the proof of Lemma 5.1 shows that $\hat{\Delta}_2$ would have to receive $\frac{\pi}{5}$ from a region Δ_4, say, across its (v_4, v_5)-edge in Figure 5.2(i), 5.1(i) forcing Δ_4 to be given by Δ of Figure 4.1(v) and we would obtain one of Figure 4.27(ii)–(iv), 4.28(ii)–(iv), a contradiction. In particular $\hat{\Delta}_1$ *does not receive positive curvature from* $\hat{\Delta}_2$, *that is,* $c(w_2, u_1) = 0$.

We return now to $\hat{\Delta}_1$ of Figure 5.3(i), 5.4(i) and first assume that $d(u_3) \geq 5$. Then $c(w_1, u_3) = \frac{\pi}{15}$ and $c(u_3, u_2) = \frac{2\pi}{15}$ by Figure 4.33(ii). Since $c(u_1, u_2) = \frac{2\pi}{15}$ it follows that $c^*(\hat{\Delta}_1) \leq c(\hat{\Delta}) + \frac{7\pi}{15}$. If $d(u_1) > 3$ or $d(u_2) > 3$ then $c(\hat{\Delta}_1) \leq c(3, 3, 3, 4, 4, 5) = -\frac{9\pi}{15}$; on the other hand if $d(u_1) = d(u_2) = 3$ then (an inspection of Figures 5.1 and 5.2 shows that) $c(u_1, u_2) = 0$ and $c^*(\hat{\Delta}_1) \leq c(3, 3, 3, 3, 4, 5) + \frac{\pi}{3} < 0$. Now let $d(u_3) = 4$. Then $c(u_1, u_2) = \frac{2\pi}{15}$, $c(u_2, u_3) = \frac{7\pi}{30}$ and $c(u_3, w_1) = 0$ so $c^*(\hat{\Delta}_1) \leq c(\hat{\Delta}_1) + \frac{\pi}{2}$. If $d(u_1) > 3$ or $d(u_2) > 3$ then $c(\hat{\Delta}_1) \leq -\frac{\pi}{2}$; on the other hand if $d(u_1) = d(u_2) = 3$ then $c^*(\hat{\Delta}_1) \leq c(3, 3, 3, 3, 4, 4) + \frac{11\pi}{30} = \frac{\pi}{30}$ which is added to $c(\hat{\Delta}_2)$ as shown in Figure 5.3(ii), 5.4(ii). Finally let $d(u_3) = 3$. Then $c(u_3, w_1) = \frac{\pi}{5}$, $c(u_2, u_1) = \frac{2\pi}{15}$ and $c(u_2, u_3) = 0$. If $d(u_1) = 3$ then $c(u_1, u_2) = 0$ and so $d(u_2) \geq 4$ would imply $c^*(\hat{\Delta}_1) \leq c(3, 3, 3, 3, 4, 4) + \frac{\pi}{3} = 0$; whereas if also $d(u_2) = 3$ then $c^*(\hat{\Delta}_1) \leq \frac{\pi}{6}$ is distributed to $\hat{\Delta}_2$ as shown in Figure 5.3(iii), 5.4(iii). Let $d(u_1) = 4$. If $d(u_2) \geq 4$ then $c^*(\hat{\Delta}_1) \leq c(3,3,3,4,4,4) + \frac{7\pi}{15} < 0$ so assume that $d(u_2) = 3$. Reading clockwise from the $\hat{\Delta}_1$ corner label if $l(u_1) = bbx^{-1}y$, $bx^{-1}yb$ in Figure 5.3(i), 5.4(i) respectively then $c(u_1, u_2) = 0$ and $c^*(\hat{\Delta}_2) \leq -\frac{\pi}{3} + \frac{\pi}{3} = 0$; otherwise $c(u_1, u_2) = \frac{\pi}{15}$ and $\hat{\Delta}_1$ is given by Figure 5.3(iv), 5.4(iv) and $c^*(\hat{\Delta}_1) \leq -\frac{\pi}{3} + \frac{6\pi}{15} = \frac{\pi}{15}$ is distributed to $\hat{\Delta}_2$ as shown. This leaves $d(u_1) \geq 5$ in which case $c(u_1, u_2) = \frac{\pi}{15}$ and $c^*(\hat{\Delta}_1) \leq c(3,3,3,3,4,5) + \frac{6\pi}{15} < 0$.

Consider Configurations C and D of Figure 4.29 and assume that $d(\hat{\Delta}_1) = 6$ (**en5.1b**). Then $\hat{\Delta}_1$ is given by Figure 5.5(i), (ii). Observe that $c(u_1, u_2) = \frac{\pi}{15}$; $c(u_2, u_3) = \frac{\pi}{10}$; $c(u_3, u_4) = \frac{2\pi}{15}$, indeed $c(u_3, u_4)$ cannot exceed $\frac{2\pi}{15}$ due to $l(u_4)$ (see Figure 4.20(viii), 4.22(vi) and 4.10(v), (ix)); and so $c^*(\hat{\Delta}_1) \leq c(\hat{\Delta}_1) + \frac{8\pi}{15}$. If $d(u_2) \geq 4$ and $d(u_3) \geq 5$ or $d(u_2) \geq 5$ and $d(u_3) \geq 4$ then $c(\hat{\Delta}_1) = c(3,3,3,4,4,5) = -\frac{9\pi}{15}$; if $d(u_2) = d(u_3) = 4$ then $c(u_2, u_3) = 0$ and $c^*(\hat{\Delta}_1) \leq -\frac{\pi}{2} + \frac{13\pi}{30} < 0$; if $d(u_2) = 3$ then $c(u_1, u_2) = 0$ so if $d(u_3) \geq 6$ then $c^*(\hat{\Delta}_1) \leq c(3,3,3,3,4,6) + \frac{7\pi}{15} < 0$; if $d(u_2) = 3$ and $d(u_3) = 5$ then $c(u_2, u_3) = \frac{\pi}{15}$ (see Figure 4.33(ii)) and $c^*(\hat{\Delta}) \leq c(3,3,3,3,4,5) + \frac{13\pi}{30} = 0$; and if $d(u_2) = 3$ and $d(u_3) = 4$ then $c(u_2, u_3) = c(u_3, u_4) = 0$ and $c^*(\hat{\Delta}_1) \leq -\frac{\pi}{3} + \frac{7\pi}{30} < 0$. This leaves $d(u_3) = 3$, $d(u_2) \geq 3$.

Let $d(u_3) = 3$. We claim that this implies $c(u_3, u_4) = 0$. The only way this may fail is if $\hat{\Delta}_1$ of Figure 5.5(i), (ii) coincides with $\hat{\Delta}_1$ in Figure 5.2(i), 5.1(i) or with $\hat{\Delta}_2$ of Figure 5.4(iv), 5.3(iv) (respectively). But then $d(u) = 4$ in Figure 5.5(i), (ii) whereas $d(u) = 3$ for the corresponding vertex u of Figure 5.1(i), 5.2(i); and the fact that $\hat{\Delta}_1$ of Figure 5.5(i), (ii) receives $\frac{\pi}{5}$ across the (w, u_1) edge means that $\hat{\Delta}_1$ cannot be $\hat{\Delta}_2$ of Figure 5.3(iv), 5.4(iv) because we would obtain Figure 4.27(ii)–(iv), Figure 4.28(ii)–(iv) a contradiction. If $d(u_2) = 3$ then $c(u_2, u_3) = c(u_1, u_2) = 0$ and $c^*(\hat{\Delta}_1) \leq c(3,3,3,3,3,4) + \frac{7\pi}{30} = \frac{\pi}{15}$ is added to $c(\hat{\Delta})$ as shown in Figure 5.5(iii), (iv); if $d(u_2) = 4$ then $c(u_1, u_2) = \frac{\pi}{30}$ (which is attained when $\hat{\Delta}_1$ coincides with $\hat{\Delta}_1$ of Figure 5.3(iv), 5.4(iv), the difference being that $\hat{\Delta}_1$ of Figure 5.5(v), (vi) receives only $\frac{\pi}{30}$ across the (b^{-1}, x^{-1})-edge, (b^{-1}, y)-edge, respectively) and $c^*(\hat{\Delta}_1) \leq -\frac{\pi}{3} + \frac{11\pi}{30} = \frac{\pi}{30}$ is added to $c(\hat{\Delta}_2)$ as shown in Figure 5.5(v), (vi); and if $d(u_2) \geq 5$ then $c(u_1, u_2) = \frac{\pi}{15}$ and $c(u_2, u_3) = \frac{2\pi}{15}$ and $c^*(\hat{\Delta}) \leq c(3,3,3,3,4,5) + \frac{13\pi}{30} = 0$.

Note 1: The upper bounds $c(u, v)$ of Figure 4.33 remain unchanged as a result of the distribution of curvature described in this section above.

Note 2: Before proceeding with Lemma 5.2 and its proof we note that an inspection of all distribution of curvature thus far described yields the following. If positive curvature is distributed across an (x, a^{-1})-edge e into a region of degree > 4 then e is given by: Figure 4.15(xvii) (two cases); Figure 4.16(i) (two cases); Figure 4.16(x); and Figure 4.28(i). In particular if the x-corner vertex has degree 4 and the a^{-1}-corner vertex has degree 3 then e is given by Figure 4.28(i) (Configuration B). If positive curvature is distributed across an (a^{-1}, y^{-1})-edge e into a region of degree > 4 then e is given by: Figure 4.15(xvii) (two cases); Figure 4.16(i) (two cases); Figure 4.17(ix); and Figure 4.27(i). In particular if the a^{-1}-corner has degree 3 and the y^{-1}-corner has degree 4 then e is given by Figure 4.27(i) (Configuration A).

Lemma 5.2 *Let $\hat{\Delta}$ be a region of degree 6 that receives positive curvature across at least one edge. Then one of the following occurs.*

(i) $c^*(\hat{\Delta}) \leq 0$;

(ii) $c^*(\hat{\Delta}) > 0$ *is distributed to a region of degree* > 6;

(iii) $c^*(\hat{\Delta}) \in \left\{\frac{\pi}{30}, \frac{\pi}{15}\right\}$ *is distributed to a region* Δ' *of degree 6 and* $c^*(\Delta') \leq 0$.

Proof. Using Lemma 5.1 and the analysis of Configurations A-F following the proof of Lemma 5.1 it is clear that if (i) and (ii) do not hold then $c^*(\hat{\Delta}) \in \left\{\frac{\pi}{30}, \frac{\pi}{15}\right)$ is distributed to $\hat{\Delta}_1$ of Figure 5.1(i), 5.2(i) or $\hat{\Delta}_2$ of Figure 5.3(iv), 5.4(iv) or $\hat{\Delta}_2$ of Figure 5.5(v), (vi). It follows that a region Δ' of degree 6 receives positive curvature from at most one region of degree 6. We treat each of these three cases in turn.

Consider $\hat{\Delta}_1$ of Figure 5.1(i), 5.2(i). Then $\hat{\Delta}_1$ is given by Figure 5.6(i), (ii). Since $\hat{\Delta}$ receives $\frac{\pi}{15}$ from $\hat{\Delta}_6$ and since $d(\hat{\Delta}_5) > 4$, the region $\hat{\Delta}_6$ coincides with $\hat{\Delta}$ of Figure 4.7(xi). It follows that $\hat{\Delta}_1$ does not receive any positive curvature from $\hat{\hat{\Delta}}$ in Figure 5.6(i), (ii). Note also that $d(w_4) > 3$; $c(w_3, w_4) = \frac{\pi}{15}$; $c(u_2, w_4) = \frac{\pi}{10}$; and $c(w_1, w_2) + c(w_2, w_3) = \frac{7\pi}{30}$ (see Note 3 following the table given earlier in this section). Therefore $c^*(\hat{\Delta}_1) \leq c(\hat{\Delta}_1) + \frac{13\pi}{30}$. If $\hat{\Delta}_1$ has at least three vertices of degree > 3 then $c(\hat{\Delta}_1) \leq -\frac{\pi}{2}$; and if $d(w_4) \geq 5$ then $c(\hat{\Delta}_1) \leq c(3,3,3,3,4,5) = -\frac{13\pi}{30}$; this leaves $d(w_i) = 3$ $(1 \leq i \leq 3)$ and $d(w_4) = 4$ in which case $c(w_1, w_2) = 0$ and $c(w_2, w_3) = \frac{\pi}{6}$. If $c(w_3, w_4) = 0$ then $c^*(\hat{\Delta}_1) \leq -\frac{\pi}{2} + \frac{\pi}{2} = 0$. On the other hand if $c(w_3, w_4) > 0$ then it follows from Note 2 above that $\hat{\Delta}_1$ must coincide with $\hat{\Delta}_1$ of Configuration A or B. But this contradicts the fact that $\hat{\Delta}_1$ does not receive positive curvature from $\hat{\Delta}_2$ in Figures 5.3 and 5.4 as noted at the end of the ***key observation*** (**en5.2**) made earlier, that is, $\hat{\Delta}$ of Figure 5.6(i), (ii) cannot coincide with $\hat{\Delta}_2$ of Figure 5.4(i), 5.3(i) (respectively).

Consider $\hat{\Delta}_2$ of Figure 5.3(iv), 5.4(iv) and assume that $d(\hat{\Delta}_2) = 6$. Then $\hat{\Delta}_2$ is given by Figure 5.7(i), (ii) in which the following hold: $c(u_2, w_3) = \frac{\pi}{10}$; $c(w_3, u_6) = \frac{\pi}{30}$ if $d(u_6) = 6$ and $c(w_3, u_6) = 0$ if $d(u_6) \neq 6$ (indeed $c(w_3, u_6) > 0$ can only occur if $\hat{\Delta}_2$ coincides with region $\hat{\Delta}_8$ of Figure 4.16(x), 4.17(xi)); $c(u_5, u_6) = \frac{2\pi}{15}$ if $d(u_6) = 6$ (see Figure 4.33(ii)) and $c(u_5, u_6) = \frac{\pi}{6}$ if $d(u_6) \neq 6$ (indeed $\frac{\pi}{6}$ cannot be exceeded since $\hat{\Delta}_2$ cannot be given by Figure 4.27(ii)-(iv), Figure 4.28(ii)-(iv) and to see this observe, for example, that Δ_3 of Figure 5.7 contributes $\frac{\pi}{3}$ to $\hat{\Delta}$ as opposed to $\frac{3\pi}{10}$); $c(u_4, u_5) = \frac{7\pi}{30}$ if $d(u_4) < 6$ and $c(u_4, u_5) = \frac{2\pi}{15}$ if $d(u_4) \geq 6$; $c(u_1, u_2) + c(u_1, u_4) = \frac{\pi}{10}$ if $d(u_4) < 6$ and $c(u_1, u_2) + c(u_1, u_4) = \frac{\pi}{6}$ if $d(u_4) \geq 6$. (Indeed if $\hat{\Delta}_1$ receives $\frac{\pi}{15}$ from Δ then Δ is given by Figure 4.7(xi) which implies $d(u) = 3$ and $d(\Delta_4) > 4$ in Figure 5.7, and an inspection of Figures 5.1-5.6 shows that $c(u_1, u_4) = 0$, so it can be assumed that $\hat{\Delta}_1$ receives $\frac{\pi}{30}$ from Δ and so $\hat{\Delta}_2$ receives $\frac{\pi}{30}$ from $\hat{\Delta}_1$. But now $d(u_4) < 6$ implies $c(u_1, u_4) = \frac{\pi}{15}$ and $d(u_4) \geq 6$ implies $c(u_1, u_4) = \frac{2\pi}{15}$ and this is given by Figure 4.12(iii).) It follows that if $d(u_4) < 6$ then $c^*(\hat{\Delta}_2) = c(\hat{\Delta}_2) + c(u_2, w_3) + c(w_3, u_6) +$

$c(u_5, u_6) + c(u_4, u_5) + (c(u_1, u_2) + c(u_1, u_4)) \leq c(\hat{\Delta}_2) + \frac{\pi}{10} + \frac{\pi}{30} + \frac{\pi}{6} + \frac{7\pi}{30} + \frac{\pi}{10} = c(\hat{\Delta}_2) + \frac{19\pi}{30}$; or if $d(u_4) \geq 6$ then $c^*(\hat{\Delta}_2) \leq c(\hat{\Delta}_2) + \frac{\pi}{10} + \frac{\pi}{30} + \frac{\pi}{6} + \frac{2\pi}{15} + \frac{\pi}{6} = c(\hat{\Delta}_2) + \frac{18\pi}{30}$.

Let $d(u_6) \geq 4$. If $d(u_4) \geq 4$ or $d(u_5) \geq 4$ then $c^*(\hat{\Delta}_2) \leq -\frac{2\pi}{3} + \frac{19\pi}{30} < 0$; on the other hand if $d(u_4) = d(u_5) = 3$ then $c(u_4, u_5) = 0$ and $c^*(\hat{\Delta}_2) \leq c(3,3,3,4,4,4) + \left(\frac{19\pi}{30} - \frac{7\pi}{30}\right) < 0$. Let $d(u_6) = 3$ so, in particular, $c(w_3, u_6) = 0$. If $d(u_4) \geq 4$ and $d(u_5) \geq 4$ or if $d(u_4) = 3$ and $d(u_5) \geq 5$ or if $d(u_4) \geq 5$ and $d(u_5) = 3$ then $c(\hat{\Delta}_2) \leq -\frac{3\pi}{5}$ and it follows that $c^*(\hat{\Delta}_2) \leq 0$. If $d(u_4) = 4$ and $d(u_5) = 3$ then $d(\Delta_5) > 4$, $c(u_4, u_5) = 0$ and $c^*(\hat{\Delta}_2) \leq -\frac{\pi}{2} + \frac{\pi}{10} + 0 + \frac{\pi}{6} + 0 + \frac{\pi}{10} < 0$; and if $d(u_4) = 3$ and $d(u_5) = 4$ then $d(\Delta_6) > 4$, $c(u_5, u_6) = 0$ and $c^*(\hat{\Delta}) \leq -\frac{\pi}{2} + \frac{\pi}{10} + 0 + 0 + \frac{7\pi}{30} + \frac{\pi}{10} < 0$. This leaves $d(u_4) = d(u_5) = 3$ in which case $c(u_4, u_5) = 0$. Moreover $d(\Delta_5) > 4$ also means that if $c(u_1, u_4) = \frac{\pi}{15}$ then Δ_4 is given by Δ of Figure 4.7(xi) forcing the region Δ of Figure 5.7 to have degree > 4, a contradiction, so $c(u_1, u_4) = \frac{\pi}{30}$. Since, as noted above, $c(u_1, u_2) = \frac{\pi}{15}$ implies $c(u_1, u_4) = 0$ it follows that $c(u_1, u_2) + c(u_1, u_4) = \frac{\pi}{15}$ and $c^*(\hat{\Delta}) \leq c(3,3,3,3,4,4) + \frac{\pi}{10} + 0 + \frac{\pi}{6} + 0 + \frac{\pi}{15} = 0$.

Finally consider $\hat{\Delta}_2$ of Figure 5.5(v), (vi) and assume that $d(\hat{\Delta}_2) = 6$. The fact that $\hat{\Delta}_1$ receives $\frac{\pi}{30}$ from Δ means that $\hat{\Delta}_1$ is given by the region $\hat{\Delta}_1$ of Figures 5.3(iv), 5.4(iv) with the difference being that in those figures $\hat{\Delta}_1$ receives $\frac{\pi}{15}$ from Δ as opposed to receiving $\frac{\pi}{30}$ from the corresponding region Δ in Figure 5.5(v), (vi). This now implies that $\hat{\Delta}_2$ is again given by Figure 5.7(i), (ii) and we are in the previous case where $\hat{\Delta}_2$ is given by Figure 5.3(iv), 5.4(iv). □

An immediate consequence of Lemma 5.2 is the following.

Proposition 5.3 *If* $\boldsymbol{K} = \boldsymbol{K}_0$ *then* $c(\boldsymbol{K}) \leq \sum_{d(\hat{\Delta}) \geq 8} c^*(\hat{\Delta})$*; or if* $\boldsymbol{K} = \boldsymbol{K}_1$ *then* $c(\boldsymbol{K}) \leq \sum_{\substack{d(\hat{\Delta}) \geq 8 \\ \hat{\Delta} \neq \Delta_0}} c^*(\hat{\Delta}) + c^*(\Delta_0)$.

Given this, it remains to consider regions $\hat{\Delta}$ of degree ≥ 8. To do this we partition such $\hat{\Delta}$ into regions of type $\mathcal{A}$ or type $\mathcal{B}$. If $\hat{\Delta}$ is given by $\hat{\Delta}_3$ or $\hat{\Delta}_4$ of Figure 4.6 then $\hat{\Delta}$ is a *region of type* $\mathcal{B}$; otherwise we will say that $\hat{\Delta}$ is a *region of type* $\mathcal{A}$.

There will be no further distribution of curvature in what follows and so we collect together in the following lemma statements that can be verified by inspecting Figures 4.1–4.32 and Figures 5.1–5.7. Further details will appear in the proof of Lemma 6.1.

Lemma 5.4 *Let* e_i *be an edge with endpoint* u, v *such that* e_i *is neither a* (b, a)*-edge nor is given by Figure 4.6.*

(i) *If* $c(e_i) := c(u, v) > \frac{2\pi}{15}$ *then* $c(e_i) \in \{\frac{\pi}{6}, \frac{\pi}{5}, \frac{7\pi}{30}\}$.

(ii) *If* $c(e_i) \in \{\frac{\pi}{6}, \frac{\pi}{5}, \frac{7\pi}{30}\}$ *then* e_i *is given by Figure 5.8.*

(iii) *If* $c(e_i) > \frac{2\pi}{15}$ *then either* $c(e_{i-1}) = 0$ *or* $c(e_{i+1}) = 0$ *except for* e_i *of Figure 5.8(vii), (xi), (xii) and (xvi).*

Now let e_i be a (b,a)-edge.

(iv) *If $c(e_i) > \frac{2\pi}{15}$ then $c(e_i) \in \{\frac{\pi}{6}, \frac{\pi}{5}, \frac{7\pi}{30}, \frac{4\pi}{15}, \frac{3\pi}{10}\}$.*

(v) *If $c(e_i) \in \{\frac{\pi}{6}, \frac{\pi}{5}, \frac{7\pi}{30}, \frac{4\pi}{15}, \frac{3\pi}{10}\}$ then c_i is given by Figure 5.9.*

(vi) *If $c(e_i) > \frac{2\pi}{15}$ then either $c(e_{i-1}) = 0$ or $c(e_{i+1}) = 0$ except for e_i of Figure 5.9(vii) and (x).*

Remarks.

1. Statement (iii) is readily verified except perhaps for Figure 5.8(xix) and (xx). But these correspond to the edges of Figure 4.30(i) and (iii), and so statement (iii) holds.

2. In Figure 5.8(vii) if $c(u,v) = \frac{\pi}{6}$ then $\hat{\Delta} = \hat{\Delta}_4$ of Figure 4.2(i)-(iii); if $c(u,v) = \frac{\pi}{5}$ then $\hat{\Delta} = \hat{\Delta}_4$ of Figure 4.2(iv); moreover the $\frac{\pi}{30}$ distributed across the e_{i+1} edge is given by Figure 5.1(viii) and (ix). In Figure 5.8(xi), $\hat{\Delta} = \hat{\Delta}_1$ of Figure 4.23(vii). In Figure 5.8(xii), $\hat{\Delta} = \hat{\Delta}_2$ of Figure 4.23(viii). In Figure 5.8(xiii), $\hat{\Delta} = \hat{\Delta}_2$ of Figure 5.3(iii). In Figure 5.8(xiv), $\hat{\Delta} = \hat{\Delta}_2$ of Figure 5.4(iii). In Figure 5.8(xvi), $\hat{\Delta} = \hat{\Delta}_3$ of Figure 4.3(vi)-(vii); moreover the $\frac{\pi}{30}$ distributed across the e_{i-1} edge is given by Figure 5.2(viii) and (ix).

3. In Figure 5.9(vii) if $c(u,v) = \frac{\pi}{5}$ then $\hat{\Delta}$ is given by Figure 4.28(i); and in Figure 5.9(x) if $c(u,v) = \frac{\pi}{5}$ then $\hat{\Delta}$ is given by Figure 4.27(i), in particular, $\hat{\Delta}$ in both cases is a type $\mathcal{B}$ region.

The next result will be used throughout later sections.

Lemma 5.5 *Let the regions $\hat{\Delta}$, Δ_i and Δ_{i+1} be given by Figure 5.10(i) or 5.10(vii).*

(i) *If $c_i = \frac{9\pi}{30}$ then $c_{i+1} = 0$.*

(ii) *If $c_i = \frac{8\pi}{30}$ then $c_{i+1} \leq \frac{5\pi}{30}$.*

(iii) *If $c_i = \frac{8\pi}{30}$ and $c_{i+1} = \frac{3\pi}{30}$ then $\hat{\Delta}$ of Figure 5.10(i) is given by Figure 5.10(ii) in which $\Delta_{i+1} = \hat{\Delta}_3$ of Figure 4.17(xii); and $\hat{\Delta}$ of Figure 5.10(vii) is given by $\hat{\Delta}$ of Figure 5.10(viii) in which $\Delta_{i+1} = \hat{\Delta}_4$ of Figure 4.16(xiii).*

(iv) *If $c_i = \frac{8\pi}{30}$ then $c_{i+1} \neq \frac{4\pi}{30}$.*

(v) *If $c_i = \frac{8\pi}{30}$ and $c_{i+1} = \frac{5\pi}{30}$ then $\hat{\Delta}$ of Figure 5.10(i) is given by Figure 5.10(iii) in which $\Delta_{i+1} = \Delta$ of Figure 4.20(viii); and $\hat{\Delta}$ of Figure 5.10(vii) is given by Figure 5.10(ix) in which $\Delta_{i+1} = \Delta$ of Figure 4.22(vi).*

(vi) *If* $c_i = \frac{7\pi}{30}$ *then* $c_{i+1} \leq \frac{7\pi}{30}$.

(vii) *If* $c_i = \frac{7\pi}{30}$ *and* $c_{i+1} = \frac{5\pi}{30}$ *then* $\hat{\Delta}$ *of Figure 5.10(i) is given by Figures 5.10(iv) and (v) in which* $\Delta_{i+1} = \hat{\Delta}_4$ *of Figure 4.9(iv) and* $\Delta_{i+1} = \Delta$ *of Figure 4.20(viii), respectively; and* $\hat{\Delta}$ *of Figure 5.10(vii) is given by Figures 5.10(x) and (xi) in which* $\Delta_{i+1} = \hat{\Delta}_2$ *of Figure 4.8(iv) and* $\Delta_{i+1} = \Delta$ *of Figure 4.22(vi), respectively.*

(viii) *If* $c_i = \frac{7\pi}{30}$ *then* $c_{i+1} \neq \frac{6\pi}{30}$.

(ix) *If* $c_i = c_{i+1} = \frac{7\pi}{30}$ *then* $\hat{\Delta}$ *of Figure 5.10(i) is given by Figure 5.10(vi) in which* $\Delta_{i+1} = \hat{\Delta}_4$ *of Figure 4.10(ix); and* $\hat{\Delta}$ *of Figure 5.10(vii) is given by Figure 5.10(xii) in which* $\Delta_{i+1} = \hat{\Delta}_2$ *of Figure 4.10(v) or of Figure 4.24(v).*

Proof. Statements (i), (vi) and (viii) follow from an inspection of Figures 5.8 and 5.9. Moreover if $\hat{\Delta}$ is given by Figure 5.10(i) and $c_i = \frac{8\pi}{30}$ then it can be assumed without any loss that either $\Delta_i = \hat{\Delta}_2$ of Figure 4.17(iv) or (xiii) or $\Delta_i = \hat{\Delta}_4$ of Figure 4.25(v); and if $\hat{\Delta}$ is given by Figure 5.10(vii) and $c_i = \frac{8\pi}{30}$ then it can be assumed without any loss that either $\Delta_i = \hat{\Delta}_4$ of Figure 4.16(v) or (xiv) or $\Delta_i = \hat{\Delta}_2$ of Figure 4.25(iii).

(ii) Let $\hat{\Delta}$ be given by Figure 5.10(i). If $c_{i+1} > \frac{5\pi}{30}$ then the only possibility is given by Figure 5.8(ix) in which case $c_{i+1} = \frac{7\pi}{30}$ and $\Delta_{i+1} = \hat{\Delta}_2$ of Figure 4.10(ix) where we note that $d(v_1) = 4$ and $d(v_2) = 3$. However if $\Delta_i = \hat{\Delta}_2$ of Figure 4.17(iv) then the vertex corresponding to v_1 is u_1 which has degree 3; or if $\Delta_i = \hat{\Delta}_4$ of Figure 4.25(v) then the vertex corresponding to v_1 is v_2 which has degree 5, in each case a contradiction. This leaves $\Delta_i = \hat{\Delta}_2$ of Figure 4.17(xiii), where $\Delta_{i+1} = \hat{\Delta}_7$ and this is shown in Figure 5.11(i). But observe that the vertex corresponding to v_2 of Figure 4.10(ix) is w_3 which has degree 4, again a contradiction.

Let $\hat{\Delta}$ be given by Figure 5.10(vii). If $c_{i+1} > \frac{5\pi}{30}$ then the only possibiilty is given by Figure 5.8(x) in which case $c_{i+1} = \frac{7\pi}{30}$ and either $\Delta_{i+1} = \hat{\Delta}_2$ of Figure 4.10(v) where $d(v_2) = 3$ and $d(v_3) = 4$ or $\Delta_{i+1} = \hat{\Delta}_2$ of Figure 4.24(v) where $d(v_2) = 5$ and $d(v_3) = 4$. However if $\Delta_i = \hat{\Delta}_4$ of Figure 4.16(v) the vertex corresponding to v_3 (both cases) is u_6 which has degree 3; or if $\Delta_i = \hat{\Delta}_2$ of Figure 4.25(iii) the vertex corresponding to v_3 (both cases) is v_2 which has degree 5, in all cases a contradiction. This leaves $\Delta_i = \hat{\Delta}_4$ of Figure 4.16(xiv) where $\Delta_{i+1} = \hat{\Delta}_7$ and this is shown in Figure 5.11(ii). But observe that the vertex corresponding to v_2 of Figures 4.10(v), 4.24(v) is w_1 which has degree 4, again a contradiction.

(iii) Checking Figures 4.1–4.32 and 5.1–5.7 shows that if $c_{i+1} = \frac{3\pi}{30}$ in Figure 5.10(i) then Δ_{i+1} must be one of Figures 4.4(vii), 4.4(viii), 4.17(iii), 4.17(xii), 4.15(iv) or 4.27(ii). Given that $\Delta_i = \hat{\Delta}_2$ of Figure 4.17(iv) or (xiii) or $\Delta_i = \hat{\Delta}_4$ of Figure 4.25(v) there is a vertex degree contradiction in each possible combination except when Δ_{i+1} is given by Figure 4.17(xii) or 4.25(iv) and these each yield Figure 5.10(ii). If $c_{i+1} = \frac{3\pi}{30}$ in

Figure 5.10(ii) then Δ_{i+1} must be one of Figure 4.5(vii), 4.5(viii), 4.16(iv), 4.16(xiii), 4.25(ii) or 4.28(ii). Given that $\Delta_i = \hat{\Delta}_4$ of Figure 4.16(v) or (xiv) or $\Delta_i = \hat{\Delta}_2$ of Figure 4.25(iii) again there is a vertex degree contradiction in each case except when Δ_{i+1} is given by Figure 4.16(xiii) or 4.25(ii) and these yield Figure 5.10(viii).

(iv) If $c_{i+1} = \frac{4\pi}{30}$ in Figure 5.10(i) then Δ_{i+1} must be one of the Figures 4.7(viii), 4.11(vii), 4.12(iii) and 4.28(i), but in each case there is a vertex degree contradiction. If $c_{i+1} = \frac{4\pi}{30}$ in Figure 5.10(vii) then Δ_{i+1} must be one of Figures 4.7(xii), 4.11(iii), 4.12(iii) or 4.27(i), and again in each case there is a vertex contradiction.

(v) The possibilities for Δ_{i+1} of Figure 5.10(i) are (see Figures 5.8(ix) and 5.9(v)) $\hat{\Delta}_4$ of Figure 4.9(iv) which yields a vertex degree contradiction (for each choice of Δ_i) and Δ of Figure 4.20(viii) which is given by Figure 5.10(iii); and for Δ_{i+1} of Figure 5.10(vii) are (see Figures 5.8(x) and 5.9(iii)) $\hat{\Delta}_2$ of Figure 4.8(iv) which yields a vertex degree contradiction and Δ of Figure 4.22(vi) which is given by Figure 5.10(ix).

Finally (vii) follows from the proof of (v) and (ix) follows from the proof of (ii). □

6 Type $\mathcal{A}$ regions

Throughout this section many assertions will be based on previous lemmas. Moreover *checking* means checking Figures 4.1–4.32 and Figures 5.1–5.7. The reader is also referred to Figures 4.33, 5.8, 5.9 and 5.10.

The *surplus* s_i of an edge e_i is defined by $s_i = c_i - \frac{2\pi}{15}$ $(1 \leq i \leq k)$ where c_i is the maximum amount of curvature that is transferred across e_i. If we add s_i to c_{i+1}, c_{i-1} we will say that e_{i+1}, e_{i-1} (respectively) *absorbs* s_i from c_i. Checking Figures 5.8 and 5.9 shows, for example, that if $d(u_i) = d(u_{i+1}) = 3$ in Figure 6.1 then $s_i \leq \frac{\pi}{15}$. The *deficit* δ_i of a vertex u_i of degree d_i is defined by $\delta_i = 2\pi(\frac{1}{d_i} - \frac{1}{3})$ and so if $d_i \geq 4$ then $\delta_i \leq -\frac{\pi}{6}$. If we add s_{i-1}, s_i (respectively) to δ_i we will say that u_i *absorbs* s_{i-1}, s_i from e_{i-1}, e_i (respectively).

Lemma 6.1 *Let $\hat{\Delta}$ be a type $\mathcal{A}$ region of degree k. Then the following hold. (i) $c^*(\hat{\Delta}) \leq (2-k) + k.\frac{2\pi}{3} + k.\frac{2\pi}{15}$. (ii) If $k \geq 10$ then $c^*(\hat{\Delta}) \leq 0$.*

Proof. (i) Denote the vertices of $\hat{\Delta}$ by v_i $(1 \leq i \leq k)$, the edges by e_i $(1 \leq i \leq k)$ and the degrees of the v_i by d_i $(1 \leq i \leq k)$. Let c_i denote the amount of curvature $\hat{\Delta}$ receives across the edge e_i $(1 \leq i \leq k)$. Consider the edge e_i of $\hat{\Delta}$ as shown in Figure 6.1. If $c_i \leq \frac{2\pi}{15}$ there is nothing to consider, so let $c_i > \frac{2\pi}{15}$. Then by Lemma 5.4, $c_i \in \{\frac{\pi}{6}, \frac{\pi}{5}, \frac{7\pi}{30}, \frac{4\pi}{15}, \frac{3\pi}{10}\}$ and $\hat{\Delta}$ is given by Figures 5.8 and 5.9. First assume that e_i is not given by Figure 4.30(i) or (iii) (**ex6.1**).

Let $\hat{\Delta}$ be given by Figure 5.8. If $\hat{\Delta}$ is given by Figure 5.8(i), (vii), (viii), (xiv) or (xv) then the edge e_{i+1} absorbs $s_i \leq \frac{\pi}{15}$ (from c_i). Note that in these cases $(d_{i+1}, c_{i+1}) \in$

$\{(3,0),(3,\frac{\pi}{30})\}$. If $\hat{\Delta}$ is given by Figure 5.8(ii), (iii), (vi), (xiii), (xvi), (xix) or (xx) then e_{i-1} absorbs $s_i \leq \frac{\pi}{15}$. Note that $(d_i, c_{i-1}) \in \{(3,0),(3,\frac{\pi}{30})\}$. If $\hat{\Delta}$ is given by Figure 5.8(iv), (x) or (xviii) then the vertex v_i absorbs $s_i \leq \frac{\pi}{10}$. Note that $(d_i, c_{i-1}) \in \{(4,0),(5,0)\}$. If $\hat{\Delta}$ is given by Figure 5.8(v), (ix) or (xvii) then v_{i+1} absorbs $s_i \leq \frac{\pi}{10}$. Note that $(d_{i+1}, c_{i+1}) \in \{(4,0),(5,0)\}$. This leaves Figure 5.8(xi) and (xii) to be considered. If $\hat{\Delta}$ is given by Figure 5.8(xi) or (xii) then v_i absorbs $s_i = \frac{\pi}{30}$. Note that $d_i = 4$.

Now let $\hat{\Delta}$ be given by Figure 5.9. If $\hat{\Delta}$ is given by Figure 5.9(i) or (ix) then the edge e_{i+1} absorbs $s_i = \frac{\pi}{30}$. Note that $(d_{i+1}, c_{i+1}) = (3,0)$. If $\hat{\Delta}$ is given by Figure 5.9(ii) or (viii) restricted to the case $c_i = \frac{5\pi}{30}$ then e_{i-1} absorbs $s_i = \frac{\pi}{30}$. Note that $(d_i, c_{i-1}) = (3,0)$. If $\hat{\Delta}$ is given by Figure 5.9(iii), (vi) or (xi) then v_{i+1} absorbs $s_i \leq \frac{\pi}{6}$. Note that $(d_{i+1}, c_{i+1}) = (4,0)$. If $\hat{\Delta}$ is given by Figure 5.9(iv), (v) or (xii) then v_i absorbs $s_i \leq \frac{\pi}{6}$. Note that $(d_i, c_{i-1}) = (4,0)$. This leaves the cases Figure 5.9(vii), (viii) with $c_i = \frac{8\pi}{30}$ and (x). If $\hat{\Delta}$ is given by Figure 5.9(vii) then v_i absorbs $s_i \leq \frac{2\pi}{15}$. Note that $d_i = 4$. If $\hat{\Delta}$ is given by Figure 5.9(viii) or (x) then v_{i+1} absorbs $s_i \leq \frac{2\pi}{15}$. Note that $d_{i+1} = 4$.

This completes absorption by edges or vertices when e_i is not given by Figure 4.30(i) or (iii) (and these correspond to cases of Figure 5.8(xix), (xx)). Observe that if an edge e_j absorbs positive curvature a_j, say, then $a_j \leq \frac{\pi}{15}$ and either $c_j = 0$ or $c_j = \frac{\pi}{30}$; moreover e_j *always absorbs across a vertex of degree 3*. If $c_j = 0$ then $c_j + a_j \leq \frac{2\pi}{15}$ so let $c_j = \frac{\pi}{30}$. We claim that in this case we also have $c_j + a_j \leq \frac{2\pi}{15}$. The only possible way this fails is if $s_{j-1} = s_{j+1} = \frac{\pi}{15}$, that is, $c_{j-1} = c_{j+1} = \frac{\pi}{5}$. Thus $e_j = e_{i+1}$ of Figure 5.8(vii) and $\hat{\Delta} = \hat{\Delta}_4$ of Figure 4.2(iv); and also $e_j = e_{i-1}$ of Figure 5.8(xvi) and $\hat{\Delta} = \hat{\Delta}_3$ of Figure 4.3(vi), (vii). But any attempt at labelling shows that this is impossible and so our claim follows. Observe further that any pair of vertices each absorbing more than $\frac{\pi}{30}$ cannot coincide. This follows immediately from the fact that either $c_{i-1} = 0$ or $c_{i+1} = 0$ or the vertex is given by v_i of Figure 5.9(vii) or v_{i+1} of Figure 5.9(x) and clearly these cannot coincide. Also observe that if a vertex v_i say absorbs more than $\frac{2\pi}{15}$ from e_i or e_{i-1} (respectively) then it absorbs 0 from e_{i-1} or e_i (respectively). Therefore any given vertex can absorb at most $\frac{\pi}{6} + 0 = \frac{\pi}{6}$ as in Figure 5.9(iv) and (vi), or at most $\frac{2\pi}{15} + \frac{\pi}{30} = \frac{\pi}{6}$. But since any vertex that absorbs curvature *has degree at least 4* and so a deficit of at most $-\frac{\pi}{6}$, statement (i) holds for these cases.

Finally let e_i be given by Figure 4.30(i) or (iii) (**en6.1**). Since $d(v) \geq 4$ in both figures it follows that e_{i-1} does not absorb any surplus from e_{i-2}. If $s_{i+1} > \frac{\pi}{15}$ then according to the above it must be absorbed by v_{i+2}, so let $s_{i+1} \leq \frac{\pi}{15}$. In this case e_{i-1} absorbs $s_i + s_{i+1} \leq \frac{2\pi}{15}$ and statement (i) follows.

(ii) This follows from (i) and the fact that $(2-k) + k.\frac{2\pi}{3} + k.\frac{2\pi}{15} \leq 0$ if and only if $k \geq 10$. □

It follows from Lemma 6.1(ii) that we need only consider type A regions of degree at most 9.

Lemma 6.2 *If $7 \leq d(\hat{\Delta}) \leq 9$ then (up to cycle-permutation and corner labelling) either*

$d(\hat{\Delta}) = 8$ *and* $\hat{\Delta}$ *is given by Figure 6.2(i)-(xi) or* $d(\hat{\Delta}) = 9$ *and* $\hat{\Delta}$ *is given by Figure 6.2(xii).*

Proof. If $7 \leq d(\hat{\Delta}) \leq 9$ then $\hat{\Delta}$ is given by Figure 3.6(iv)–(xi). It turns out that there is (up to cyclic permutation and inversion) exactly one way to label $\hat{\Delta}$ of Figure 3.6(iv), (v), (ix) and (xi); four ways to label $\hat{\Delta}$ of (vi); six ways to label $\hat{\Delta}$ of (vii); and two ways to label $\hat{\Delta}$ of (viii) and (x). The resulting set of seventeen labelled regions contains some repeats with respect to corner labelling and deleting these leaves the twelve $\hat{\Delta}$ of Figure 6.2(i)-(xii). □

Notation Let $d(\hat{\Delta}) = k$ and suppose that the vertices of $\hat{\Delta}$ are u_i $(1 \leq i \leq k)$. We write $cv(\hat{\Delta}) = (a_1, \dots, a_k)$, where each a_i is a non-negative integer, to denote the fact that the total amount of curvature $\hat{\Delta}$ receives is bounded above by $(a_1 + \dots + a_k)\frac{\pi}{30}$ with the understanding that $a_i \frac{\pi}{30}$ is transferred to $\hat{\Delta}$ across the (u_i, u_{i+1})-edge (subscripts mod k).

Notation In the proof of Proposition 6.3 we will use non-negative integers a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2, e_1, e_2, h_1, h_2 where: $a_1 + a_2 = 7$; $b_1 + b_2 = 8$; $c_1 + c_2 = 9$; $d_1 + d_2 = 10$; $e_1 + e_2 = 11$; and $h_1 + h_2 = 14$.

$\boldsymbol{c(\Delta) = c(d_1, \dots, d_m)}$ Let $m = m_1 + m_2 + m_3 = 8 + k$ where $k \geq 0$ and suppose that Δ contains m_1, m_2, m_3 vertices of degree 3, 4, 5 (respectively). Then we will use the formula (here and in the next section)

$$c(\Delta) = c(3, \dots, 3, 4, \dots, 4, 5, \dots, 5) = -\frac{(20 + 10k + 5m_2 + 8m_3)\pi}{30}.$$

Proposition 6.3 *If* $\hat{\Delta}$ *is a type* $\mathcal{A}$ *region and* $7 \leq d(\hat{\Delta}) \leq 9$ *then* $c^*(\hat{\Delta}) \leq 0$.

Proof. It follows from Lemma 6.2 that we need only consider $\hat{\Delta}$ of Figure 6.2 in which the label $\alpha(\beta)$ at the edge with endpoints u, v indicates $c(u,v) = \frac{\alpha\pi}{30}$ and $c(u,v) = \frac{\beta\pi}{30}$ when $d(u) = d(v) = 3$. We treat each of the twelve cases of Figure 6.2 in turn. (We will make extensive use of *checking* and Figures 5.8–5.10. Some details of this will be given, mainly in Cases 1 and 4.)

Case 1 Let $\hat{\Delta}$ be given by Figure 6.2(i). If $c(u_1, u_2) > \frac{2\pi}{15}$ then (see Figure 5.8(xvii)) $c(u_8, u_1) = 0$; and if $\frac{\pi}{15} < c(u_1, u_2) < \frac{\pi}{5}$ then $c(u_1, u_2) \in \left\{\frac{2\pi}{15}, \frac{\pi}{10}\right\}$ and (see Figures 4.7(iii), 4.10(vii), 4.18(vii), (xi), 4.31(iv), (vii) and 5.2(v), (ix)) either $c(u_8, u_1) = 0$ or $c(u_2, u_3) = 0$. (Note that Figure 5.8(xiv) does not apply to $\hat{\Delta}$.) Similar statements hold for each of (u_2, u_3), (u_3, u_4), (u_7, u_8) and (u_8, u_1). In particular it follows that $c(u_7, u_8) + c(u_8, u_1) + c(u_1, u_2) + c(u_2, u_3) + c(u_3, u_4) \leq \frac{2\pi}{3}$. If $c(u_4, u_5) > \frac{2\pi}{15}$ then (see Figure 5.8(ix)) $c(u_3, u_4) = 0$; if $c(u_6, u_7) > \frac{2\pi}{15}$ then (see Figure 5.8(i), (vi)) either $c(u_5, u_6) = 0$ or $c(u_7, u_8) = 0$; and by Lemma 5.5 (see Figure 5.10(vi)), $c(u_4, u_5) + c(u_5, u_6) \leq \frac{7\pi}{15}$. Therefore if $c(u_4, u_5) > \frac{2\pi}{15}$ then $cv(\hat{\Delta}) = (0, 6, 0, 7, 7, 6, 0, 6)$; and if $c(u_4, u_5) \leq \frac{2\pi}{15}$ then $cv(\hat{\Delta}) = (4, 0, 6, e_1, e_2, 6, 0, 6)$ (see Figure 5.10). So if $\hat{\Delta}$ has at least three vertices of degree > 3 then $c^*(\hat{\Delta}) \leq -\frac{35\pi}{30} + \frac{33\pi}{30} < 0$. If $d(u_1) = d(u_2) = 3$ and $c(u_1, u_2) > 0$ then checking shows that $\hat{\Delta}$ is given by $\hat{\Delta}_1$ of Figure 5.2(v) or (ix) and $c(u_1, u_2) = \frac{2\pi}{15}$. Again similar statements hold for (u_2, u_3), (u_3, u_4),

(u_7,u_8) and (u_8,u_1). Suppose that $\hat{\Delta}$ has no vertices of degree >3. Then $c(u_4,u_5)=0$ and $c(u_5,u_6)=\frac{\pi}{15}$ (see Figure 5.1) so it follows that $cv(\hat{\Delta})=(4,0,4,0,2,4,4,0)$ and $c^*(\hat{\Delta})\leq -\frac{2\pi}{3}+\frac{3\pi}{5}<0$. Suppose that $\hat{\Delta}$ has exactly one vertex of degree >3. If $d(u_5)=3$ then $c(u_4,u_5)=0$, $c(u_5,u_6)=\frac{\pi}{15}$ and it follows that $cv(\hat{\Delta})=(0,6,4,0,2,6,0,4)$; and if $d(u_5)>3$ then $c(u_4,u_5)=\frac{\pi}{30}$ (see Figures 5.2 and 5.5) so $cv(\hat{\Delta})=(4,0,4,c_1,c_2,4,4,0)$. Therefore $c^*(\hat{\Delta})\leq -\frac{5\pi}{6}+\frac{5\pi}{6}=0$. Finally suppose that $\hat{\Delta}$ has exactly two vertices u_i,u_j of degree >3. If $d(u_5)=3$ then $c^*(\hat{\Delta})<0$ so it can be assumed without any loss that $i=5$. If $j=1$ then $c(u_8,u_1)=0$ and $c(u_4,u_5)>\frac{\pi}{30}$ forces $c(u_5,u_6)=0$ so $cv(\hat{\Delta})=(6,0,4,c_1,c_2,4,4,0)$; if $j=2$ then $c(u_1,u_2)=0$ and $cv(\hat{\Delta})=(0,6,4,c_1,c_2,6,0,4)$; if $j=3$ then $c(u_2,u_3)=0$ and $cv(\hat{\Delta})=(4,0,6,c_1,c_2,4,4,0)$; if $j=4$ then $c(u_3,u_4)=0$ and $cv(\hat{\Delta})=(0,4,0,7,7,6,0,4)$; if $j=6$ then $c(u_4,u_5)=\frac{\pi}{6}$, $c(u_5,u_6)=0$ and $cv(\hat{\Delta})=(4,0,4,0,5,0,4,0)$; if $j=7$ then $cv(\hat{\Delta})=(4,0,4,c_1,c_2,4,4,0)$; and if $j=8$ then $c(u_7,u_8)=0$ and $cv(\hat{\Delta})=(4,0,4,c_1,c_2,6,0,6)$. It follows that $c^*(\hat{\Delta})\leq -\pi+\frac{29\pi}{30}<0$.

Case 2 Let $\hat{\Delta}$ be given by Figure 6.2(ii). If $c(u_3,u_4)>\frac{2\pi}{15}$ then $(d(u_3),d(u_4))=(4,4)$ and $c(u_2,u_3)=0$; and if $c(u_5,u_6)>\frac{2\pi}{15}$ then $(d(u_5),d(u_6))=(3,4)$ and $c(u_6,u_7)=0$. It follows that if at least three of u_i have degree ≥ 4 then $c^*(\hat{\Delta})\leq -\frac{7\pi}{6}+\frac{7\pi}{6}=0$, so assume otherwise. If $\hat{\Delta}$ has no vertices of degree >3 then we see (from Figure 6.2(ii) and Figure 5.4(iii)) that $cv(\hat{\Delta})=(0,4,0,0,0,4,6,0)$ and $c^*(\hat{\Delta})\leq -\frac{2\pi}{3}+\frac{7\pi}{15}<0$. Let $\hat{\Delta}$ have exactly one vertex u_i of degree >3. Then the following holds. If $i=1$ then $cv(\hat{\Delta})=(3,5,0,0,0,4,6,2)$; if $i=2$ then $cv(\hat{\Delta})=(3,6,0,0,0,4,6,0)$; if $i=3$ then $cv(\hat{\Delta})=(0,0,4,0,0,4,6,0)$; if $i=4$ then $cv(\hat{\Delta})=(0,4,4,0,0,4,6,0)$; if $i=5$ then $cv(\hat{\Delta})=(0,4,0,4,4,4,6,0)$; if $i=6$ then $cv(\hat{\Delta})=(0,4,0,0,d_1,d_2,6,0)$; if $i=7$ then $cv(\hat{\Delta})=(0,4,0,0,0,6,6,0)$; and if $i=8$ then $cv(\hat{\Delta})=(0,4,0,0,0,4,6,2)$. It follows that $c^*(\hat{\Delta})\leq -\frac{5\pi}{6}+\frac{11\pi}{15}<0$. Let $\hat{\Delta}$ have exactly two vertices u_i,u_j of degree >3. If $d(u_3)=d(u_4)=d(u_5)=3$ or $d(u_4)=d(u_5)=d(u_6)=3$ then $c^*(\hat{\Delta})\leq -\pi+\pi=0$. This leaves 14 out of 28 cases to be considered. If $(i,j)=(1,4)$ then $cv(\hat{\Delta})=(3,5,4,4,0,4,6,2)$; if $(i,j)=(1,5)$ then $cv(\hat{\Delta})=(3,5,0,4,4,4,6,2)$; if $(i,j)=(2,4)$ then $cv(\hat{\Delta})=(3,6,4,4,0,4,6,0)$; if $(i,j)=(2,5)$ then $cv(\hat{\Delta})=(3,6,0,4,4,4,6,0)$; if $(i,j)=(3,4)$ then $cv(\hat{\Delta})=(0,d_1,d_2,4,0,4,6,0)$; if $(i,j)=(3,5)$ then $cv(\hat{\Delta})=(0,6,4,4,4,4,6,0)$; if $(i,j)=(3,6)$ then $cv(\hat{\Delta})=(0,6,4,0,d_1,d_2,6,0)$; if $(i,j)=(4,5)$ then $cv(\hat{\Delta})=(0,4,4,4,4,4,6,0)$; if $(i,j)=(4,6)$ then $cv(\hat{\Delta})=(0,4,4,4,d_1,d_2,6,0)$; if $(i,j)=(4,7)$ then $cv(\hat{\Delta})=(0,4,4,4,0,6,6,0)$; if $(i,j)=(4,8)$ then $cv(\hat{\Delta})=(0,4,4,4,0,4,6,2)$; if $(i,j)=(5,6)$ then $cv(\hat{\Delta})=(0,4,0,4,4,6,6,0)$; if $(i,j)=(5,7)$ then $cv(\hat{\Delta})=(0,4,0,4,4,6,6,0)$; and if $(i,j)=(5,8)$ then $cv(\hat{\Delta})=(0,4,0,4,4,4,6,2)$. It follows that $c^*(\hat{\Delta})\leq -\pi+\frac{14\pi}{15}<0$.

Case 3 Let $\hat{\Delta}$ be given by Figure 6.2(iii). If $c(u_2,u_3)>\frac{2\pi}{15}$ then $(d(u_2),d(u_3))=(4,3)$ and $c(u_1,u_2)=0$; if $c(u_4,u_5)>\frac{2\pi}{15}$ then $(d(u_4),d(u_5))=(4,4)$ and $c(u_3,u_4)=0$; if $c(u_5,u_6)>\frac{2\pi}{15}$ then $(d(u_5),d(u_6))=(4,4)$ and $c(u_6,u_7)=0$; if $c(u_7,u_8)>\frac{2\pi}{15}$ then $(d(u_7),d(u_8))=(3,4)$ and $c(u_8,u_1)=0$. Moreover if $c(u_1,u_2)>\frac{2\pi}{15}$ then $d(u_2)=3$ and $c(u_1,u_2)=0$; and if $c(u_8,u_1)>\frac{2\pi}{15}$ then $d(u_8)=3$ and $c(u_7,u_8)=0$. It follows that $c(u_1,u_2)+c(u_2,u_3)\leq\frac{4\pi}{15}$; $c(u_3,u_4)+c(u_4,u_5)\leq\frac{7\pi}{30}$; $c(u_5,u_6)+c(u_6,u_7)\leq\frac{7\pi}{30}$; and $c(u_7,u_8)+c(u_8,u_1)\leq\frac{4\pi}{15}$. Therefore if $\hat{\Delta}$ has at least two vertices of degree > 3 then $c^*(\hat{\Delta}) \leq -\pi + \pi = 0$. Suppose that $\hat{\Delta}$ contains no vertices of degree > 3. Then we see (from Figure 6.2(iii)) that $cv(\hat{\Delta}) = (6,0,0,0,0,0,0,6)$ and $c^*(\hat{\Delta}) \leq -\frac{2\pi}{3} + \frac{2\pi}{5} < 0$. Let $\hat{\Delta}$ have exactly one vertex u_i of degree > 3. Then the following holds: if $i = 1$ then $cv(\hat{\Delta}) = (6,0,0,0,0,0,0,6)$; if $i = 2$ then $cv(\hat{\Delta}) = (b_1, b_2, 0,0,0,0,0,6)$; if $i = 3$ then $cv(\hat{\Delta}) = (6,0,3,0,0,0,0,6)$; if $i = 4$ then $cv(\hat{\Delta}) = (6,0,3,4,0,0,0,6)$; if $i = 5$ then $cv(\hat{\Delta}) = (6,0,0,4,4,0,0,6)$; if $i = 6$ then $cv(\hat{\Delta}) = (6,0,0,0,4,3,0,6)$; if $i = 7$ then $cv(\hat{\Delta}) = (6,0,0,0,0,3,0,6)$; and if $i = 8$ then $cv(\hat{\Delta}) = (6,0,0,0,0,0,b_1,b_2)$. Therefore $c^*(\hat{\Delta}) \leq -\frac{5\pi}{6} + \frac{2\pi}{3} < 0$.

Case 4 Let $\hat{\Delta}$ be given by Figure 6.2(iv). If $c(u_1,u_2) > \frac{2\pi}{15}$ then $c(u_2,u_3) = 0$; if $c(u_2,u_3) > \frac{2\pi}{15}$ then $c(u_1,u_2) = 0$; if $c(u_8,u_1) > \frac{2\pi}{15}$ then $c(u_7,u_8) = 0$; if $c(u_7,u_8) > \frac{2\pi}{15}$ then $c(u_8,u_1) = 0$; if $c(u_4,u_5) = \frac{3\pi}{10}$ then $c(u_3,u_4) = 0$; if $c(u_4,u_5) = \frac{4\pi}{15}$ then $c(u_3,u_4) = \frac{\pi}{15}$ (see Figure 5.10); if $c(u_5,u_6) = \frac{3\pi}{10}$ then $c(u_6,u_7) = 0$; and if $c(u_5,u_6) = \frac{4\pi}{15}$ then $c(u_6,u_7) = \frac{\pi}{15}$ (see Figure 5.10). It follows that $c(u_3,u_4) + c(u_4,u_5) + c(u_5,u_6) + c(u_6,u_7) = \frac{11\pi}{15}$. Therefore $c^*(\hat{\Delta}) \leq c(\Delta) + \frac{19\pi}{15}$ so if $\hat{\Delta}$ has at least four vertices of degree > 3 then $c^*(\hat{\Delta}) < 0$. Let $\hat{\Delta}$ have no vertices of degree > 3. Then $cv(\hat{\Delta}) = (6,0,0,2,2,0,0,6)$ and $c^*(\hat{\Delta}) \leq -\frac{2\pi}{3} + \frac{8\pi}{15} < 0$. Let $\hat{\Delta}$ have exactly one vertex u_i of degree > 3. If $d(u_4) = 3$ then $c(u_3,u_4) = 0$ and $c(u_4,u_5) = \frac{\pi}{15}$; and if $d(u_6) = 3$ then $c(u_5,u_6) = \frac{\pi}{15}$ and $c(u_6,u_7) = 0$. Thus if $d(u_4) = d(u_6) = 3$ then $c^*(\hat{\Delta}) \leq -\frac{5\pi}{6} + \frac{2\pi}{3} < 0$; if $d(u_4) > 3$ then $cv(\hat{\Delta}) = (6,0,e_1,e_2,2,0,0,6)$; and if $d(u_6) > 3$ then $cv(\hat{\Delta}) = (6,0,0,2,e_1,e_2,0,6)$. Therefore $c^*(\hat{\Delta}) \leq -\frac{5\pi}{6} + \frac{5\pi}{6} = 0$. Let $\hat{\Delta}$ have exactly two vertices of degree > 3. If $d(u_4) = 3$ or $d(u_6) = 3$ then $c^*(\hat{\Delta}) \leq -\pi + \frac{29\pi}{30} < 0$ so it can be assumed that $d(u_4) > 3$ and $d(u_6) > 3$. Then $d(u_2) = 3$ implies $d(\Delta_2) > 4$ and $c(u_2,u_3) = 0$; and $d(u_8) = 3$ implies $d(\Delta_7) > 4$ and $c(u_7,u_8) = 0$. This then prevents $c(u_3,u_4) = \frac{2\pi}{15}$ or $c(u_6,u_7) = \frac{2\pi}{15}$ (see Figure 4.7(xii) and (xiii)) so $c(u_3,u_4) = c(u_7,u_8) = \frac{\pi}{15}$. Since $c(u_1,u_2) = c(u_8,u_1) = \frac{\pi}{5}$ it follows that if $c(u_4,u_5) \neq \frac{4\pi}{15}$ and $c(u_5,u_6) \neq \frac{4\pi}{15}$ then $cv(\hat{\Delta}) = (6,0,2,7,7,2,0,6)$ and $c^*(\hat{\Delta}) \leq -\pi + \pi = 0$. Suppose that $c(u_4,u_5) = \frac{4\pi}{15}$, say. Then we see from Figure 4.24(vi) and 4.25(iii) that $d(v_2) = 5$ and so $c(u_3,u_4) > 0$ implies that $\Delta_3 = \Delta$ of Figure 4.23. But $d(\Delta_2) > 4$ in fact forces $\Delta_3 = \Delta$ of Figure 4.23(x) and $c(u_3,u_4) = \frac{\pi}{30}$. Similarly if $c(u_5,u_6) = \frac{4\pi}{15}$ then we see from Figure 4.25(v) and Figure 4.23(x), (xii) that $c(u_6,u_7) = \frac{\pi}{30}$. It follows that if $c(u_4,u_5) = \frac{4\pi}{15}$ or $c(u_5,u_6) = \frac{4\pi}{15}$ then $c^*(\hat{\Delta}) \leq -\pi + \pi = 0$. Finally let $\hat{\Delta}$ have exactly three vertices u_i, u_j, j_k of degree > 3. Then $c(\hat{\Delta}) \leq -\frac{7\pi}{6}$. If $d(u_2) = d(u_8) = 3$ then $c(u_2,u_3) = c(u_7,u_8) = 0$ and $cv(\hat{\Delta}) = (6,0,e_1,e_2,e_1,e_2,0,6)$; if $d(u_4) = 3$ then $cv(\hat{\Delta}) = (b_1,b_2,0,2,e_1,e_2,b_1,b_2)$; and if $d(u_6) = 3$ then $cv(\hat{\Delta}) = (b_1,b_2,e_1,e_2,2,0,b_1,b_2)$. So it can be assumed that $(i,j,k) = (2,4,6)$ or $(4,6,8)$ and in both cases $c^*(\hat{\Delta}) \leq c(\hat{\Delta}) + \frac{36\pi}{30}$. If $d(u_i)$ or $d(u_j)$ or $d(u_k)$ is greater than 4 then $c^*(\hat{\Delta}) < 0$ so assume otherwise. But now $d(u_2) = 4$ forces $c(u_1,u_2) = 0$ and $d(u_8) = 4$ forces $c(u_8,u_1) = 0$. It follows that $cv(\hat{\Delta}) = (0,7,e_1,e_2,e_1,e_2,0,6)$ or $(6,0,e_1,e_2,e_1,e_2,7,0)$ so $c^*(\hat{\Delta}) \leq -\frac{7\pi}{6} + \frac{7\pi}{6} = 0$.

Case 5 Let $\hat{\Delta}$ be given by Figure 6.2(v). If $c(u_1,u_2) = \frac{9\pi}{30}$ then $c(u_8,u_1) = 0$; if $c(u_1,u_2) = \frac{4\pi}{15}$ then $c(u_8,u_1) = \frac{\pi}{10}$ (see Figure 5.10); if $c(u_8,u_1) = \frac{3\pi}{10}$ then $c(u_1,u_2) = 0$; if $c(u_8,u_1) = \frac{4\pi}{15}$ then $c(u_1,u_2) = \frac{\pi}{10}$; if $c(u_4,u_5) > \frac{2\pi}{15}$ then $c(u_3,u_4) = 0$; and if $c(u_5,u_6) > \frac{2\pi}{15}$ then

$c(u_6, u_7) = 0$. It follows that $c(u_8, u_1) + c(u_1, u_2) = \frac{7\pi}{15}$; $c(u_3, u_4) + c(u_4, u_5) = \frac{7\pi}{30}$; and $c(u_5, u_6) + c(u_6, u_7) = \frac{7\pi}{30}$ so $c^*(\hat{\Delta}) \leq c(\hat{\Delta}) + \frac{16\pi}{15}$. Therefore if $\hat{\Delta}$ has at least three vertices of degree > 3 then $c^*(\hat{\Delta}) < 0$. If $\hat{\Delta}$ has no vertices of degree > 3 then we see (from Figure 6.2(v)) that $c^*(\hat{\Delta}) \leq \frac{2\pi}{3} + \frac{2\pi}{15} < 0$. Observe that if $d(u_1) = 3$ then $c(u_8, u_1) + c(u_1, u_2) = \frac{2\pi}{15}$; and if $d(u_5) = 3$ then $c(u_4, u_5) = c(u_5, u_6) = 0$. It follows that if $d(u_1) = 3$ or $d(u_5) = 3$ then $c^*(\hat{\Delta}) \leq c(\hat{\Delta}) + \frac{11\pi}{15} < 0$ and so if $\hat{\Delta}$ has exactly one vertex of degree > 3 then $c^*(\hat{\Delta}) < 0$. If $\hat{\Delta}$ has exactly two vertices u_i, u_j of degree > 3 it can be assumed that $(i, j) = (1, 5)$ in which case $cv(\hat{\Delta}) = (h_1, 0, 0, 7, 7, 0, 0, h_2)$. Therefore $c^*(\hat{\Delta}) \leq -\pi + \frac{14\pi}{15} < 0$.

Case 6 Let $\hat{\Delta}$ be given by Figure 6.2(vi). If $c(u_4, u_5) = \frac{3\pi}{10}$ then $c(u_3, u_4) = 0$; if $c(u_4, u_5) = \frac{4\pi}{15}$ then $c(u_3, u_4) = \frac{\pi}{15}$ (see Figure 5.10); if $c(u_5, u_6) = \frac{3\pi}{10}$ then $c(u_6, u_7) = 0$; if $c(u_5, u_6) = \frac{4\pi}{15}$ then $c(u_6, u_7) = \frac{\pi}{15}$; and as in Case 5, $c(u_8, u_1) + c(u_1, u_2) = \frac{7\pi}{15}$. It follows that $c^*(\hat{\Delta}) \leq c(\hat{\Delta}) + \frac{4\pi}{3}$ so if $\hat{\Delta}$ has at least four vertices of degree > 3 then $c^*(\hat{\Delta}) \leq 0$. Let $\hat{\Delta}$ have no vertices of degree > 3. Then $cv(\hat{\Delta}) = (2, 0, 0, 2, 2, 0, 0, 2)$ and $c^*(\hat{\Delta}) \leq -\frac{2\pi}{3} + \frac{4\pi}{15} < 0$. Let $\hat{\Delta}$ have exactly one vertex u_i of degree > 3. If $i = 1$ then $cv(\hat{\Delta}) = (h_1, 0, 0, 2, 2, 0, 0, h_2)$; if $i = 2$ then $cv(\hat{\Delta}) = (0, 2, 0, 2, 2, 0, 0, 2)$; if $i = 3$ then $cv(\hat{\Delta}) = (2, 2, 4, 2, 2, 0, 0, 2)$; if $i = 4$ then $cv(\hat{\Delta}) = (2, 0, e_1, e_2, 2, 0, 0, 2)$; if $i = 5$ then $cv(\hat{\Delta}) = (2, 0, 0, 9, 9, 0, 0, 2)$; if $i = 6$ then $cv(\hat{\Delta}) = (2, 0, 0, 2, e_1, e_2, 0, 2)$; if $i = 7$ then $cv(\hat{\Delta}) = (2, 0, 0, 2, 2, 4, 2, 2)$; and if $i = 8$ then $cv(\hat{\Delta}) = (2, 0, 0, 2, 2, 0, 2, 0)$. Therefore $c^*(\hat{\Delta}) \leq -\frac{5\pi}{6} + \frac{11\pi}{15} < 0$. Let $\hat{\Delta}$ have exactly two vertices u_i, u_j of degree > 3. Then $c(\hat{\Delta}) \leq -\pi$. If $d(u_1) = 3$ then $c(u_8, u_1) = c(u_1, u_2) = \frac{\pi}{15}$ and $c^*(\hat{\Delta}) \leq 0$ so it can be assumed that $i = 1$. If $j = 2$ then $cv(\hat{\Delta}) = (h_1, 2, 0, 2, 2, 0, 0, h_2)$; if $j = 3$ then $cv(\hat{\Delta}) = (h_1, 2, 4, 2, 2, 0, 0, h_2)$; if $j = 4$ then $cv(\hat{\Delta}) = (h_1, 0, e_1, e_2, 2, 0, 0, h_2)$; if $j = 5$ then $cv(\hat{\Delta}) = (h_1, 0, 0, 2, 2, 0, 0, h_2)$; if $j = 6$ then $cv(\hat{\Delta}) = (h_1, 0, 0, 2, e_1, e_2, 0, h_2)$; if $j = 7$ then $cv(\hat{\Delta}) = (h_1, 0, 0, 2, 2, 4, 2, h_2)$; and if $j = 8$ then $cv(\hat{\Delta}) = (h_1, 0, 0, 2, 2, 0, 2, h_2)$. Therefore $c^*(\hat{\Delta}) \leq -\pi + \frac{9\pi}{10} < 0$. Let $\hat{\Delta}$ have exactly three vertices of degree > 3 so that $c(\hat{\Delta}) \leq -\frac{7\pi}{6}$. If $d(u_1) = 3$ then $c^*(\hat{\Delta}) \leq -\frac{7\pi}{6} + \pi$; if $d(u_4) = 3$ then $c^*(\hat{\Delta}) \leq -\frac{7\pi}{6} + \frac{31\pi}{30}$; and if $d(u_6) = 3$ then $c^*(\hat{\Delta}) \leq \frac{7\pi}{6} + \frac{31\pi}{30}$. So it can be assumed that $d(u_1) > 3$, $d(u_4) > 3$ and $d(u_6) > 3$ in which case $cv(\hat{\Delta}) = (h_1, 0, e_1, e_2, e_1, e_2, 0, h_2)$. If $d(u_1) > 4$ then $c^*(\hat{\Delta}) \leq -\frac{19}{15}\pi + \frac{18}{15}\pi < 0$; whereas if $d(u_1) = 4$ then the fact that $d(u_2) = d(u_8) = 3$ means that $l(u_1) = bbx^{-1}y$ forces either $c(u_1, u_2) = 0$ or $c(u_8, u_1) = 0$ and $c^*(\hat{\Delta}) \leq -\frac{7\pi}{6} + \frac{31\pi}{30} < 0$.

Case 7 Let $\hat{\Delta}$ be given by Figure 6.2(vii). If $c(u_1, u_2) > \frac{2\pi}{15}$ then $d(u_1) = 3$ and $c(u_8, u_1) = \frac{\pi}{30}$; if $c(u_4, u_5) > \frac{2\pi}{15}$ then $c(u_5, u_6) = 0$; if $c(u_5, u_6) > \frac{2\pi}{15}$ then $d(u_5) = 3$ and $c(u_4, u_5) = \frac{\pi}{30}$; and if $c(u_8, u_1) > \frac{2\pi}{15}$ then $c(u_1, u_2) = 0$. It follows that $c(u_8, u_1) + c(u_1, u_2) \leq \frac{4\pi}{15}$ and $c(u_4, u_5) + c(u_5, u_6) \leq \frac{4\pi}{15}$. If $\hat{\Delta}$ has at least two vertices of degree > 3 then $c^*(\hat{\Delta}) \leq -\pi + \frac{14\pi}{15} < 0$. If $\hat{\Delta}$ has no vertices of degree > 3 then we see (from Figure 6.2(vii)) that $c^*(\hat{\Delta}) \leq -\frac{2\pi}{3} + \frac{8\pi}{15} < 0$. Let $\hat{\Delta}$ have exactly one vertex of degree > 3. If $d(u_3) = d(u_4) = 3$ then $c(u_3, u_4) = 0$ and $c^*(\hat{\Delta}) \leq -\frac{5\pi}{6} + \frac{4\pi}{5} < 0$; if $d(u_3) > 3$ then $cv(\hat{\Delta}) = (6, 2, 4, 0, 6, 0, 0, 0)$; if $d(u_4) > 3$ then $cv(\hat{\Delta}) = (6, 0, 4, b_1, b_2, 0, 0, 0)$; and it follows that $c^*(\hat{\Delta}) \leq -\frac{5\pi}{6} + \frac{18\pi}{30} < 0$.

Case 8 Let $\hat{\Delta}$ be given by Figure 6.2(viii). Then $cv(\hat{\Delta}) = (4, 4, 6, 2, 2, 4, 9, 2)$ so if $\hat{\Delta}$ has at least three vertices of degree 2 then $c^*(\hat{\Delta}) \leq -\frac{7\pi}{6} + \frac{11\pi}{10} < 0$. Note that if $d(u_2) = 3$ then $c(u_1, u_2) = c(u_2, u_3) = 0$ and if $d(u_7) = 3$ then $c(u_6, u_7) = 0$. If $\hat{\Delta}$ has no vertices of degree > 3 then $cv(\hat{\Delta}) = (0, 0, 6, 0, 0, 0, 2, 0)$ and $c^*(\hat{\Delta}) \leq -\frac{2\pi}{3} + \frac{4\pi}{15} < 0$. Let $\hat{\Delta}$ have exactly one vertex u_i of degree > 3. If $i = 1$ then $cv(\hat{\Delta}) = (0, 0, 6, 0, 0, 0, 2, 2)$; if $i = 2$ then $cv(\hat{\Delta}) = (4, 4, 6, 0, 0, 0, 2, 0)$; if $i = 3$ then $cv(\hat{\Delta}) = (0, 0, 6, 0, 0, 0, 2, 0)$; if $i = 4$ then $cv(\hat{\Delta}) = (0, 0, 6, 2, 0, 0, 2, 0)$; if $i = 5$ then $cv(\hat{\Delta}) = (0, 0, 6, 2, 2, 0, 2, 0)$; if $i = 6$ then $cv(\hat{\Delta}) = (0, 0, 6, 0, 2, 4, 2, 0)$; if $i = 7$ then $cv(\hat{\Delta}) = (0, 0, 0, 6, 0, 0, 4, 9, 0)$; and if $i = 8$ then $cv(\hat{\Delta}) = (0, 0, 6, 0, 0, 0, 9, 2)$. Therefore $c^*(\hat{\Delta}) \leq -\frac{5\pi}{6} + \frac{19\pi}{30} < 0$. Let $\hat{\Delta}$ have exactly two vertices of degree > 3. If $d(u_2) = 3$ or $d(u_7) = 3$ then $c^*(\hat{\Delta}) \leq -\pi + \frac{29\pi}{30} < 0$ so assume that $d(u_2) > 3$ and $d(u_7) > 3$. Then $cv(\hat{\Delta}) = (4, 4, 6, 0, 0, 4, 9, 0)$ and $c^*(\hat{\Delta}) \leq -\pi + \frac{9\pi}{10} < 0$.

Case 9 Let $\hat{\Delta}$ be given by Figure 6.2(ix). If $c(u_4, u_5) > \frac{2\pi}{15}$ then $d(u_4) = 4$ and $c(u_3, u_4) = 0$; and if $c(u_6, u_7) > \frac{2\pi}{15}$ then $(d(u_6), d(u_7)) = (4, 4)$ and $c(u_7, u_8) = 0$. It follows that if at least three of the u_i have degree ≥ 4 then $c^*(\hat{\Delta}) \leq -\frac{7\pi}{6} + \frac{7\pi}{6} = 0$, so assume otherwise. If $\hat{\Delta}$ has no vertices of degree > 3 then we see (from Figure 6.2(ix) and Figure 5.3(iii)) that $cv(\hat{\Delta}) = (0, 6, 4, 0, 0, 0, 4, 0)$ and $c^*(\hat{\Delta}) \leq -\frac{2\pi}{3} + \frac{7\pi}{30} < 0$. Let $\hat{\Delta}$ have exactly one vertex u_i of degree > 3. Then the following holds. If $i = 1$ then $cv(\hat{\Delta}) = (2, 6, 4, 0, 0, 0, 5, 3)$; if $i = 2$ then $cv(\hat{\Delta}) = (2, 6, 4, 0, 0, 0, 4, 0)$; if $i = 3$ then $cv(\hat{\Delta}) = (0, 6, 6, 0, 0, 0, 4, 0)$; if $i = 4$ then $cv(\hat{\Delta}) = (0, 6, d_1, d_2, 0, 0, 4, 0)$; if $i = 5$ then $cv(\hat{\Delta}) = (0, 6, 4, 4, 4, 0, 4, 0)$; if $i = 6$ then $cv(\hat{\Delta}) = (0, 6, 4, 0, 4, 4, 4, 0)$; if $i = 7$ then $cv(\hat{\Delta}) = (0, 6, 4, 0, 0, 4, 6, 0)$; and if $i = 8$ then $cv(\hat{\Delta}) = (0, 6, 4, 0, 0, 0, 6, 3)$. It follows that $c^*(\hat{\Delta}) \leq -\frac{5\pi}{6} + \frac{11\pi}{15} < 0$. Let $\hat{\Delta}$ have exactly two vertices u_i, u_j of degree > 3. If $d(u_4) = d(u_5) = d(u_6) = 3$ or $d(u_5) = d(u_6) = d(u_7) = 3$ then $c^*(\hat{\Delta}) \leq -\pi + \pi = 0$. This leaves 14 out of 28 cases to be considered. If $(i, j) = (1, 5)$ then $cv(\hat{\Delta}) = (2, 6, 4, 4, 4, 0, 5, 3)$; if $(i, j) = (1, 6)$ then $cv(\hat{\Delta}) = (2, 6, 4, 0, 4, 4, 5, 3)$; if $(i, j) = (2, 5)$ then $cv(\hat{\Delta}) = (2, 6, 4, 4, 4, 0, 4, 0)$; if $(i, j) = (2, 6)$ then $cv(\hat{\Delta}) = (2, 6, 4, 0, 4, 4, 4, 0)$; if $(i, j) = (3, 5)$ then $cv(\hat{\Delta}) = (0, 6, 4, 4, 4, 0, 4, 0)$; if $(i, j) = (3, 6)$ then $cv(\hat{\Delta}) = (0, 6, 6, 0, 4, 4, 4, 0)$; if $(i, j) = (4, 5)$ then $cv(\hat{\Delta}) = (0, 6, 4, 4, 4, 0, 4, 0)$; if $(i, j) = (4, 6)$ then $cv(\hat{\Delta}) = (0, 6, d_1, d_2, 4, 4, 4, 0)$; if $(i, j) = (4, 7)$ then $cv(\hat{\Delta}) = (0, 6, d_1, d_2, 0, 4, 4, 0)$; if $(i, j) = (5, 6)$ then $cv(\hat{\Delta}) = (0, 6, 4, 4, 4, 4, 4, 0)$; if $(i, j) = (5, 7)$ then $cv(\hat{\Delta}) = (0, 6, 4, 4, 4, 4, 0, 0)$; if $(i, j) = (5, 8)$ then $cv(\hat{\Delta}) = (0, 6, 4, 4, 4, 0, 4, 3)$; if $(i, j) = (6, 7)$ then $cv(\hat{\Delta}) = (0, 6, 4, 0, 4, d_1, d_2, 0)$; and if $(i, j) = (6, 8)$ then $cv(\hat{\Delta}) = (0, 6, 4, 0, 4, 4, 4, 3)$. It follows that $c^*(\hat{\Delta}) \leq= -\pi + \frac{14\pi}{15} < 0$.

Case 10 Let $\hat{\Delta}$ be given by Figure 6.2(x). If $c(u_1, u_2) > \frac{2\pi}{15}$ then $(d(u_1), d(u_2)) = (4, 4)$ and $c(u_8, u_1) = 0$; if $c(u_7, u_8) > \frac{2\pi}{15}$ then $(d(u_7), d(u_8)) = (4, 3)$ and $c(u_6, u_7) = 0$; and if $c(u_6, u_7) > \frac{2\pi}{15}$ then $d(u_7) = 3$ forcing $c(u_7, u_8) = 0$. It follows that $c(u_8, u_1) + c(u_1, u_2) \leq \frac{7\pi}{30}$; and $c(u_6, u_7) + c(u_7, u_8) \leq \frac{4\pi}{15}$. If $\hat{\Delta}$ has at least three vertices of degree > 3 then $c^*(\hat{\Delta}) \leq -\frac{7\pi}{6} + \frac{11\pi}{10} < 0$. If $\hat{\Delta}$ has no vertices of degree > 3 then we see (from Figure 6.2(x)) that $cv(\hat{\Delta}) = (0, 0, 6, 6, 0, 6, 0, 0)$ and $c^*(\hat{\Delta}) \leq -\frac{2\pi}{3} + \frac{18\pi}{30} < 0$. Let $\hat{\Delta}$ have exactly one vertex u_i of degree > 3. Then the following holds. If $i = 1$ then $d(u_2) = 3$ and so $cv(\hat{\Delta}) = (0, 0, 6, 6, 0, 6, 0, 3)$; if $i = 2$ then $d(u_1) = 3$ and so $cv(\hat{\Delta}) = (0, 4, 6, 6, 0, 6, 0, 0)$; if $i = 3$ then $cv(\hat{\Delta}) = (0, 4, 6, 6, 0, 6, 0, 0)$; if $i = 4$ then $cv(\hat{\Delta}) = (0, 0, 6, 6, 0, 6, 0, 0)$; if $i = 5$

then $cv(\hat{\Delta}) = (0,0,6,6,2,6,0,0)$; if $i = 6$ then $cv(\hat{\Delta}) = (0,0,6,6,2,6,0,0)$; if $i = 7$ then $cv(\hat{\Delta}) = (0,0,6,6,0,b_1,b_2,0)$; and if $i = 8$ then $d(u_7) = 3$ so $cv(\hat{\Delta}) = (0,0,6,6,0,6,0,3)$. Therefore $c^*(\hat{\Delta}) \leq -\frac{5\pi}{6} + \frac{11\pi}{15} < 0$. Let $\hat{\Delta}$ have exactly two vertices of degree > 3. If $d(u_1) = 3$ or $d(u_2) = 3$ then $cv(\hat{\Delta}) = (0,4,6,6,2,b_1,b_2,3)$ and $c^*(\hat{\Delta}) \leq -\pi + \frac{29\pi}{30} < 0$. On the other hand if $d(u_1) > 3$ and $d(u_2) > 3$ then $cv(\hat{\Delta}) = (a_1,4,6,6,0,6,0,a_2)$ and $c^*(\hat{\Delta}) \leq -\pi + \frac{29\pi}{30} < 0$.

Case 11 Let $\hat{\Delta}$ be given by Figure 6.2(xi). If $c(u_1,u_2) > \frac{2\pi}{15}$ then $d(u_8) = 3$ and $c(u_8,u_1) = 0$; if $c(u_8,u_1) > \frac{2\pi}{15}$ then $c(u_1,u_2) = 0$; and if $c(u_6,u_7) > \frac{2\pi}{15}$ then $c(u_7,u_8) = 0$. It follows that $c^*(\hat{\Delta}) \leq c(\Delta) + \frac{11\pi}{10}$ and so if $\hat{\Delta}$ has at least three vertices of degree > 3 then $c^*(\hat{\Delta}) < 0$. If $\hat{\Delta}$ has no vertices of degree > 3 then we see (from Figure 6.2(xi)) that $c^*(\hat{\Delta}) \leq -\frac{2\pi}{3} + \frac{3\pi}{5} < 0$. Let $\hat{\Delta}$ have exactly one vertex u_i of degree > 3. If $d(u_6) = d(u_7) = d(u_8) = 3$ then $c(u_5,u_6) = c(u_6,u_7) = c(u_7,u_8) = 0$; if $i = 6$ then $cv(\hat{\Delta}) = (6,0,6,6,4,0,0,0)$; if $i = 7$ then $cv(\hat{\Delta}) = (6,0,6,6,0,0,3,0)$; and if $i = 8$ then $cv(\hat{\Delta}) = (6,0,6,6,0,0,3,0)$. It follows that $c^*(\hat{\Delta}) \leq -\frac{5\pi}{6} + \frac{11\pi}{15} < 0$. Let $\hat{\Delta}$ have exactly two vertices of degree > 3. If $d(u_6) = 3$ then $c(u_5,u_6) = 0$; if $d(u_7) = 3$ then $c(u_6,u_7) = 0$; if $d(u_6) > 3$ and $d(u_7) > 3$ then $cv(\hat{\Delta}) = (6,0,6,6,4,b_1,b_2,0)$. It follows that $c^*(\hat{\Delta}) \leq -\pi + \pi = 0$.

Case 12 Suppose that $\hat{\Delta}$ has at least one vertex of degree > 4. Using a similar analysis as done for Case 1, it follows that $c^*(\hat{\Delta}) \leq -\frac{38\pi}{30} + \frac{36\pi}{30} < 0$. Indeed the maximum $\frac{36\pi}{30}$ can only be obtained when $cv(\hat{\Delta}) = (0,6,0,7,7,6,0,6,4,0)$. Suppose that $\hat{\Delta}$ has no vertices of degree > 4 and at least one vertex of degree 4. Then (see Figure 5.8(xvii)) $c(u_i,u_j) = \frac{2\pi}{15}$ for $(i,j) \in \{(7,8),(8,1),(1,2),(2,3),(3,4)\}$. It follows that $c^*(\hat{\Delta}) \leq -\frac{35\pi}{30} + \frac{32\pi}{30} < 0$, the maximum $\frac{32\pi}{30}$ being obtained when $cv(\hat{\Delta}) = (0,4,0,7,7,6,0,4,4)$. But if $\hat{\Delta}$ has no vertices of degree > 3 then $c(u_4,u_5) = 0$, $c(u_5,u_6) = \frac{\pi}{15}$, $cv(\hat{\Delta}) = (4,0,4,0,2,6,0,4,0)$ and $c^*(\hat{\Delta}) \leq -\pi + \frac{2\pi}{3} < 0$ and this completes the proof. □

7 Regions of Type $\mathcal{B}$

Let $\hat{\Delta}$ be a type $\mathcal{B}$ region. Therefore $\hat{\Delta}$ is given by Figure 4.6 and in particular $d(\hat{\Delta}) \geq 8$. A *b-segment* of $\hat{\Delta}$ of length k is a sequence of edges $e_1,\ldots,e_k$ of $\hat{\Delta}$ maximal with respect to each vertex having degree 3 with vertex label $a(a\lambda)(b^{-1}\mu) = axy^{-1}$ and which (up to inversion) contribute one of four possible alternating sequences to the corner labelling of $\hat{\Delta}$, namely: $x^{-1},y^{-1},\ldots,x^{-1},y^{-1}$; $x^{-1},y^{-1},\ldots,y^{-1},x^{-1}$; y^{-1},x^{-1}; $y^{-1},x^{-1},\ldots,y^{-1},x^{-1}$; $y^{-1},x^{-1},\ldots,x^{-1},y^{-1}$. An example showing the first sequence is given in Figure 7.1(i) and so maximal in this case means that either $d(u_0) > 3$ or $d(u_0) = 3$ but does not extend the sequence to $\bar{y},\bar{x},\bar{y},\ldots,\bar{x},\bar{y}$; and that either $d(u_{k+1}) > 3$ or $d(u_{k+1}) = 3$ but does not extend the sequence to $\bar{x},\bar{y},\ldots,\bar{x},\bar{y},\bar{x}$. Since $\hat{\Delta}$ is of type $\mathcal{B}$, it must contain at least one b-segment in which at least one of the regions Δ_i $(1 \leq i \leq k)$ is given by the region Δ in Figure 4.6 and we will from now on call such a region Δ_i a *b-region*. Therefore a b-region contributes up to $\frac{\pi}{3}$ to $\hat{\Delta}$. (If Δ_i is not a b-region then it contributes up to $\frac{\pi}{5}$ to $\hat{\Delta}$.)

The absorption rules for edges and vertices described in §6 apply also to $\hat{\Delta}$. Since $\hat{\Delta}$ is a type $\mathcal{B}$ region Figures 4.27–4.29 must also be considered. In Figures 4.27 and 4.28 $\hat{\Delta}$ receives $\frac{\pi}{5}, \frac{2\pi}{15}$ from Δ_1, Δ_2 so the vertex of degree 4 with label $b^{-1}b^{-1}y^{-1}x$ is used to absorb $\frac{\pi}{15}$; and in Figure 4.29 $\hat{\Delta}$ receives $\frac{\pi}{5}$ across an edge, e say, but checking Figures 5.1-5.5 shows that $\hat{\Delta}$ receives no curvature from $\hat{\Delta}_1$ across the neighbouring edge which is used to absorb $\frac{\pi}{15}$ noting from Figure 4.29 that this is all the curvature that this edge will absorb (relative to curvature transferred to $\hat{\Delta}$).

It follows from the above paragraph and the proof of Lemma 6.1 that if the b-segments containing at least one b-region of $\hat{\Delta}$ contribute a total of n_1 edges to $\hat{\Delta}$ then putting $n = n_1 + n_2$,

$$c^*(\hat{\Delta}) \leq (2 - (n_1 + n_2))\pi + 2(n_1 + n_2)\frac{\pi}{3} + n_1\frac{\pi}{3} + n_2\frac{2\pi}{15} = \pi\left(2 - \frac{n_2}{5}\right). \qquad (\dagger)$$

Therefore if $n_2 \geq 10$ then $c^*(\hat{\Delta}) \leq 0$. The next result improves this bound slightly.

Lemma 7.1 *If $n_2 \geq 9$ and $\hat{\Delta}$ is not given by Figure 7.2 (in which the b-segment contains at least one b-region) then $c^*(\hat{\Delta}) \leq 0$.*

Proof. We will show that the existence of a b-segment in which at least one Δ_i $(1 \leq i \leq k)$ is a b-region allows us to decrease the upper bound (†) for $c^*(\hat{\Delta})$ given above. First consider the region Δ_0 of Figure 7.1(i) or (ii).

In each case if Δ_1 is not a b-region then $\hat{\Delta}$ receives at most $\frac{\pi}{5}$ from Δ_1 and the upper bound for $c^*(\hat{\Delta})$ is reduced by at least $\frac{\pi}{3} - \frac{\pi}{5} = \frac{2\pi}{15}$, so assume the Δ_1 is a b-region. In particular e_0 absorbs no positive curvature from Δ_1 (in the sense described in the proof of Lemma 6.1). Let $d(u_0) \geq 5$ and so u_0 can absorb at least $\frac{2\pi}{3} - \frac{2\pi}{5} = \frac{4\pi}{15}$. Since $\hat{\Delta}$ then receives at most $\frac{\pi}{15}$ from Δ_0 (see Figure 4.33(ii)) and since the maximum amount any vertex, in particular u_0 absorbs from Δ_{-1} is $\frac{\pi}{6}$, u_0 can absorb the $\frac{\pi}{15}$ crossing e_0 and so $c^*(\hat{\Delta})$ is reduced by at least $\frac{2\pi}{15}$. Let $d(u_0) = 4$ and so u_0 can absorb $\frac{2\pi}{3} - \frac{\pi}{2} = \frac{\pi}{6}$. If the total curvature $\hat{\Delta}$ receives across e_0 and e_{-1} is at most $\frac{3\pi}{10}$ then $c^*(\hat{\Delta})$ is reduced by at least $\frac{2\pi}{15}$, so assume otherwise. In particular $\hat{\Delta}$ must receive curvature from Δ_0 which forces $l(u_0)$ to be as shown in Figure 7.1(iii) and (iv) and so (see Figure 4.33(i)) $\hat{\Delta}$ receives at most $\frac{2\pi}{15}$ from Δ_0. To exceed a total of $\frac{3\pi}{10}$, therefore, it follows that $\hat{\Delta}$ must receive at least $\frac{\pi}{5}$ across e_{-1} and so (see Figure 5.8) $l(u_{-1})$ must be as shown in Figures 7.1(iii) and (iv) and in these figures the maximum combination $\hat{\Delta}$ can receive across e_{-1}, e_0 is $\frac{7\pi}{30}, \frac{2\pi}{15}$ (see Figure 5.10), therefore $c^*(\hat{\Delta})$ is reduced by at least $\frac{\pi}{15}$. Let $d(u_0) = 3$. Then labelling shows that $d(\Delta_0) \geq 6$ and $d(\Delta_{-1}) \geq 6$ and checking Figures 5.1-5.5 shows that $\hat{\Delta}$ does not receive curvature across e_0 and at most $\frac{2\pi}{15}$ across e_{-1} so $c^*(\hat{\Delta})$ is reduced by at least $\frac{2\pi}{15}$. Note that we use the fact that $l(u_0) \neq axy^{-1}$ in Figure 7.1(i) or (ii) for otherwise the b-segment would be extended, a contradiction.

Now consider the region $\hat{\Delta}_{k+1}$ of Figure 7.1(i) and (v). Again if Δ_k is not a b-region then $c^*(\hat{\Delta})$ is reduced by $\frac{2\pi}{15}$ so assume otherwise. In particular e_{k+1} absorbs no positive curvature from Δ_k. Moreover, if Δ_k is given by Δ_1 of Figure 4.30(i) or (iii) (Configuration E, F) then $c^*(\hat{\Delta})$ is again reduced by $\frac{\pi}{5}$, so assume otherwise, in particular u_{k+2} is not given by Figure 4.30(i) or (iii). Let $d(u_{k+2}) \geq 5$ and so u_{k+2} can absorb $\frac{4\pi}{15}$. Since $\hat{\Delta}$ then receives at most $\frac{\pi}{15}$ from Δ_{k+1} and since the maximum amount u_{k+1} absorbs from Δ_{k+2} is $\frac{\pi}{6}$, u_{k+2} can absorb the $\frac{\pi}{15}$ crossing e_{k+1} and so $c^*(\hat{\Delta})$ is reduced by at least $\frac{2\pi}{15}$. Let $d(u_{k+2}) = 4$ and so u_{k+2} can absorb $\frac{\pi}{6}$. If $\hat{\Delta}$ does not receive curvature from Δ_{k+1} then $c^*(\hat{\Delta})$ is reduced by $\frac{2\pi}{15}$; otherwise checking possible vertex labels for u_{k+2} shows that $l(u_{k+2}) = aaz\mu$ and $\hat{\Delta}$ receives at most $\frac{7\pi}{30}$ in total across e_{k+1} and 0 across e_{k+2}, so $c^*(\hat{\Delta})$ is reduced by $\frac{2\pi}{15}$. Let $d(u_{k+2}) = 3$ and so using the maximality of the b-segment and the fact that u_{k+2} is not given by Figure 4.30(i) or (iii) it follows that $l(u_{k+2})$ must be as shown in Figures 7.1(vi) and (vii). Then $d(\Delta_{k+1}) \geq 6$ and checking Figures 5.1-5.5 shows that $\hat{\Delta}$ does not receive curvature from Δ_{k+1}. It follows that $c^*(\hat{\Delta})$ is reduced by $\frac{2\pi}{15}$ except possibly when $d(u_{k+3}) = 3$ and $\hat{\Delta}$ receives $\frac{\pi}{6}$ or $\frac{\pi}{5}$ from Δ_{k+2} (see Figure 5.8). There are four cases. Two (see Figures 5.8(i), (ii), (vi) and (xv)) are given by Figure 7.1(vi) and (vii) where $\hat{\Delta}$ can receive $\frac{\pi}{5}$ from Δ_{k+2} and $c^*(\hat{\Delta})$ is reduced by $\frac{\pi}{15}$; and two (see Figures 5.8(xiii) and (xiv)) are given by Figures 5.3(iii) and 5.4(iii) in which the region $\hat{\Delta}_2$, $\hat{\Delta}_1$, Δ_2 (respectively) plays the role of the region $\hat{\Delta}_1$, Δ_{k+2}, Δ_{k+3} (respectively) which implies, in particular, that $d(u_{k+4}) = 4$ in Figures 7.1(vi) and (vii). In each of these last two cases $\hat{\Delta}$ receives $\frac{\pi}{6}$ from Δ_{k+2} and $\frac{\pi}{10}$ from Δ_{k+3}, and since $d(u_{k+4}) = 4$ it follows that $c^*(\hat{\Delta})$ is reduced by $\frac{2\pi}{15}$.

It follows from the above that if the b-segment of Figure 7.1(i) is not given by Figure 7.2 then there is a reduction of $\frac{\pi}{15} + \frac{2\pi}{15} = \frac{3\pi}{15}$ to $c^*(\hat{\Delta})$ (if $e_{k+2} = e_0$ the reduction is also $\frac{3\pi}{35}$) therefore $c^*(\hat{\Delta}) \leq \pi\left(2 - \frac{n_2}{5}\right) - \frac{3\pi}{15}$ and so $n_2 \geq 9 \Rightarrow c^*(\hat{\Delta}) \leq 0$. $\square$

Lemma 7.2 *Let $\hat{\Delta}$ be a type $\mathcal{B}$ region such that $d(\hat{\Delta}) \geq 10$.*

(i) *If $\hat{\Delta}$ has exactly three b-segments that contain a b-region then $n_2 \geq 8$.*

Assume now that $\hat{\Delta}$ has exactly two b-segments B_1 and B_2 that contain a b-region as shown in Figure 7.3(i) and assume that $(m,n) \in \{(2,j)\ (2 \leq j \leq 6), (3,3), (3,4), (3,5), (4,4)\}$ where m, n are given by Figure 7.3(i).

(ii) *$\hat{\Delta}$ must contain a shadow edge with an endpoint in B_1 and the other endpoint in B_2 except when $\hat{\Delta}$ is given by Figure 7.3(ii)-(v).*

(iii) *If $v \in \hat{\Delta}$ is a vertex of B_1 or B_2 and $(m,n) \neq (2,6)$ then $i\deg(v) = 1$ where $i\deg(v)$ denotes the number of shadow edges in $\hat{\Delta}$ incident at v.*

(iv) *If $(m,n) \in \{(3,3),(3,4),(3,5),(4,4)\}$ and $\hat{\Delta}$ is not given by Figure 7.3(ii)-(v) there must be a shadow edge in $\hat{\Delta}$ either from 1 to B_2 or from 4 to B_1; and there must be a shadow edge in $\hat{\Delta}$ either from 2 to B_2 or from 3 to B_1.*

Finally assume that $\hat{\Delta}$ has exactly one b-segment containing a b-region.

(v) *If* $n_2 \leq 8$ *then* $\hat{\Delta}$ *is given by Figure 7.4.*

(vi) *If* $n_2 = 9$ *and* $\hat{\Delta}$ *is given by Figure 7.2 then* $\hat{\Delta}$ *is one of the regions of 7.5.*

Proof. See Appendix. □

Notation Throughout the following proofs we will use non-negative integers $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2, e_1, e_2, f_1, f_2$ where: $a_1+a_2 = 7$; $b_1+b_2 = 8$; $c_1+c_2 = 9$; $d_1+d_2 = 10$; $e_1 + e_2 = 11$; and $f_1 + f_2 = 12$.

Proposition 7.3 *Let* $\hat{\Delta}$ *be a type* $\mathcal{B}$ *region. If* $d(\hat{\Delta}) < 10$ *then* $c^*(\hat{\Delta}) \leq 0$.

Proof. If $d(\hat{\Delta}) < 10$ then $\hat{\Delta}$ is given by Figure 6.2(viii), (x) or (xi).

Case 1 Let $\hat{\Delta}$ be given by Figure 6.2(viii) in which it is now assumed that $d(u_3) = d(u_4) = 3$. Then $cv(\hat{\Delta}) = (4, 4, 10, 2, 2, e_1, e_2, 2)$ so $c^*(\hat{\Delta}) \leq c(\hat{\Delta}) + \frac{7\pi}{6}$ and if $\hat{\Delta}$ has at least three vertices of degree ≥ 4 then $c^*(\hat{\Delta}) \leq 0$. If $\hat{\Delta}$ has no vertices of degree > 3 then $cv(\hat{\Delta}) = (0, 0, 10, 0, 0, 0, 2, 0)$ and $c^*(\hat{\Delta}) \leq -\frac{2\pi}{3} + \frac{2\pi}{5} < 0$. Let $\hat{\Delta}$ have exactly one vertex u_i of degree > 3. If $i = 1$ then $cv(\hat{\Delta}) = (4, 0, 10, 0, 0, 0, 2, 2)$; if $i = 2$ then $cv(\hat{\Delta}) = (4, 4, 10, 0, 0, 0, 2, 0)$; if $i = 5$ then $cv(\hat{\Delta}) = (0, 0, 10, 2, 2, 0, 2, 0)$; if $i = 6$ then $cv(\hat{\Delta}) = (0, 0, 10, 0, 2, 4, 2, 0)$; if $i = 7$ then $cv(\hat{\Delta}) = (0, 0, 10, 0, 0, e_1, e_2, 0)$; and if $i = 8$ then $cv(\hat{\Delta}) = (0, 0, 10, 0, 0, 0, 9, 2)$. It follows that $c^*(\hat{\Delta}) \leq -\frac{5\pi}{6} + \frac{21\pi}{30} < 0$. Let $\hat{\Delta}$ have exactly two vertices u_i, u_j of degree > 3. If $d(u_7) = 3$ then $cv(\hat{\Delta}) = (4, 4, 10, 2, 2, 4, 2, 2)$ and $c^*(\hat{\Delta}) \leq -\pi + \pi = 0$ so assume $i = 7$. If $j = 1$ then $cv(\hat{\Delta}) = (4, 0, 10, 0, 0, e_1, e_2, 2)$; if $j = 2$ then $cv(\hat{\Delta}) = (4, 4, 10, 0, 0, e_1, e_2, 0)$; if $j = 5$ then $cv(\hat{\Delta}) = (0, 0, 10, 2, 2, e_1, e_2, 0)$; if $j = 6$ then $cv(\hat{\Delta}) = (0, 0, 10, 0, 2, e_1, e_2, 0)$; and if $j = 8$ then $cv(\hat{\Delta}) = (0, 0, 10, 0, 0, e_1, e_2, 2)$. It follows that $c^*(\hat{\Delta}) \leq -\pi + \frac{29\pi}{30} < 0$.

Remark The next case will involve the first use of Configurations E and F in Figure 4.30. If $d(u) > 4$ in Figure 4.30(i), (iii) then $c(\Delta_1) \leq c(3, 3, 4, 5) = \frac{7\pi}{30}$ is added to $c(\hat{\Delta})$; and if $d(u) > 4$ in Figure 4.30(ii), (iv) then (see Figure 5.8) at most $\frac{\pi}{5}$ is added to $c(\hat{\Delta})$ from $c(\Delta_1)$.

Case 2 Let $\hat{\Delta}$ be given by Figure 6.2(x) in which it is now assumed that $d(u_4) = 3$ and at least one of $d(u_3)$, $d(u_5)$ equals 3. Then $cv(\hat{\Delta}) = (a_2, 4, 10, 10, 2, b_1, b_2, a_1)$, so $c^*(\hat{\Delta}) \leq c(\hat{\Delta}) + \frac{41\pi}{30}$ and if $\hat{\Delta}$ has at least four vertices of degree > 3 then $c^*(\hat{\Delta}) \leq 0$. Note also that if $d(u_5) = d(u_6) = 3$, $d(u) = 4$ and $\hat{\Delta}$ receives more than $\frac{2\pi}{15}$ across the (u_4, u_5)-edge then $\frac{\pi}{5}$ is distributed from $\hat{\Delta}$ according to Configuration E in Figure 4.30. If $\hat{\Delta}$ has no vertices of degree > 3 then either $d(u) = 4$, $cv(\hat{\Delta}) = (0, 0, 10, 10, 0, 6, 0, 0)$ and $c^*(\hat{\Delta}) \leq -\frac{2\pi}{3} + \frac{13\pi}{15} - \frac{\pi}{5} = 0$ or $d(u) > 4$, $cv(\hat{\Delta}) = (0, 0, 7, 7, 0, 6, 0, 0)$ and $c^*(\hat{\Delta}) \leq -\frac{2\pi}{3} + \frac{2\pi}{3} = 0$. Let $\hat{\Delta}$ have exactly one vertex u_i of degree > 3 and assume that $d(u) = 4$. If $i = 1$ then $cv(\hat{\Delta}) = (0, 0, 10, 10, 0, 6, 0, 3)$; if $i = 2$ then $cv(\hat{\Delta}) = (x_1, y_1, 10, 10, 0, 6, 0, 0)$ ($x_1 + y_1 = 4$, since $l(u_1) = b\mu z$, $l(u_2) = bbx^{-1}y$ and $l(u_3) = axy^{-1}$); if $i = 3$ then $cv(\hat{\Delta}) = (0, 0, 6, 10, 0, 6, 0, 0)$; if $i = 5$ then $cv(\hat{\Delta}) = (0, 0, 10, 6, 2, 6, 0, 0)$; if $i = 6$ and $d(u_6) = 4$ then $cv(\hat{\Delta}) = (0, 0, 10, 10, 2, 0, 0, 0)$; if $i = 6$ and $d(u_6) > 4$ then $cv(\hat{\Delta}) = (0, 0, 10, 10, 2, 2, 0, 0)$;

if $i = 7$ then $cv(\hat{\Delta}) = (0, 0, 10, 10, 0, 6, 4)$; and if $i = 8$ then $cv(\hat{\Delta}) = (0, 0, 10, 10, 0, 6, 0, 3)$. It follows that if $d(u_5) = d(u_6) = 3$ then $c^*(\hat{\Delta}) \leq -\frac{5\pi}{6} + \pi - \frac{\pi}{5} < 0$; if $d(u_5) > 3$ or $d(u_6) = 4$ then $c^*(\hat{\Delta}) \leq -\frac{5\pi}{6} + \frac{22\pi}{30} < 0$; and if $d(u_6) > 4$ then $c^*(\hat{\Delta}) \leq -\frac{14\pi}{15} + \frac{24\pi}{30} < 0$. If now $d(u) > 4$ then each $cv(\hat{\Delta})$ is altered by replacing each 10 by 7 and it follows that $c^*(\hat{\Delta}) \leq -\frac{5\pi}{6} + \frac{24\pi}{30} < 0$. Let $\hat{\Delta}$ have exactly two vertices u_i, u_j of degree > 3 and assume that $d(u) = 4$. If $(i, j) = (1, 2)$ then $cv(\hat{\Delta}) = (a_2, 4, 10, 10, 0, 6, 0, a_1)$ and so if $d(u_1) > 4$ or $d(u_2) > 4$ then $c^*(\hat{\Delta}) \leq -\frac{11\pi}{10} + \frac{37\pi}{30} - \frac{\pi}{5} < 0$; and if $d(u_1) = d(u_2) = 4$ then $c(u_8, u_1) = 0$ and, moreover, $c(u_1, u_2) > \frac{2\pi}{15}$ implies $c(u_2, u_3) = 0$ so $cv(\hat{\Delta}) = (b_1, b_2, 10, 10, 0, 6, 0, 0)$ and $c^*(\hat{\Delta} \leq -\pi + \frac{17\pi}{15} - \frac{\pi}{5} < 0$. If $(i, j) = (1, 3)$ then $cv(\hat{\Delta}) = (0, 0, 6, 10, 0, 6, 0, 3)$; if $(i, j) = (1, 5)$ then $cv(\hat{\Delta}) = (0, 0, 10, 6, 2, 6, 0, 3)$; if $(i, j) = (1, 6)$ and $d(u_6) = 4$ then $cv(\hat{\Delta}) = (0, 0, 10, 10, 2, 0, 0, 3)$; if $(i, j) = (1, 6)$ and $d(u_6) > 4$ then $cv(\hat{\Delta}) = (0, 0, 10, 10, 2, 2, 0, 3)$; if $(i, j) = (1, 7)$ then $cv(\hat{\Delta}) = (0, 0, 10, 10, 0, b_1, b_2, 3)$; if $(i, j) = (1, 8)$ then $cv(\hat{\Delta}) = (0, 0, 10, 10, 0, 6, 0, 3)$; if $(i, j) = (2, 3)$ then $cv(\hat{\Delta}) = (x_1, y_1, 6, 10, 0, 6, 0, 0)$; if $(i, j) = (2, 5)$ then $cv(\hat{\Delta}) = (x_1, y_1, 10, 6, 2, 6, 0, 0)$; if $(i, j) = (2, 6)$ and $d(u_6) = 4$ then $cv(\hat{\Delta}) = (x_1, y_1, 10, 10, 2, 0, 0, 0)$; if $(i, j) = (2, 6)$ and $d(u_6) > 4$ then $cv(\hat{\Delta}) = (x_1, y_1, 10, 10, 2, 2, 0, 3)$; if $(i, j) = (2, 7)$ then $cv(\hat{\Delta}) = (x_1, y_1, 10, 10, 0, b_1, b_2, 0)$; if $(i, j) = (2, 8)$ then $cv(\hat{\Delta}) = (x_1, y_1, 10, 10, 0, 6, 0, 3)$; if $(i, j) = (3, 6)$ then $cv(\hat{\Delta}) = (0, 0, 6, 10, 2, 6, 0, 0)$; if $(i, j) = (3, 7)$ then $cv(\hat{\Delta}) = (0, 0, 6, 10, 0, b_1, b_2, 0)$; if $(i, j) = (3, 8)$ then $cv(\hat{\Delta}) = (0, 0, 6, 10, 0, 6, 0, 3)$; if $(i, j) = (5, 6)$ then $cv(\hat{\Delta}) = (0, 0, 10, 6, 2, 6, 0, 0)$; if $(i, j) = (5, 7)$ then $cv(\hat{\Delta}) = (0, 0, 10, 6, 2, b_1, b_2, 0)$; if $(i, j) = (5, 8)$ then $cv(\hat{\Delta}) = (0, 0, 10, 6, 2, 6, 0, 3)$; if $(i, j) = (6, 7)$ then $cv(\hat{\Delta}) = (0, 0, 10, 10, 2, b_1, b_2, 0)$; if $(i, j) = (6, 8)$ and $d(u_6) = 4$ then $cv(\hat{\Delta}) = (0, 0, 10, 10, 2, 0, 0, 3)$; if $(i, j) = (6, 8)$ and $d(u_6) > 4$ then $cv(\hat{\Delta}) = (0, 0, 10, 10, 2, 2, 0, 3)$; and if $(i, j) = (7, 8)$ then $cv(\hat{\Delta}) = (0, 0, 10, 10, 0, b_1, b_2, 3)$. It follows that if $(i, j) \neq (1, 2)$ and if $d(u_5) = d(u_6) = 3$ then $c^*(\hat{\Delta}) \leq -\pi + \frac{11\pi}{10} - \frac{\pi}{5} < 0$; if $d(u_5) > 3$ or $d(u_6) = 4$ then $c^*(\hat{\Delta}) \leq -\pi + \pi = 0$; and if $d(u_6) > 4$ then $c^*(\hat{\Delta}) \leq -\frac{11\pi}{10} + \frac{31\pi}{30} < 0$. If now $d(u) > 4$ then, as before, replacing each 10 by 7 in the above yields $c^*(\hat{\Delta}) \leq -\pi + \frac{28\pi}{30} < 0$ except when $(i, j) = (1, 2)$ and either $d(u_1) > 4$ or $d(u_2) > 4$ and $c^*(\hat{\Delta}) \leq -\frac{11\pi}{10} + \frac{31\pi}{30} < 0$. Let $\hat{\Delta}$ have exactly three vertices u_i, u_j, u_k of degree > 3. If $d(u_2) = 3$ then $cv(\hat{\Delta}) = (0, 0, 10, 10, 2, b_1, b_2, 3)$ and $c^*(\hat{\Delta}) \leq -\frac{7\pi}{6} + \frac{11\pi}{10} < 0$; and if $d(u_5) = d(u_6) = 3$ then $c^*(\hat{\Delta}) \leq -\frac{7\pi}{6} + \frac{41\pi}{30} - \frac{\pi}{5} = 0$, so assume otherwise. If $d(u_3) = 4$ then $c(u_3, u_4) = 0$ and if $d(u_3) \geq 5$ then $c(u_3, u_4) = \frac{\pi}{15}$, and in both cases $c^*(\hat{\Delta} \leq 0$. Similarly if $d(u_5) \neq 3$ then $c^*(\hat{\Delta}) \leq 0$, so it can be assumed that $d(u_3) = d(u_5) = 3$. If $(i, j, k) = (1, 2, 6)$ and $d(u_6) = 4$ then $cv(\hat{\Delta}) = (a_2, 4, 10, 10, 2, 0, 0, a_1)$ and $c^*(\hat{\Delta}) \leq -\frac{7\pi}{6} + \frac{11\pi}{10} < 0$; and if $d(u_6) > 4$ then $cv(\hat{\Delta}) = (a_2, 4, 10, 10, 2, 2, 0, a_1)$ and $c^*(\hat{\Delta}) \leq -\frac{19\pi}{15} + \frac{7\pi}{6} < 0$. If $(i, j, k) = (2, 6, 7)$ then $cv(\hat{\Delta}) = (0, 4, 10, 10, 2, b_1, b_2, 0)$; and if $(i, j, k) = (2, 6, 8)$ then $cv(\hat{\Delta}) = (0, 4, 10, 10, 2, 6, 0, 3)$. In both cases $c^*(\hat{\Delta}) \leq 0$. Finally let $\hat{\Delta}$ have exactly four vertices of degree > 3. If $d(u_3) \neq 3$ or $d(u_5) \neq 3$ or if any vertex has degree > 4 or if any of u_1, u_2, u_6 or u_7 has degree 3 then clearly $c^*(\hat{\Delta}) \leq 0$, so assume otherwise. But this forces the assumption $d(u_1) = 4$ which implies $c(u_8, u_1) = 0$ and $c^*(\hat{\Delta}) < 0$.

Case 3 Let $\hat{\Delta}$ be given by Figure 6.2(xi). Then as noted in Case 11 of the proof of

Lemma 6.3, $c(u_8,u_1)+c(u_1,u_2)\leq \frac{8\pi}{30}$, $c(u_8,u_1)+c(u_1,u_2)+c(u_2,u_3)\leq \frac{\pi}{3}$ and $c(u_6,u_7)+c(u_7,u_8)\leq \frac{7\pi}{30}$; so $c^*(\hat{\Delta})\leq c(\hat{\Delta})+\frac{41\pi}{30}$. Moreover if $d(u_2)=d(u_3)=3$, $d(u)=4$ and $\hat{\Delta}$ receives more than $\frac{2\pi}{15}$ across the (u_4,u_5)-edge then $\frac{\pi}{5}$ is distributed from $\hat{\Delta}$ according to Configuration F of Figure 4.30. If $\hat{\Delta}$ has no vertices of degree >3 then either $d(u)=4$, $cv(\hat{\Delta})=(6,0,10,10,0,0,0,0)$ and $c^*(\hat{\Delta})\leq -\frac{2\pi}{3}+\frac{13\pi}{15}-\frac{\pi}{5}=0$ or $d(u)>4$, $cv(\hat{\Delta})=(6,0,7,7,0,0,0,0)$ and $c^*(\hat{\Delta})\leq -\frac{2\pi}{3}+\frac{2\pi}{3}-0$. Let $\hat{\Delta}$ have exactly one vertex u_i of degree >3 and assume that $d(u)=4$. If $i=1$ then $cv(\hat{\Delta})=(b_2,0,10,10,0,0,0,b_1)$; if $i=2$ and $d(u_2)=4$ then $c(u_1,u_2)=0$ and $cv(\hat{\Delta})=(0,2,10,10,0,0,0,0)$; if $i=2$ and $d(u_2)>4$ then $cv(\hat{\Delta})=(2,2,10,10,0,0,0,0)$; if $i=3$ then $cv(\hat{\Delta})=(6,2,6,10,0,0,0,0)$; if $i=5$ then $cv(\hat{\Delta})=(6,0,10,6,0,0,0,0)$; if $i=6$ then $cv(\hat{\Delta})=(6,0,10,10,x_1,y_1,0,0)$ ($x_1+y_1=4$, since $l(u_5)=axy^{-1}$, $l(u_6)=bbx^{-1}y$ and $l(u_7)=b\mu z$); if $i=7$ then $cv(\hat{\Delta})=(6,0,10,10,0,0,3,0)$; if $i=8$ and $d(u_8)=4$ then $cv(\hat{\Delta})=(6,0,10,10,0,0,3,0)$; and if $i=8$ and $d(u_8)>4$ then $cv(\hat{\Delta})=(6,0,10,10,0,0,2,2)$. In each case $c^*(\hat{\Delta})\leq 0$ when $(d(u_2),d(u_3))\neq(3,3)$; and if $d(u_2)=d(u_3)=3$ then $c^*(\hat{\Delta})\leq -\frac{5\pi}{6}+\pi-\frac{\pi}{5}<0$. If now $d(u)>4$ then replacing each 10 by 7 in the above yields $c^*(\hat{\Delta})\leq -\frac{5\pi}{6}+\frac{24\pi}{30}<0$. Let $\hat{\Delta}$ have exactly two vertices u_i,u_j of degree >3 and assume that $d(u)=4$. If $(i,j)=(1,2)$ then $cv(\hat{\Delta})=(b_2,2,10,10,0,0,0,b_1)$; if $(i,j)=(1,3)$ then $cv(\hat{\Delta})=(b_2,2,6,10,0,0,0,b_1)$; if $(i,j)=(1,5)$ then $cv(\hat{\Delta})=(b_2,0,10,6,0,0,0,b_1)$; if $(i,j)=(1,6)$ then $cv(\hat{\Delta})=(b_2,0,10,10,x_1,y_1,0,b_1)$; if $(i,j)=(1,7)$ then $cv(\hat{\Delta})=(b_2,0,10,10,0,a_1,a_2,b_1)$; if $(i,j)=(1,8)$ then $cv(\hat{\Delta})=(b_2,0,10,10,0,0,3,b_1)$; if $(i,j)=(2,3)$ then $cv(\hat{\Delta})=(6,2,6,10,0,0,0,0)$; if $(i,j)=(2,5)$ then $cv(\hat{\Delta})=(6,2,10,6,0,0,0,0)$; if $(i,j)=(2,6)$ and $d(u_2)=4$ then $cv(\hat{\Delta})=(0,2,10,10,x_1,y_1,0,0)$; if $(i,j)=(2,6)$ and $d(u_2)>4$ then $cv(\hat{\Delta})=(2,2,10,10,x_1,y_1,0,0)$; if $(i,j)=(2,7)$ and $d(u_2)=4$ then $cv(\hat{\Delta})=c(0,2,10,10,0,0,3,0)$; if $(i,j)=(2,7)$ and $d(u_2)>4$ then $cv(\hat{\Delta})=c(2,2,10,10,0,0,3,0)$; if $(i,j)=(2,8)$ and $d(u_2)=4$ then $cv(\hat{\Delta})=(0,2,10,10,0,0,3,0)$; if $(i,j)=(2,8)$ and $d(u_2)>4$ then $cv(\hat{\Delta})=(2,2,10,10,0,0,3,0)$; if $(i,j)=(3,6)$ then $cv(\hat{\Delta})=(6,2,6,10,x_1,y_1,0,0)$; if $(i,j)=(3,7)$ then $cv(\hat{\Delta})=(6,2,6,10,0,0,3,0)$; if $(i,j)=(3,8)$ then $cv(\hat{\Delta})=(6,2,6,10,0,0,3,0)$; if $(i,j)=(5,6)$ then $cv(\hat{\Delta})=(6,0,10,6,x_1,y_1,0,0)$; if $(i,j)=(5,7)$ then $cv(\hat{\Delta})=(6,0,10,6,0,0,3,0)$; if $(i,j)=(5,8)$ then $cv(\hat{\Delta})=(6,0,10,6,0,0,3,0)$; if $(i,j)=(6,7)$ and $d(u_6)\neq 4$ or $d(u_7)\neq 4$ then $cv(\hat{\Delta})=(6,0,10,10,z_1,z_2,z_3,0)$ ($z_1+z_2+z_3=10$, see Figure 4.33(ii)); if $(i,j)=(6,7)$ and $d(u_6)=d(u_7)=4$ then $cv(\hat{\Delta})=(6,0,10,10,a_1,a_2,0,0)$; if $(i,j)=(6,8)$ then $cv(\hat{\Delta})=(6,0,10,10,x_1,y_1,3,0)$; and if $(i,j)=(7,8)$ then $cv(\hat{\Delta})=(6,0,10,10,0,0,3,0)$. In each case either $c^*(\hat{\Delta})\leq c(\Delta)+\pi\leq 0$ or $c^*(\hat{\Delta})\leq c(\Delta)+\frac{6\pi}{5}-\frac{\pi}{5}\leq 0$. If now $d(u)>4$ then replacing each 10 by 7 in the above yields $c^*(\hat{\Delta})\leq -\pi+\pi=0$. Let $\hat{\Delta}$ have exactly three vertices u_i,u_j,u_k of degree >3. If $d(u_6)=3$ then $cv(\hat{\Delta})=(b_2,2,10,10,0,0,3,b_1)$; if $d(u_3)=4$ then $c(u_3,u_4)=0$ or if $d(u_3)>4$ then $c(u_3,u_4)=\frac{\pi}{15}$; if $d(u_5)=4$ then $c(u_4,u_5)=0$ or if $d(u_5)>4$ then $c(u_4,u_5)=\frac{\pi}{15}$; and if $d(u_2)=d(u_3)=3$, $d(u)=4$ and $\hat{\Delta}$ receives more than $\frac{2\pi}{15}$ across the (u_2,u_3)-edge then $\frac{\pi}{5}$ is distributed from $\hat{\Delta}$. If $d(u_2)=d(u_3)=3$ and $d(u_4)>4$ then $cv(\hat{\Delta})=(z_1,z_2,7,7,4,a_1,a_2,z_3)=\frac{35\pi}{30}$ (where $z_1+z_2+z_3=10$). It follows that $c^*(\hat{\Delta})\leq 0$ in each case, so assume otherwise. If $(i,j,k)=(6,2,7)$ and

$d(u_2) = 4$ then $cv(\hat{\Delta}) = (0, 2, 10, 10, 4, a_1, a_2, 0)$; if $(i, j, k) = (6, 2, 7)$ and $d(u_2) > 4$ then $cv(\hat{\Delta}) = (2, 2, 10, 10, 4, a_1, a_2, 0)$; if $(i, j, k) = (6, 2, 8)$ then $cv(\hat{\Delta}) = (6, 2, 10, 10, 4, 0, 3, 0)$; and if $(i, j, k) = (6, 2, 1)$ then $cv(\hat{\Delta}) = (b_2, 2, 10, 10, 4, 0, 0, b_1)$. In each case $c^*(\hat{\Delta}) \leq 0$ so assume that $\hat{\Delta}$ has exactly four vertices of degree > 3. It is clear that if $d(u_3) \neq 3$ or $d(u_5) \neq 3$ or any of u_1, u_2, u_6, u_7 has degree 3 or at least 5 then $c^*(\hat{\Delta}) \leq 0$; otherwise $d(u_1) = 4$ which implies $c(u_1, u_2) = 0$ and $c^*(\hat{\Delta}) \leq 0$. $\square$

Proposition 7.4 Let $\hat{\Delta}$ be a type $\mathcal{B}$ region. If $\hat{\Delta}$ is given by Figure 7.3(ii)-(v), 7.4 or 7.5 then $c^*(\hat{\Delta}) \leq 0$.

Proof. Let $\hat{\Delta}$ be given by Figure 7.3(ii)-(v). Then (up to cyclic permutation and inversion) there are two ways to label each of (ii) and (iii); and one way to label each of (iv) and (v) and so $\hat{\Delta}$ is given by Figure 7.6. There are six cases.

Case 1 (7.6) Let $\hat{\Delta}$ be given by Figure 7.6(i) in which (it can be seen from Figure 7.3(ii) that) $d(u_1) = d(u_2) = d(u_3) = d(u_7) = d(u_8) = 3$ and $d(u_6) > 3$. Then $cv(\hat{\Delta}) = (10, 10, 2, d_1, d_2, 6, 10, 4, a_1, a_2)$ so $c^*(\hat{\Delta}) \leq c(\hat{\Delta}) + \frac{59\pi}{30}$. Let $\hat{\Delta}$ have exactly one vertex u_6 of degree > 3. Then $cv(\hat{\Delta}) = (10, 10, 0, 6, 0, 6, 10, 0, 0, 0)$ and $c^*(\hat{\Delta}) \leq -\frac{45\pi}{30} + \frac{42\pi}{30} < 0$. Let $\hat{\Delta}$ have exactly two vertices u_6, u_i of degree > 3. If $i = 4$ then $cv(\hat{\Delta}) = (10, 10, 2, 6, 0, 6, 10, 0, 0, 0)$; if $i = 5$ then $cv(\hat{\Delta}) = (10, 10, 0, d_1, d_2, 6, 10, 0, 0, 0)$; if $i = 9$ then $cv(\hat{\Delta}) = (10, 10, 0, 6, 0, 6, 10, 4, 2, 0)$; and if $i = 10$ then $cv(\hat{\Delta}) = (10, 10, 0, 6, 0, 6, 10, 0, a_1, a_2)$. It follows that $c^*(\hat{\Delta}) \leq -\frac{50\pi}{30} + \frac{49\pi}{30} < 0$. Let $\hat{\Delta}$ have exactly three vertices u_6, u_i, u_j of degree > 3. If $d(u_5) = 3$ then $cv(\hat{\Delta}) = (10, 10, 2, 6, 0, 6, 10, 4, a_1, a_2)$; if $d(u_{10}) = 3$ then $cv(\hat{\Delta}) = (10, 10, 2, d_1, d_2, 6, 10, 4, 2, 0)$; and if $(i, j) = (5, 10)$ then $cv(\hat{\Delta}) = (10, 10, 0, d_1, d_2, 6, 10, 0, a_1, a_2)$. It follows that $c^*(\hat{\Delta}) \leq -\frac{55\pi}{30} + \frac{55\pi}{30} = 0$. If $\hat{\Delta}$ has more than three vertices of degree > 3 then $c^*(\hat{\Delta}) \leq -\frac{60\pi}{30} + \frac{59\pi}{30} < 0$.

Case 2 (7.6) Let $\hat{\Delta}$ be given by Figure 7.6(ii) in which $d(u_1) = d(u_2) = d(u_3) = d(u_7) = d(u_8) = 3$ and $d(u_6) > 3$. Then $cv(\hat{\Delta}) = (10, 10, 4, a_1, a_2, 6, 10, 2, d_1, d_2)$ so $c^*(\hat{\Delta}) \leq c(\hat{\Delta}) + \frac{59\pi}{30}$. Let $\hat{\Delta}$ have exactly one vertex u_6 of degree > 3. Then $cv(\hat{\Delta}) = (10, 10, 0, 0, 3, 6, 10, 0, 6, 0)$ and $c^*(\hat{\Delta}) \leq -\frac{45\pi}{30} + \frac{45\pi}{30} = 0$. Let $\hat{\Delta}$ have exactly two vertices u_6, u_i of degree > 3. If $i = 4$ and $d(u_6) = 4$ then $cv(\hat{\Delta}) = (10, 10, 4, 2, 3, 0, 10, 0, 6, 0)$; if $i = 4$ and $d(u_6) > 4$ then $cv(\hat{\Delta}) = (10, 10, 4, 2, 2, 2, 10, 0, 6, 0)$; if $i = 5$ then $cv(\hat{\Delta}) = (10, 10, 0, a_1, a_2, 6, 10, 0, 6, 0)$; if $i = 9$ then $cv(\hat{\Delta}) = (10, 10, 0, 0, 3, 6, 10, 2, 6, 0)$; and if $i = 10$ then $cv(\hat{\Delta}) = (10, 10, 0, 0, 3, 6, 10, 0, d_1, d_2)$. It follows that either $c^*(\hat{\Delta}) \leq -\frac{50\pi}{30} + \frac{49\pi}{30} < 0$ or $c^*(\hat{\Delta}) \leq -\frac{53\pi}{30} + \frac{46\pi}{30} < 0$. Let $\hat{\Delta}$ have exactly three vertices u_6, u_i, u_j of degree > 3. If $d(u_{10}) = 3$ then $cv(\hat{\Delta}) = (10, 10, 4, a_1, a_2, 6, 10, 2, 6, 0)$ and $c^*(\hat{\Delta}) \leq -\frac{55\pi}{30} + \frac{55\pi}{30} = 0$, so assume $i = 10$ and $j \in \{4, 5, 9\}$. If $j = 4$ then $cv(\hat{\Delta}) = (10, 10, 4, 2, 3, 6, 10, 0, d_1, d_2)$; if $j = 5$ then $cv(\hat{\Delta}) = (10, 10, 0, a_1, a_2, 6, 10, 0, d_1, d_2)$; and if $j = 9$ then $cv(\hat{\Delta}) = (10, 10, 0, 0, 3, 6, 10, 2, d_1, d_2)$. It follows that $c^*(\hat{\Delta}) \leq -\frac{55\pi}{30} + \frac{55\pi}{30} = 0$. If $\hat{\Delta}$ has more than three vertices of degree > 3 then $c^*(\hat{\Delta}) \leq -\frac{60\pi}{30} + \frac{59\pi}{30} = 0$.

Case 3 (7.6) Let $\hat{\Delta}$ be given by Figure 7.6(iii) in which (see Figure 7.3(iii)) $d(u_1) =$

$d(u_2) = d(u_7) = d(u_8) = 3$, $d(u_3) > 3$ and $d(u_6) > 3$. Then $cv(\hat{\Delta}) = (10, 6, a_1, a_2, 4, 6, 10, d_1, d_2, 2)$ and $c^*(\hat{\Delta}) \leq c(\hat{\Delta}) + \frac{55\pi}{30}$. If $\hat{\Delta}$ has at least three vertices of degree > 3 then $c^*(\hat{\Delta}) \leq -\frac{55\pi}{30} + \frac{55\pi}{30} = 0$. This leaves the case $d(u_3) > 3$ and $d(u_6) > 3$ only. Then $cv(\hat{\Delta}) = (10, 6, 3, 0, 0, 6, 10, 0, 6, 0)$ and $c^*(\hat{\Delta}) \leq -\frac{50\pi}{30} + \frac{41\pi}{30} < 0$.

Case 4 (7.6) Let $\hat{\Delta}$ be given by Figure 7.6(iv) in which $d(u_1) = d(u_2) = d(u_7) = d(u_8) = 3$, $d(u_3) > 3$ and $d(u_6) > 3$. Then $cv(\hat{\Delta}) = (10, 6, d_1, d_2, 2, 6, 10, a_1, a_2, 4)$ and $c^*(\hat{\Delta}) \leq c(\hat{\Delta}) + \frac{55\pi}{30}$. If $\hat{\Delta}$ has at least three vertices of degree > 3 then $c^*(\hat{\Delta}) \leq -\frac{55\pi}{30} + \frac{55\pi}{30} = 0$. This leaves the case $d(u_3) > 3$ and $d(u_6) > 3$ only. Then $cv(\hat{\Delta}) = (10, 6, 0, 6, 2, 6, 10, 0, 0, 0)$ and $c^*(\hat{\Delta}) \leq -\frac{50\pi}{30} + \frac{40\pi}{30} < 0$.

Case 5 (7.6) Let $\hat{\Delta}$ be given by Figure 7.6(v) in which (see Figure 7.3(iv)) $d(u_1) = d(u_2) = d(u_3) = d(u_7) = d(u_8) = d(u_9) = 3$. Then $cv(\hat{\Delta}) = (10, 10, d_1, d_2, 6, 2, 10, 10, 3, d_1, d_2, 4)$ so $c^*(\hat{\Delta}) \leq c(\hat{\Delta}) + \frac{75\pi}{30}$. If $\hat{\Delta}$ has no vertices of degree > 3 then $cv(\hat{\Delta}) = (10, 10, 0, 0, 6, 0, 10, 10, 0, 0, 0, 0)$ and $c^*(\hat{\Delta}) \leq -\frac{60\pi}{30} + \frac{46\pi}{30} < 0$. Let $\hat{\Delta}$ have exactly one vertex u_i of degree > 3. If $i = 4$ then $cv(\hat{\Delta}) = (10, 10, d_1, d_2, 6, 0, 10, 10, 0, 0, 0, 0)$; if $i = 5$ then $cv(\hat{\Delta}) = (10, 10, 0, 6, 6, 0, 10, 10, 0, 0, 0, 0)$; if $i = 6$ then $cv(\hat{\Delta}) = (10, 10, 0, 0, 6, 2, 10, 10, 0, 0, 0, 0)$; if $i = 10$ then $cv(\hat{\Delta}) = (10, 10, 0, 0, 6, 0, 10, 10, 3, 6, 0, 0)$; if $i = 11$ then $cv(\hat{\Delta}) = (10, 10, 0, 0, 6, 0, 10, 10, 0, d_1, d_2, 0)$; and if $i = 12$ then $cv(\hat{\Delta}) = (10, 10, 0, 0, 6, 0, 10, 10, 0, 0, 7, 4)$. It follows that $c^*(\hat{\Delta}) \leq -\frac{65\pi}{30} + \frac{57\pi}{30} < 0$. Let $\hat{\Delta}$ have exactly two vertices u_i, u_j of degree > 3. If $d(u_4) = d(u_5) = 3$ then $cv(\hat{\Delta}) = (10, 10, 0, 0, 6, 2, 10, 10, 3, d_1, d_2, 4)$; if $d(u_{10}) = d(u_{11}) = 3$ then $cv(\hat{\Delta}) = (10, 10, d_1, d_2, 6, 2, 10, 10, 3, 0, 2, 4)$; and if $d(u_{12}) = 3$ then $cv(\hat{\Delta}) = (10, 10, d_1, d_2, 6, 2, 10, 10, 3, 6, 0, 0)$. It follows that $c^*(\hat{\Delta}) \leq -\frac{70\pi}{30} + \frac{67\pi}{30}$. If $\hat{\Delta}$ has at least three vertices of degree > 3 then $c^*(\hat{\Delta}) \leq -\frac{75\pi}{30} + \frac{75\pi}{30} = 0$.

Case 6 (7.6) Let $\hat{\Delta}$ be given by Figure 7.6(vi) in which (see Figure 7.3(v)) $d(u_1) = d(u_2) = d(u_6) = d(u_7) = 3$, $d(u_3) > 3$ and $d(u_8) > 3$. Then $cv(\hat{\Delta}) = (10, 6, d_1, d_2, 2, 10, 6, a_1, a_2, 4)$ and $c^*(\Delta) \leq c(\hat{\Delta}) + \frac{55\pi}{30}$. If $\hat{\Delta}$ has at least three vertices of degree > 3 then $c^*(\hat{\Delta}) \leq -\frac{55\pi}{30} + \frac{55\pi}{30} = 0$. This leaves the case $d(u_3) > 3$ and $d(u_8) > 3$ only. Then $cv(\hat{\Delta}) = (10, 6, 0, 6, 0, 10, 6, 3, 0, 0)$ and $c^*(\hat{\Delta}) \leq -\frac{50\pi}{30} + \frac{41\pi}{30} < 0$.

Now let $\hat{\Delta}$ be one of the regions of Figure 7.4. It turns out that (up to cyclic permutation and inversion) there are two ways to label each of Figure 7.4(i), (ii), (iii) and (iv); four ways to label (v); two ways to label each of (vi) and (vii); and four ways to label (viii). However the labelled regions produced by (vii) already appear in those produced by (vi); and two of the labelled regions produced by (viii) already appear in those produced by (ii), leaving a total of 16 regions and $\hat{\Delta}$ is given by Figure 7.7. The table below gives $c(u_i, u_{i+1})$ $(1 \leq i \leq 8)$ in multiples of $\frac{\pi}{30}$ for each of the sixteen regions of Figure 7.7 with the total plus the contribution made via the b-segment in the final column.

(i)	2	d_1	d_2	6	6	a_1	a_2	4	$35+40=75$
(ii)	4	a_1	a_2	6	6	d_1	d_2	2	$35+40=75$
(iii)	4	d_1	d_2	3	d_1	d_2	d_1	d_2	$37+20=57$
(iv)	2	6	d_1	d_2	e_1	e_2	e_1	e_2	$40+20=60$
(v)	4	d_1	d_2	3	d_1	d_2	6	2	$35+40=75$
(vi)	2	6	d_1	d_2	3	d_1	d_2	4	$35+40=75$
(vii)	d_1	d_2	2	6	6	a_1	a_2	4	$35+20=55$
(viii)	2	d_1	d_2	6	6	4	a_1	a_2	$35+20=55$
(ix)	e_1	e_2	e_1	e_2	d_1	d_2	6	2	$40+20=60$
(x)	a_1	a_2	a_1	a_2	b_1	b_2	6	2	$30+20=50$
(xi)	a_1	a_2	a_1	a_2	3	c_1	c_2	4	$36+20=56$
(xii)	d_1	d_2	d_1	d_2	3	c_1	c_2	4	$36+20=56$
(xiii)	a_1	a_2	4	6	6	d_1	d_2	2	$35+20=55$
(xiv)	d_1	d_2	2	6	6	a_1	a_2	4	$35+20=55$
(xv)	4	d_1	d_2	3	d_1	d_2	d_1	d_2	$37+20=57$
(xvi)	2	6	d_1	d_2	a_1	a_2	a_1	a_2	$32+20=52$

The regions in Figure 7.7(i), (ii), (v) and (vi) each have degree 12 and so $c(\hat{\Delta}) \leq (2-12)\pi + \frac{24\pi}{3} = -2\pi$; whereas the rest have degree 10 and in these cases $c(\hat{\Delta}) \leq -\frac{4\pi}{3}$. It follows from the table above that if $\hat{\Delta}$ has at least two vertices of degree > 3 then $c^*(\hat{\Delta}) \leq 0$ for (x); if at least three then $c^*(\hat{\Delta}) \leq 0$ for (i), (ii), (v), (vi), (vii), (viii), (xiii), (xiv) and (xvi); and if at least four then $c^*(\hat{\Delta}) \leq -\frac{40\pi}{30} + \frac{40\pi}{30} = 0$.

If $\hat{\Delta}$ has no vertices of degree > 3 then we see from Figure 7.7 that $c^*(\hat{\Delta}) \leq -\frac{20\pi}{30} + \frac{18\pi}{30} < 0$.

We consider each of the sixteen cases in turn.

Case 1 (7.7) Let $\hat{\Delta}$ be given by Figure 7.7(i). Suppose that $\hat{\Delta}$ has exactly one vertex u_i of degree > 3. If $d(u_3) = d(u_8) = 3$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 2, 6, 0, 6, 6, 3, 0, 0)$; if $i = 3$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 0, d_1, d_2, 6, 6, 0, 0, 0)$; and if $i = 8$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 0, 6, 0, 6, 6, 0, 2, 4)$. It follows that $c^*(\hat{\Delta}) \leq -\frac{65\pi}{30} + \frac{64\pi}{30} < 0$. Let $\hat{\Delta}$ have exactly two vertices u_i, u_j of degree > 3. If $d(u_3) = d(u_7) = 3$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 2, 6, 0, 6, 6, 3, 2, 4)$; if $d(u_8) = 3$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 2, d_1, d_2, 6, 6, 3, 0, 0)$; if $(i, j) = (3, 8)$ then

$c^*(\hat{\Delta}) = (10,10,10,10,0,d_1,d_2,6,6,0,2,4)$; and if $(i,j) = (7,8)$ then $cv(\hat{\Delta}) = (10,10,10,10,0,6,0,6,6,a_1,a_2,4)$. It follows that $c^*(\hat{\Delta}) \leq -\frac{70\pi}{30} + \frac{69\pi}{30} < 0$.

Case 2 (7.7) Let $\hat{\Delta}$ be given by Figure 7.7(ii). Suppose that $\hat{\Delta}$ has exactly one vertex u_i of degree > 3. If $d(u_2) = d(u_7) = 3$ then $cv(\hat{\Delta}) = (10,10,10,10,0,0,3,6,6,0,6,2)$; if $i = 2$ then $cv(\hat{\Delta}) = (10,10,10,10,4,2,0,6,6,0,6,0)$; and if $i = 7$ then $cv(\hat{\Delta}) = (10,10,10,10,0,0,0,6,6,d_1,d_2,2)$. It follows that $c^*(\hat{\Delta}) \leq -\frac{65\pi}{30} + \frac{64\pi}{30} < 0$. Let $\hat{\Delta}$ have exactly two vertices u_i, u_j of degree > 3. If $d(u_2) = 3$ then $cv(\hat{\Delta}) = (10,10,10,10,0,0,3,6,6,d_1,d_2,2)$; if $d(u_3) = d(u_7) = 3$ then $cv(\hat{\Delta}) = (10,10,10,10,4,2,3,6,6,0,6,2)$; if $(i,j) = (2,3)$ then $cv(\hat{\Delta}) = (10,10,10,10,4,a_1,a_2,6,6,0,6,0)$; and if $(i,j) = (2,7)$ then $cv(\hat{\Delta}) = (10,10,10,10,4,2,0,6,6,d_1,d_2,0)$. It follows that $c^*(\hat{\Delta}) \leq -\frac{70\pi}{30} + \frac{69\pi}{30} < 0$.

Case 3 (7.7) Let $\hat{\Delta}$ be given by Figure 7.7(iii). Suppose that $\hat{\Delta}$ has exactly one vertex u_i of degree > 3. If $d(u_7) = 3$ then $cv(\hat{\Delta}) = (10,10,4,d_1,d_2,3,2,2,2,2)$; if $i = 7$ then $cv(\hat{\Delta}) = (10,10,0,0,5,0,0,9,9,0)$. It follows that $c^*(\hat{\Delta}) \leq -\frac{45\pi}{30} + \frac{43\pi}{30}$. Let $\hat{\Delta}$ have exactly two vertices u_i, u_j of degree > 3. If $d(u_7) = 3$ then $c^*(\hat{\Delta}) < 0$; if $d(u_2) = d(u_8) = 3$ then $cv(\hat{\Delta}) = (10,10,0,0,6,3,d_1,d_2,9,0)$; if $(i,j) = (7,2)$ then $cv(\hat{\Delta}) = (10,10,4,2,5,0,0,9,9,0)$; and if $(i,j) = (7,8)$ then $cv(\hat{\Delta}) = (10,10,0,0,5,0,0,9,d_1,d_2)$. It follows that $c^*(\hat{\Delta}) \leq -\frac{50\pi}{30} + \frac{48\pi}{30} < 0$. Let $\hat{\Delta}$ have exactly three vertices u_i, u_j, j_k of degree > 3. If $d(u_2) = 3$ or $d(u_7) = 3$ then $c^*(\hat{\Delta}) < 0$; if $d(u_3) = d(u_8) = 3$ then $cv(\hat{\Delta}) = (10,10,4,2,6,3,d_1,d_2,9,0)$; if $(i,j,k) = (2,7,3)$ then $cv(\hat{\Delta}) = (10,10,4,d_1,d_2,0,0,9,9,0)$; and if $(i,j,k) = (2,7,8)$ then $cv(\hat{\Delta}) = (10,10,4,2,5,0,0,9,d_1,d_2)$. It follows that $c^*(\hat{\Delta}) \leq -\frac{55\pi}{30} + \frac{54\pi}{30} < 0$.

Case 4 (7.7) Let $\hat{\Delta}$ be given by Figure 7.7(iv). Suppose that $\hat{\Delta}$ has exactly one vertex u_i of degree > 3. If $d(u_4) = d(u_6) = d(u_8) = 3$ then $cv(\hat{\Delta}) = c(10,10,2,6,6,0,0,2,2,0)$; if $i = 4$ then $cv(\hat{\Delta}) = (10,10,0,6,0,7,0,2,2,0)$; if $i = 6$ then $cv(\hat{\Delta}) = (10,10,0,6,5,0,e_1,e_2,2,0)$; and if $i = 8$ then $cv(\hat{\Delta}) = (10,10,0,6,5,0,0,2,e_1,e_2)$. Therefore $c^*(\hat{\Delta}) \leq -\frac{45\pi}{30} + \frac{44\pi}{30}$. Let $\hat{\Delta}$ have exactly two vertices u_i, u_j of degree > 3. If $d(u_2) = d(u_6) = 3$ then $cv(\hat{\Delta}) = (10,10,0,6,d_1,d_2,0,2,e_1,e_2)$; if $d(u_4) = d(u_8) = 3$ then $cv(\hat{\Delta}) = (10,10,2,6,5,0,e_1,e_2,2,0)$; if $(i,j) = (2,4)$ then $cv(\hat{\Delta}) = (10,10,2,6,0,7,0,2,2,0)$; if $(i,j) = (2,8)$ then $cv(\hat{\Delta}) = (10,10,2,6,5,0,0,2,e_1,e_2)$; if $(i,j) = (6,4)$ then $cv(\hat{\Delta}) = (10,10,0,6,0,7,e_1,e_2,2,0)$; and if $(i,j) = (6,8)$ then $cv(\hat{\Delta}) = (10,10,0,6,5,0,e_1,e_2,e_1,e_2)$. It follows that either $c^*(\hat{\Delta}) \leq -\frac{50\pi}{30} + \frac{49\pi}{30} < 0$ or $(i,j) = (6,8)$, but here either $d(u) = 4$ and $\frac{\pi}{5}$ is distributed from $\hat{\Delta}$ across the (u_1,u_2) edge according to Configuration E of Figure 4.30 and so $c^*(\hat{\Delta}) \leq -\frac{50\pi}{30} + \frac{53\pi}{30} - \frac{\pi}{5} < 0$ or $d(u) > 4$, $cv(\hat{\Delta}) = (7,7,0,6,5,0,e_1,e_2,e_1,e_2)$ and so $c^*(\hat{\Delta}) \leq -\frac{50\pi}{30} + \frac{47\pi}{30} < 0$. Let $\hat{\Delta}$ have exactly three vertices u_1, u_j, u_k of degree > 3. If $d(u_4) = 3$ or $d(u_6) = 3$ or $d(u_8) = 3$ then $c^*(\hat{\Delta}) \leq -\frac{55\pi}{30} + \frac{53\pi}{30} < 0$; and if $(i,j,k) = (4,6,8)$ then $cv(\hat{\Delta}) = (10,10,0,6,0,7,e_1,e_2,e_1,e_2)$ and $c^*(\hat{\Delta}) \leq -\frac{55\pi}{30} + \frac{55\pi}{30} = 0$.

Case 5 (7.7) Let $\hat{\Delta}$ be given by Figure 7.7(v). Suppose that $\hat{\Delta}$ has exactly one vertex u_i of degree > 3. If $d(u_2) = d(u_6) = 3$ then $cv(\hat{\Delta}) = (10,10,10,10,0,0,6,3,0,4,6,2)$;

if $i = 2$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 4, 2, 5, 0, 0, 4, 6, 0)$; and if $i = 6$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 0, 0, 5, 0, 7, 0, 6, 0)$. It follows that $c^*(\hat{\Delta}) \leq -\frac{65\pi}{30} + \frac{61\pi}{30} < 0$. Let $\hat{\Delta}$ have exactly two vertices u_i, u_j of degree > 3. If $d(u_2) = 3$ or $d(u_6) = 3$ then $c^*(\hat{\Delta}) \leq -\frac{70\pi}{30} + \frac{69\pi}{30} < 0$; and if $(i, j) = (2, 6)$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 4, 2, 5, 0, 7, 0, 6, 0)$ and $c^*(\hat{\Delta}) < 0$.

Case 6 (7.7) Let $\hat{\Delta}$ be given by Figure 7.7(vi). Suppose that $\hat{\Delta}$ has exactly one vertex u_i of degree > 3. If $d(u_4) = d(u_8) = 3$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 2, 6, 6, 0, 3, 6, 0, 0)$; if $i = 4$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 0, 6, d_1, d_2, 0, 5, 0, 0)$; and if $i = 8$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 0, 6, 5, 0, 0, 5, 2, 4)$. It follows that $c^*(\hat{\Delta}) \leq -\frac{65\pi}{30} + \frac{63\pi}{30} < 0$. Let $\hat{\Delta}$ have exactly two vertices u_i, u_j of degree > 3. If $d(u_8) = 3$ then
$cv(\hat{\Delta}) = (10, 10, 10, 10, 2, 6, d_1, d_2, 3, 6, 0, 0)$; if $d(u_2) = d(u_4) = 3$ then
$cv(\hat{\Delta}) = (10, 10, 10, 10, 0, 6, 6, 0, 3, d_1, d_2, 4)$; if $(i, j) = (8, 2)$ then
$cv(\hat{\Delta}) = (10, 10, 10, 10, 2, 6, 5, 0, 0, 5, 2, 4)$; and if $(i, j) = (8, 4)$ then
$cv(\hat{\Delta}) = (10, 10, 10, 10, 0, 6, d_1, d_2, 0, 5, 2, 4)$. It follows that $c^*(\hat{\Delta}) \leq -\frac{70\pi}{30} + \frac{69\pi}{30} < 0$.

Case 7 (7.7) Let $\hat{\Delta}$ be given by Figure 7.7(vii). Suppose that $\hat{\Delta}$ has exactly one vertex u_i of degree > 3. If $d(u_2) = d(u_8) = 3$ then $cv(\hat{\Delta}) = (10, 10, 0, 6, 2, 6, 6, 3, 0, 0)$; if $i = 2$ then $cv(\hat{\Delta}) = (10, 10, d_1, d_2, 0, 6, 6, 0, 0, 0)$; and if $i = 8$ then $cv(\hat{\Delta}) = (10, 10, 0, 6, 0, 6, 6, 0, 2, 4)$. It follows that $c^*(\hat{\Delta}) \leq -\frac{45\pi}{30} + \frac{44\pi}{30} < 0$. Let $\hat{\Delta}$ have exactly two vertices u_i, u_j of degree > 3. If $d(u_8) = 3$ then $cv(\hat{\Delta}) = (10, 10, d_1, d_2, 2, 6, 6, 3, 0, 0)$; if $d(u_2) = d(u_7) = 3$ then $cv(\hat{\Delta}) = (10, 10, 0, 6, 2, 6, 6, 3, 2, 4)$; if $(i, j) = (8, 2)$ then $cv(\hat{\Delta}) = (10, 10, d_1, d_2, 0, 6, 6, 0, 2, 4)$; and if $(i, j) = (8, 7)$ then $cv(\hat{\Delta}) =$
$(10, 10, 0, 6, 0, 6, 6, a_1, a_2, 4)$. It follows that $c^*(\hat{\Delta}) \leq -\frac{50\pi}{30} + \frac{49\pi}{30} < 0$.

Case 8 (7.7) Let $\hat{\Delta}$ be given by Figure 7.7(viii). Suppose that $\hat{\Delta}$ has exactly one vertex u_i of degree > 3. If $d(u_3) = d(u_7) = 3$ then $cv(\hat{\Delta}) = (10, 10, 2, 6, 0, 6, 6, 0, 0, 3)$; if $i = 3$ then $cv(\hat{\Delta}) = (10, 10, 0, d_1, d_2, 6, 6, 0, 0, 0)$; and if $i = 7$ then $cv(\hat{\Delta}) = (10, 10, 0, 6, 0, 6, 6, 4, 2, 0)$. It follows that $c^*(\hat{\Delta}) \leq -\frac{45\pi}{30} + \frac{44\pi}{30} < 0$. Let $\hat{\Delta}$ have exactly two vertices u_i, u_j of degree > 3. If $d(u_7) = 3$ then $cv(\hat{\Delta}) = (10, 10, 2, d_1, d_2, 6, 6, 0, 0, 3)$; if $d(u_2) = d(u_3) = 3$ then $cv(\hat{\Delta}) = (10, 10, 0, 6, 0, 6, 6, 4, a_1, a_2)$; if $(i, j) = (7, 2)$ then $cv(\hat{\Delta}) = (10, 10, 2, 6, 0, 6, 6, 4, 2, 0)$; and if $(i, j) = (7, 3)$ then $cv(\hat{\Delta}) =$
$(10, 10, 0, d_1, d_2, 6, 6, 4, 2, 0)$. It follows that $c^*(\hat{\Delta}) \leq -\frac{50\pi}{30} + \frac{49\pi}{30} < 0$.

Case 9 (7.7) Let $\hat{\Delta}$ be given by Figure 7.7(ix). Suppose that $\hat{\Delta}$ has exactly one vertex u_i of degree > 3. If $d(u_2) = d(u_4) = 3$ then $cv(\hat{\Delta}) = (10, 10, 0, 2, 2, 0, d_1, d_2, 6, 2)$; if $i = 2$ then $cv(\hat{\Delta}) = (10, 10, e_1, e_2, 2, 0, 0, 4, 6, 0)$; and if $i = 4$ then
$cv(\hat{\Delta}) = (10, 10, 0, 2, e_1, e_2, 0, 4, 6, 0)$. It follows that $c^*(\hat{\Delta}) \leq -\frac{45\pi}{30} + \frac{43\pi}{30} < 0$.

Let $\hat{\Delta}$ have exactly two vertices u_i, u_j of degree > 3. If $d(u_2) = d(u_4) = 3$ then $c^*(\hat{\Delta}) < 0$; if $d(u_2) = d(u_8) = 3$ then $cv(\hat{\Delta}) = (10, 10, 0, 2, e_1, e_2, d_1, d_2, 6, 0)$; if $d(u_4) = d(u_8) = 3$ then $cv(\hat{\Delta}) = (10, 10, e_1, e_2, 2, 0, d_1, d_2, 6, 0)$; if $(i, j) = (2, 4)$ then $cv(\hat{\Delta}) = (10, 10, e_1, e_2, e_1, e_2, 0, 4, 6, 0)$; if $(i, j) = (2, 8)$ then $cv(\hat{\Delta}) = (10, 10, e_1, e_2, 2, 0, 0, 4, 6, 2)$; and if $(i, j) = (4, 8)$ then $cv(\hat{\Delta}) = (10, 10, 0, 2, e_1, e_2, 0, 4, 6, 2)$. It follows that $c^*(\hat{\Delta}) \leq -\frac{50\pi}{30} + \frac{49\pi}{30} < 0$ except for $(i, j) = (2, 4)$, in which case either $d(u) = 4$ and $\frac{\pi}{5}$ is distributed from $\hat{\Delta}$ across the

(u_8, u_9) edge according to Configuration F of Figure 4.30 and $c^*(\hat{\Delta}) \leq -\frac{50\pi}{30} + \frac{52\pi}{30} - \frac{\pi}{5} < 0$ or $d(u) > 4$, $cv(\hat{\Delta}) = (7, 7, e_1, e_2, e_1, e_2, 0, 4, 6, 0)$ and $c^*(\hat{\Delta}) \leq -\frac{50\pi}{30} + \frac{46\pi}{30} < 0$. Let $\hat{\Delta}$ have exactly three vertices u_i, u_j, u_k of degree > 3. If $d(u_2) = 3$ or $d(u_4) = 3$ then $c^*(\hat{\Delta}) < 0$; if $d(u_6) = d(u_8) = 3$ then $cv(\hat{\Delta}) = (10, 10, e_1, e_2, e_1, e_2, 0, 6, 6, 0)$; if $(i, j, k) = (2, 4, 6)$ then $cv(\hat{\Delta}) = (10, 10, e_1, e_2, e_1, e_2, 7, 0, 6, 0)$; and if $(i, j, k) = (2, 4, 8)$ then $cv(\hat{\Delta}) = (10, 10, e_1, e_2, e_1, e_2, 0, 4, 6, 2)$. It follows that $c^*(\hat{\Delta}) \leq -\frac{55\pi}{30} + \frac{55\pi}{30} = 0$.

Case 10 (7.7) Let $\hat{\Delta}$ be given by Figure 7.7(x). Let $\hat{\Delta}$ have exactly one vertex u_i of degree > 3. If $d(u_3) = d(u_6) = 3$ then $cv(\hat{\Delta}) = c(10, 10, 3, 0, 0, 3, 0, 6, 6, 2)$; if $i = 3$ then $cv(\hat{\Delta}) = (10, 10, 0, 2, 2, 0, 0, 4, 6, 0)$; and if $i = 6$ then $cv(\hat{\Delta}) = (10, 10, 0, 0, 0, 0, 7, 0, 6, 0)$. It follows that $c^*(\hat{\Delta}) \leq -\frac{45\pi}{30} + \frac{40\pi}{30} < 0$.

Case 11 (7.7) Let $\hat{\Delta}$ be given by Figure 7.7(xi). Let $\hat{\Delta}$ have exactly one vertex u_i of degree > 3. If $d(u_2) = d(u_4) = d(u_8) = 3$ then $cv(\hat{\Delta}) = (10, 10, 0, 6, 6, 0, 3, 6, 2, 0)$; if $i = 2$ then $cv(\hat{\Delta}) = (10, 10, d_1, d_2, 6, 0, 0, 5, 0, 0)$; if $i = 4$ then $cv(\hat{\Delta}) = (10, 10, 0, 6, d_1, d_2, 0, 5, 0, 0)$; and if $i = 8$ then $cv(\hat{\Delta}) = (10, 10, 0, 6, 6, 0, 0, 5, 2, 4)$. It follows that $c^*(\hat{\Delta}) \leq -\frac{45\pi}{30} + \frac{43\pi}{30} < 0$. Let $\hat{\Delta}$ have exactly two vertices u_i, u_j of degree > 3. If $d(u_2) = d(u_4) = 3$ then $cv(\hat{\Delta}) = (10, 10, 0, 6, 6, 0, 3, c_1, c_2, 4)$; if $d(u_2) = d(u_8) = 3$ then $cv(\hat{\Delta}) = (10, 10, 0, 6, d_1, d_2, 3, 6, 0, 0)$; if $d(u_4) = d(u_8) = 3$ then $cv(\hat{\Delta}) = (10, 10, d_1, d_2, 6, 0, 3, 6, 0, 0)$; if $(i, j) = (2, 4)$ then $cv(\hat{\Delta}) = (10, 10, d_1, d_2, d_1, d_2, 0, 5, 0, 0)$; if $(i, j) = (2, 8)$ then $cv(\hat{\Delta}) = (10, 10, d_1, d_2, 6, 0, 0, 5, 2, 4)$; and if $(i, j) = (4, 8)$ then $cv(\hat{\Delta}) =$
$(10, 10, 0, 6, d_1, d_2, 0, 5, 2, 4)$. It follows that
$c^*(\hat{\Delta}) \leq -\frac{50\pi}{30} + \frac{48\pi}{30} < 0$. Let $\hat{\Delta}$ have exactly three vertices u_i, u_j, u_k of degree > 3. If $d(u_2) = 3$ or $d(u_4) = 3$ or $d(u_7) = 3$ or $d(u_8) = 3$ then $c^*(\hat{\Delta}) \leq -\frac{55\pi}{30} + \frac{55\pi}{30} = 0$.

Case 12 (7.7) Let $\hat{\Delta}$ be given by Figure 7.7(xii). Suppose that $\hat{\Delta}$ has exactly one vertex u_i of degree > 3. If $d(u_3) = 3$ then $cv(\hat{\Delta}) = (10, 10, 2, 2, 2, 2, 3, c_1, c_2, 4)$; and if $i = 3$ then $cv(\hat{\Delta}) = (10, 10, 0, 9, 9, 0, 0, 5, 0, 0)$. It follows that $c^*(\hat{\Delta}) \leq -\frac{45\pi}{30} + \frac{44\pi}{30} < 0$. Let $\hat{\Delta}$ have exactly two vertices u_3, u_j of degree > 3. If $d(u_7) = d(u_8) = 3$ then $cv(\hat{\Delta}) = (10, 10, d_1, d_2, d_1, d_2, 3, 6, 0, 0)$; if $j = 7$ then $cv(\hat{\Delta}) = (10, 10, 0, 9, 9, 0, 0, 6, 0, 0)$; and if $j = 8$ then $cv(\hat{\Delta}) = (10, 10, 0, 9, 9, 0, 0, 5, 2, 4)$. It follows that $c^*(\hat{\Delta}) \leq -\frac{50\pi}{30} + \frac{49\pi}{30} < 0$. Let $\hat{\Delta}$ have exactly three vertices u_3, u_j, u_k of degree > 3. If $d(u_8) = 3$ then $c^*(\hat{\Delta}) < 0$; if $d(u_5) = d(u_6) = 3$ then $cv(\hat{\Delta}) = (10, 10, d_1, d_2, d_1, d_2, 0, c_1, c_2, 4)$; if $(j, k) = (8, 5)$ then $cv(\hat{\Delta}) = (10, 10, 0, 9, d_1, d_2, 3, 5, 2, 4)$; and if $(j, k) = (8, 6)$ then $cv(\hat{\Delta}) = (10, 10, 0, 9, 9, 0, 3, 6, 2, 4)$. It follows that $c^*(\hat{\Delta}) \leq -\frac{55\pi}{30} + \frac{53\pi}{30} < 0$.

Case 13 (7.7) Let $\hat{\Delta}$ be given by Figure 7.7(xiii). Suppose that $\hat{\Delta}$ has exactly one vertex u_i of degree > 3. If $d(u_2) = d(u_3) = d(u_8) = 3$ then $cv(\hat{\Delta}) = (10, 10, 0, 0, 0, 6, 6, d_1, d_2, 0)$; if $i = 2$ then $cv(\hat{\Delta}) = (10, 10, 3, 0, 0, 6, 6, 0, 6, 0)$; if $i = 3$ then $cv(\hat{\Delta}) = (10, 10, 0, 2, 4, 6, 6, 0, 6, 0)$; and if $i = 8$ then $cv(\hat{\Delta}) = (10, 10, 0, 0, 0, 6, 6, 0, 6, 2)$. It follows that $c^*(\hat{\Delta}) \leq -\frac{45\pi}{30} + \frac{44\pi}{30} < 0$. Let $\hat{\Delta}$ have exactly two vertices u_i, u_j of degree > 3. If $d(u_2) = 3$ then $cv(\hat{\Delta}) = (10, 10, 0, 2, 4, 6, 6, d_1, d_2, 2)$; if $d(u_3) = 3$ then $cv(\hat{\Delta}) = (10, 10, 3, 0, 0, 6, 6, d_1, d_2, 2)$; and if $(i, j) = (2, 3)$ then $cv(\hat{\Delta}) = (10, 10, a_1, a_2, 4, 6, 6, 0, 6, 0)$. It follows that $c^*(\hat{\Delta}) \leq$

$-\frac{50\pi}{30} + \frac{50\pi}{30} < 0$.

Case 14 (7.7) Let $\hat{\Delta}$ be given by Figure 7.7(xiv). Suppose that $\hat{\Delta}$ has exactly one vertex u_i of degree > 3. If $d(u_2) = d(u_8) = 3$ then $cv(\hat{\Delta}) = (10, 10, 0, 6, 2, 6, 6, 3, 0, 0)$; if $i = 2$ then $cv(\hat{\Delta}) = (10, 10, d_1, d_2, 0, 6, 6, 0, 0, 0)$; and if $i = 8$ then $cv(\hat{\Delta}) = (10, 10, 0, 6, 0, 6, 6, 0, 2, 4)$. It follows that $c^*(\hat{\Delta}) \leq -\frac{45\pi}{30} + \frac{44\pi}{30} < 0$. Let $\hat{\Delta}$ have exactly two vertices u_i, u_j of degree > 3. If $d(u_8) = 3$ then $cv(\hat{\Delta}) = (10, 10, d_1, d_2, 2, 6, 6, 3, 0, 0)$; if $d(u_6) = d(u_7) = 3$ then $cv(\hat{\Delta}) = (10, 10, d_1, d_2, 2, 6, 6, 0, 2, 4)$; if $(i, j) = (8, 6)$ then $cv(\hat{\Delta}) = (10, 10, 0, 6, 0, 6, 6, 3, 2, 4)$; and if $(i, j) = (8, 7)$ then
$cv(\hat{\Delta}) = (10, 10, 0, 6, 0, 6, 6, a_1, a_2, 4)$. It follows that $cv(\hat{\Delta}) \leq -\frac{50\pi}{30} + \frac{50\pi}{30} = 0$.

Case 15 (7.7) Let $\hat{\Delta}$ be given by Figure 7.7(xv). Suppose that $\hat{\Delta}$ has exactly one vertex u_i of degree > 3. If $d(u_2) = d(u_6) = 3$ then $cv(\hat{\Delta}) = (10, 10, 0, 0, 6, 3, 0, 6, d_1, d_2)$; if $i = 2$ then $cv(\hat{\Delta}) = (10, 10, 4, 2, 5, 0, 0, 6, 6, 0)$; and if $i = 6$ then $cv(\hat{\Delta}) = (10, 10, 0, 0, 5, 0, d_1, d_2, 6, 0)$. It follows that $c^*(\hat{\Delta}) \leq -\frac{45\pi}{30} + \frac{45\pi}{30} = 0$. Let $\hat{\Delta}$ have exactly two vertices u_i, u_j of degree > 3. If $d(u_2) = 3$ then $cv(\hat{\Delta}) = (10, 10, 0, 0, 6, 3, d_1, d_2, d_1, d_2)$; if $d(u_6) = d(u_8) = 3$ then $cv(\hat{\Delta}) = (10, 10, 4, d_1, d_2, 3, 0, 6, 6, 0)$, if $(i, j) = (2, 6)$ then $cv(\hat{\Delta}) = (10, 10, 4, 2, 5, 0, d_1, d_2, 6, 0)$; and if $(i, j) = (2, 8)$ then $cv(\hat{\Delta}) = (10, 10, 4, 2, 5, 0, 0, 6, d_1, d_2)$. It follows that $c^*(\hat{\Delta}) \leq -\frac{50\pi}{30} + \frac{49\pi}{30} < 0$. Let $\hat{\Delta}$ have exactly three vertices of degree > 3. If $d(u_2) = 3$ or $d(u_6) = 3$ or $d(u_8) = 3$ then $c^*(\hat{\Delta}) < 0$; and if $(i, j, k) = (2, 6, 8)$ then $cv(\hat{\Delta}) = (10, 10, 4, 2, 5, 0, d_1, d_2, d_1, d_2)$ and $c^*(\hat{\Delta}) \leq -\frac{55\pi}{30} + \frac{51\pi}{30} < 0$.

Case 16 (7.7) Let $\hat{\Delta}$ be given by Figure 7.7(xvi). Suppose $\hat{\Delta}$ has exactly one vertex u_i of degree 3. If $d(u_4) = d(u_8) = 3$ then $c^*(\hat{\Delta}) = (10, 10, 2, 6, 6, 0, a_1, a_2, 2, 0)$; if $i = 4$ then $cv(\hat{\Delta}) = (10, 10, 0, 6, d_1, d_2, 0, 0, 0, 0)$; and if $i = 8$ then $cv(\hat{\Delta}) = (10, 10, 0, 6, 5, 0, 0, 0, 2, 3)$. It follows that $c^*(\hat{\Delta}) \leq -\frac{45\pi}{30} + \frac{43\pi}{30} < 0$. Let $\hat{\Delta}$ have exactly three vertices u_i, u_j, u_k of degree > 3. If $d(u_4) = 3$ or $d(u_6) = 3$ or $d(u_8) = 3$ then $c^*(\hat{\Delta}) \leq -\frac{50\pi}{30} + \frac{50\pi}{30} = 0$; and if $(i, j, k) = (4, 6, 8)$ then $cv(\hat{\Delta}) = (10, 10, 0, 6, d_1, d_2, a_1, a_2, a_1, a_2)$ and $c^*(\hat{\Delta}) \leq 0$.

Finally let $\hat{\Delta}$ be given by one of the regions of Figure 7.5. It turns out that (up to cyclic permutations and inversion) there is one way to label each of Figure 7.5(i), (ii), (iii), (v) and (vi); two ways to label (iv); five ways to label (vii) or (viii); three ways to label (ix) or (x); seven ways to label (xi) or (xii) or (xiii); and seven ways to label (xiv) or (xv) or (xvi) or (xvii). This yields a total of twenty-nine regions. There are however several coincidences amongst these regions resulting in $\hat{\Delta}$ being one of the eight regions given by Figure 7.8. The table below gives $c(u_i, u_{i+1})$ $(1 \leq i \leq 9)$ in multiples of $\pi/30$ for each of the eight regions of Figure 7.8 with the total plus the contribution via the b-segment in the final column.

We claim that, in the table below, $x_1 + y_1 + z_1 = 15$. To see this let $\hat{\Delta}$ be given by Figure 7.8(i). If $c(u_5, u_6) = 0$ then $x_1 + y_1 + z_1 = 14$, so assume otherwise, in which case $c(u_4, u_5) = \frac{2\pi}{15}$ (Figure 5.8(ix)). If now $c(u_5, u_6) = \frac{2\pi}{15}$ then $x_1 + y_1 + z_1 = 15$. On the other hand if $c(u_5, u_6) > 4$ then $d(u_5) = 3$ (see Figure 5.8) forcing $c(u_4, u_5) = \frac{\pi}{30}$ and $x_1 + y_1 + z_1 = 15$. Note that we use here and below the fact that labelling prevents $\hat{\Delta} = \hat{\Delta}_2$

of Figure 5.4(iv). The arguments for $\hat{\Delta}$ of figures (iii), (iv), (v), (vii) and (viii) are similar although for (v), (vii) and (viii) we use the fact, again both here and below, that $\hat{\Delta} = \hat{\Delta}_2$ of Figure 5.3(iv).

(i)	d_1	d_2	x_1	y_1	z_1	6	6	e_1	e_2	$48 + 10 = 58$
(ii)	e_1	e_2	6	d_1	d_2	d_1	d_2	d_1	d_2	$47 + 30 = 77$
(iii)	d_1	d_2	x_1	y_1	z_1	e_1	e_2	e_1	e_2	$47 + 30 = 77$
(iv)	d_1	d_2	f_1	f_2	x_1	y_1	z_1	e_1	e_2	$48 + 30 = 78$
(v)	d_1	d_2	f_1	f_2	x_1	y_1	z_1	e_1	e_2	$48 + 50 = 98$
(vi)	e_1	e_2	d_1	d_2	d_1	d_2	6	d_1	d_2	$47 + 50 = 97$
(vii)	d_1	d_2	x_1	y_1	z_1	f_1	f_2	e_1	e_2	$48 + 50 = 98$
(viii)	d_1	d_2	6	6	x_1	y_1	z_1	e_1	e_2	$48 + 70 = 118$

Observe that $d(\hat{\Delta}) = 10$ in (i); $d(\hat{\Delta}) = 12$ in (ii)-(iv); $d(\hat{\Delta}) = 14$ in (v)-(vii); and $d(\hat{\Delta}) = 16$ in (viii). It follows that if $\hat{\Delta}$ has at least four vertices of degree > 3 then $c^*(\hat{\Delta}) \leq 0$. If $\hat{\Delta}$ has no vertices of degree > 3 then we see from Figure 7.8 that $c^*(\hat{\Delta}) \leq -7\pi + \frac{18\pi}{3} + \frac{26\pi}{30} < 0$.

We deal with each of the eight cases in turn.

Case 1 (7.8) Let $\hat{\Delta}$ be given by Figure 7.8(i). Suppose $\hat{\Delta}$ has exactly one vertex u_i of degree > 3. If $d(u_4) = d(u_9) = 3$ then $cv(\hat{\Delta}) = (10, d_1, d_2, 2, 0, 6, 6, 6, 2, 0)$; if $i = 4$ then $cv(\hat{\Delta}) = (10, 0, 6, c_1, c_2, 4, 4, 6, 2, 0)$ (**Note**: the c_1, c_2 here follows from the fact that $d(u_4) > 3$, $d(u_5) = 3$ implies $\hat{\Delta}$ receives at most $\frac{\pi}{30}$ across the (u_4, u_5)-edge (see Figures 5.1 and 5.5)); and if $i = 9$ then $cv(\hat{\Delta}) = (10, 0, 6, 2, 0, 4, 4, 6, e_1, e_2)$. It follows that $c^*(\hat{\Delta}) \leq -\frac{45\pi}{30} + \frac{43\pi}{30} < 0$. Let $\hat{\Delta}$ have exactly two vertices u_i, u_j of degree > 3. If $d(u_2) = d(u_4) = 3$ then $cv(\hat{\Delta}) = (10, 0, 6, 2, 0, 6, 6, 6, e_1, e_2)$; if $d(u_2) = d(u_9) = 3$ then $cv(\hat{\Delta}) = (10, 0, 6, x_1, y_1, z_1, 6, 6, 2, 0)$; if $d(u_4) = d(u_9) = 3$ then $c^*(\hat{\Delta}) \leq -\frac{50\pi}{30} + \frac{42\pi}{30} < 0$; if $(i, j) = (2, 4)$ then $cv(\hat{\Delta}) = (10, d_1, d_2, c_1, c_2, 4, 4, 6, 2, 0)$; if $(i, j) = (2, 9)$ then $cv(\hat{\Delta}) = (10, d_1, d_2, 2, 0, 4, 4, 6, e_1, e_2)$; if $(i, j) = (4, 9)$ then $cv(\hat{\Delta}) = (10, 0, 6, c_1, c_2, 4, 4, 6, e_1, e_2)$. It follows that $c^*(\hat{\Delta}) \leq -\frac{50\pi}{30} + \frac{50\pi}{30} = 0$. Let $\hat{\Delta}$ have exactly three vertices u_i, u_j, u_k of degree > 3. If $d(u_4) = 3$ or $d(u_9) = 3$ then $c^*(\hat{\Delta}) < 0$; if $d(u_2) = d(u_7) = 3$ then $cv(\hat{\Delta}) = (10, 0, 6, x_1, y_1, z_1, 4, 6, e_1, e_2)$; if $(i, j, k) = (4, 9, 2)$ then $cv(\hat{\Delta}) = (10, d_1, d_2, c_1, c_2, 4, 4, 6, e_1, e_2)$; and if $(i, j, k) = (4, 9, 7)$ then $cv(\hat{\Delta}) = (10, 0, 6, c_1, c_2, 4, 4, 6, e_1, e_2)$. It follows that $c^*(\hat{\Delta}) \leq -\frac{55\pi}{30} + \frac{54\pi}{30} < 0$.

Case 2 (7.8) Let $\hat{\Delta}$ be given by Figure 7.8(ii). Suppose that $\hat{\Delta}$ has exactly one vertex u_i of degree > 3. If $d(u_2) = d(u_9) = 3$ then $cv(\hat{\Delta}) = (10, 10, 10, 0, 2, 6, d_1, d_2, d_1, d_2, 6, 0)$; if $i = 2$ then $cv(\hat{\Delta}) = (10, 10, 10, e_1, e_2, 6, 4, 0, 0, 6, 6, 0)$; and if $i = 9$ then $cv(\hat{\Delta}) = (10, 10, 10, 0, 2, 6, 4, 0, 0, 6, d_1, d_2)$. It follows that $c^*(\hat{\Delta}) \leq -\frac{65\pi}{30} + \frac{64\pi}{30} < 0$. Let $\hat{\Delta}$ have exactly two vertices u_i, u_j of degree > 3. If $d(u_2) = 3$ then
$cv(\hat{\Delta}) = (10, 10, 10, 0, 2, 6, d_1, d_2, d_1, d_2, d_1, d_2)$; if $d(u_6) = 3$ then

$cv(\hat{\Delta}) = (10, 10, 10, e_1, e_2, 6, 6, 0, 0, 6, d_1, d_2)$; and if $(i, j) = (2, 6)$ then $cv(\hat{\Delta}) = (10, 10, 10, e_1, e_2, 6, 4, 2, 2, 6, 6, 0)$. It follows that $c^*(\hat{\Delta}) \leq -\frac{70\pi}{30} + \frac{69\pi}{30}$. Let $\hat{\Delta}$ have exactly three vertices u_i, u_j, u_k of degree > 3. If $d(u_2) = 3$ or $d(u_6) = 3$ or $d(u_9) = 3$ then $c^*(\hat{\Delta}) \leq -\frac{75\pi}{30} + \frac{73\pi}{30} < 0$; and if $(i, j, k) = (2, 6, 9)$ then $cv(\hat{\Delta}) = (10, 10, 10, e_1, e_2, 6, 4, 2, 2, 6, d_1, d_2)$ and $c^*(\hat{\Delta}) \leq -\frac{75\pi}{30} + \frac{71\pi}{30} < 0$.

Case 3 (7.8) Let $\hat{\Delta}$ be given by Figure 7.8(iii). Suppose that $\hat{\Delta}$ has exactly one vertex u_i of degree > 3. If $d(u_7) = d(u_9) = 3$ then $cv(\hat{\Delta}) = (10, 10, 10, d_1, d_2, x_1, y_1, z_1, 2, 2, 2, 0)$; if $i = 7$ then $cv(\hat{\Delta}) = (10, 10, 10, 0, 6, 2, 0, 6, e_1, e_2, 2, 0)$; and if $i = 9$ then $cv(\hat{\Delta}) = (10, 10, 10, 0, 6, 2, 0, 6, 2, 2, e_1, e_2)$. It follows that $c^*(\hat{\Delta}) \leq -\frac{65\pi}{30} + \frac{61\pi}{30} < 0$. Let $\hat{\Delta}$ have exactly two vertices u_i, u_j of degree > 3. If $d(u_2) = d(u_7) = 3$ then $cv(\hat{\Delta}) = (10, 10, 10, 0, 6, x_1, y_1, z_1, 2, 2, e_1, e_2)$; if $d(u_9) = 3$ then $cv(\hat{\Delta}) = (10, 10, 10, d_1, d_2, x_1, y_1, z_1, e_1, e_2, 2, 0)$; if $(i, j) = (9, 2)$ then $cv(\hat{\Delta}) = (10, 10, 10, d_1, d_2, 2, 0, 6, 2, 2, e_1, e_2)$; and if $(i, j) = (9, 7)$ then $cv(\hat{\Delta}) = (10, 10, 10, 0, 6, 2, 0, 6, e_1, e_2, e_1, e_2)$. It follows that $c^*(\hat{\Delta}) \leq -\frac{70\pi}{30} + \frac{68\pi}{30} = 0$. Let $\hat{\Delta}$ have exactly three vertices u_i, u_j, u_k of degree > 3. If $d(u_2) = 3$ or $d(u_7) = 3$ or $d(u_9) = 3$ then $c^*(\hat{\Delta}) \leq 0$; and if $(i, j, k) = (2, 7, 9)$ then $cv(\hat{\Delta}) = (10, 10, 10, d_1, d_2, 2, 0, 6, e_1, e_2, e_1, e_2)$ and $c^*(\hat{\Delta}) \leq -\frac{75\pi}{30} + \frac{70\pi}{30} < 0$.

Case 4 (7.8) Let $\hat{\Delta}$ be given by Figure 7.8(iv). Suppose that $\hat{\Delta}$ has exactly one vertex u_i of degree > 3. If $d(u_4) = d(u_9) = 3$ then $c^*(\hat{\Delta}) = (10, 10, 10, d_1, d_2, 2, 2, x_1, y_1, z_1, 2, 0)$; if $i = 4$ then $cv(\hat{\Delta}) = (10, 10, 10, 0, 6, f_1, f_2, 2, 0, 6, 2, 0)$; and if $i = 9$ then $c^*(\hat{\Delta}) = (10, 10, 10, 0, 6, 2, 2, 2, 0, 6, e_1, e_2)$. It follows that $c^*(\hat{\Delta}) \leq -\frac{65\pi}{30} + \frac{61\pi}{30} < 0$. Let $\hat{\Delta}$ have exactly two vertices u_i, u_j of degree > 3. If $d(u_9) = 3$ then $cv(\hat{\Delta}) = (10, 10, 10, d_1, d_2, f_1, f_2, x_1, y_1, z_1, 2, 0)$; if $d(u_2) = d(u_4) = 3$ then $cv(\hat{\Delta}) = (10, 10, 10, 0, 6, 2, 2, x_1, y_1, z_1, e_1, e_2)$; if $(i, j) = (9, 2)$ then $cv(\hat{\Delta}) = (10, 10, 10, d_1, d_2, 2, 2, 2, 0, 6, e_1, e_2)$; and if $(i, j) = (9, 4)$ then $cv(\hat{\Delta}) = (10, 10, 10, 0, 6, f_1, f_2, 2, 0, 6, e_1, e_2)$. It follows that $c^*(\hat{\Delta}) \leq -\frac{70\pi}{30} + \frac{69\pi}{30} = 0$. Let $\hat{\Delta}$ have exactly three vertices u_i, u_j, u_k of degree > 3. If $d(u_2) = 3$ or $d(u_4) = 3$ or $d(u_9) = 3$ then $c^*(\hat{\Delta}) \leq 0$; and if $(i, j, k) = (2, 4, 9)$ then $cv(\hat{\Delta}) = (10, 10, 10, d_1, d_2, f_1, f_2, 2, 0, 6, e_1, e_2)$ and $c^*(\hat{\Delta}) \leq -\frac{75\pi}{30} + \frac{71\pi}{30} < 0$.

Case 5 (7.8) Let $\hat{\Delta}$ be given by Figure 7.8(v). Suppose that $\hat{\Delta}$ has exactly one vertex u_i of degree > 3. If $d(u_4) = d(u_9) = 3$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 10, d_1, d_2, 2, 2, x_1, y_1, z_1, 2, 0)$; if $i = 4$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 10, 0, 6, f_1, f_2, 6, 0, 2, 2, 0)$; and if $i = 9$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 10, 0, 6, 2, 2, 6, 0, 2, e_1, e_2)$. It follows that $c^*(\hat{\Delta}) \leq -\frac{85\pi}{30} + \frac{81\pi}{30} < 0$. Let $\hat{\Delta}$ have exactly two vertices u_i, u_j of degree > 3. If $d(u_9) = 3$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 10, d_1, d_2, f_1, f_2, x_1, y_1, z_1, 2, 0)$; if $d(u_2) = d(u_4) = 3$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 10, 0, 6, 2, 2, x_1, y_1, z_1, e_1, e_2)$; if $(i, j) = (9, 2)$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 10, d_1, d_2, 2, 2, 6, 0, 2, e_1, e_2)$; and if $(i, j) = (9, 4)$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 10, 0, 6, f_1, f_2, 6, 0, 2, e_1, e_2)$. It follows that $c^*(\hat{\Delta}) \leq -\frac{90\pi}{30} + \frac{89\pi}{30} = 0$. Let $\hat{\Delta}$ have exactly three vertices u_i, u_j, u_k of degree > 3. If $d(u_2) = 3$ or $d(u_4) = 3$ or $d(u_9) = 3$ then $c^*(\hat{\Delta}) \leq 0$; and if $(i, j, k) = (2, 4, 9)$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 10, d_1, d_2, f_1, f_2, 6, 0, 2, e_1, e_2)$

and $c^*(\hat{\Delta}) \leq -\frac{95\pi}{30} + \frac{91\pi}{30} < 0$.

Case 6 (7.8) Let $\hat{\Delta}$ be given by Figure 7.8(vi). Suppose that $\hat{\Delta}$ has exactly one vertex u_i of degree > 3. If $d(u_2) = d(u_9) = 3$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 10, 0, 2, d_1, d_2, d_1, d_2, 6, 6, 0)$; if $i = 2$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 10, e_1, e_2, 6, 0, 0, 4, 6, 6, 0)$; and if $i = 9$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 10, 0, 2, 6, 0, 0, 4, 6, d_1, d_2)$. It follows that $c^*(\hat{\Delta}) \leq -\frac{85\pi}{30} + \frac{84\pi}{30} < 0$. Let $\hat{\Delta}$ have exactly two vertices u_i, u_j of degree > 3. If $d(u_2) = 3$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 10, 0, 2, d_1, d_2, d_1, d_2, 6, d_1, d_2)$; if $d(u_5) = 3$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 10, e_1, e_2, 6, 0, 0, 6, 6, d_1, d_2)$; and if $(i, j) = (2, 5)$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 10, e_1, e_2, 6, 2, 2, 4, 6, 6, 0)$. It follows that $c^*(\hat{\Delta}) \leq -\frac{90\pi}{30} + \frac{89\pi}{30}$. Let $\hat{\Delta}$ have exactly three vertices u_i, u_j, u_k of degree > 3. If $d(u_2) = 3$ or $d(u_5) = 3$ or $d(u_9) = 3$ then $c^*(\hat{\Delta}) \leq 0$; and if $(i, j, k) = (2, 5, 9)$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 10, e_1, e_2, 6, 2, 2, 4, 6, d_1, d_2)$ and $c^*(\hat{\Delta}) \leq -\frac{95\pi}{30} + \frac{91\pi}{30} < 0$.

Case 7 (7.8) Let $\hat{\Delta}$ be given by Figure 7.8(vii). Suppose that $\hat{\Delta}$ has exactly one vertex u_i of degree > 3. If $d(u_7) = d(u_9) = 3$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 10, d_1, d_2, x_1, y_1, z_1, 2, 2, 2, 0)$; if $i = 7$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 10, 0, 6, 6, 0, 2, f_1, f_2, 2, 0)$; and if $i = 9$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 10, 0, 6, 6, 0, 2, 2, 2, e_1, e_2)$. It follows that $c^*(\hat{\Delta}) \leq -\frac{85\pi}{30} + \frac{81\pi}{30} < 0$. Let $\hat{\Delta}$ have exactly two vertices u_i, u_j of degree > 3. If $d(u_9) = 3$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 10, d_1, d_2, x_1, y_1, z_1, f_1, f_2, 2, 0)$; if $d(u_2) = d(u_7) = 3$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 10, 0, 6, x_1, y_1, z_1, 2, 2, e_1, e_2)$; if $(i, j) = (9, 2)$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 10, d_1, d_2, 6, 0, 2, 2, 2, e_1, e_2)$; and if $(i, j) = (9, 7)$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 10, 0, 6, 6, 0, 2, f_1, f_2, e_1, e_2)$. It follows that $c^*(\hat{\Delta}) \leq -\frac{90\pi}{30} + \frac{89\pi}{30} = 0$. Let $\hat{\Delta}$ have exactly three vertices u_i, u_j, u_k of degree > 3. If $d(u_2) = 3$ or $d(u_8) = 3$ or $d(u_9) = 3$ then $c^*(\hat{\Delta}) \leq 0$; if $(i, j, k) = (2, 8, 9)$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 10, d_1, d_2, 6, 0, 2, f_1, f_2, e_1, e_2)$ and $c^*(\hat{\Delta}) \leq -\frac{95\pi}{30} + \frac{91\pi}{30} < 0$.

Case 8 (7.8) Let $\hat{\Delta}$ be given by Figure 7.8(viii). Suppose that $\hat{\Delta}$ has exactly one vertex u_i of degree > 3. If $d(u_7) = d(u_9) = 3$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 10, 10, 10, d_1, d_2, 6, 6, 6, 0, 2, 2, 0)$; if $i = 7$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 10, 10, 10, 0, 6, 6, 4, 4, c_1, c_2, 2, 0)$ (for the c_1, c_2 see the note in Case 1); and if $i = 9$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 10, 10, 10, 0, 6, 6, 4, 4, 0, 2, e_1, e_2)$. It follows that $c^*(\hat{\Delta}) \leq -\frac{105\pi}{30} + \frac{103\pi}{30} = 0$. Let $\hat{\Delta}$ have exactly two vertices, u_i, u_j of degree > 3. If $d(u_2) = d(u_7) = 3$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 10, 10, 10, 0, 6, 6, 6, 6, 0, 2, e_1, e_2)$; if $d(u_2) = d(u_9) = 3$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 10, 10, 10, 0, 6, 6, 6, x_1, y_1, z_1, 2, 0)$; if $d(u_7) = d(u_9) = 3$ then $c^*(\hat{\Delta}) \leq -\frac{110\pi}{30} + \frac{100\pi}{30}$; if $(i, j) = (2, 7)$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 10, 10, 10, d_1, d_2, 6, 4, 4, c_1, c_2, 2, 0)$; if $(i, j) = (2, 9)$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 10, 10, 10, d_1, d_2, 6, 4, 4, 0, 2, e_1, e_2)$; and if $(i, j) = (7, 9)$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 10, 10, 10, 0, 6, 6, 4, 4, c_1, c_2, e_1, e_2)$. It follows that $c^*(\hat{\Delta}) \leq -\frac{110\pi}{30} + \frac{110\pi}{30} = 0$. Let $\hat{\Delta}$ have exactly three vertices u_i, u_j, u_k of degree > 3. If $d(u_7) = 3$ or $d(u_9) = 3$ then $c^*(\hat{\Delta}) < 0$; if $d(u_2) = d(u_4) = 3$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 10, 10, 10, 0, 6, 6, 4, x_1, y_1, z_1, e_1, e_2)$; if $(i, j, k) = (7, 9, 2)$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 10, 10, 10, d_1, d_2, 6, 4, 4, c_1, c_2, e_1, e_2)$; and if $(i, j, k) = (7, 9, 4)$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 10, 10, 10, 0, 6, 6, 6, 4, c_1, c_2, e_1, e_2)$. It follows that $cv(\hat{\Delta}) \leq -\frac{115\pi}{30} + \frac{114\pi}{30} < 0$.

□

Proposition 7.5 *If $\hat{\Delta}$ is a type $\mathcal{B}$ region and $d(\hat{\Delta}) \geq 10$ then $c^*(\hat{\Delta}) \leq 0$.*

Proof. It can be assumed that $d(\hat{\Delta}) \geq 10$ and that $\hat{\Delta}$ is not one of the regions of Figures 7.3(ii)-(v), 7.4 or 7.5, otherwise Proposition 7.4 applies. Moreover if $n_2 \geq 10$ then $c^*(\hat{\Delta}) \leq 0$ so assume that $n_2 \leq 9$. It follows from the proof of Lemma 7.1 that the upper bound (†) is reduced by at least $\frac{2\pi}{15}$ for each gap between two b-segments that contain b-regions so if there are at least three such b-segments then $c^*(\hat{\Delta}) \leq \pi(2 - \frac{n_2}{5}) - 3\left(\frac{2\pi}{15}\right)$ implying $c^*(\hat{\Delta}) \leq 0$ for $n_2 \geq 8$. Since there are at least two edges between b-segments it follows that if $\hat{\Delta}$ contains more than three such b-segments then $c^*(\hat{\Delta}) \leq 0$ or if exactly three then $n_2 \geq 8$ by Lemma 7.2(i) and again $c^*(\hat{\Delta}) \leq 0$. If $\hat{\Delta}$ has exactly one b-segment that contains a b-region then $c^*(\hat{\Delta}) \leq 0$ by Proposition 7.3 together with Lemma 7.2(v), (vi) so suppose from now on that $\hat{\Delta}$ contains exactly two such segments. Then $c^*(\hat{\Delta}) \leq \pi\left(2 - \frac{n_2}{5}\right) - 2\left(\frac{2\pi}{15}\right)$ which implies $c^*(\hat{\Delta}) \leq 0$ for $n_2 \geq 9$, so assume $n_2 \leq 8$ in which case $\hat{\Delta}$ is given by Figure 7.3(i) where $(m, n) \in \{(2,2), (2,3), (2,4), (2,5), (2,6), (3,3), (3,4), (3,5), (4,4)\}$. Applying Proposition 7.3 and Lemma 7.2(ii) shows it can be assumed that there is at least one shadow edge in $\hat{\Delta}$ between the two b-segments.

Let $m = 2$. It follows from the statement at the end of the above paragraph that $\hat{\Delta}$ contains the shadow edge (14) and $\hat{\Delta}$ is given by Figure 7.9(i)-(ii). If $(m, n) \neq (2, 6)$ then $i\deg(1) = i\deg(4) = 1$ by Lemma 7.2(iii) and there is a length contradiction so let $(m, n) = (2, 6)$. We claim that there is a reduction to (†) of $\frac{4\pi}{15}$ between vertices 1 and 4. Given this and the fact that there is a reduction of $\frac{2\pi}{15}$ between 2 and 3 we obtain $c^*(\hat{\Delta}) \leq \pi\left(2 - \frac{n_2}{5}\right) - \frac{6\pi}{15}$ and $c^*(\hat{\Delta}) \leq 0$ for $n_2 \geq 8$, in particular when $(m, n) = (2, 6)$. To prove the claim observe that if $d(a_1) = 3$ in Figure 7.9(i) or (ii) then $c_1 = c_2 = 0$; and if $d(a_1) \geq 4$ then $c_1 + c_2 \leq \frac{2\pi}{15}$ (see Figure 4.33). In the first instance there is a deficit of at least $\left(\frac{2\pi}{3} + 2\left(\frac{2\pi}{15}\right)\right) - \frac{2\pi}{3} = \frac{4\pi}{15}$; and in the second case the deficit is at least $\left(\frac{2\pi}{3} + 2\left(\frac{2\pi}{15}\right)\right) - \left(\frac{2\pi}{4} + \frac{2\pi}{15}\right) = \frac{3\pi}{10}$.

Let $m = 3$ or 4. Applying Lemma 7.2(ii)-(iv) and Proposition 7.3 it can be assumed that $\hat{\Delta}$ is given by Figure 7.10 with the understanding that the segment of $\hat{\Delta}$ between vertices 2 and 3 is also one of these nine possibilities. (Note that in Figure 7.10 the length of the shadow edge incident at vertex 1 is shown.) We claim that if $m = 3$ then the edges between 1 and 4 produce a deficit of at least $\frac{2\pi}{5}$; and if $m = 4$ then the reduction is at least $\frac{\pi}{5}$. Given this, if $(m, n) = (3, 3)$ then $c^*(\hat{\Delta}) \leq \pi\left(2 - \frac{6}{5}\right) - \frac{4\pi}{5} = 0$; if $(m, n) = (3, 4)$ then $c^*(\hat{\Delta}) \leq \pi\left(2 - \frac{7}{5}\right) - \frac{3\pi}{5} = 0$; if $(m, n) = (3, 5)$ then $c^*(\hat{\Delta}) \leq \pi\left(2 - \frac{8}{5}\right) - \left(\frac{2\pi}{5} + \frac{2\pi}{15}\right) < 0$; and if $(m, n) = (4, 4)$ then $c^*(\hat{\Delta}) \leq \pi\left(2 - \frac{8}{5}\right) - 2\left(\frac{\pi}{5}\right) = 0$, so it remains to prove the claim for the possible labellings of the regions of Figure 7.10 and these are shown in Figure 7.11(i)-(xx). Indeed there are four ways to label each of Figure 7.10(iv) and (vi); and two ways to label each of the others. However the labelling obtained from Figure 7.10(vii) already appears in Figure 7.10(vi).

Let $m = 3$. Then the tables below (in which $\kappa_1, \kappa_2, \kappa_3$ and deficit are given as multiples of $\frac{\pi}{30}$) show that the deficit in each case is $\frac{2\pi}{5}$, as required, except for $d(v_1) = d(v_2) = 3$ in (i) and we consider this below. Note that in (i) and (iii) when $d(u)=4$ and either $d(v_1)=d(v_2)=3$ or $d(v_1)=3$, $d(v_2)=4$, $\frac{\pi}{5}$ is distributed from $\hat{\Delta}$ according to Configuration E, F of Figure 4.30(i), (iii). Note that in (ii) when $d(v_1)=3$ and $d(v_2)=4$ the region $\hat{\Delta}$ cannot be $\hat{\Delta}_2$ of Figure 5.4(iv) because $d(w)=3$ in Figure 7.11 but the corresponding vertex in Figure 5.4(ii) has degree 4. Similarly in (iv) when $d(v_1)=4$ and $d(v_2)=(3)$ the region $\hat{\Delta}$ cannot be $\hat{\Delta}_2$ of Figure 5.3(iv).

Suppose that $d(u)>4$ in Figure 7.11(i) in which the vertex v corresponds to the vertex 4 of Figure 7.10(i). If there are at least two regions in the b-segment between vertices 4 and 3 then $2\left(\frac{10\pi}{30}-\frac{7\pi}{30}\right)=\frac{\pi}{5}$ is contributed to the deficit and so the totals $\frac{12\pi}{30}$, $\frac{16\pi}{30}$ remain the same. If however there is exactly one region in the b-segment then only $\frac{10\pi}{30}-\frac{7\pi}{30}=\frac{\pi}{10}$ is contributed to the deficit and so the total is $\frac{9\pi}{30}$ when $d(v_1)=d(v_2)=3$ and $\frac{13\pi}{30}$ when $d(v_1)=3$, $d(v_2)=4$. If $(m,n)=(3,5)$ then $c^*(\hat{\Delta})\leq\pi(2-\frac{8}{5})-\left(\frac{9\pi}{30}+\frac{2\pi}{15}\right)<0$ so it can be assumed that $n\in\{3,4\}$. But given that there are no vertices between 4 and 3, it follows immediately from length considerations that (i) of Figure 7.10 can only be combined with (iv) or (viii), and so, in particular, $n=4$. Any attempt at labelling shows that (i) with (viii) is impossible and the unique region $\hat{\Delta}$ obtained from (i) with (iv) is given by Figure 7.12 in which the segment of vertices from 2 to 3 corresponds to Figure 7.11(x). We show below that for Figure 7.11(x), the deficit is at least $\frac{9\pi}{30}$ and so $c^*(\hat{\Delta})\leq\pi\left(2-\frac{7}{5}\right)-2\left(\frac{9\pi}{30}\right)=0$. If $d(u)>4$ in Figure 7.11(iii) in which the vertex v corresponds to the vertex 4 of Figure 7.10(iv), then since there are at least two regions in the b-segment between 4 and 3 it follows that, as in the above case, the total deficit remains unchanged.

(i)

$d(v_1)$	$d(v_2)$	κ_1	κ_2	κ_3	deficit	
3	3	0	6	0	12 (9)	(Note)
4	3	2	0	0	15	
3	4	0	0	7	16 (13)	(Note)
5+	3	2	2	0	16	
3	5+	0	2	2	16	
4	4	2	0	7	13	
4	5+	2	4	2	17	
5+	4	2	4	7	12	
5+	5+	2	2	2	22	

(ii)

$d(v_1)$	$d(v_2)$	κ_1	κ_2	κ_3	deficit	
3	3	0	0	0	12	
4	3	0	0	0	17	
3	4	0	1	4	12	(Note)
5+	3	2	2	0	16	
3	5+	0	2	2	16	
4	4	0	7	0	15	
4	4	0	0	4	18	
4	5+	0	4	2	19	
5+	4	2	4	4	15	
5+	5+	2	2	2	22	

(iii)

$d(v_1)$	$d(v_2)$	κ_1	κ_2	κ_3	deficit	
3	3	0	6	0	12	(Note)
4	3	2	0	0	15	
3	4	0	0	7	16	(Note)
5+	3	2	2	0	16	
3	5+	0	2	2	16	
4	4	2	0	7	13	
4	5+	2	4	2	17	
5+	4	2	4	7	12	
5+	5+	2	2	2	22	

(iv)	$d(v_1)$	$d(v_2)$	κ_1	κ_2	κ_3	deficit	
	3	3	0	0	0	12	
	4	3	4	1	0	12	(Note)
	3	4	0	0	0	17	
	5+	3	2	2	0	16	
	3	5+	0	2	2	16	
	4	4	4	0	0	18	
	4	4	0	7	0	15	
	4	5+	4	4	2	15	
	5+	4	2	4	0	15	
	5+	5+	2	4	2	20	

Now let $m = 4$ and consider Figure 7.11. Checking Figures 4.33, 5.1–5.5, 5.8 and 5.9 and Lemma 5.4 shows that $\kappa_1 + \kappa_2 + \kappa_3 + \kappa_4 \leq \frac{11\pi}{15}$ for (xiv); and $\kappa_1 + \kappa_2 + \kappa_3 + \kappa_4 \leq \frac{9\pi}{15}$ in all other cases. Indeed the upper bounds are shown in the following table.

	κ_1	κ_2	κ_3	κ_4			κ_1	κ_2	κ_3	κ_4	
(v)	b_1	b_2	6	2	16	(xiii)	x_1	x_2	x_3	x_4	18
(vi)	3	d_1	d_2	4	17	(xiv)	e_1	e_2	e_1	e_2	22
(vii)	2	6	7	2	17	(xv)	y_1	y_2	y_3	y_4	17
(viii)	3	7	4	2	16	(xvi)	a_1	a_2	a_1	a_2	14
(ix)	7	6	2	2	17	(xvii)	4	7	3	2	16
(x)	4	7	3	2	16	(xviii)	2	6	7	2	17
(xi)	2	2	6	7	17	(xix)	2	6	d_1	d_2	18
(xii)	2	4	7	3	16	(xx)	4	d_1	d_2	3	17

Note that $\kappa_4 \leq 2$ in (vii)–(x), (xvii) and (xviii) follows from the fact that $d(v_3) \geq 4$; $\kappa_1 \leq 2$ in (xi) and (xii) follows from the fact that $d(v_1) \geq 4$; that $x_1 + x_2 + x_3 + x_4 \leq 18$ in (xiii) follows from the fact that $d(v_1) = 3$ implies $\kappa_1 = 0$, $d(v_1) > 3$ implies $\kappa_2 = 5$, $d(v_3) = 3$ implies $\kappa_4 = 0$ and $d(v_3) > 3$ implies $\kappa_3 = 5$; in (xv) the fact that $\kappa_1 > 4$ implies $\kappa_2 = 0$, $\kappa_2 > 4$ implies $\kappa_1 = 0$ or $\kappa_3 = 0$, $\kappa_3 > 4$ implies $\kappa_2 = 0$ or $\kappa_4 = 0$ and $\kappa_4 > 4$ implies $\kappa_3 = 0$ forces $y_1 + y_2 + y_3 + y_4 \leq 17$.

In the cases other than (x) where $\kappa_1 + \kappa_2 + \kappa_3 + \kappa_4 \leq \frac{9\pi}{15}$ if at least one of v_1, v_2, v_3 has degree ≥ 5 then there is a deficit of at least $\left(\frac{2\pi}{3} + \frac{8\pi}{15}\right) - \left(\frac{2\pi}{5} + \frac{9\pi}{15}\right) = \frac{\pi}{5}$; and if at least two have degree ≥ 4 then the deficit is at least $\left(\frac{4\pi}{3} + \frac{8\pi}{15}\right) - \left(\pi + \frac{9\pi}{15}\right) = \frac{4\pi}{15}$, so it can be assumed that $(d(v_1), d(v_2), d(v_3)) \in \{(3,3,3), (4,3,3), (3,4,3), (3,3,4)\}$. For cases (vii)–(ix) and (xvii)–(xviii), $d(v_3) = 4$ and checking shows that $\kappa_1 + \kappa_2 + \kappa_3 + \kappa_4 \leq \frac{4\pi}{15}$ which gives a deficit of at least $\frac{13\pi}{30}$. For (xi) and (xii), $d(v_1) = 4$ and $\kappa_1 + \kappa_2 + \kappa_3 + \kappa_4 \leq \frac{4\pi}{15}$ which gives a deficit of at least $\frac{13\pi}{30}$.

Tables for (v), (vi), (xiii), (xv), (xvi), (xix) and (xx) are as follows.

	$d(v_1)$	$d(v_2)$	$d(v_3)$	κ_1	κ_2	κ_3	κ_4	def	
(v)	3	3	3	0	4	6	0	6	
	4	3	3	7	0	6	0	8	
	3	4	3	0	0	0	0	21	
	3	3	4	0	4	0	2	15	
(vi)	3	3	3	0	5	0	0	11	
	4	3	3	0	0	0	0	21	
	3	4	3	0	0	0	0	21	
	3	3	4	0	5	2	4	10	
(xiii)	3	3	3	0	2	2	0	12	
	4	3	3	2	0	2	0	17	
	3	4	3	0	9	0	0	12	
	3	4	3	0	0	9	0	12	
	3	3	4	0	2	0	2	17	
(xv)	3	3	3	0	6	6	0	6	(Note)
	4	3	3	7	0	6	0	8	
	3	4	3	0	0	0	0	21	
	3	3	4	0	6	0	7	8	
(xvi)	3	3	3	0	0	0	0	16	
	4	3	3	0	0	0	0	21	
	3	4	3	0	2	2	0	17	
	3	3	4	0	0	0	0	21	
(xix)	3	3	3	0	6	4	0	6	
	4	3	3	2	0	4	0	15	
	3	4	3	0	0	0	0	21	
	3	3	4	0	6	0	7	8	

(xx)	3	3	3	0	0	5	0	11
	4	3	3	4	2	5	0	10
	3	4	3	0	0	0	0	21
	3	3	4	0	0	0	0	21

For these cases it remains to explain the first row for (xv).

Consider (xv) with $d(v_1) = d(v_2) = d(v_3) = 3$. Then $\kappa_1 = \kappa_4 = 0$, $\kappa_2 \leq \frac{\pi}{5}$ and $\kappa_3 \leq \frac{\pi}{5}$. If $\kappa_1 + \kappa_2 \leq \frac{\pi}{3}$ then the deficit is at least $\frac{\pi}{5}$ so assume otherwise. If $\hat{\Delta}$ receives less than $\frac{\pi}{5}$ from each of Δ_2 and Δ_3 then deficit $\geq \frac{\pi}{5}$, so assume otherwise. If $\hat{\Delta}$ receives $\frac{\pi}{5}$ from Δ_2 then Δ_2 is given by Δ of Figure 4.1(v) or Figure 4.2(iv). But if Δ_2 is Δ of Figure 4.2(iv) then $\hat{\Delta}$ does not receive any curvature from Δ_3 and we are done; and if Δ_2 is Δ of Figure 4.1(v) then according to Configuration D, $\hat{\Delta}$ receives $\frac{3\pi}{10}$ from Δ_0. If $\hat{\Delta}$ receives $\frac{\pi}{5}$ from Δ_3 then Δ_3 is given by Figure 4.1(v) or Figure 4.3(vi), (vii). But if Δ_3 is Δ of Figure 4.3(vi), (vii) then $\hat{\Delta}$ does not receive any curvature from Δ_2 and we are done; and if Δ_3 is Δ of Figure 4.1(v) then according to Configuration C, $\hat{\Delta}$ receives $\frac{3\pi}{10}$ from Δ_5. It now follows that deficit $\geq \frac{\pi}{5}$ as required.

For case (x), $d(v_3) \geq 4$. If at least one of v_1 or v_2 has degree ≥ 4 then the deficit is at least $\frac{\pi}{3}$; if $d(v_1) = d(v_2) = 3$ and $d(v_3) \geq 5$ then $\kappa_3 = 2$ and the deficit is at least $\frac{3\pi}{10}$; and if $d(v_1) = d(v_2) = 3$ and $d(v_3) = 4$ then $\kappa_1 = \kappa_2 = 0$ and the deficit is $\frac{8\pi}{15}$.

Finally the table for (xiv) is given below.

(xiv)	3	3	3	0	2	2	0	12	
	4	3	3	e_1	e_2	2	0	8	
	3	4	3	0	0	0	0	21	
	3	3	4	0	2	e_1	e_2	8	
	5+	3	3	2	2	2	0	18	
	3	5+	3	0	2	2	0	20	
	3	3	5+	0	2	2	2	18	
	4	4	3	c_1	c_2	0	0	17	
	4	3	4	e_1	e_2	e_1	e_2	6	(Note)
	3	4	4	0	0	c_1	c_2	17	

Since, as mentioned earlier, $\kappa_1 + \kappa_2 + \kappa_3 + \kappa_4 \leq \frac{11\pi}{15}$, if there is a vertex of degree ≥ 4 and one of degree ≥ 5 it follows that the deficit is at least $\frac{7\pi}{30}$; and if there are at least three vertices of degree ≥ 4 then the deficit $\geq \frac{3\pi}{10}$ and so to complete the proof it remains to explain the penultimate row for (xiv), that is, case (xiv) with $d(v_1) = d(v_3) = 4$ and $d(v_2) = 3$. If $\kappa_1 + \kappa_2 = \frac{\pi}{3}$ and $\kappa_3 + \kappa_4 = \frac{\pi}{3}$ then deficit $\geq \frac{\pi}{5}$, so assume otherwise. If

$\kappa_1 + \kappa_2 > \frac{\pi}{3}$ then the only way this can occur (see Figure 5.10) is if $\kappa_1 = \frac{2\pi}{15}$ and $\kappa_2 = \frac{7\pi}{30}$ forcing Δ_1 to be given by Δ of Figure 4.15(v) and Δ_2 to be given by Δ of Figure 4.7(xiii). But either this gives Configuration B of Figure 4.28, a contradiction or $d(u) > 4$ in Figure 4.28 and $\frac{\pi}{3} - \frac{7\pi}{30} = \frac{3\pi}{30}$ is added to the deficit. If $\kappa_3 + \kappa_4 > \frac{\pi}{3}$ then the only way this can occur is if $\kappa_3 = \frac{7\pi}{30}$ and $\kappa_4 = \frac{2\pi}{15}$ forcing Δ_3 to be given by Δ of Figure 4.15(vi) and Δ_4 to be given by Δ of Figure 4.7(xii). But either this gives Configuration A of Figure 4.27 or $d(u) > 4$ in Figure 4.27 and $\frac{3\pi}{30}$ is added to the deficit. It now follows that deficit $\geq \frac{\pi}{5}$, as required. □

8 Proof of Theorem 1.2

In section 3 it was assumed by way of contradiction that there is a reduced spherical picture $\boldsymbol{P}$ over $\mathcal{P}_n$. The dual $\boldsymbol{D}$ of $\boldsymbol{P}$ was amended to produce $\boldsymbol{K}$ and the map $\boldsymbol{K}_0$ is a connected component of $\boldsymbol{K}$.

Let $\boldsymbol{K}_0 = \boldsymbol{K}$. Using the same curvature method as in [8] we obtain $c(\boldsymbol{K}_0) = \sum_{\Delta \in \boldsymbol{K}_0} c(\Delta) = 4\pi$. If $c(\Delta) \leq 0$ for each $\Delta \in \boldsymbol{K}_0$ this immediately yields a contradiction. If $c(\Delta) > 0$ then $d(\Delta) = 4$ and $c(\Delta)$ is distributed to a near region $\hat{\Delta}$ as described in Section 4. If $d(\hat{\Delta}) = 4$ then, by Lemma 4.1, either $c^*(\hat{\Delta}) \leq 0$ or $c^*(\hat{\Delta})$ is distributed to a region of degree > 4 or $c^*(\hat{\Delta})$ is distributed to a region Δ' of degree 4 where $c^*(\Delta') \leq 0$ or $c^*(\Delta') > 0$ and is distributed to a region of degree > 4. An immediate consequence (see Proposition 4.2) is that $c(\boldsymbol{K}_0) \leq \sum_{\hat{\Delta} \in \boldsymbol{K}_0} c^*(\hat{\Delta})$ where the sum is taken over regions of $\hat{\Delta}$ of degree ≥ 6. If $d(\hat{\Delta}) = 6$ then, by Lemma 5.2, either $c^*(\hat{\Delta}) \leq 0$ or $c^*(\hat{\Delta})$ is distributed to a region of degree ≥ 8 or $c^*(\hat{\Delta})$ is distributed to a region Δ' of degree 6 and $c^*(\Delta') \leq 0$. An immediate consequence (see Proposition 5.3) is that $c(\boldsymbol{K}_0) \leq \sum_{\hat{\Delta} \in \boldsymbol{K}_0} c^*(\hat{\Delta})$ where each $\hat{\Delta}$ has degree ≥ 8 and has received positive curvature possibly from regions of degree 4 or 6. Finally in Sections 6 and 7 it is shown that if $d(\hat{\Delta}) \geq 8$ then $c^*(\hat{\Delta}) \leq 0$, a contradiction to $c(\boldsymbol{K}_0) = 4\pi$ that yields the result.

Now suppose that $\boldsymbol{K}_0 \neq \boldsymbol{K}$. In this case (as described at the end of Section 3) delete all vertices and edges in $\boldsymbol{K} \backslash \boldsymbol{K}_0$ to produce a tessellation $\boldsymbol{K}_1$ of S^2 consisting of the union of $\boldsymbol{K}_0$ and a single distinguished region Δ_0, say. Then $c(\boldsymbol{K}_0) + c(\Delta_0) = 4\pi$. Apply the same distribution of curvature as in the above to $\Delta \neq \Delta_0$ with the difference being that if at any stage positive curvature is transferred to Δ_0 then it remains with Δ_0. It follows in exactly the same way (see Propositions 4.2 and 5.3) that $4\pi = c(\boldsymbol{K}_1) = c(\boldsymbol{K}_0) + c(\Delta_0) \leq c^*(\Delta_0)$. But if $d(\Delta_0) = k$ then, since $\frac{\pi}{3}$ is the maximum amount of curvature transferred across any edge, $c(\Delta_0) \leq (2-k)\pi + k\left(\frac{2\pi}{3}\right) + k\left(\frac{\pi}{3}\right) = 2\pi$. This final contradiction proves the theorem.

9 References

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10 Appendix

In what follows we use LEC to denote a length contradiction as defined in Section 3; and we use LAC to denote labelling contradiction, which will often be a basic labelling contradiction corresponding to Figure 3.4.

Proof of Lemma 3.4 The proof is immediate for $n = 3, 4$ and has been given for $n = 6$ in Section 3 so let $n = 5$ and $\hat{\Delta}$ be given by Figure A.1(i). If $\hat{\Delta}$ contains no shadow edges then LEC. If $\hat{\Delta}$ contains exactly one shadow edge up to symmetry it is (13) and this yields LEC. Up to symmetry this leaves the case (13), (14) forcing LEC.

Let $n = 7$ and $\hat{\Delta}$ be given by Figure A.1(ii). If only (13) or only (14) or only (14), (15) occurs this forces LEC so it can be assumed without any loss that (13), (14) occurs. If there are no more shadow edges then LEC; if exactly one more then it is one of (15), (16), (46), (47), (57) and each forces LEC; and if two more then they are one of the pairs (15), (16); (15), (57); (16), (46); (46), (47); or (47), (57). But (15) yields LAC; (16), (46) yields LAC; (46), (47) yields LAC; and (47), (57) yields LEC.

Let $n = 8$ and $\hat{\Delta}$ be given by Figure A.1(iii). If there are no shadow edges then $\hat{\Delta}$ is given by Figure 3.6(iv). If (15) occurs only then $\hat{\Delta}$ is given by Figure 3.6(v). If there are now no shadow edges $(i\ i+2)$ (subscripts modulo 8) then up to symmetry one of (14); (14), (15); (14), (16); (14), (58); (14), (15), (16); or (14), (15), (58) occurs. But (14), (15), (16) yields LAC; (14), (15), (58) is given by Figure 3.6(vi); and the other cases each yield LEC. It can be assumed without loss that (13), (14) occurs. The remaining possible shadow edges are: (15), (16), (17), (46), (47), (48), (57), (58) and (68). If (15) then LAC so assume otherwise. No more shadow edges yields LEC. Exactly one more shadow edge yields LEC except for (47) given by Figure 3.6(vii) or (16) given by Figure 3.6(viii) or (58) given by Figure 3.6(x). If there are exactly two more shadow edges then it is one of the pairs (16), (17); (16), (46); (16), (68); (17), (46); (17), (47); (17), (57); (46), (47); (46), (48); (46), (68); (47), (48); (47), (57); (48), (58); (48), (68); (57), (58); or (58), (68). But each case forces LEC.

If there are exactly three more shadow edges then it is one of the triples (16), (17), (46); (16), (46), (68); (17), (46), (47); (17), (47), (57); (46), (47), (48); (46), (48), (68); (47), (48), (57); (48), (57), (58); or (48), (58), (68). Each of these forces LAC except for (13), (14), (47), (48), (57) which is given by Figure 3.6(ix).

Finally let $n = 9$ and $\hat{\Delta}$ be given by Figure A.1(iv). If there are no shadow edges then $\hat{\Delta}$ is given by Figure 3.6(xi). If there is exactly one shadow edge then (up to symmetry) it is one of (13), (14) or (15) and each forces LEC. If the girth of $\hat{\Delta}$ together with shadow edges is five then either (15) only or (15), (16) only occurs and each forces LEC. Suppose (14) occurs (the girth four case). If at most one more shadow edge occurs then it is one of (15), (16), (17), (58), (59), (69) and each case forces LEC. If there are exactly two more shadow edges then LAC or one of (15), (17); (15), (58); (15), (59), (15), (69); (16), (17); (16), (69);

$(17),(47)$; $(47),(48)$; $(47),(49)$; $(48),(49)$; $(48),(58)$; or $(59),(69)$ occur forcing LEC. If there are three more then this forces LAC or $(14),(15),(59),(69)$ which yields LEC.

It can now be assumed without any loss that $(13),(14)$ occurs. If there are no more shadow edges then LEC. If there is exactly one more then it is (15) and LAC or one of (16), (17), (18), (46), (47), (48), (49), (57), (58), (59), (68), (69) or (79) and LEC. If exactly two more then LAC or one of $(16),(17)$; $(16),(18)$; $(16),(46)$; $(16),(68)$; $(16),(69)$; $(16,(79)$; $(17),(18)$; $(17),(46)$; $(17),(47)$; $(17),(57)$; $(17),(79)$; $(18),(46)$; $(18),(47)$; $(18),(48)$; $(18),(57)$; $(18),(58)$; $(18),(68)$; $(46),(47)$; $(46),(48)$; $(46),(49)$; $(46),(68)$; $(46),(69)$; $(46),(79)$; $(47),(48)$; $(47),(49)$; $(47),(57)$; $(47),(79)$; $(48),(57)$; $(48),(58)$; $(48),(68)$; $(49),(57)$; $(49),(58)$; $(49),(59)$; $(49),(68)$; $(49),(69)$; $(49),(79)$; $(57),(58)$; $(57),(59)$; $(57),(79)$; $(58),(59)$; $(58),(68)$; $(59),(68)$; $(59),(69)$; $(59),(79)$; $(68),(69)$; or $(69),(79)$ occur and forcing LEC.

If there are three or four more shadow edges this forces either LAC or one of $(16),(17),(79)$; $(16),(69),(79)$; $(17),(18),(46)$; $(17),(18),(57)$; $(17),(46),(79)$; $(17),(57),(79)$; $(18),(46),(47)$; $(18),(46),(68)$; $(18),(47),(57)$; $(18),(57),(58)$; $(18),(58),(68)$; $(46),(47),(49)$; $(46),(47),(79)$; $(46),(49),(68)$; $(46),(49),(79)$; $(46),(68),(69)$; $(46),(69),(79)$; $(47),(48),(57)$; $(47),(49),(57)$; $(47),(57),(79)$; $(48),(57),(58)$; $(49),(78),(58)$; $(49),(78),(59)$; $(49),(57),(79)$; $(49),(58),(59)$; $(49),(58),(68)$; $(49),(45),(68)$; $(49),(59),(79)$; $(49),(68),(69)$; $(49),(69),(79)$; $(57),(58),(59)$; $(58),(59),(68)$; $(59),(68),(69)$; or $(49),(58),(59),(68)$ occurs each forcing LEC. $\square$

Remark 1 If the corner label at the vertex v of the region $\hat{\Delta}$ is x or y then it follows from equations (3.1) in Section 3 that there must be an odd number of shadow edges in $\hat{\Delta}$ incident at v.

Remark 2 Let v_1, v_2 be vertices of the same b-segment of the region $\hat{\Delta}$. It follows from Remark 1 that there are no shadow edges in $\hat{\Delta}$ from v_1 to v_2.

Proof of Lemma 7.2

(i) Consider the regions $\hat{\Delta}$ of Figure A.2(i)-(ii) in which 2, 6, 10 refer to the (possibly empty) set of vertices between vertices 1 and 3, 5 and 7, 9 and 11.

We write (ab) to indicate that there was a 2-segment between vertices a and b in D with the understanding that if $a = 2$, for example, we mean a vertex belonging to a. By remark 1 above the number of (ab) involving each of 1, 3, 5, 7, 9 and 11 must be odd and at least one. First consider Figure A.2(i). It follows from remark 2 above that if $\{a, b\} \subseteq \{12, 1, 2, 3, 4\}$ or $\{4, 5, 6, 7, 8\}$ or $\{8, 9, 10, 11, 12\}$ then (ab) does not occur. Moreover (18) forces (19), (1 11) and this yields LAC. It follows that the only pairs involving 4, 8 or 12 are (4 10), (28) and (6 12). First assume that none of (35), (79) or (1 11) occur. Then since (15), (16) and (17) each forces (35), and (19), (1 10) each forces (1 11), we get a contradiction. Assume exactly one of (35), (79), (1 11) occurs – without any loss (79). Then again (15), (16) and (17) each force (35), and (19) and (1 10) each force (1 11), a contradiction. Assume exactly two of (35), (75), (1 11) occur – without any loss (35) and (79). Then (19) and (1 10) each forces (1 11), a contradiction; and (16) and (17) each forces LEC at (35) or forces either (52), (52) or (52), (51) or (36), (36) or (36), (37) yielding LAC. This leaves (15). Since the number of (ab) involving 5 must be odd at least one of (59), (5 10) or (5 11) occurs. But (59) forces (11 5) and (5 10); (5 10) forces (11 5) and another (5 10); and (5 11) forces either a length contradiction at (79) or forces (95), (96) or (96), (96) or (7 10), (7 10) or (7 10), (7 11) yielding LAC in all cases. Finally assume that (1 11), (35) and (79) occur. Since the length of each is $n-1$ we must have more pairs otherwise there is a length contradiction. Assume without any loss that 1 is involved in further pairs. Since (16) and (19) each forces either (36), (36) or (36)(37) or (52), (52) or (52)(51) yielding LAC it follows that at least two of (15), (19) and (1 10) occur. However (19), (1 10) and (1 10), (1 10) yield LAC and (15), (19) forces (59) and LAC. This leaves (15), (1 10) together with at least one of (25), (59), (5 10). But (25) yields LAC; (59) forces (19) or (69) and LAC; and finally (5 10) forces either a length contradiction or one of (7 10), (7 10) or (59)(69) or (69)(69) and LAC.

Now consider Figure A.2(ii). Here if $\{a, b\} \subseteq \{3, 4, 5, 6, 7\}$ or $\{7, 8, 9, 10, 11\}$ or $\{11, 12, 13, 1, 2\}$ then (ab) does not occur. First assume that (68) and (10 12) do not occur. Then (69), (6 10) and (6 11) force (68); (6 12) forces (68) or (10 12); (6 13) and (61) each force (68) or (10 12); and (62) forces one of (12 6), (12 7), (12 8) or (12 9) and each forces (68) or (10 12) – in all cases a contradiction.

Now assume that exactly one of (68) and (10 12) occurs – without any loss (68). Since this segment has length $n-1$ this forces at least one of (69), (6 10), (58), (48) to occur. If (68) and (69) occur then at least one (69), (6 10), (6 11), (6 12), (6 13), (61) or (62) occurs. But (69) and (6 10) yields LAC; and each of (6 11), (6 12), (6 13), (61) and (62) forces (6 10) or (10 12) a contradiction. If (68) and (6 10) occurs then at least one of (6 11), (6 12), (6 13), (61) and (62) occurs. But (6 11) yields LAC and the rest force (10 12) or (6 12) and LAC. If (68) and (58) occur then at least one of (85), (84), (83), (82), (81), (8 13) and (8 12) occurs. But (85) and (84) yield LAC; (83) forces (84); and each of the rest forces (10 12). If (68) and (48) occur then at least one of (83), (82), (81), (8 13) or (8 12) occurs. But (83) yields LAC and the rest force (10 12).

Finally assume that (68) and (10 12) occur. Then length implies that at least one of (69), (6 10), (85), (84) occurs and at least one of (10 13), (10 1), (12 9), (12 8) occurs. Let (69) and (10 13) occur. Then at least one of (10 13), (10 1), (10 2), (10 3), (10 4), (10 5) and (10 6) also occurs. But (10 13) and (10 1) yield LAC; (10 2) forces (10 1); and each of (10 3), (10 4) and (10 5) forces another (69) or (6 10) and LAC. Let (69) and (10 1) occur. Then at least one of (10 2), (10 3), (10 4) and (10 5) occurs. But (10 2) yields LAC; and each of (10 3), (10 4) and (10 5) forces (69) and LAC. Let (69) and (12 9) occur. Then at least one of (6 12), (6 13), (61) and (62) occurs. But (6 12) yields LAC; and (6 13), (61) and (62) each forces either (6 12) or (12 9) and LAC. The segments (69) and (12 8) cannot both occur. Let (6 10) and (10 13) occur. Then at least one of (10 13), (10 1), (10 2), (10 3), (10 4) and (10 5) must also occur. But (10 13) and (10 1) each force LAC; (10 2) forces (10 1); and each of (10 3), (10 4) and

(10 5) forces (69), and LAC. Let (6 10) and (10 1) occur. Then at least one of (10 2), (10 3), (10 4) and (10 5) occurs and there is a contradiction as in the subcase above. The segment (6 10) cannot occur with (12 9) or (12 8). The subcases (84), (12 9); (84), (12 8); (85), (12 9); and (85), (12 8) follow by symmetry. Let (84) and (10 13) occur. Then at least one of (85), (83), (82), (81) and (8 13) occurs. But (85) and (83) yield LAC; and the others force either (10 13) or (10 1) and LAC. Let (84) and (10 1) occur. Then at least one of (85), (83), (82) and (81) occurs and similarly there is a labelling contradiction. Let (85) and (10 13) occur. Then at least one of (85), (84), (83), (82), (81) and (8 13) also occurs. But (85), (84) and (83) yield LAC; and the rest forces either (10 13) or (10 1) and LAC. Finally if (85) and (10 1) occur then at least one of (85), (84), (83), (82) and (81) occurs and similarly there is a labelling contradiction.

(ii) Let $m = 2$. Then $\hat{\Delta}$ is given by Figure A.3(i). If $n = 2$ as in Figure A.3(ii) then $(1b)$ is forced yielding (2?); and if $n = 3$ as in Figure A.3(iii) then $(1b)$ forces (2?) and $(1c)$ forces $(3a)$ and (4?). Let $n = 4$ as in Figure A.3(iv). Then $(1b)$ forces (2?) and $(1d)$ forces $(3a), (4?)$ so $(1c)$ must occur. This forces $(2c)$ and no vertices between 1 and 2 otherwise LAC. If $(3a)$ then (4?) so must have $(3c)$ and $(4c)$. Thus there are no vertices between 3 and 4, otherwise LAC, and so $d(\hat{\Delta}) = 8$, a contradiction. Let $n = 5$ as in Figure A.3(v). If $(1b)$ then (2?) and if $(1e)$ then $(3a)$ and (4?) so must have $(1c)$ or $(1d)$. Suppose $(1c)$ occurs. This forces $(2c)$ and no vertices between 1 and 2. If $(4d)$ then $(3d)$ must occur and no vertices between 3 and 4. But then $d(\hat{\Delta}) = 9$, a contradiction, so must have $(4c)$ since $(4e)$ forces (3?). If now $(3c)$ then $d(\hat{\Delta}) = 9$ as before so $(3d)$ must occur. If (ac) occurs or there are at least three vertices between 3 and 4 then LAC, so assume otherwise. If 5 is the only vertex between 3 and 4 then $(5c)$ yields LEC and $(5d)$ yields LAC; and if 5 and 6 are between 3 and 4 then must have $(5d)$ and $(6c)$ otherwise LAC. There are no more shadow edges and the resulting region yields LEC. Now suppose $(1d)$ occurs. This forces $(3d), (4d)$ and no vertices between 3 and 4. If $(2d)$ then $d(\hat{\Delta}) = 9$ so $(2c)$ occurs. By symmetry we can now argue as in the above to obtain a contradiction.

Finally let $n = 6$ as in Figure A.3(vi). Note that if $v \notin \{a, b, c, d, e, f\}$ then $i\deg(v) = 1$ otherwise LAC. If $(1b)$ then (2?); if $(1f)$ then $(3a)$ and (4?); if $(4f)$ then (3?); and if $(4b)$ then $(2a)$ and (1?). Thus there are six cases: $(1c), (4c)$; $(1c), (4d)$; $(1c), (4e)$; $(1d), (4d)$; $(1d), (4e)$; and $(1e), (4e)$.

Consider $(1c), (4c)$. This forces $(2c)$ and no vertices between 1 and 2. If (ac) then LAC. If $(3c)$ then there are no vertices between 3 and 4 and LAC so must have $(3d)$ or $(3e)$. Suppose $(3d)$. If there are at least three vertices between 3 and 4 then LAC; and if there are less than three this forces LEC in each case. Suppose $(3e)$. Then $(5e)$ must occur for some vertex 5 between 3 and 4 otherwise LEC. If there are no other vertices between 3 and 4 then LEC; and if there is at least one more vertex between 3 and 4 then LAC.

Consider $(1c), (4d)$. This forces $(2c)$ and no vertices between 1 and 2. Observe that

(ac) forces LAC. There must be $(3d)$ or $(3e)$. Suppose $(3d)$ occurs. Then there are no vertices between 3 and 4. If (da) or (df) then LAC so $\hat{\Delta}$ has no more vertices and this yields LEC. Suppose $(3e)$ occurs. Then must have $(5e)$ where 5 is a vertex between 3 and 4, otherwise LAC. If there are no other vertices between 3 and 4 then either (da) occurs or does not occur, but in both cases there is LEC; if there is exactly one more, 6 say, then $(6e)$ yields LAC and if $(6d)$ then (da) yields LAC and if not (da) then LEC.

Consider $(1c), (4e)$. This forces $(2c), (3e)$, no vertices between 1 and 2 and no vertices between 3 and 4. If there are no other shadow edges then LEC so at least one of (ac), (ad), (ae) or (ce) occurs. But (ac), (ad) or (ae) forces LAC; and (ce) yields LEC.

Consider $(1d), (4d)$. If (ad) then LAC. By symmetry there are three cases: $(2c), (3e)$; $(2d), (3d)$; and $(2c), (3d)$. If $(2c), (3e)$ then length forces at least one vertex between $1, 2$ and $3, 4$ and labelling implies at most two. If there is exactly one vertex 5, say, between 1 and 2 and one vertex 6, say, between 3 and 4 then must have $(5c), (6e)$ yielding LEC; and if exactly two vertices, between 1 and 2 and either one or two vertices between 3 and 4 then LAC. If $(2d), (3d)$ then there are no vertices between 1 and 2 or 3 and 4 yielding LAC. If $(2c), (3d)$ then there are no vertices between 3 and 4; (df) yields LAC; and there is at least one vertex 5 say between 1 and 2 with $(5c)$, otherwise LEC. If only 5 occurs between 1 and 2 then this forces LEC; and if there are any more vertices between 1 and 2 this forces LAC.

Case $(1d), (4e)$ is the same as $(1c), (4d)$ by symmetry; and case $(1e), (4e)$ is the same as $(1c), (4c)$ by symmetry.

Let $m = 3$. Then $\hat{\Delta}$ is given by Figure A.4(i). If $n = 3$ as in Figure A.4(ii) then $(1c)$ forces $(2?)$ and $(3b)$ forces $(4?)$. So must have either $(1d)$ or $(1b)$. Suppose $(1d)$ occurs. This forces $(2d)$, $(3a)$, $(4a)$ and no vertices between 1 and 2 or between 3 and 4 otherwise LAC. But then $d(\hat{\Delta}) = 8$, a contradiction. Suppose $(1b)$ occurs. This forces $(3c), (4c), (2b)$, no vertices between 1 and 2 or between 3 and 4 and again $d(\hat{\Delta}) = 8$.

Let $n = 4$ as in Figure A.4(iii). If $(1c)$ then $(2?)$ and if $(4e)$ then $(3?)$. The possible cases up to symmetry are $(1b), (4c)$; $(1b), (4d)$; and $(1d), (4d)$.

Consider $(1b), (4c)$. This forces $(2b), (3d)$ and no vertices between 1 and 2. If (bc) then LAC. If there are no vertices between 3 and 4 then $d(\hat{\Delta}) = 9$, a contradiction; if there is exactly one vertex, 5 say, between 3 and 4 then must have $(5d)$ otherwise LEC, there are no more shadow edges and $\hat{\Delta}$ is given by Figure 7.3(iv); and if there are at least two vertices between 3 and 4 then LAC.

Consider $(1b), (4d)$. This forces $(3d)$ and no vertices between 3 and 4. If now $(2b)$ then there are no vertices between 1 and 2 so $d(\hat{\Delta}) = 9$, a contradiction. This leaves $(2d)$ and either LEC or there are two vertices, 5 and 6 say, between 1 and 2 with $(5b)$ and $(6d)$. Any other shadow edges yields LAC and no more shadow edges yields LAC.

Consider $(1d), (4d)$. This forces $(2d), (3d)$, no vertices between 1 and 2 or between 3 and 4 and so $d(\hat{\Delta}) = 9$, a contradiction.

Let $n = 5$ as in Figure A.4(iv). As before $(1c)$, $(2a)$, $(3b)$ and $(4f)$ yield contradictions. Up to symmetry the cases are $(2b), (3c)$; $(2b), (3d)$; $(2b), (3e)$; $(2d), (3d)$; and $(2d), (3e)$.

Consider $(2b), (3c)$. This forces $(1b), (4c)$ and no vertices between 1 and 2 or between 3 and 4. If (bc) occurs then LAC otherwise there is still a labelling contradiction.

Consider $(2b), (3d)$. This forces $(1b)$ and no vertices between 1 and 2. The subcases are $(4c)$ and $(4d)$. Consider first $(4d)$. Then there are no vertices between 3 and 4, (bc) yields LAC, (bd) yields LAC and (df) yields LAC. Therefore there are no more shadow edges and this yields LEC. Now suppose $(4c)$ occurs. If (bc) then LAC and any other shadow edge incident at vertex c forces LAC. Suppose that there are no vertices between 3 and 4. If (df) does not occur then LEC and if (df) occurs then $\hat{\Delta}$ is given by Figure 7.3(v). If there is more than one vertex between 3 and 4 then LAC and if there is exactly one, 5 say, then must have $(5d)$ yielding LEC with or without (df).

Consider $(2b), (3e)$. This forces $(1b)$ and no vertices between 1 and 2. The subcases are $(4c)$, $(4d)$ and $(4e)$. Consider first $(4c)$. If (bc) then LAC; or if there any further shadow edges at c then LAC. If $i\deg(3) > 1$ then LAC so there must be a vertex, 5 say, between 3 and 4 and $(5e)$ otherwise LEC. If there are no more shadow edges then LEC; if (ce) occurs then LEC; and if there is either one more vertex, 6 say between 4 and 5 with $(6d)$ or two more, 6 and 7 say, between 4 and 5 with $(6d), (7d)$ then this forces LEC in both cases. Now suppose $(4d)$ occurs. If (bc) then LAC or if $(4c)$ and $(4e)$ then LAC. If there is exactly one vertex, 5 say, between 3 and 4 then $(5e)$ occurs, otherwise LEC. If there are no more shadow edges then LEC and the only other possibility is (db) and again LEC; and if there are exactly two vertices, 5 and 6, say between 3 and 4 then must have $(5e)$ and $(6d)$. Any further shadow edges yields LAC and no more yield LEC. Now suppose $(4e)$ occurs. This forces $(3e)$ and no vertices between 3 and 4. Now (bc) and (be) each yield LAC and if (ce) then LEC. If there are no more shadow edges then LEC or if (bd) then LAC.

Consider $(2d), (3d)$. The subcases are $(1d), (4d)$; $(1d), (4a)$; and $(1b), (4d)$. Suppose $(1d), (4d)$ occurs. Then there are no vertices between 1 and 2 or between 3 and 4, and each of (df), (da) and (db) forces LAC. Thus there are no more shadow edges and this forces LEC. Suppose $(1d), (4a)$ occurs. Then there are no vertices between 1 and 2. If there are no vertices between 3 and 4 then LEC or if there are at least three vertices between 3 and 4 then LAC. If there is exactly one vertex, 5 say, between 3 and 4 then must have $(5a)$ otherwise LEC, and if (da) then LAC or if not (da) then LAC. If there are two vertices, 5 and 6 say, between then must have $(5a)$ and $(6d)$. Again if (da) then LAC and if not (da) then LAC. Suppose $(1b), (4d)$ occurs. There are no vertices between 3 and 4 and either LEC or there are vertices 5 and 6 between 1 and 2 with $(5b)$ and $(6d)$. If now (bd) then LAC or if not (bd) then LAC.

Consider $(2d), (3e)$. Up to symmetry the subcases are $(1b), (4d)$; $(1b), (4e)$; $(1d), (4d)$;

and $(1d), (4e)$. Suppose $(1b), (4d)$ occurs. Then either LEC or there are vertices 5 and 6 between 1 and 2 with $(5b), (6d)$. If now (bd) then LAC. There must be a vertex 7 between 3 and 4 with $(7e)$ otherwise LEC. If there are no more shadow edges then LEC; otherwise $(d8)$ occurs where 8 is between 7 and 4, and this yields LEC. Suppose $(1b), (4e)$ occurs. Then there are no vertices between 3 and 4, and either LEC or there are variances 5 and 6 between 1 and 2 with $(5b)$ and $(6d)$. If now (bd) then LAC. If there are no more shadow edges then LEC; if there is one more shadow edge $(e7)$ where 7 lies between 5 and 6 then LAC; and if there is a further shadow edge $(e8)$ where 8 lies between 7 and 6 then again LAC. Suppose $(1d), (4d)$ occurs. Then there are no vertices between 1 and 2. If (ad) then LAC. There must be $(5e)$ where vertex 5 is between 3 and 4 otherwise LEC. If (bd) then LAC; if no other shadow edges then LEC; and if $(d6)$ where 6 lies between 5 and 4 then LEC.

Finally let $m = n = 4$ as shown in Figure A.4(v). Up to symmetry the subcases are $(1b), (4b), (2e), (3e)$; $(1b), (4b), (2b), (3d)$; $(1b), (4b), (2b), (3c)$; $(1b), (4b), (2b), (3b)$; $(1b), (3e), (4e), (2b)$; $(1b), (3e), (4d), (2c)$; $(1b), (3e), (4e), (2c)$; $(1c), (3d), (2c), (4d)$; $(1b), (3d), (4d), (2b)$; $(1b), (3d), (4d), (2c)$. Note that $(1d)$, $(2a)$, $(3c)$ and $(4f)$ each yield a contradiction.

Suppose that $i\deg(1) > 1$. Then $i\deg(1) = 3$ and the three cases are $(1b), (1c), (1e)$; $(1b), (1e), (1f)$; and $(1c), (1e), (1f)$. Let $(1b), (1c), (1e)$ occur. This forces $(2e), (3e), (4e)$ and no vertices between 1 and 2 or between 3 and 4. If (ce) then LAC and if not (ce) then LEC. Let $(1b), (1e), (1f)$ occur. This forces $(3b), (4b)$ and LEC. If $(1c), (1e), (1f)$ occurs this forces $(3c)$ and $(4?)$. Now suppose that $i\deg(5) > 1$ where 5 is a vertex between 1 and 2. Up to symmetry this forces $(5b), (5e), (5f)$ and then $(2e), (1b), (3b), (4b)$ with no more vertices between 1 and 2 and no vertices between 3 and 4. If (bf) then LAC and not (bf) yields LEC.

By symmetry it can be assumed from now on that $i\deg(P) = 1$ where $P \in \{1, 2, 3, 4\}$ or P is a vertex between 1 and 2 or between 3 and 4.

Consider $(1b), (4b), (2e), (3e)$. This forces $(5e), (6b), (7e), (8b)$ where $5, 6$ lie between 1 and 2 and $7, 8$ lie between 3 and 4. If now (be) then LAC and if not (be) then again LAC.

Consider $(1b), (4b), (2b), (3d)$. This forces $(5b)$ where 5 lies between 3 and 4 and there are no vertices between 1 and 2. If (bd) then LAC so assume not (bd). If there are no more shadow edges then LEC. The remaining possibilities are: $(d6)$ only where 6 lies between 5 and 3, yielding LAC; (df) only, yielding LAC; and $(d6), (df)$ yielding LAC.

Consider $(1b), (4b), (2b), (3e)$. This forces $(5e), (6b)$ where $5, 6$ lie between 3 and 4; and there are no vertices between 1 and 2. If (bd) then LAC; if (be) then LAC; if $(7d)$ then LAC or if $(7d), (8d)$ then LAC, where $7, 8$ lie between 5 and 6.

Consider $(1b), (4b), (2b), (3b)$. If there are no more shadow edges then LEC; if (bd) or (bf) then LAC; if (df) then LEC; and if (be) then LAC.

Consider $(1b), (3e), (4e), (2b)$. If there are no more shadow edges then LEC; if (bd) or (ce) then LAC; if (be) then LAC; and if (cd) only then this is $\hat{\Delta}$ of Figure 7.3(vi).

Consider $(1b), (3e), (4d), (2c)$. If there are no more shadow edges then LAC; if $(c7)$ or $(d8)$ only then LEC; if $(c7), (d8)$ only then LAC; if $(c7), (cd)$ or $(d8), (cd)$ only then LAC; if $(c7), (d8), (cd)$ then LAC; and if (cd) only then we obtain $\hat{\Delta}$ of Figure 7.3(vii).

Consider $(1b), (3e), (4e), (2c)$. This forces $(5b)$ where 5 lies between 1 and 2 and there are no vertices between 3 and 4. If (ec) then LAC; if $(c6)$ only then LEC, where 6 lies between 5 and 2; if (cd) only then LEC; and if $(c6), (cd)$ then LAC.

Consider $(1c), (3d), (2c), (4d)$. Then there are no vertices between 1 and 2 or between 3 and 4. If (df) or (dc) or (ca) then LAC. If there are no more shadow edges then LAC.

Consider $(1b), (3d), (4d), (2b)$. Then there are no vertices between 1 and 2 or between 3 and 4. If (df) or (dc) or (bd) then LAC. If there are no more shadow edges then LEC.

Finally consider $(1b), (3d), (4d), (2c)$. Then there are no vertices between 3 and 4. There must be $(5b)$ where 5 lies between 1 and 2. If (df) or (cd) then LAC; and if there are no more shadow edges then LEC.

(iii) Let $m = 2$ and so $\hat{\Delta}$ is given by Figure A.3(i). It follows from (ii) that (14) must occur. It can be assumed therefore that $i\deg(1) > 1$ otherwise LEC. If $n = 2$ or 3 then this forces LAC. Let $n = 4$ and so $\hat{\Delta}$ is given by Figure A.3(i), (iv). If $(1c)$ does not occur then this forces $(1d)$, (13) and LAC. Let $(1c)$ occur. If $(1d)$ occurs then (13) and LAC is forced so assume $(1d)$ does not occur. Since $(2c)$ is forced there are no vertices between 1 and 2 and this forces LAC. Let $n = 5$ and so $\hat{\Delta}$ is given by Figure A.3(i), (v). Suppose that (13) does not occur. Then must have $(1c), (1d), (3d), (2c)$, no vertices between 1 and 2 and either LEC or $(5d)$ where 5 lies between 3 and 4 which yields LAC. Let (13) occur so that there are no vertices between 3 and 4. If $(1c)$ then $d(\hat{\Delta}) = 9$, a contradiction, if $(1e)$ then LAC, so assume $(1d)$ occurs. Then $(2d)$ forces $d(\hat{\Delta}) = 9$ so assume $(2c)$ occurs. This forces LEC or $(5c)$ where 5 lies between 1 and 2. If there are no more shadow edges then LAC; and if $(6d)$ occurs where 6 lies between 1 and 5 then LEC.

Let $m = n = 3$ and so $\hat{\Delta}$ is given by Figure A.4(i), (ii). Up to symmetry there are two cases, namely $i\deg(3) > 1$ and $i\deg(5) > 1$ where 5 lies between 3 and 4. But any triple from $(5a), (51), (56), (57), (52), (5c)$ or from $(3a), (31), (36), (37), (32), (3c)$ yields LAC.

Let $m = 3$ and $n = 4$ and so $\hat{\Delta}$ is given by Figure A.4(i), (iii). It can be assumed without any loss that $i\deg(5) > 1$ or $i\deg(4) > 1$ or $i\deg(3) > 1$ where 5 lies between 3 and 4. Suppose $i\deg(5) > 1$. Let 6 lie between 1 and 2. Then each of the pairs $(5a), (56)$; $(5a), (52)$; $(5a), (5c)$; $(51), (5c)$; $(56), (5c)$; and $(52), (5c)$ forces LAC. If $i\deg(5) > 3$ then LAC. This leaves the cases $(5a), (51), (5d)$; $(51), (56), (52)$; $(51), (56), (5d)$; $(51), (52), (5d)$; $(56), (52), (5d)$; and $(56), (57)$ where 7 lies between 6 and 2. If $(5a),(51),(5d)$ occurs then LEC or $(2d)$ and (26), and LAC. If $(51),(56),(52)$ occurs then LAC. If $(51),(56),(5d)$ occurs this forces $(2d),(27)$ and LAC.

If $(51),(52),(5d)$ occurs this forces $(3d)$. If now $i\deg(1) > 3$ then LAC so $(4a),(a8)$ must occur where 8 lies between 4 and 5. Any further shadow edges yields LAC, and no more yields LEC. If $(56),(52),(5d)$ occurs this forces $(3d)$. But now $i\deg(6) > 1$ yields LAC and $i\deg(6) = 1$ yields LAC. If $(56),(57)$ occurs then this immediately forces LAC except for $(5d)$. But this forces $(2d),(8d)$ and LAC where 8 lies between 7 and 2.

Now suppose $i\deg(4) > 1$. Similarly to the above the cases are $(4a),(41),(4d)$; $(41),(46),(42)$ yielding LAC; $(41),(46),(4d)$; $(41),(42),(4d)$; $(46),(42),(4d)$; and $(46),(47)$ where 6 lies between 1 and 2 and 7 lies between 2 and 6. If $(4a),(41),(4d)$ occurs this forces $(2d),(6d)$ and LAC. If $(41),(46),(4d)$ occurs this forces $(2d),(7d)$ and LAC. If $(41),(42),(4d)$ occurs this forces $(3d)$ and no vertices between 1 and 2 or between 3 and 4. Any other shadow edges yields LAC and no more yields LEC. If $(46),(44),(4d)$ occurs this forces $(3d)$, no vertices between 3 and 4 and LAC. If $(46),(47)$ occurs this forces LAC except possibly for $(4d)$. But $(2d)$ is then forced yielding LEC or $(8d)$, where 8 lies between 7 and 2, yielding LAC.

Suppose $i\deg(3) > 1$. Similarly to the above the cases are $(3a),(31),(3d)$; $(31),(36),(3c)$ yielding LAC; $(31),(36),(3d)$; $(31),(32),(3d)$; $(36),(32),(3d)$; and $(36),(37)$. If $(3a),(31),(3d)$ occurs this forces $(4a)$ and LAC. If $(31),(36),(3d)$ occurs this forces $(2d)$ and LAC. If any of the remaining cases occur they immediately force LEC or LAC.

Let $m = 3$, $n = 5$ and so $\hat{\Delta}$ is given by Figure A.4(i), (iv). Again up to symmetry the cases are $i\deg(5) > 1$ or $i\deg(4) > 1$ or $i\deg(3) > 1$ where 5 has between 3 and 4.

Suppose $i\deg(5) > 1$. The pairs $(5a),(56)$; $(5a),(52)$; $(5a),(5c)$; $(51),(5c)$; $(56),(5c)$ where 6 has between 1 and 2 each force LAC. If $(51),(52)$ or $(51),(56)$ occurs this forces $(4a)$ then LEC or $(4a),(a8)$, where 8 has between 4 and 5, then LAC. If $(5a),(5e)$ occurs this forces $(3d),(4a)$ and LAC. This leaves the triples $(56),(57),(5d)$; $(56),(57),(5e)$; $(5a),(51),(5d)$; $(51),(5d),(5e)$; $(56),(52),(5d)$; $(56),(52),(5e)$; $(56),(5d),(5e)$; $(52),(5c),(5d)$ and LAC; $(52),(5c),(5e)$; $(52),(5d),(53)$; $(5c),(5d),(5e)$ and LAC, where 6 lies between 1 and 2 and 7 lies between 1 and 6. If $(56),(57),(5d)$ or $(5a),(51),(5d)$ occurs this forces $(2d)$ and LEC or $(2d),(d8)$, where 8 lies between 6 and 2, and LAC. If $(56),(57),(5e)$ or $(56),(52),(5e)$ or $(52),(5c),(5e)$ occurs this forces $(3e)$ and LAC. If $(51),(5d),(5e)$ occurs this forces $(4a)$ and LEC or $(4a),(a8)$, where 8 lies between 4 and 5, and LAC. If $(56),(5d),(5e)$ occurs this forces $(2d),(d8)$, where 8 lies between 6 and 2, and LAC. If $(52),(5d),(5e)$ occurs this forces $(3e)$ and LEC. Finally suppose that $(56),(52),(5d)$ occurs. This forces $(3d)$ or $(3e)$. If $(3e)$ then LEC or $(e8)$, where 8 lies between 5 and 3. If there are no further shadow edges at d then LAC; if $(2d)$ then LAC; or if $(d8)$ where 8 lies between 5 and 3, then LAC. If $(3d)$ occurs then there are no vertices between 3 and 5 and LAC.

Now suppose $i\deg(4) > 1$. The pairs $(4a),(46)$; $(4a),(42)$; $(4a),(4c)$; $(41),(4c)$

and $(46),(4c)$ where 6 lies between 1 and 2 each yield LAC, and $(4a),(4e)$ forces $(3e)$ and either LEC or (41), yielding LAC. This leaves the triples $(4a),(41),(4d)$; $(41),(42),(4d)$; $(41),(42),(4e)$; $(41),(4d),(4e)$; $(46),(47),(4d)$ where 6 lies between 1 and 2, and 7 between 6 and 2; $(46),(47),(43)$; $(46),(4d),(4e)$; $(42),(4c),(4d)$ and LAC; $(42),(4c),(4e)$; $(42),(4d),(4e)$; and $(4c),(4d),(4e)$ yielding LAC. If $(4a),(41),(4d)$ or $(41),(4d),(4e)$ occurs this forces $(2d)$ and either LEC or $(6d)$, where 6 lies between 1 and 2, and LAC. If $(41),(42),(4d)$ occurs this forces $(3d)$ or $(3e)$. If $(3d)$ then there are no vertices between 1 and 2 or between 3 and 4, any further shadow edges yields LAC and no more yields LEC. If $(3e)$ then either LEC or $(5e)$ where 5 lies between 3 and 4 and if there are no more shadow edges at d then LEC otherwise $(6d)$ occurs where 6 lies between 5 and 4, and again LEC or there are further shadow edges and LAC. If $(41),(42),(4e)$ occurs this forces $(3e)$ and there are no vertices between $1,2$ or $3,4$. If now $(2e)$ then LAC or if either $(2d)$ or (ec) then either LEC or $(2e)$. So there are no more shadow edges and LEC. If $(46),(47),(4d)$ occurs this forces $(2d)$ and either LEC or $(8d)$ where 8 lies between 7 and 2 yielding LAC. If $(46),(47),(4e)$ or $(46),(4d),(4e)$ or $(42),(4c),(4e)$ or $(42),(4d),(4e)$ occurs this forces no more vertices between 3 and 4, $(1b)$ and either LEC or (68) where 8 lies between 1 and 6, yielding LAC.

Suppose that $i\deg(3) > 1$. The pairs $(3a),(36)$; $(3a),(32)$; $(3a),(3c)$; $(31),(3c)$; $(36),(3c)$ where 6 lies between 1 and 2 each yield LAC; and the pair $(3a),(3e)$ forces $(4a)$, no vertices between 3 and 4, and either LEC or $(3d)$ which yields LAC. This leaves the triples $(3a),(31),(3d)$; $(31),(36),(32)$ and LAC; $(31),(36),(3d)$; $(31),(36),(3e)$; $(31),(32),(3d)$; $(31),(32),(3e)$ and LEC; $(31),(3d),(3e)$; $(36),(37),(3d)$; $(36),(37),(3e)$ and LEC; $(36),(32),(3d)$; $(36),(32),(3e)$ and LEC; $(36),(3d),(3e)$; $(32),(3c),(3d)$ and LAC; $(32),(3c),(3e)$ and LEC; $(32),(3d),(3e)$ and LAC; $(3c),(3d),(3e)$ and LAC, where 6 lies between 1 and 2 and 7 between 6 and 2. If $(3a),(31),(3d)$ or $(31),(36),(3d)$ or $(31),(3d),(3e)$ or $(36),(37),(3d)$ or $(36),(3d),(3e)$ occurs this forces $(2d)$ and either LEC or $(d8)$ where 8 lies between 6 and 2 and LAC. If $(31),(36),(3e)$ occurs this forces LEC or $(3d)$ which forces LEC or LAC as in the previous cases. If $(31),(32),(3d)$ occurs this yields LEC, if (df) also occurs then again LEC or $(2d)$ which yields LAC. If $(36),(32),(3d)$ occurs then either (df) occurs and LAC, otherwise LEC.

Finally let $m = n = 4$ and so $\hat{\Delta}$ is given by Figure A.4(v). Up to symmetry there are the two cases $i\deg(5) > 1$ and $i\deg(4) > 1$. Let $i\deg(5) > 1$. The pairs $(5a),(56)$; $(5a),(52)$; $(5a),(5d)$; $(51),(5d)$; $(56),(5d)$, where 6 lies between 1 and 2 each yield LAC. Up to symmetry this leaves the triples $(5b),(5a),(51)$ and LAC; $(5b),(5a),(5e)$; $(5b),(51),(56)$; $(5b),(51),(52)$; $(5b),(51),(5e)$; $(5b),(56),(52)$; $(5b),(56),(5e)$; $(5b),(52),(5d)$; $(56),(57),(5b)$; $(51),(56),(52)$ and LAC; where 6 lies between 1 and 2 and 7 between 6 and 2. If $(5b),(5a),(5e)$ occurs this forces $(2e),(1e)$ and LAC. If $(5b),(51),(56)$ occurs this forces $(4b)$ and LAC when $i\deg(6) = 1$ or $i\deg(6) = 3$. If $(5b),(51),(52)$ occurs this forces (32) or $(3d)$ or $(3e)$: if (32) then LAC; if $(3d)$ then LAC; and if $(3e)$ then LEC. If $(5b),(51),(5e)$ occurs this forces $(2e)$

and LEC or $(2e), (6e)$ and LAC, where 6 lies between 1 and 2. If $(5b), (56), (52)$ or $(5b), (56), (5e)$ or $(5b), (52), (5d)$ occurs this forces $(4b), (1b)$ and LEC or $(4b), (1b), (7b)$ and LAC, where 7 lies between 1 and 6.

Let $i\deg(4) > 1$. The pairs $(4c), (46)$; $(4a), (42), (4a), (4d)$; $(41), (4d)$; $(46), (4d)$ where 6 lies between 1 and 2 each forces LAC. The pair $(4b), (41)$ force LEC or LAC and the pairs $(4b), (46)$; $(4b), (42)$; $(4b), (4d)$; and $(4b), (4e)$ each force LEC. This leaves the triples $(4b), (4a), (41)$ and LAC; $(4b), (4a), (4e)$; $(4a), (41), (4e)$; $(41), (46), (42)$ and LAC; $(41), (46), (4e)$; $(41), (42), (4e)$; $(46), (42), (4e)$; and $(42), (4d), (4e)$ which yields LAC. If $(4b), (4a), (4e)$ or $(4a), (41), (4e)$ or $(41), (46), (4e)$ occurs this forces $(1e)$ and LAC. If $(41), (42), (4e)$ occurs with no more shadows then LEC; and any further shadow edge forces LAC or LEC. If $(46), (42), (4e)$ occurs this forces $(3e)$ and no vertices between 3 and 4; and either $(1b)$ or $(1c)$. If $(1b)$ occurs then LEC or $(b7)$ where 7 lies between 1 and 6 again forcing LEC. If $(1c)$ or $(1c), (c7)$ occurs then LAC.

(iv) By (iii) it can be assumed that $i\deg(v) = 1$ for $v \in B_1 \cup B_2$; and by (iii) the statement clearly holds when $(m, n) = (3, 3)$. Let $(m, n) = (3, 4)$ and so $\hat{\Delta}$ is given by Figure A.4(i), (iii). Clearly the statement holds for 1 and 4. Up to symmetry there are two cases: $(14), (2d), (3d)$; and $(1b), (2d), (3d)$. Suppose $(14), (2d), (3d)$ occurs. This forces LEC or $(5d), (6d)$ where 5 lies between 3 and 4 and 6 lies between 1 and 2 and this forces LAC. Suppose $(1b), (2d), (3d)$ occurs. This forces LEC or $(5d), (6d)$ as before and $(47), (b8)$ where $7, 8$ lie between 1 and 6, and then LAC.

Let $(m, n) = (3, 5)$ and so $\hat{\Delta}$ is given by Figure A.4(i), (iv). Up to symmetry the cases are $(2d), (3d), (1b)$; $(2d), (3d), (14)$; $(2d), (3d), (4a)$; $(2d), (3e), (1b)$; and $(2d), (3e), (14)$. Suppose $(2d), (3d), (1b)$ occurs. This forces $(b8), (47), (6d)$ where $6, 7, 8$ lie between 1 and 2, otherwise LEC. Other possible shadow edges are (df) and those between additional vertices between $6, 7$ and $3, 4$; but in all cases this forces LEC. A similar argument forces LEC when $(2d), (3d), (14)$ or $(2d), (3d), (4a)$ occurs. Suppose $(2d), (3e), (1b)$ occurs. This forces $(b8), (47), (6d), (5e)$ where $6, 7, 8$ lie between 1 and 2 and 5 lies between 3 and 4, otherwise LEC.

If there are no more shadow edges from d this forces LEC; if $(d9)$ occurs only, where 9 lies between 4 and 5, this forces LAC; if $(d9), (d10)$ occurs where 10 lies between 4 and 9 this forces LEC; and if there are more than two further shadow edges at d this forces LAC. The argument is the same when $(2d), (3e), (14)$ occurs.

Let $(m, n) = (4, 4)$ and so $\hat{\Delta}$ is given by Figure A.4(v). Up to symmetry it can be assumed that $(1b), (4b)$ occurs. This forces $(6b), (5b)$ where 6 lies between 1 and 2, and 5 lies between 3 and 4, otherwise LEC. If any of (23), (27) or (38) now occurs where 7 lies between 5 and 3, and 8 lies between 6 and 2 this forces LAC. This leaves $(2e), (3e)$ forcing LEC or $(7e), (8e)$ which again forces LAC.

(v) The region $\hat{\Delta}$ is given by Figure A.5 in which $\hat{\Delta}$ has been partitioned into three regions $\hat{\Delta}_i$ $(1 \leq i \leq 3)$; $l_1 \geq 2$ denotes the number of edges between u_1 and v_1; $l_2 \geq 2$

denotes the number between u_{k+1} and v_{l+1}; l the number between v_1 and v_{l+1}; and k the number between u_1 and u_{k+1}. Moreover, as shown, there is a shadow edge in $\hat{\Delta}$ between u_1 and v_1 and between u_{k+1} and v_{l+1}; and, further to this, all shadow edges from u_i $(1 \leq i \leq k+1)$ within $\hat{\Delta}$ are connected to some v_j where $1 \leq j \leq l+1$. Thus $n_2 = l_1 + l + l_2 \geq l + 4$ and so $4 \leq n_2 \leq 9$.

Using LAC and LEC we list in Figure A.6 all the possibilities for $\hat{\Delta}_j$ $(1 \leq j \leq 2)$ for $2 \leq l_j \leq 5$; listed in Figure A.7 are the possibilities for $\hat{\Delta}_3$ when $0 \leq l \leq 3$; and when $l = 4$ $\hat{\Delta}_3$ is given by Figure A.8 under the assumption that at least one $\hat{\Delta}_1, \hat{\Delta}_2$ has degree 2.

In what follows we use the labelling of Figures A.6-8 and work up to the symmetry of $\hat{\Delta}_3$.

Let $n_2 \leq 7$. If $l = 0$ then, using the fact that $i\deg(u_i) \geq 1$, $k > 1$ forces LAC and $k = 1$ forces $d(\hat{\Delta}) = n_2 + 1 \leq 8$; if $l = 1$ then $k > 3$ forces LAC and if $k \leq 3$ then $n_2 \leq 6$ implies $d(\hat{\Delta}) \leq 9$; if $l = 2$ then $k \leq 5$ and $n_2 \geq 6$. Thus we are left with the cases $l = 1$, $n_2 = 7$; $l = 2$, $n_2 = 6$; $l = 2$, $n_2 = 7$; and $l = 3$, $n_2 = 7$.

Let $l = 1$ and $n_2 = 7$. Then $k \leq 3$ and since $k \leq 2$ implies $d(\hat{\Delta}) < 10$ it can be assumed that $\hat{\Delta}_3 = C13$. Since each of $R3B, R4B, R4C$ yields LAC it follows that $(\hat{\Delta}_1, \hat{\Delta}_2) \in \{(R3A, R3A), (R2, R4A)\}$ and LEC.

Let $l = 2$ and $n_2 = 6$. Then $k \leq 5$ and since $k \leq 3$ implies $d(\hat{\Delta}) < 10$ it can be assumed that $\hat{\Delta}_3 \in \{C24, C25\}$. But $l_1 = l_2 = 2$ implies $\hat{\Delta}_1 = \hat{\Delta}_2 = R2$ and this forces LEC.

Let $l = 2$ and $n_2 = 7$. This forces $k \leq 5$, and since $k \leq 2$ implies $d(\hat{\Delta}) < 10$ it can be assumed that $3 \leq k \leq 5$. Therefore $\hat{\Delta}_3 \in \{C23A, C23B, C23C, C24, C25\}$; and $(\hat{\Delta}_1, \hat{\Delta}_2) \in \{(R2, R3A), (R2, R3B), (R3A, R2), (R3B, R2)\}$. Each case either yields LAC or yields LEC.

Let $l = 3$ and $n_2 = 7$. Then $k \leq 7$ and since $k \leq 2$ implies $d(\hat{\Delta}) < 10$ it can be assumed that $3 \leq k \leq 7$. Therefore $\hat{\Delta}_1 = \hat{\Delta}_2 = R2$ and this yields LAC except when $\hat{\Delta}_3 \in \{C33C, C35A, C35B, C37\}$ and each of these four cases yields LEC.

Now let $n_2 = 8$. Then $d(\hat{\Delta}) \geq 10$ forces $k \geq 2$. Since $l = 0$ implies $d(\hat{\Delta}) < 10$ it follows that $1 \leq l \leq 4$.

Let $l = 4$. Then $\hat{\Delta}_1 = \hat{\Delta}_2 = R2$ and so $\hat{\Delta}_3$ is one of $C43B, C, D, E$ or $C45A, B, C, D$ or $C47B, C, D$ or $C49$ and each case forces LEC.

Let $l = 3$. Then $(l_1, l_2) \in \{(2,3), (3,2)\}$. Checking each $C3$ for $\hat{\Delta}_3$ yields LEC except for $\hat{\Delta}_3 = C34D$ and the region $\hat{\Delta}$ is given by Figure 7.4(i).

Let $l = 2$. Then $(l_1, l_2) \in \{(2,4), (3,3), (4,2)\}$. Checking each C2 case for $\hat{\Delta}_3$ yields LEC except when
$(\hat{\Delta}_1, \hat{\Delta}_3, \hat{\Delta}_2) \in \{(R2, C22A, R4A), (R2, C24, R4A), (R3A, C22B, R3A), (R3B, C22B, R3B)\}$.
Now $(R2, C22A, R4A), (R2, C24, R4A)$ yields the region $\hat{\Delta}$ of Figure 7.4(ii)–(iii);

$(R3A, C22B, R3A)$ yields the region $\hat{\Delta}$ of Figure 7.4(iv); and $(R3B, C22B, R3B)$ forces LAC.

Let $l = 1$. Then $(l_1, l_2) \in \{(2,5),(3,4),(4,3),(5,2)\}$ and $\hat{\Delta}_3 \in \{C12, C13\}$. Each case forces LAC or LEC except for
$(\hat{\Delta}_1, \hat{\Delta}_3, \hat{\Delta}_2) \in \{(R2, C12, R5E),(R2, C12, R5F),(R2, C12, R59),(R3A, C12, R4)\}$ and the four resulting regions $\hat{\Delta}$ are given by Figure 7.4(v)-(viii).

(vi) As in (v) the region $\hat{\Delta}$ is given by Figure A.5. In this case however Lemma 7.1 and Figure 7.2 can be applied. Therefore if $l_j \leq 5$ $(j = 1, 2)$ then
$\hat{\Delta}_j \in \{R2, R3A, R4A, R4C, R5A, R5E, R5F, R5G\}$. A further simple check now shows that if $l_j = 6$ then $\hat{\Delta}_j$ is given by Figure A.9; and if $l_1 = 7$, say, then $l_2 = 2$, $\hat{\Delta}_3 = CO1$ and this forces $\hat{\Delta}_1 = R7$ of Figure A.9.

Let $l = 0$ so that $\hat{\Delta}_3 = C01$. Then $(l_1, l_2) \in \{(2,7),(3,6),(4,5)\}$. If $(l_1, l_2) = (2,7)$ we see from the previous paragraph that $\hat{\Delta}$ is given by Figure 7.5(i). If $(l_1, l_2) = (3,6)$ then $\hat{\Delta}_1 = R3A$ and this forces $\hat{\Delta}_3 = R6J$ and $\hat{\Delta}$ to be given by Figure 7.5(ii). If $(l_1, l_2) = (4,5)$ then there is no choice for $\hat{\Delta}_3$ when $\hat{\Delta}_1 = R4C$ but if $\hat{\Delta}_1 = R4A$ then $\hat{\Delta}_3 = R5A$ and $\hat{\Delta}$ is given by Figure 7.5(iii).

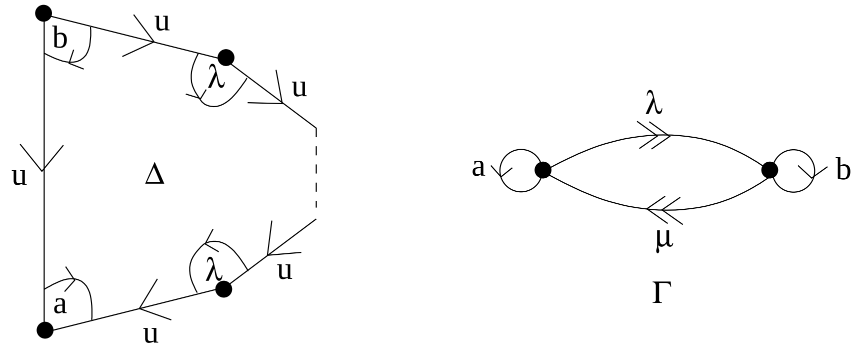
Let $l = 1$ and so $(l_1, l_2) \in \{(2,6),(3,5),(5,3),(6,2),(4,4)\}$. If $(l_1, l_2) \in \{(2,6),(6,2)\}$ then $\hat{\Delta}_3 = C12$ or $C13$ and up to symmetry $\hat{\Delta}_1 = R2$. But now checking the $C6$ case yields LEC for each choice of $\hat{\Delta}_2$. If $(l_1, l_2) = (3,5)$ then $\hat{\Delta}_1 = R3A$ and now any pairing of $\hat{\Delta}_3, \hat{\Delta}_2$ forces LEC except for $(\hat{\Delta}_3, \hat{\Delta}_2) = (C13, R5A)$ and $\hat{\Delta}$ is given by Figure 7.5(iv). If $(l_1, l_2) = (5,3)$ then it can be assumed that $\hat{\Delta}_3 = C12$ and each choice of $\hat{\Delta}_1$ forces LEC. If $(l_1, l_2) = (4,4)$ this forces LEC except for $(\hat{\Delta}_1, \hat{\Delta}_3, \hat{\Delta}_2) = (R4A, C13, R4A)$ and $\hat{\Delta}$ is given by Figure 7.5(v).

Let $l = 2$ and so $(l_1, l_2) \in \{(2,5),(3,4),(4,3),(5,2)\}$. If $(l_1, l_2) \in \{(2,5),(5,2)\}$ then it can be assumed that $\hat{\Delta}_1 = R2$ and now each choice of $\hat{\Delta}_2$ forces LEC except for when $\hat{\Delta}_3 = C23A$, $\hat{\Delta}_2 = R5A$ and $\hat{\Delta}$ is given by Figure 7.5(vi). If $(l_1, l_2) \in \{(3,4),(4,3)\}$ then each case forces LEC except when $\hat{\Delta}_3 = C23A$ or $C25$ and up to symmetry $\hat{\Delta}$ is given by Figure 7.5(vii), (viii).

Let $l = 3$ and so $(l_1, l_2) \in \{(2,4),(3,3),(4,2)\}$. If $(l_1, l_2) \in \{(2,4),(4,2)\}$ it can be assumed that $\hat{\Delta}_1 = R2$ except when $\hat{\Delta}_3 = C35B$. When $\hat{\Delta}_1 = R2$ each case yields LEC except for $(\hat{\Delta}_3, \hat{\Delta}_2) = C33C, R4A)$ and $\hat{\Delta}$ is given by Figure 7.5(ix); and $(\hat{\Delta}_3, \hat{\Delta}_2) = (C35B, R4A)$ or $(C37, R4A)$ each forcing LAC. When $\hat{\Delta}_2 = R2$ then LEC except when $\hat{\Delta}_1 = R4A$ and $\hat{\Delta}$ is given by Figure 7.5(x). If $(l_1, l_2) = (3,3)$ then $\hat{\Delta}_1 = \hat{\Delta}_2 = R3A$ forcing $\hat{\Delta}_3 \in \{C33C, C35B, C37\}$ and $\hat{\Delta}$ is given by Figure 7.5(xi)-(xiii).

Let $l = 4$ so that $(l_1, l_2) = (2,3)$ or $(3,2)$. Then length forces
$\hat{\Delta}_3 \in \{C43E, C45A, C45B, C47B, C47C, C49\}$. But $C47C$ and $C49$ each forces LAC; $C45B$ forces LAC when $(l_1, l_2) = (3,3)$; and $C47B$ forces LAC when $(l_1, l_2) = (2,3)$. It follows that up to symmetry $\hat{\Delta}$ is given by Figure 7.5(xiv)-(xvii).

Finally let $l = 5$. This forces $(l_1, l_2) = (2, 2)$ and Figure 7.2 immediately yields LAC.
$\square$



(i) (ii)

Figure 3.1

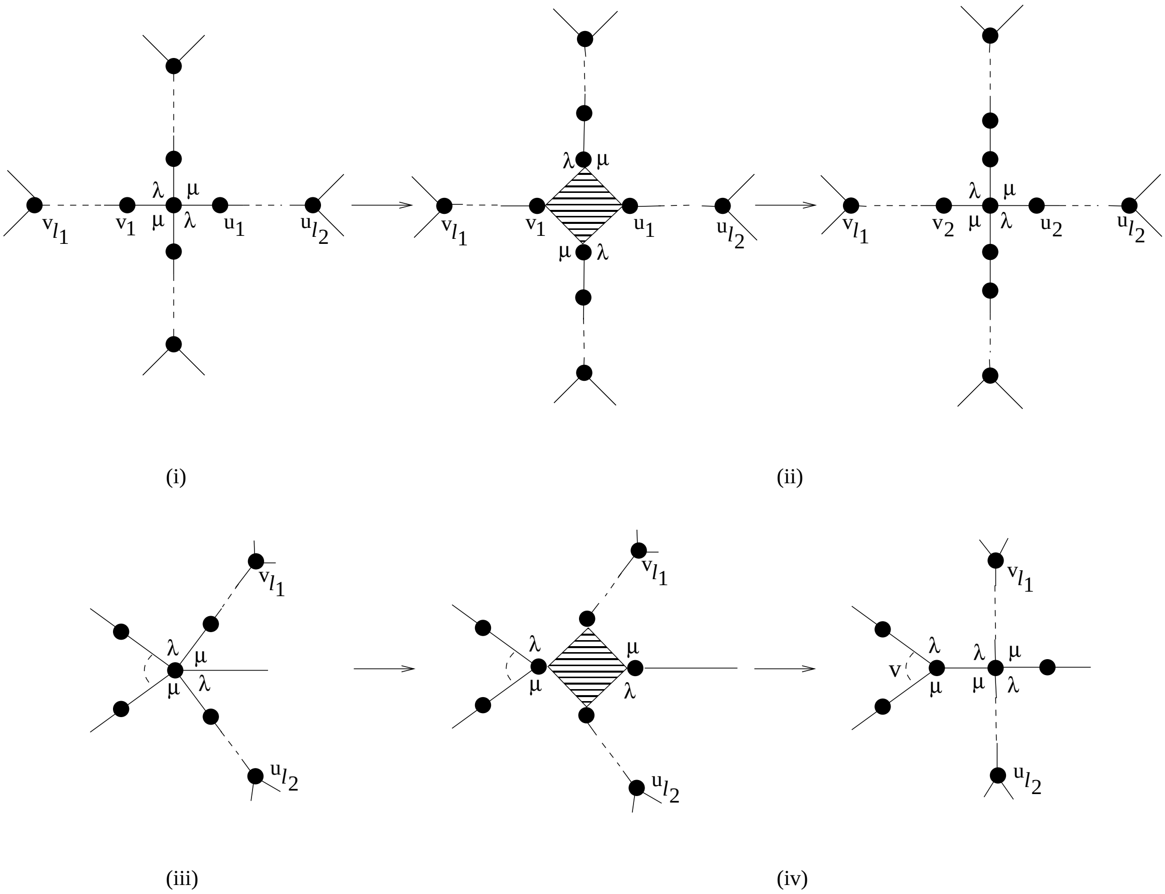
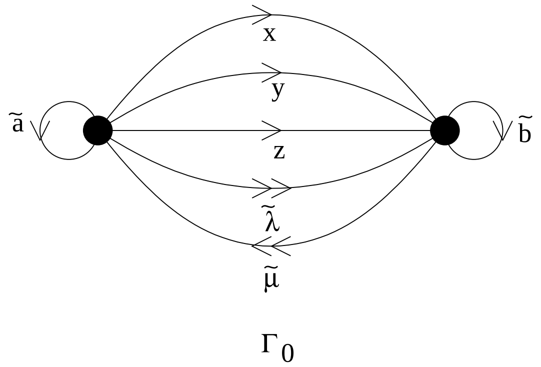


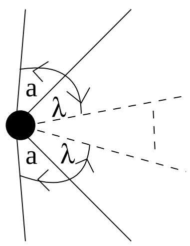
Figure 3.2



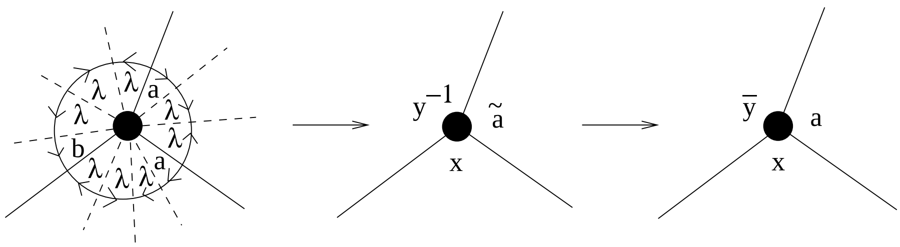
(i)

$\tilde{a}$	$\tilde{b}$	x	y	z	$\tilde{\lambda}$	$\tilde{\mu}$
1	2	1	2	3	0	0

(ii)



(iii)



(iv)

Figure 3.3

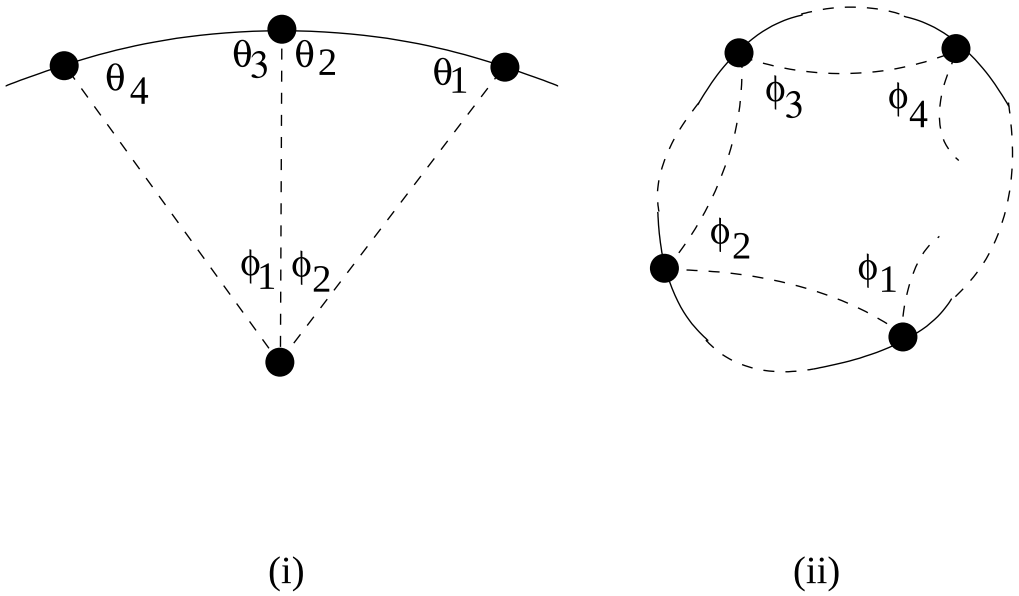


Figure 3.4

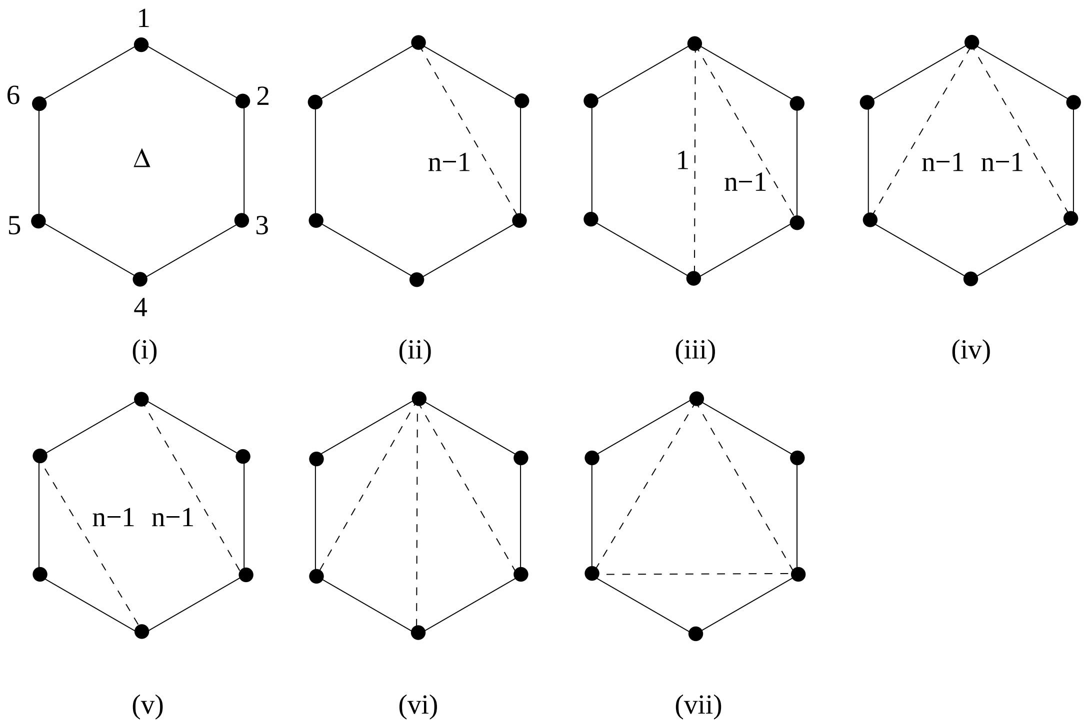


Figure 3.5

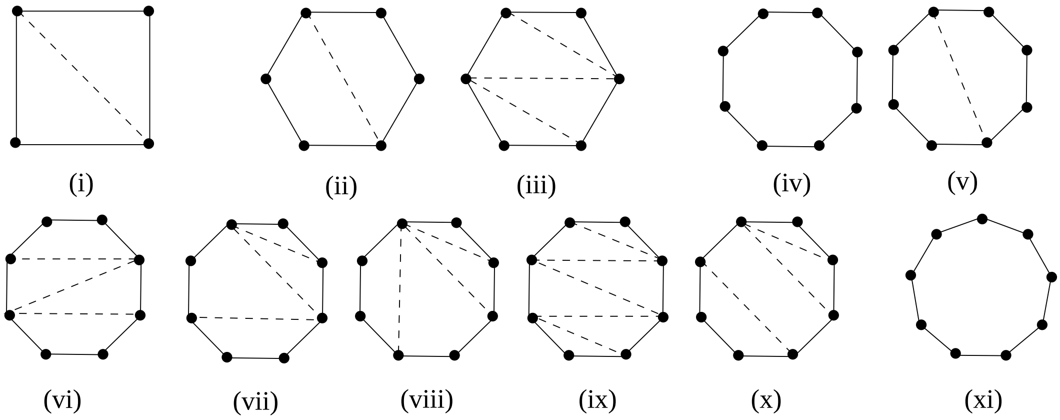
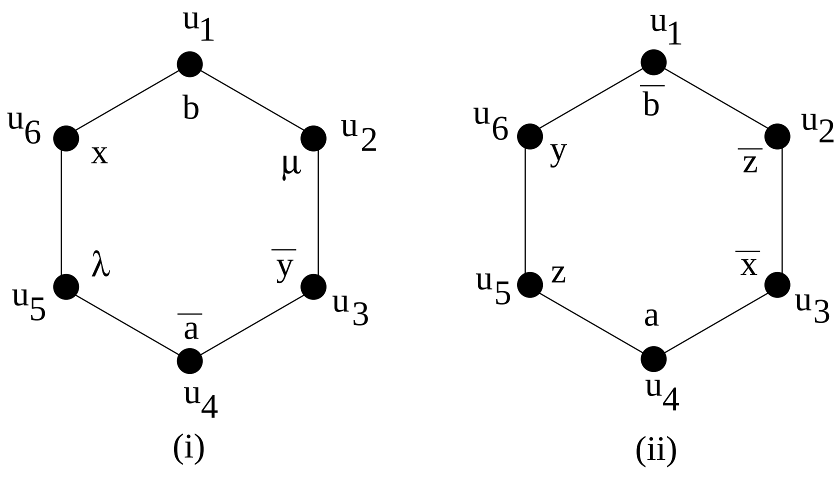


Figure 3.6



u_1
u_2
u_3
u_4
u_5
u_6
b
μ
$\bar{y}$
$\bar{a}$
λ
x
(i)
u_1
u_2
u_3
u_4
u_5
u_6
$\bar{b}$
$\bar{z}$
$\bar{x}$
a
z
y
(ii)

Figure 3.7

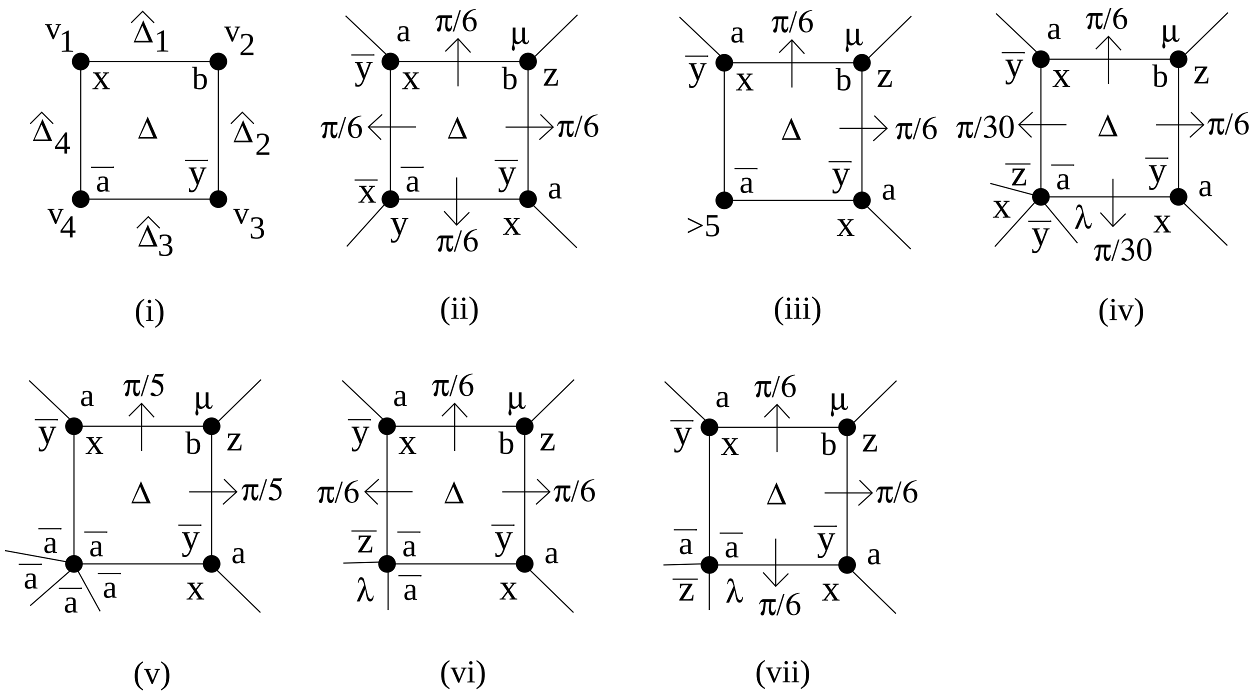


Figure 4.1

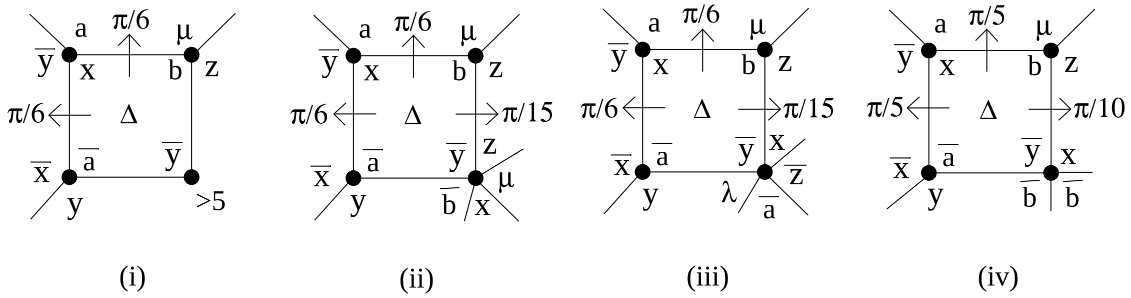


Figure 4.2

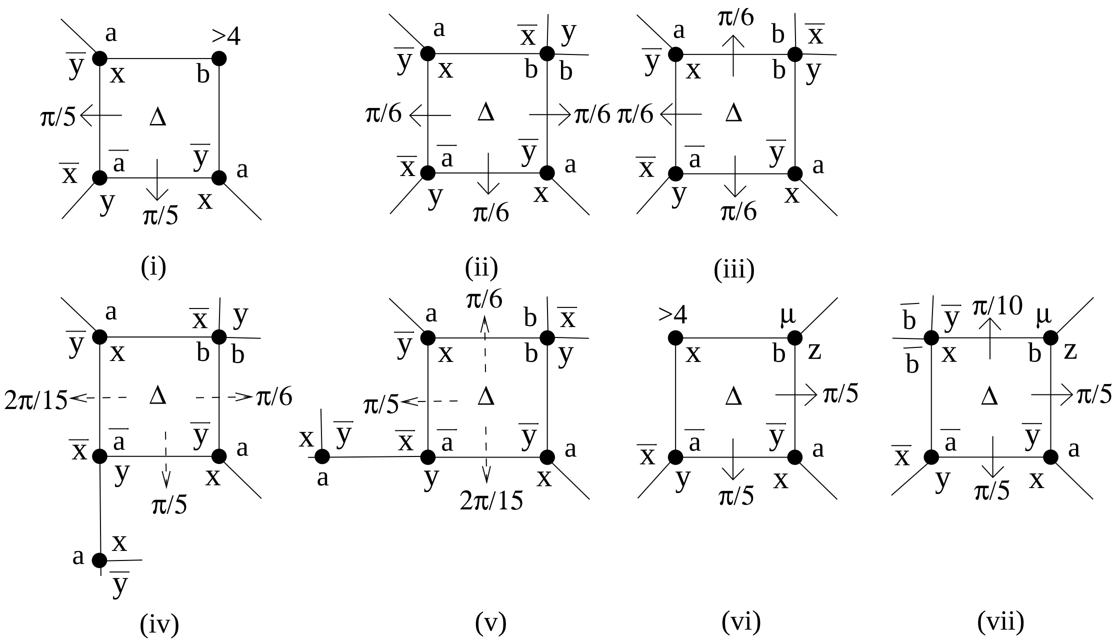


Figure 4.3

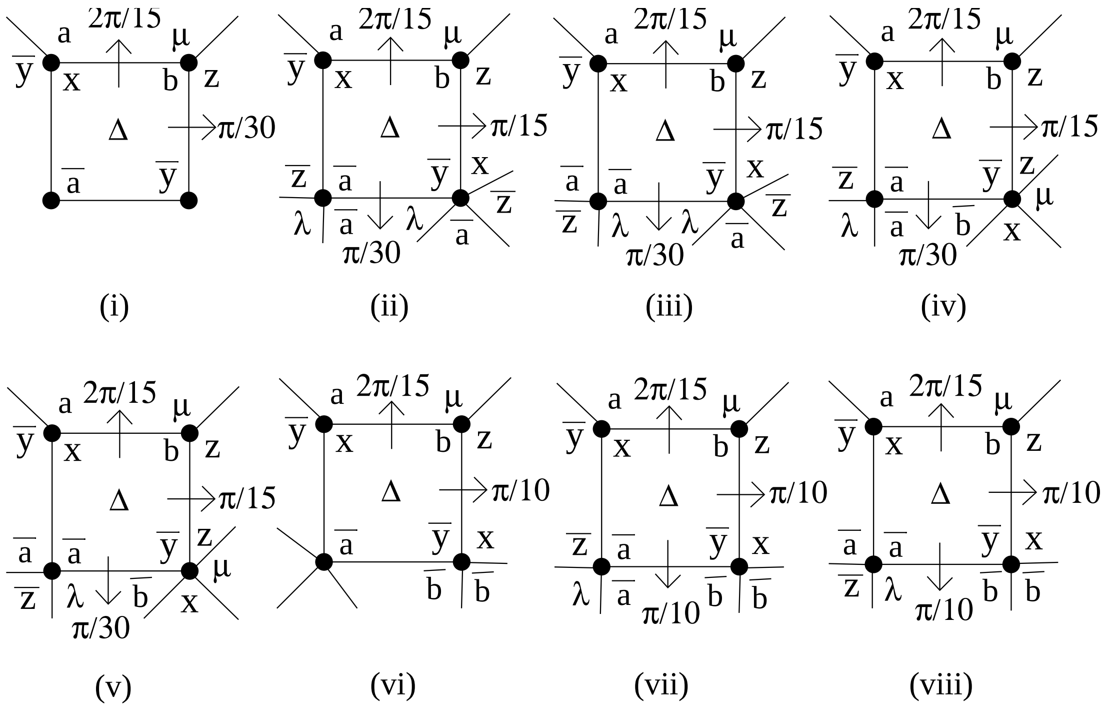


Figure 4.4

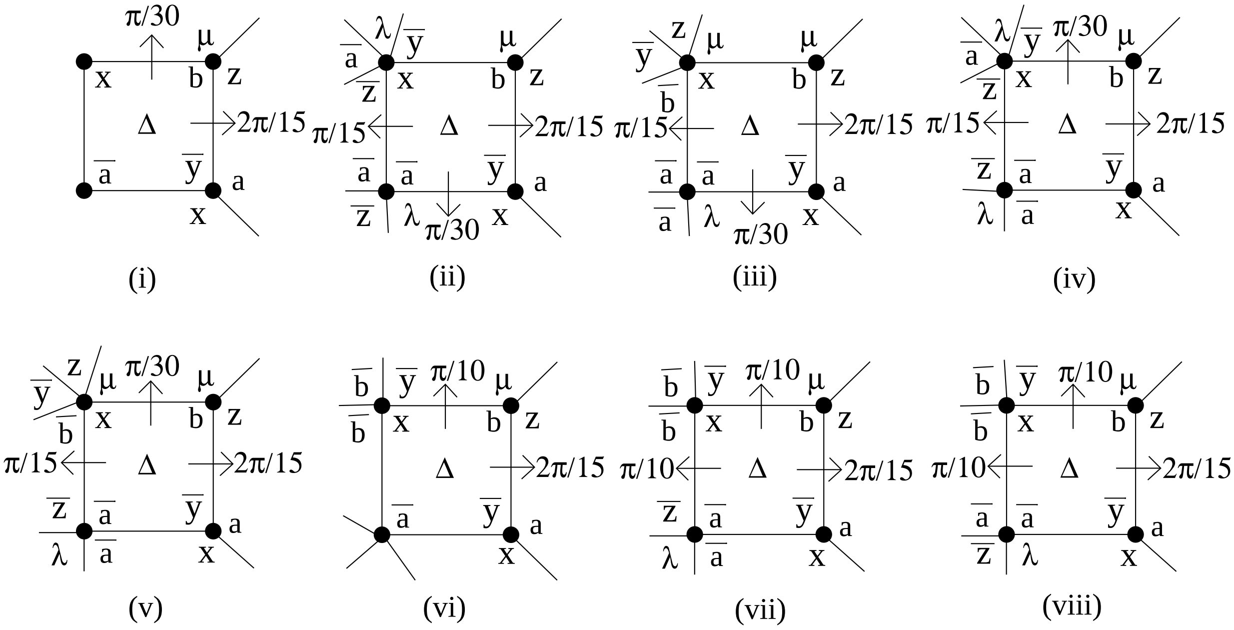
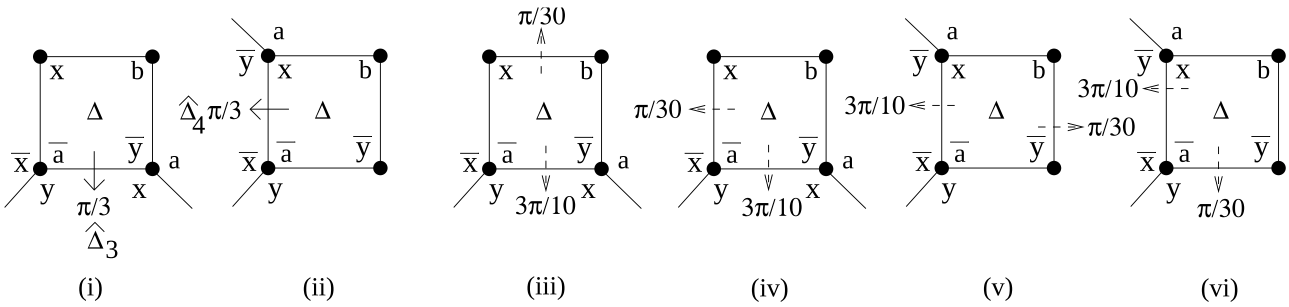


Figure 4.5



x
b
Δ
$\bar{x}$
$\bar{a}$
$\bar{y}$
a
y
x
π/3
$\hat{\Delta}_3$
(i)
(ii)
(iii)
(iv)
(v)
(vi)
$\hat{\Delta}_4$ π/3
π/30
3π/10

Figure 4.6

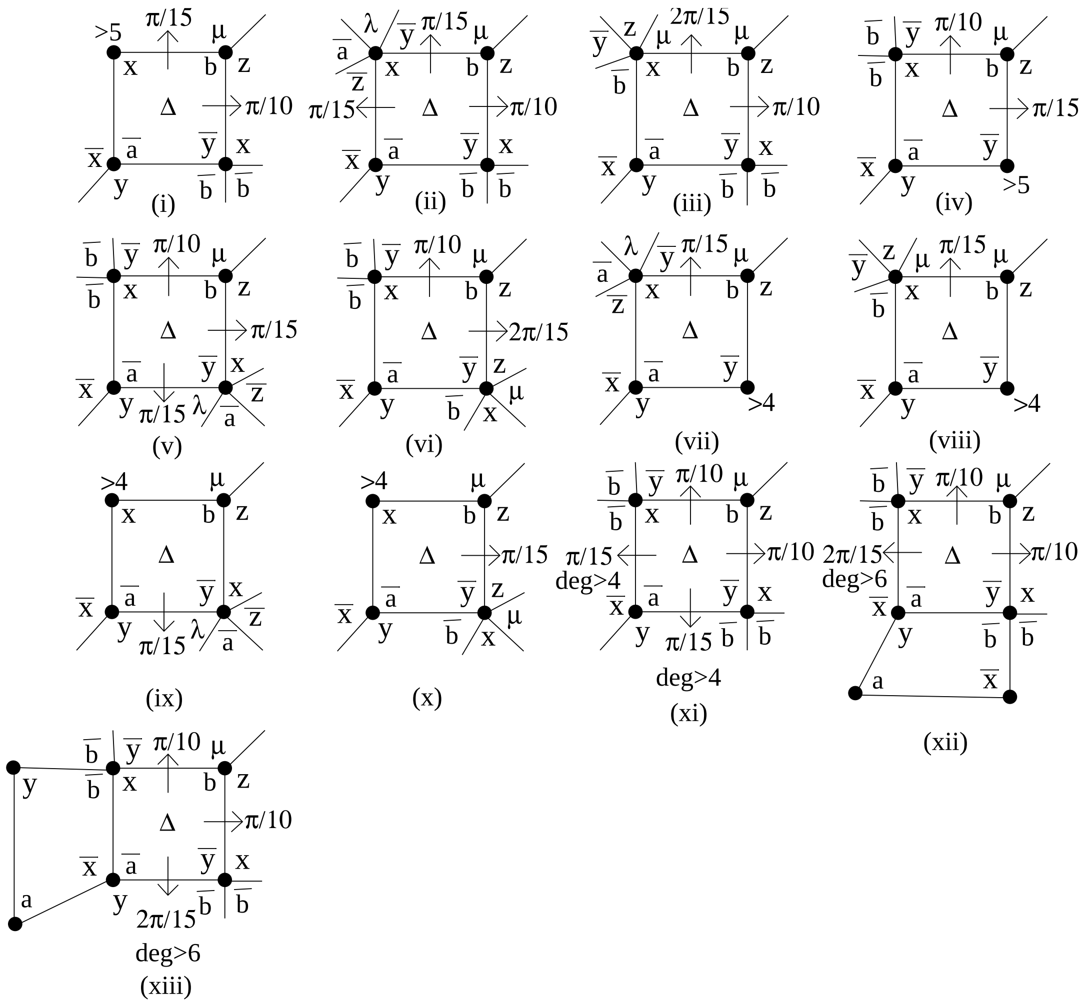


Figure 4.7

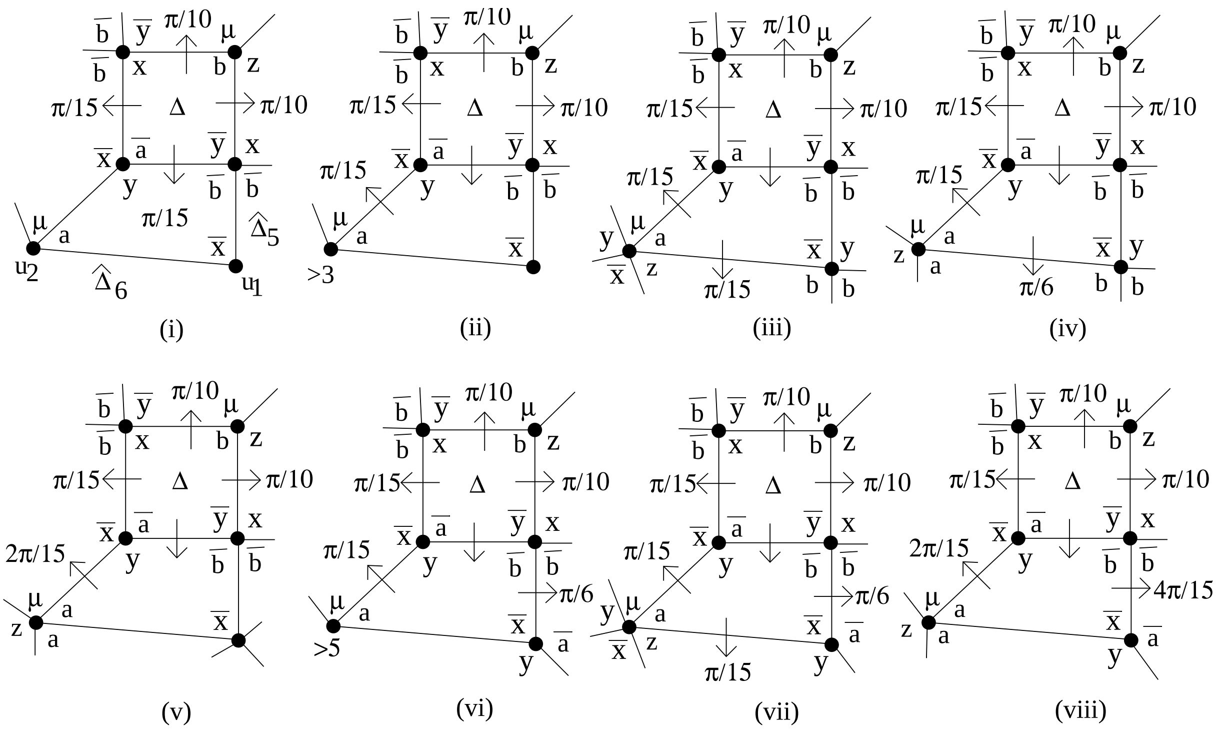


Figure 4.8

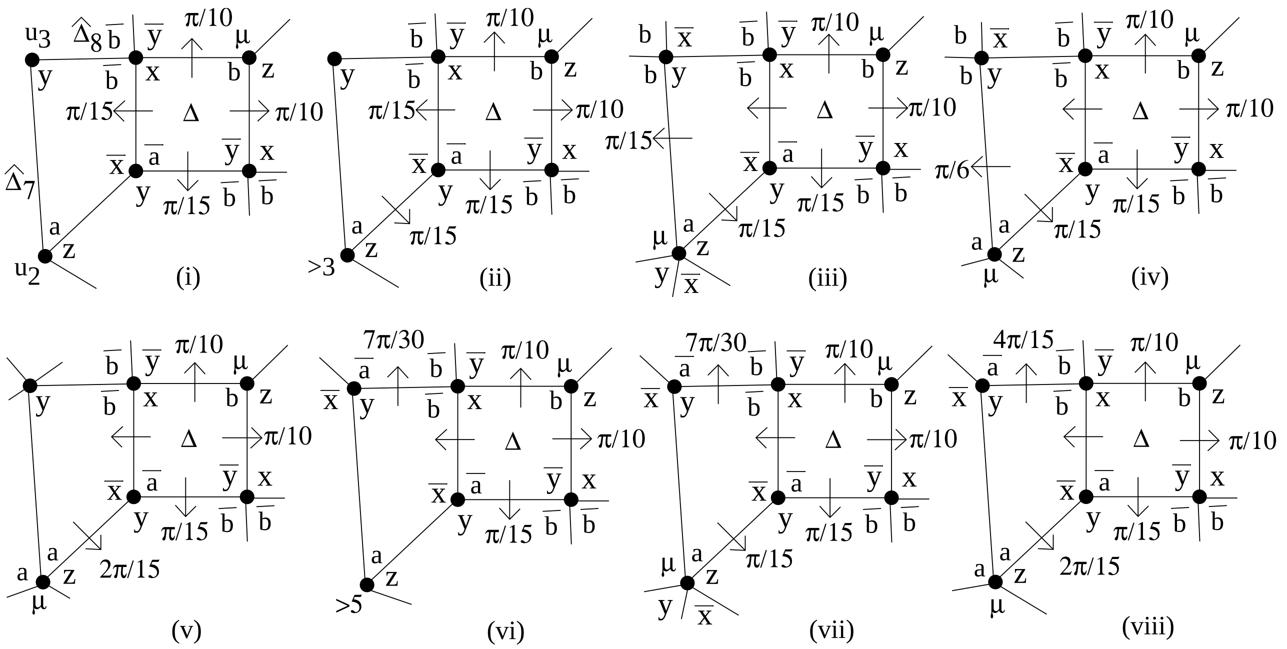


Figure 4.9

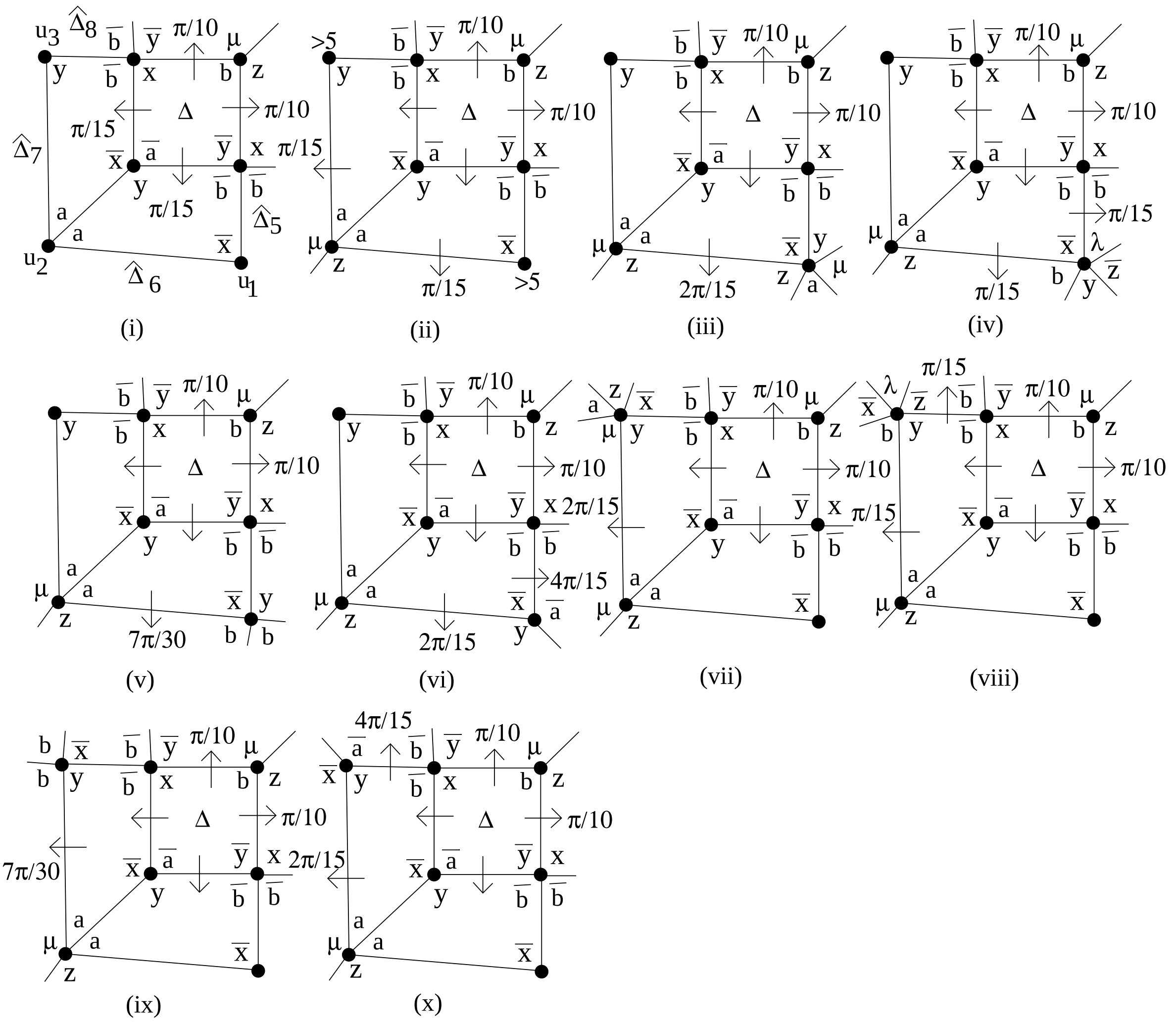
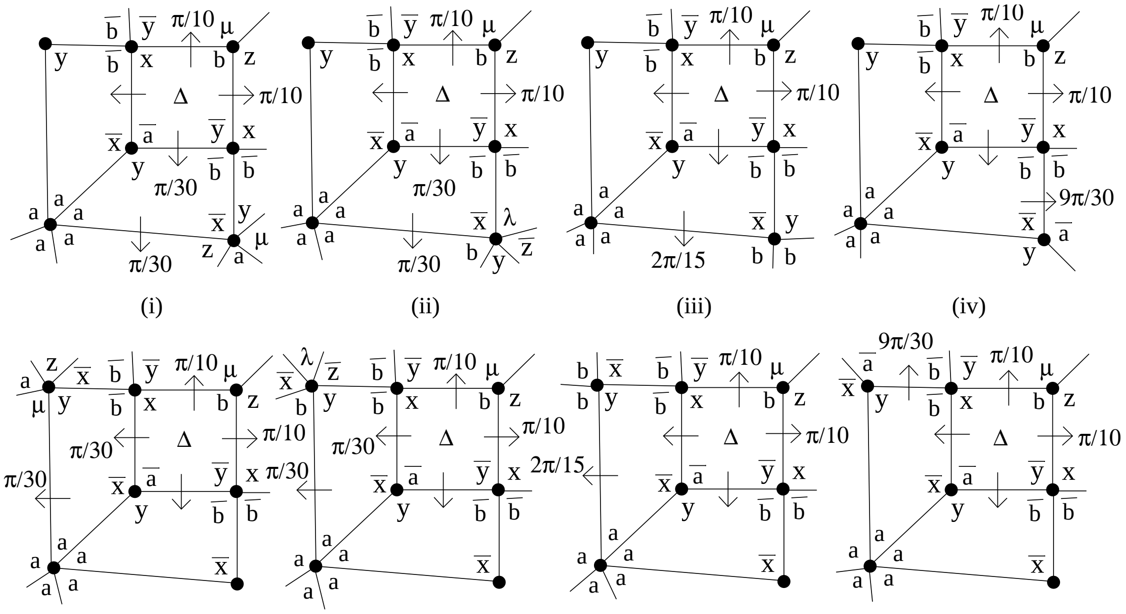


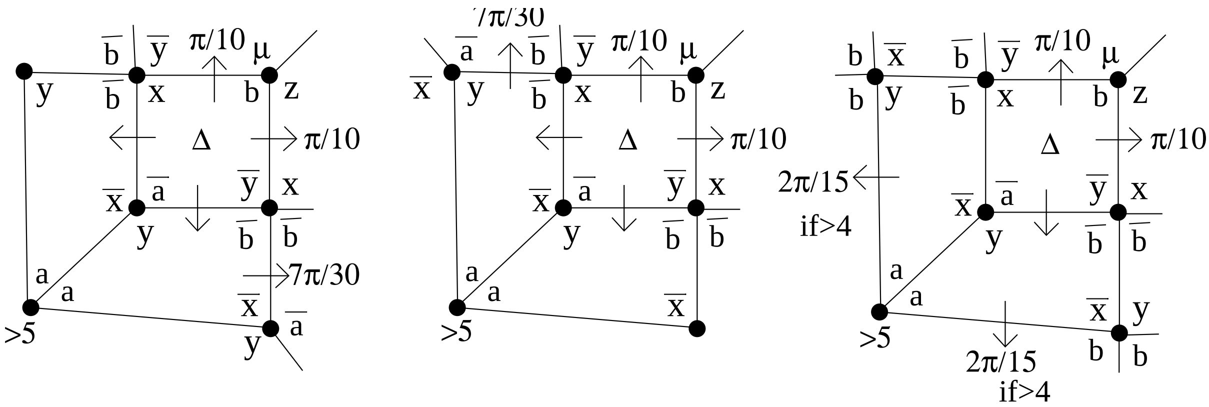
Figure 4.10



(i) (ii) (iii) (iv)

(v) (vi) (vii) (viii)

Figure 4.11



(i) (ii) (iii)

Figure 4.12

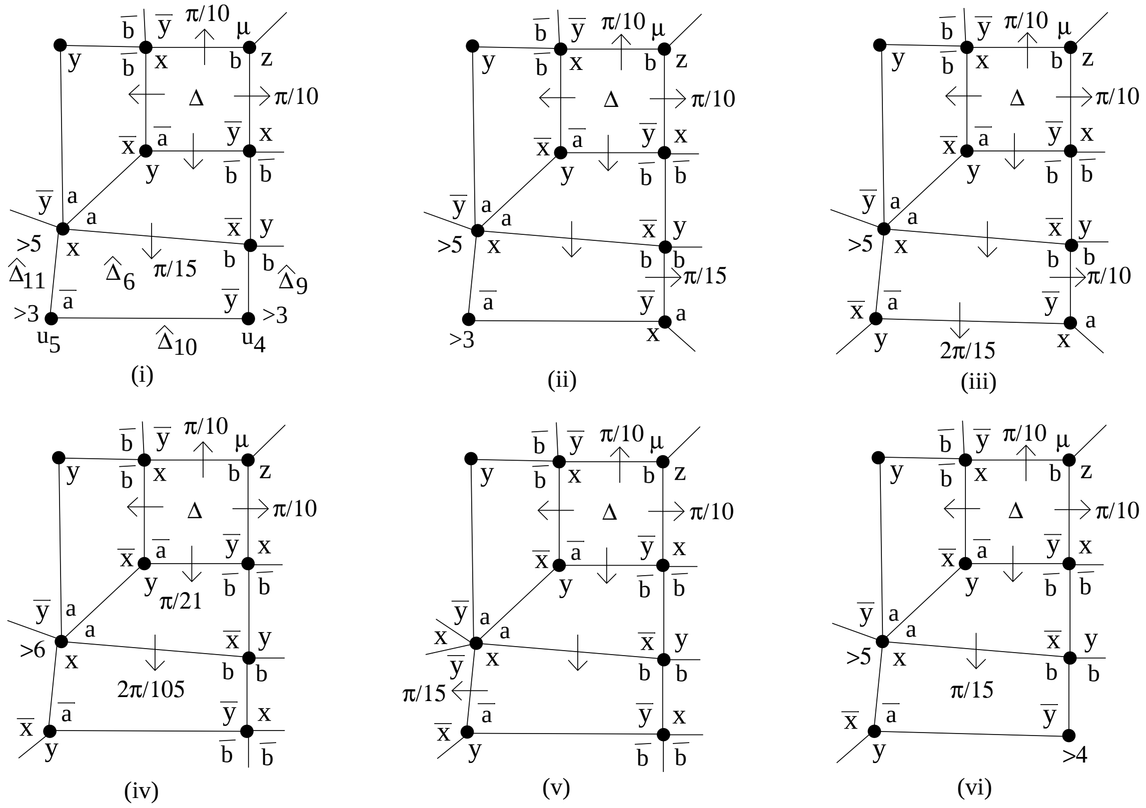

π/10
μ
Δ
π/15
$\hat{\Delta}_6$
$\hat{\Delta}_9$
$\hat{\Delta}_{10}$
$\hat{\Delta}_{11}$
u_4
u_5
>5
>3
(i)
(ii)
2π/15
(iii)
π/21
2π/105
>6
(iv)
(v)
>4
(vi)

Figure 4.13

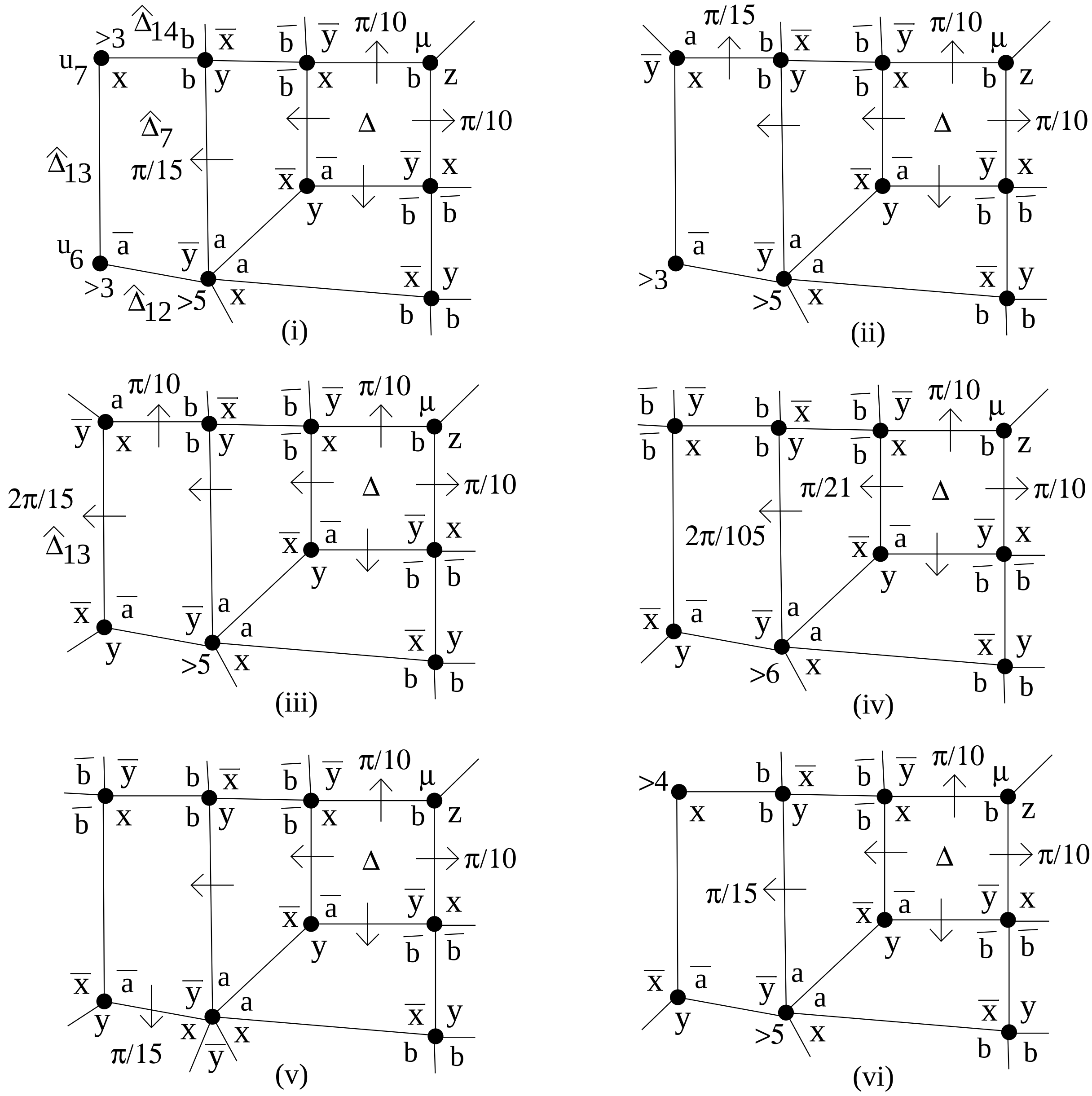


Figure 4.14

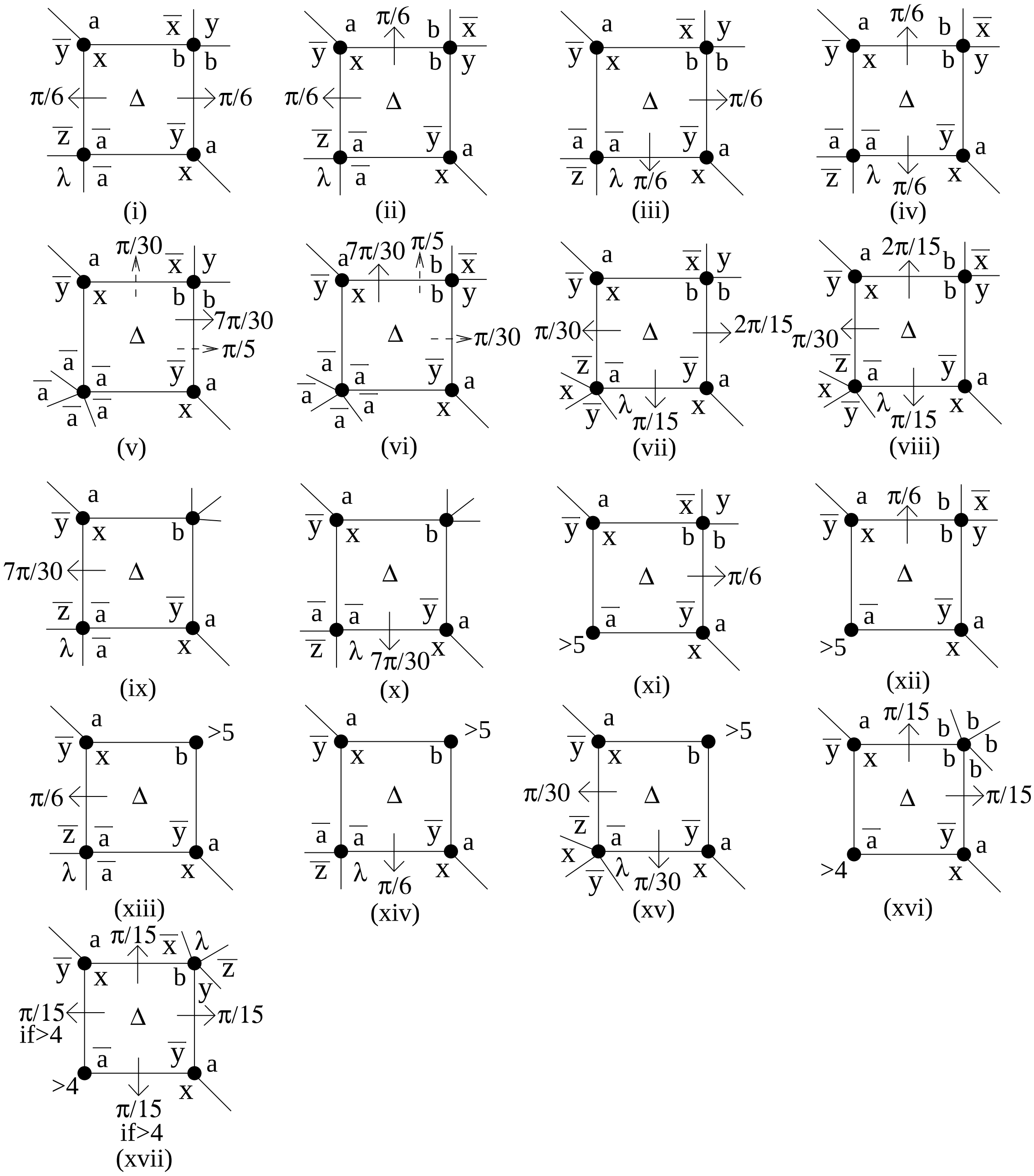


Figure 4.15

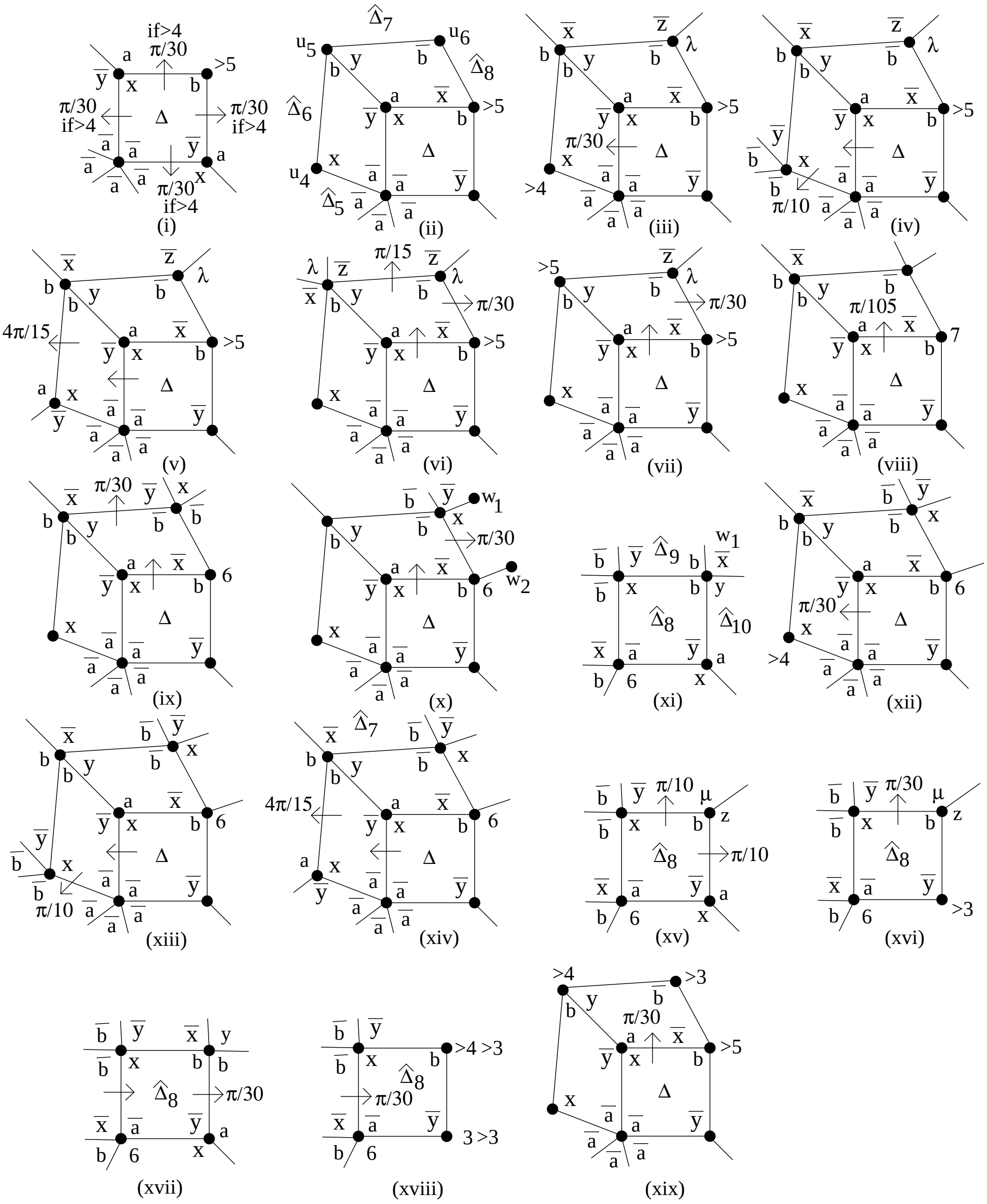

(i)
(ii)
(iii)
(iv)
(v)
(vi)
(vii)
(viii)
(ix)
(x)
(xi)
(xii)
(xiii)
(xiv)
(xv)
(xvi)
(xvii)
(xviii)
(xix)

Figure 4.16

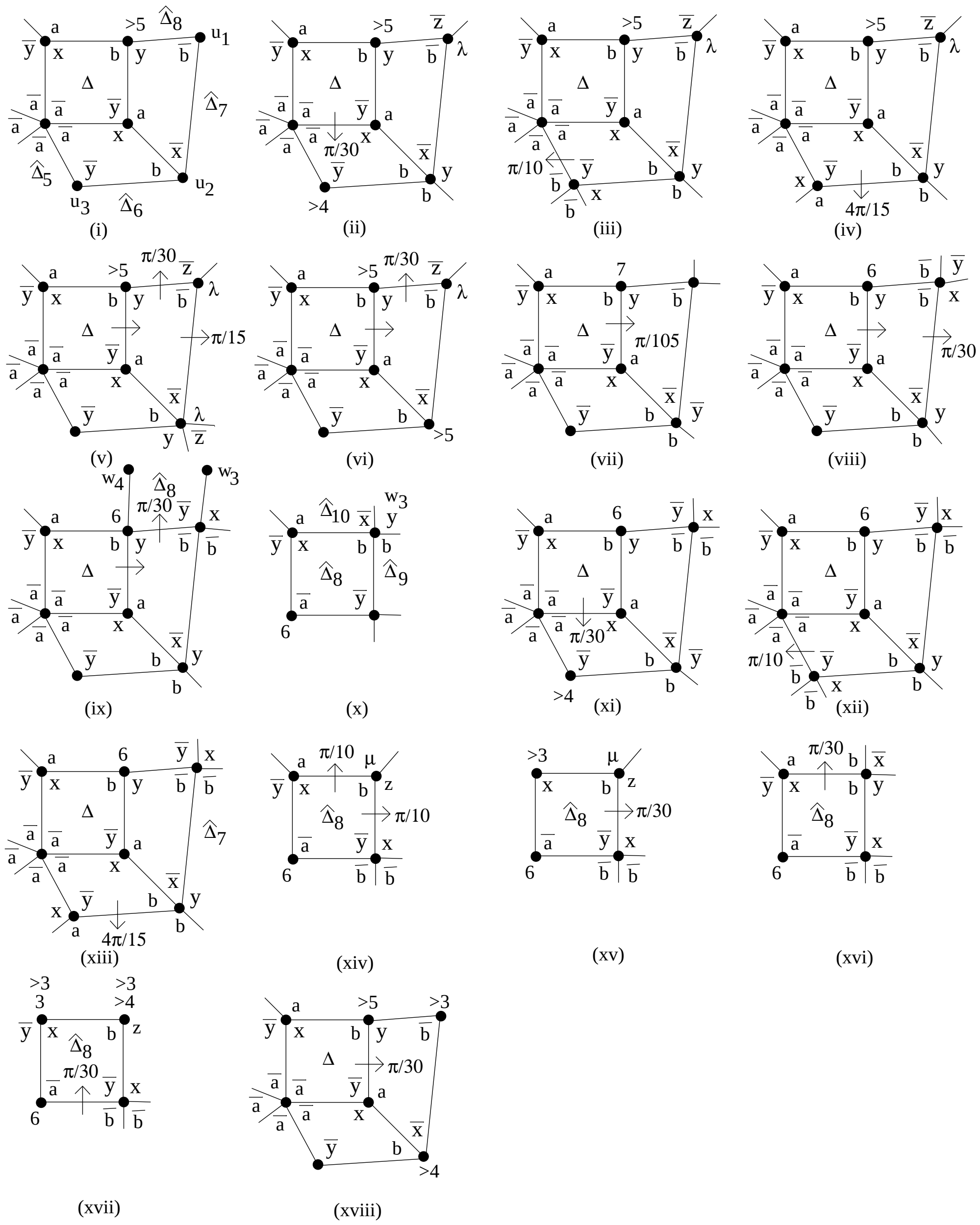


Figure 4.17

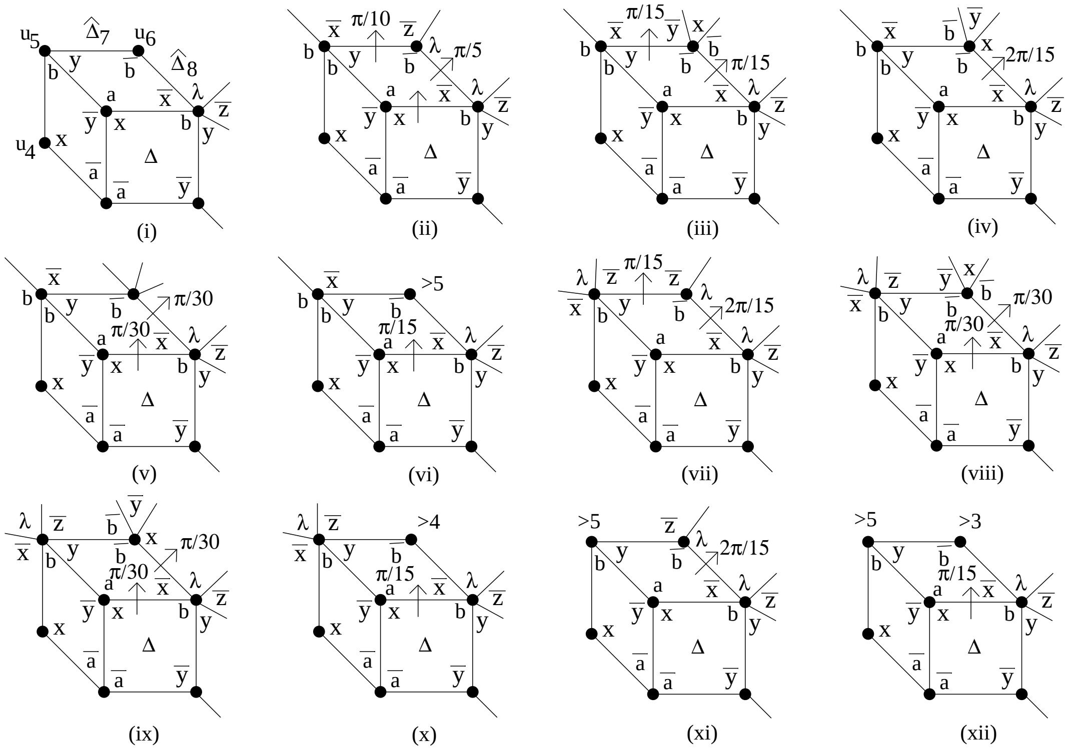


Figure 4.18

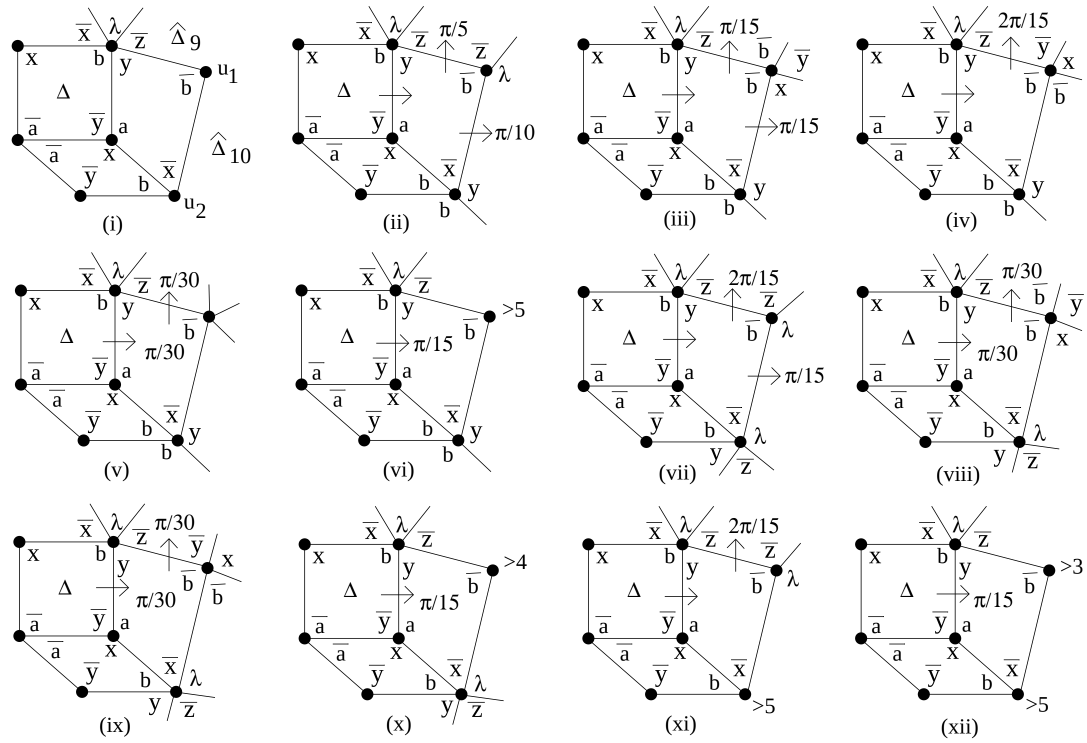


Figure 4.19

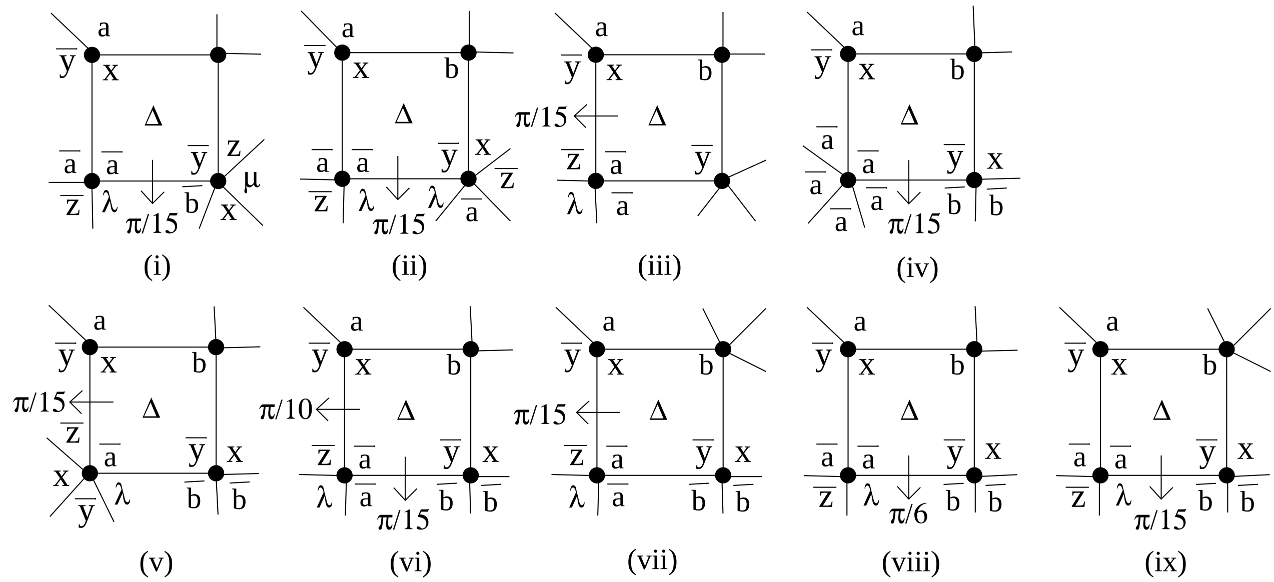


Figure 4.20

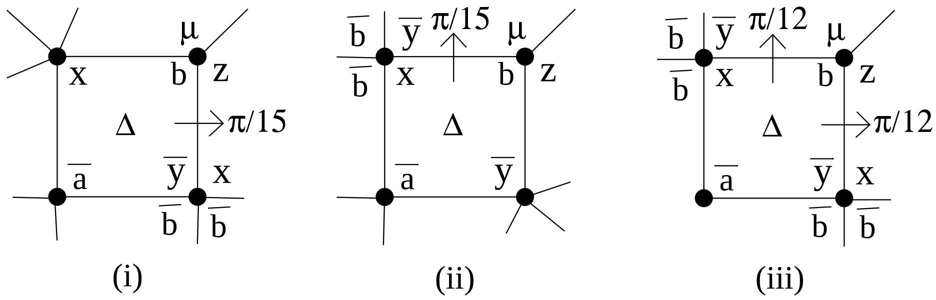


Figure 4.21

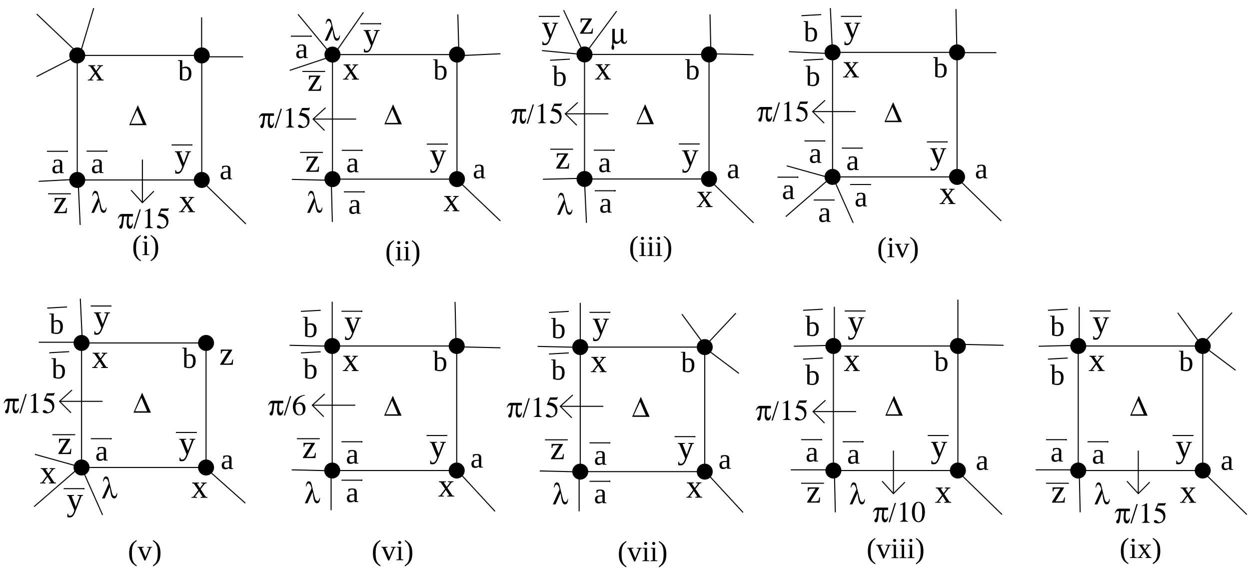
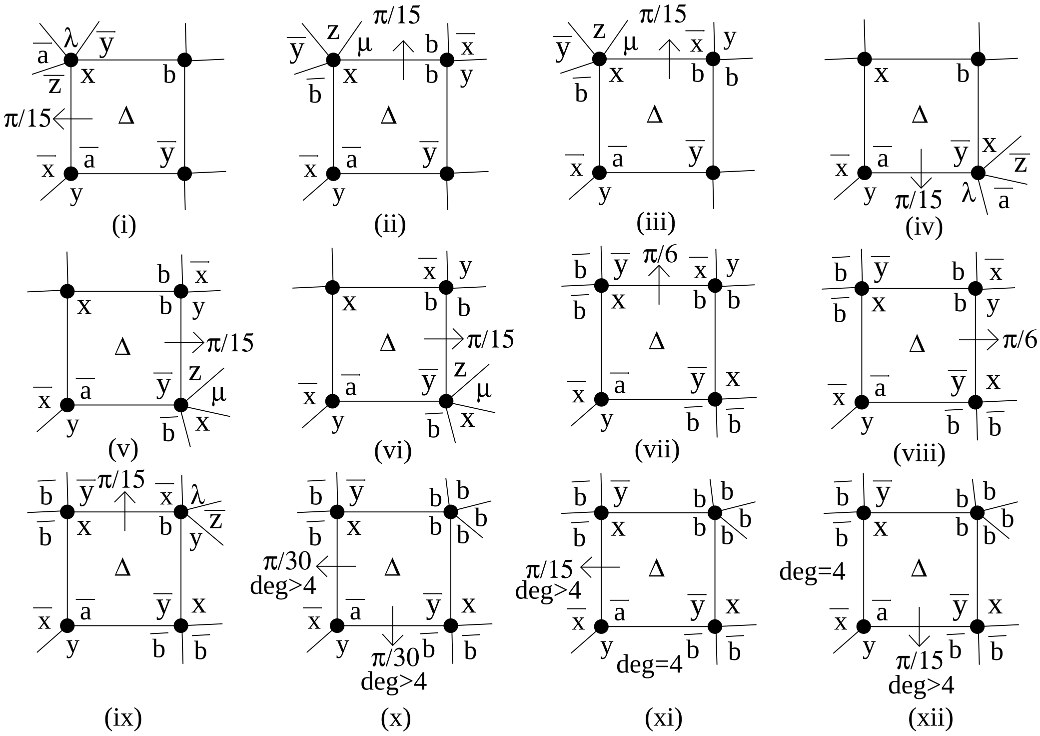


Figure 4.22



(i)
π/15
(ii)
π/15
(iii)
π/15
(iv)
π/15
(v)
π/15
(vi)
π/15
(vii)
π/6
(viii)
π/6
(ix)
π/15
(x)
π/30
deg>4
π/30
deg>4
(xi)
π/15
deg>4
deg=4
(xii)
deg=4
π/15
deg>4

Figure 4.23

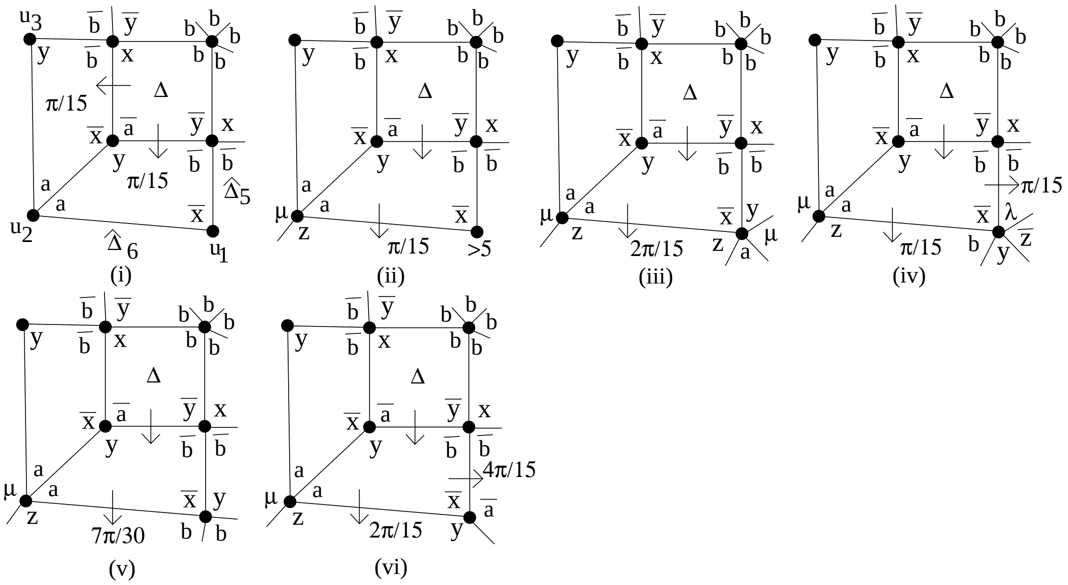


Figure 4.24

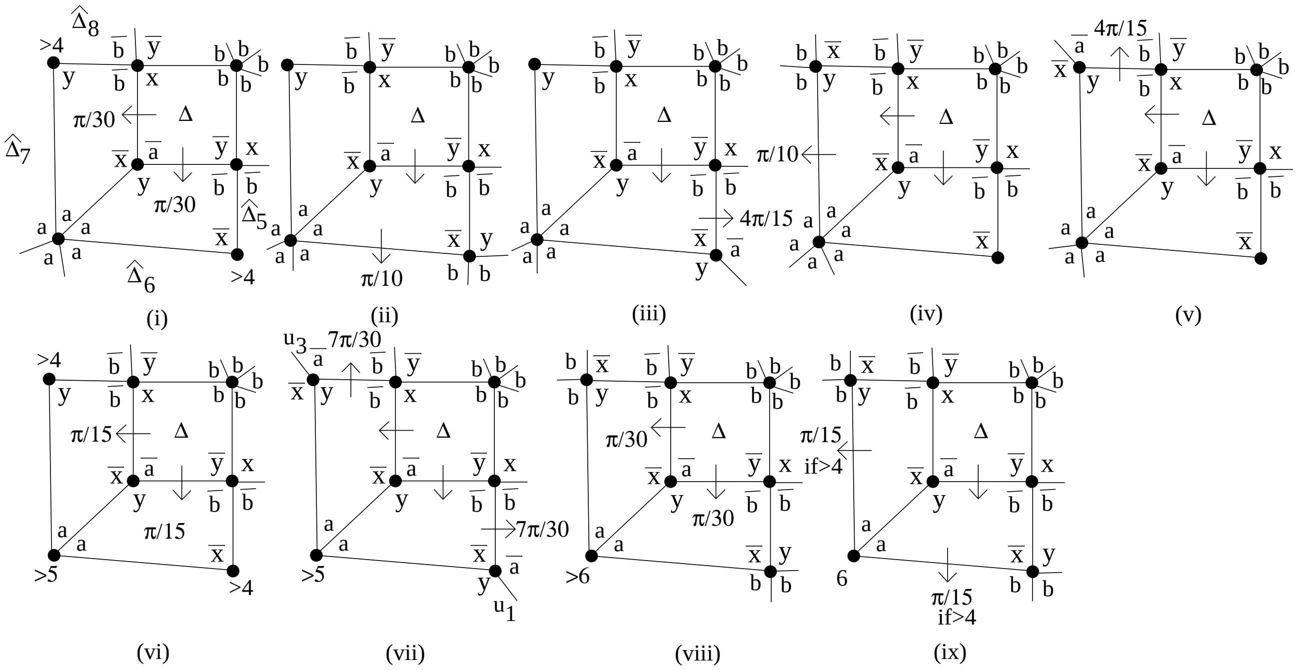


Figure 4.25

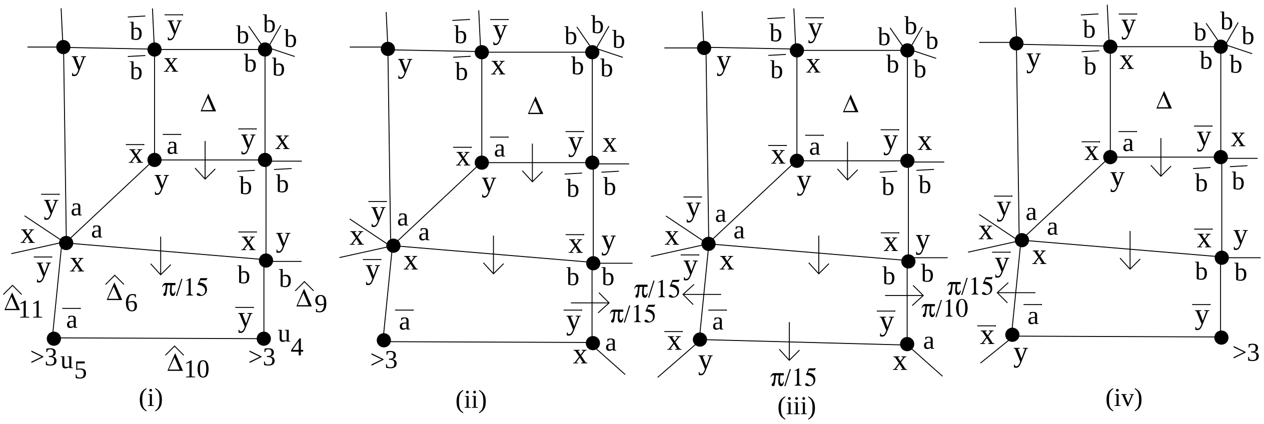


Figure 4.26

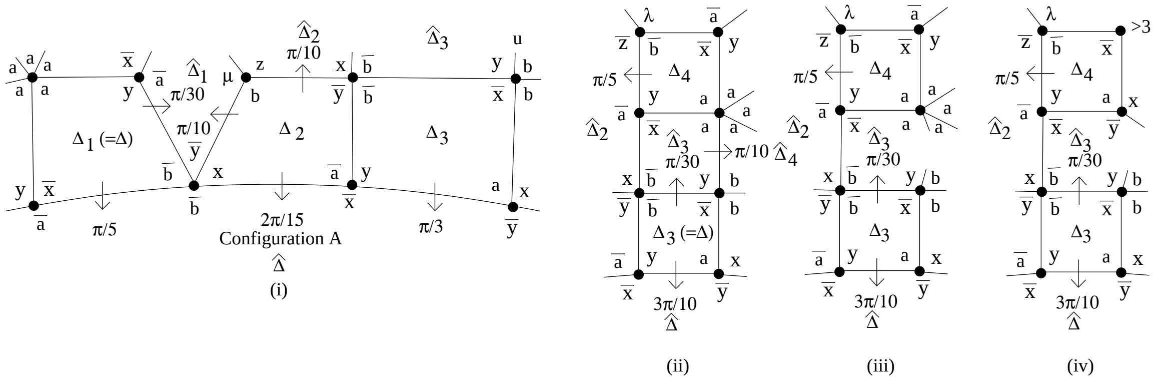


Figure 4.27

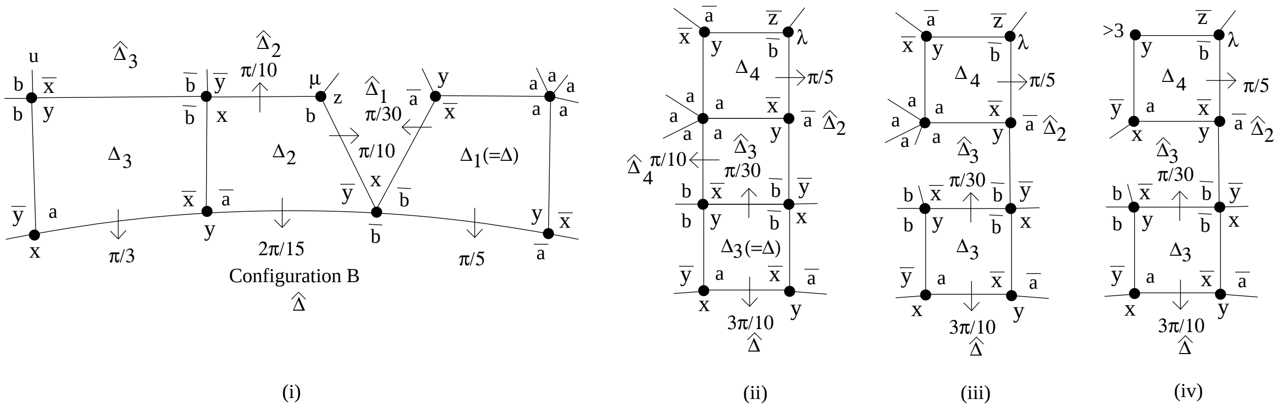
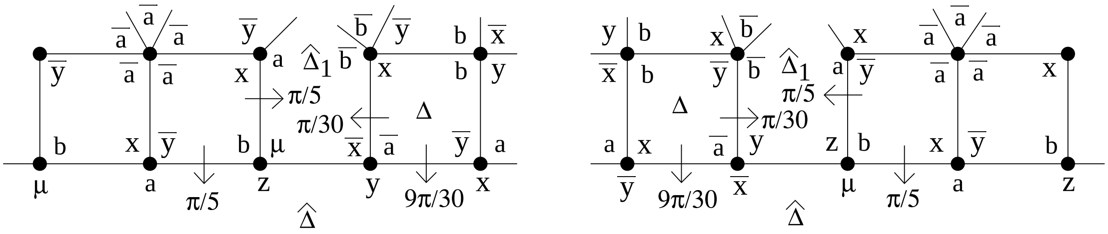


Figure 4.28



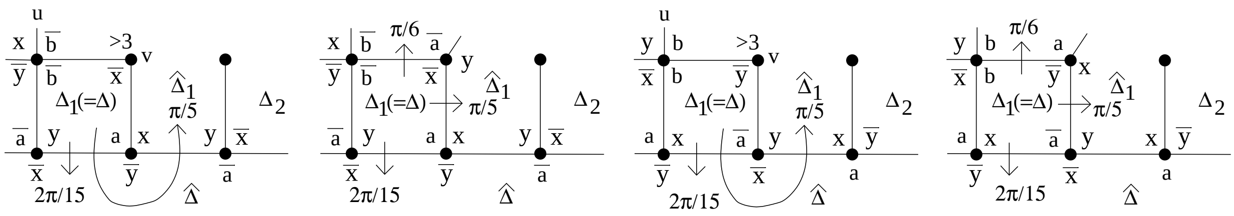
Configuration C

(i)

Configuration D

(ii)

Figure 4.29



(i) (ii) (iii) (iv)

Configuration E Configuration F

Figure 4.30

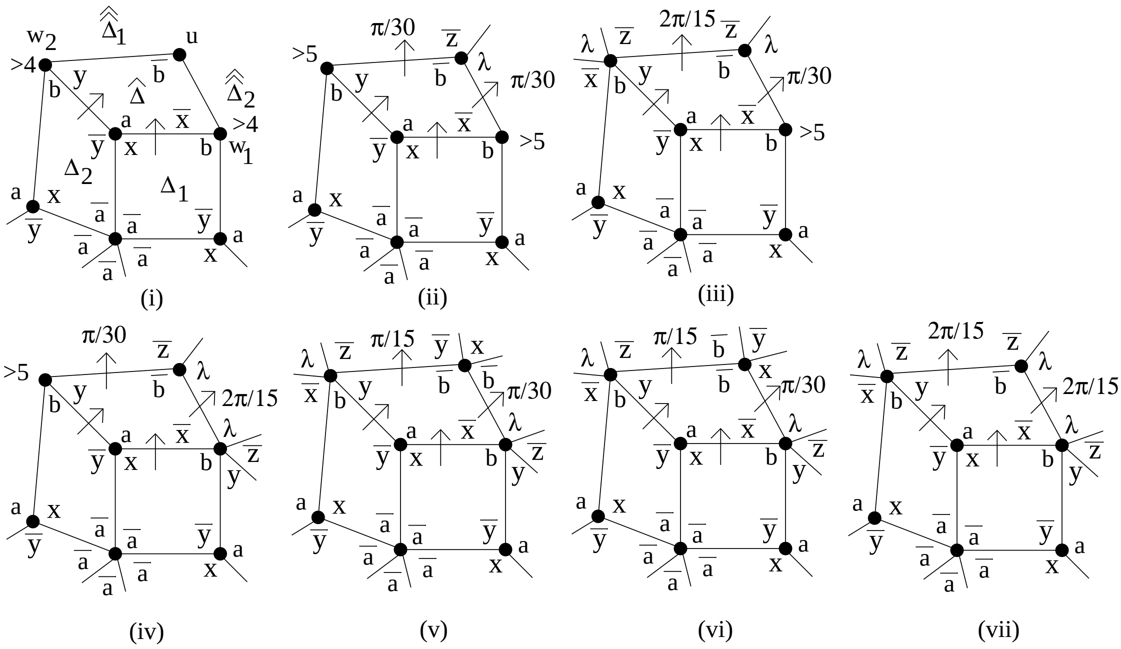
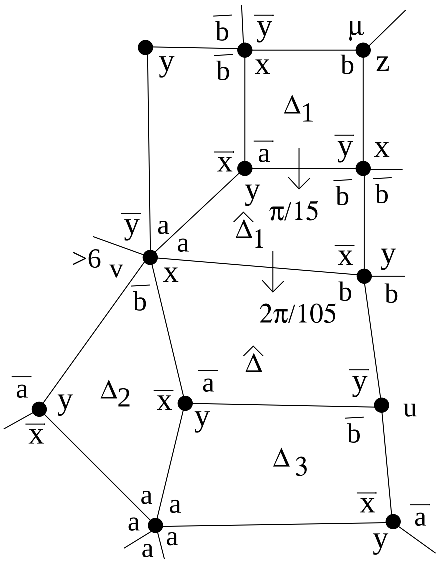
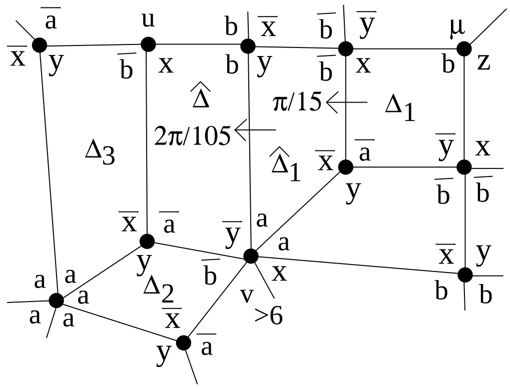


Figure 4.31



(i)



(ii)

Figure 4.32

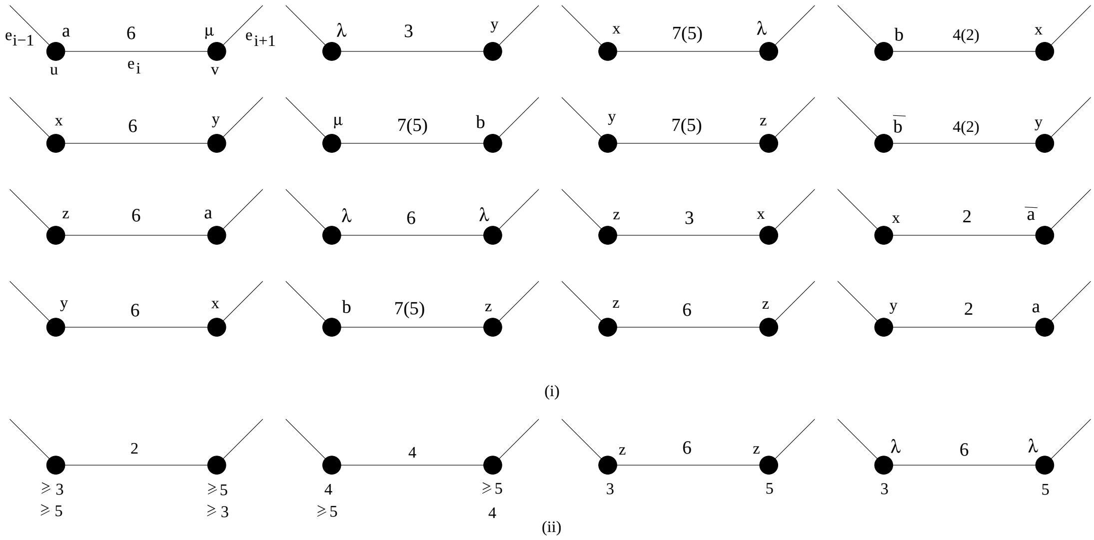


Figure 4.33

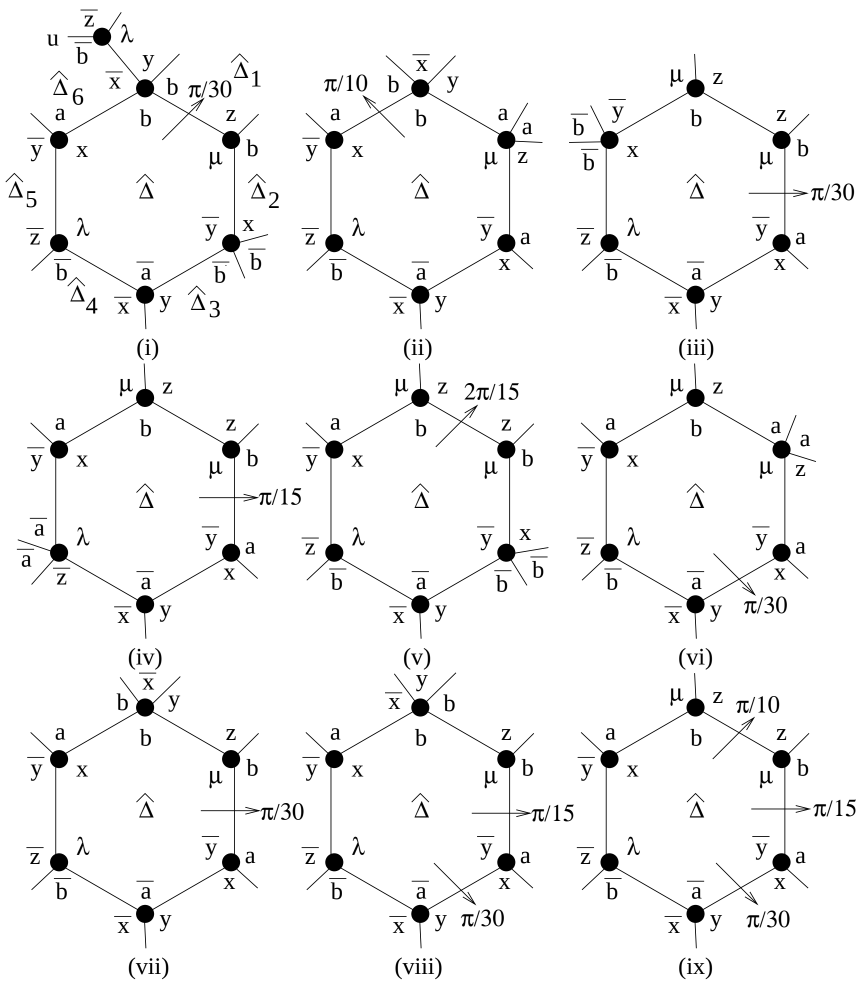


Figure 5.1

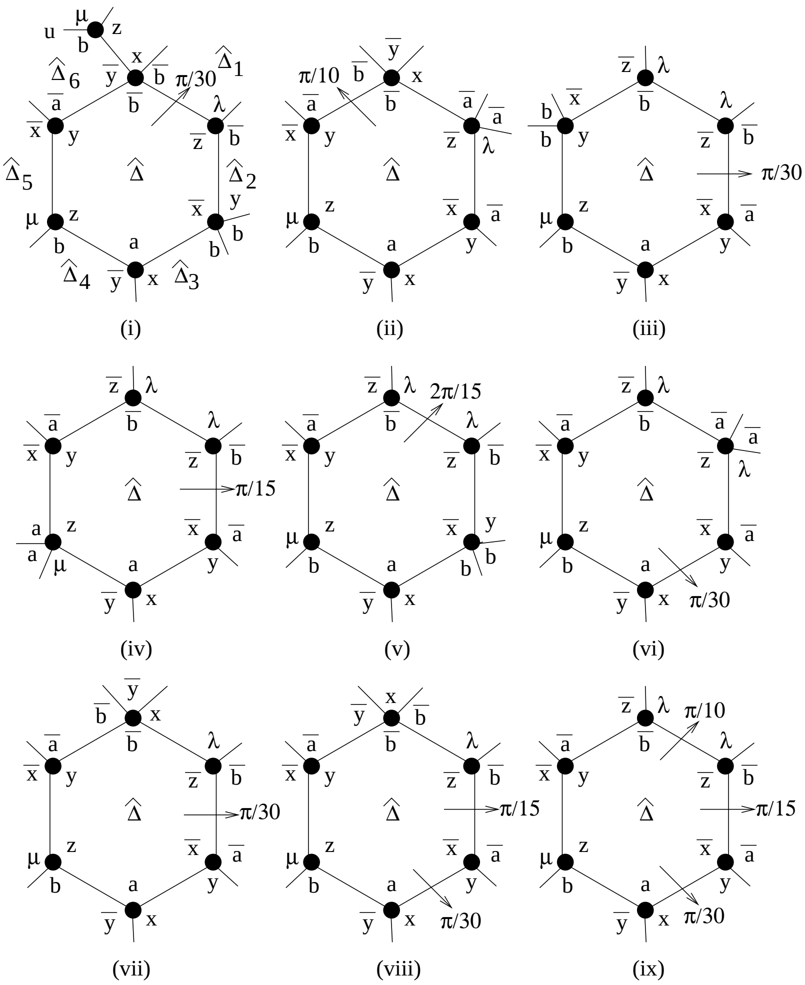

π/30
(i)
π/10
(ii)
π/30
(iii)
π/15
(iv)
2π/15
(v)
π/30
(vi)
π/30
(vii)
π/15
π/30
(viii)
π/10
π/15
π/30
(ix)

Figure 5.2

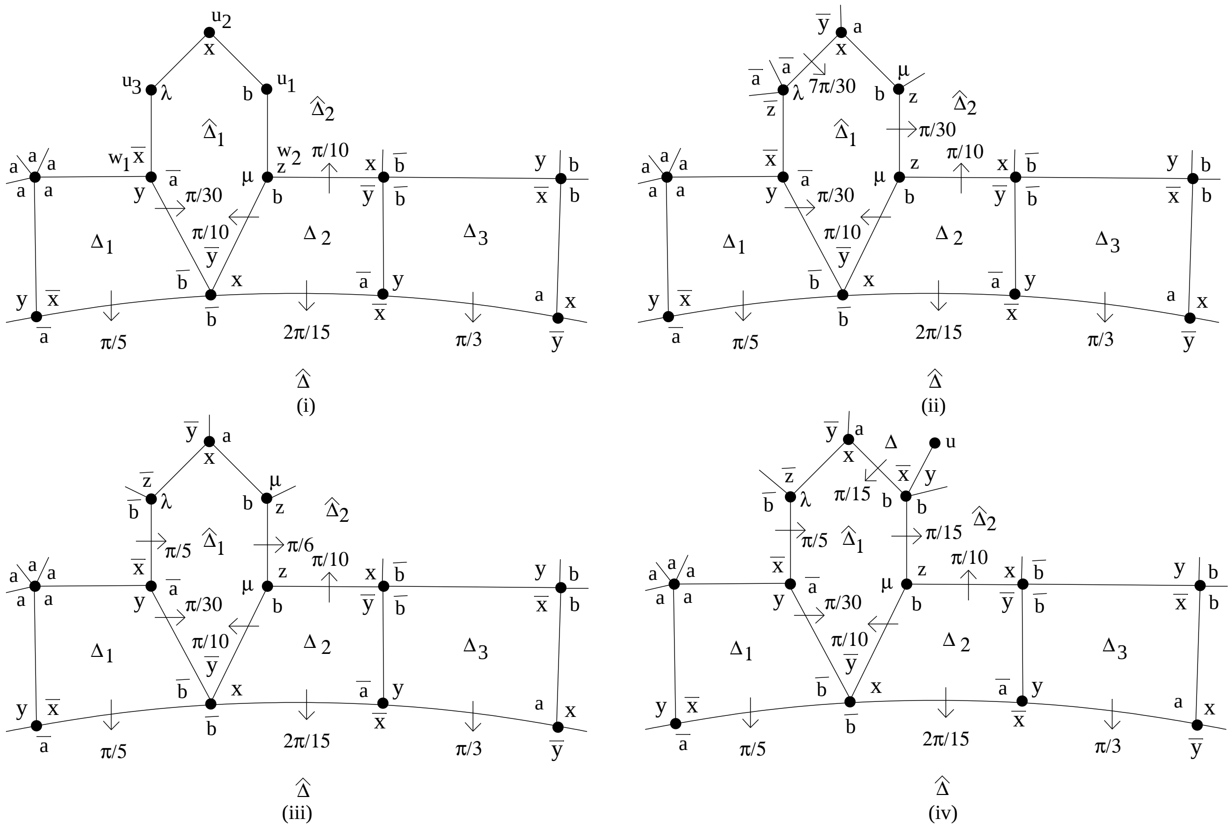


Figure 5.3

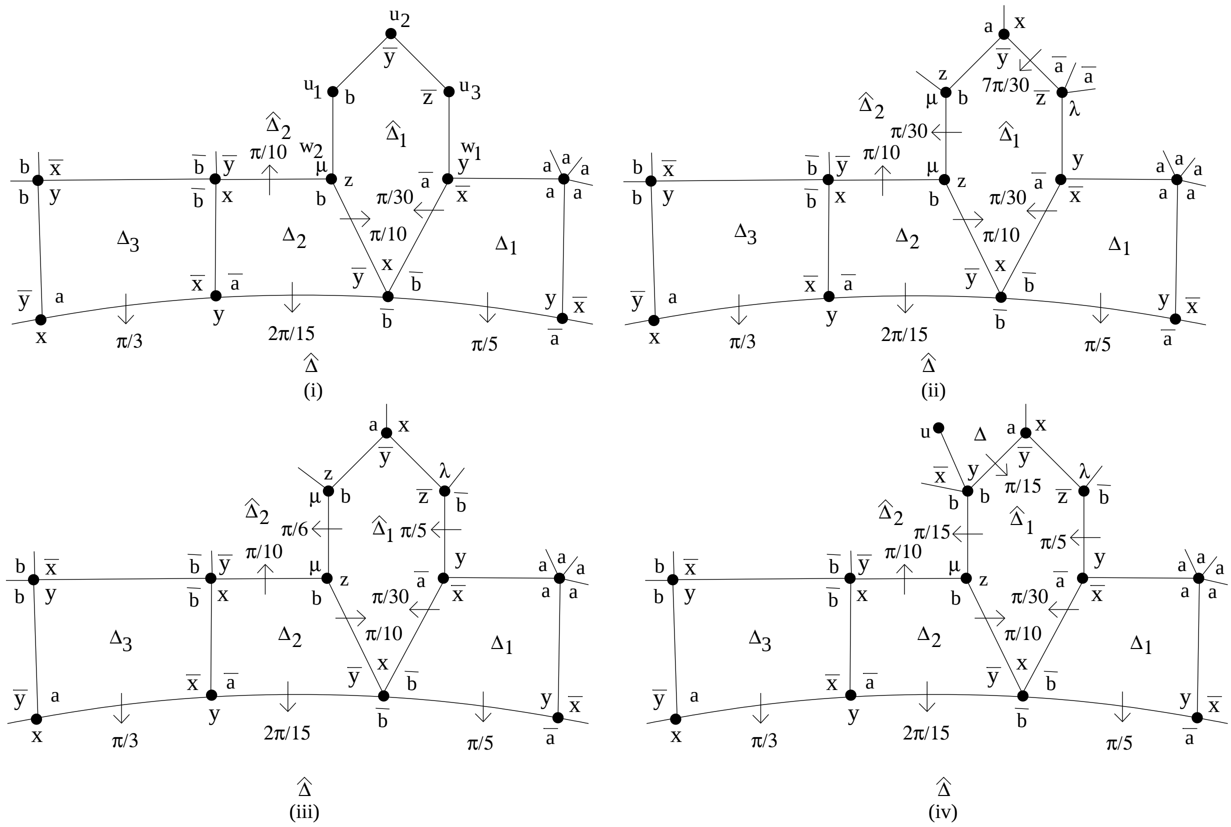


Figure 5.4

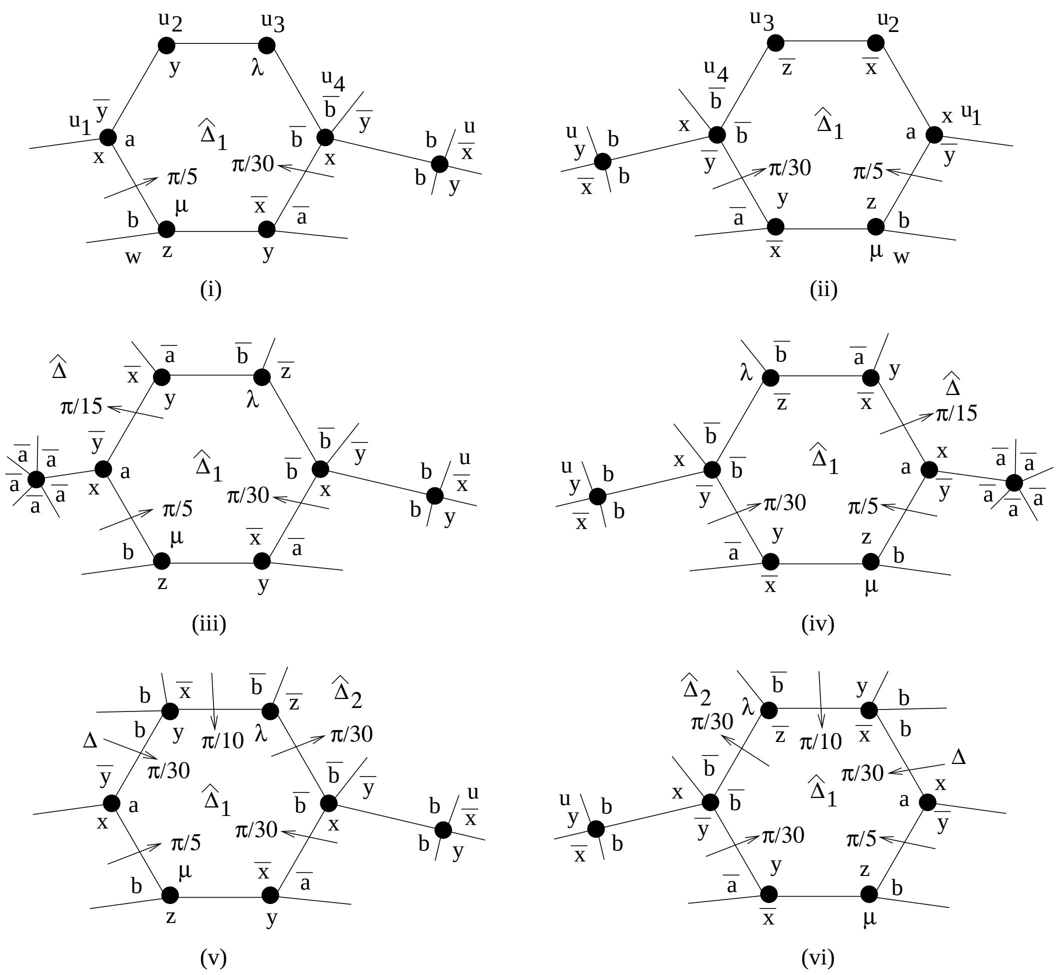


Figure 5.5

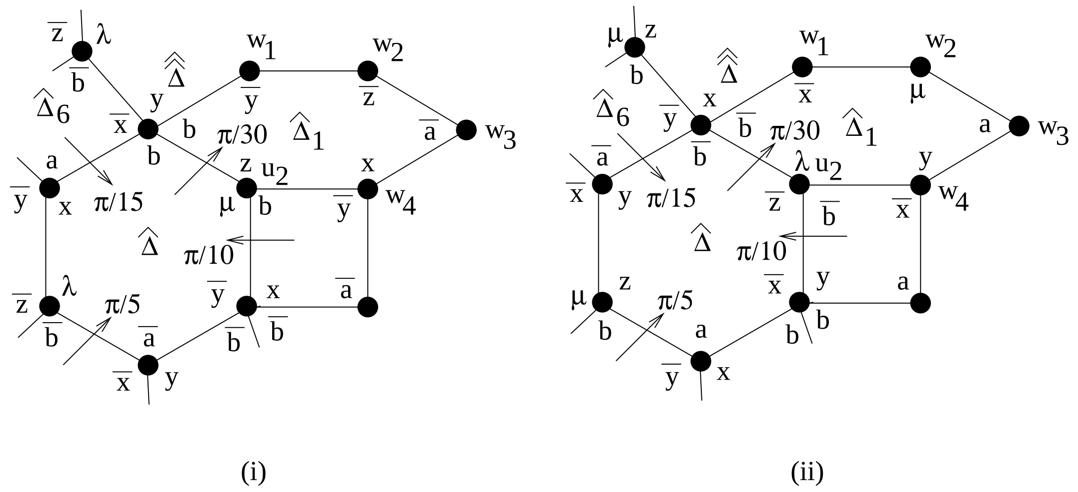


Figure 5.6

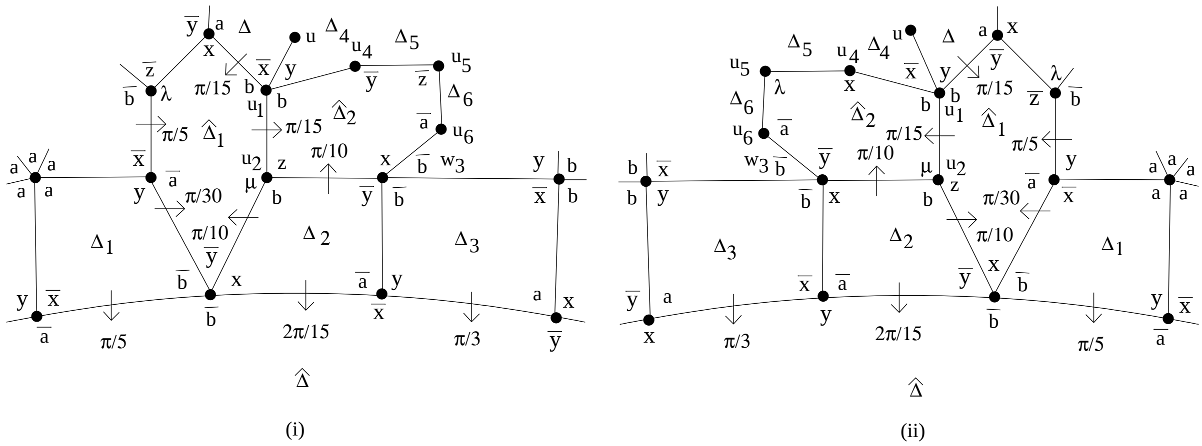


Figure 5.7

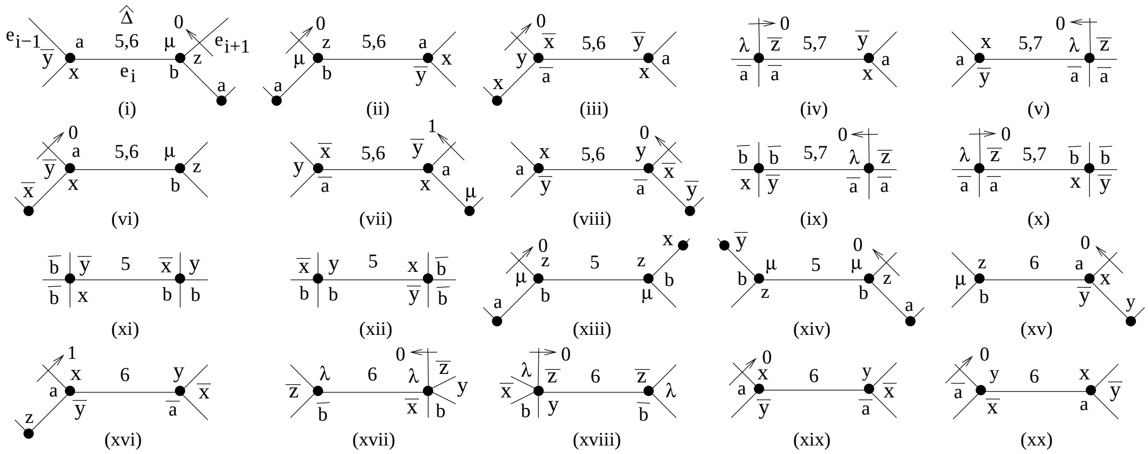

(i)
(ii)
(iii)
(iv)
(v)
(vi)
(vii)
(viii)
(ix)
(x)
(xi)
(xii)
(xiii)
(xiv)
(xv)
(xvi)
(xvii)
(xviii)
(xix)
(xx)

Figure 5.8

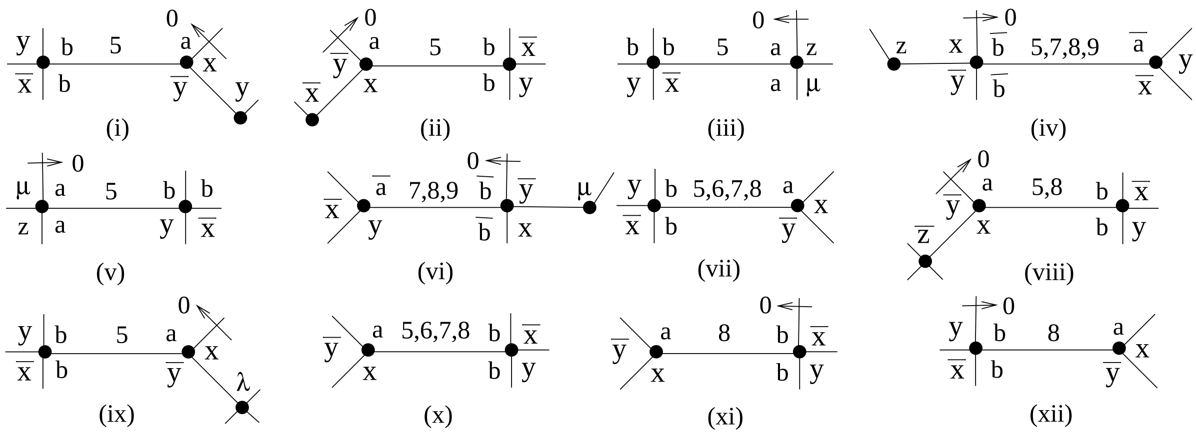


Figure 5.9

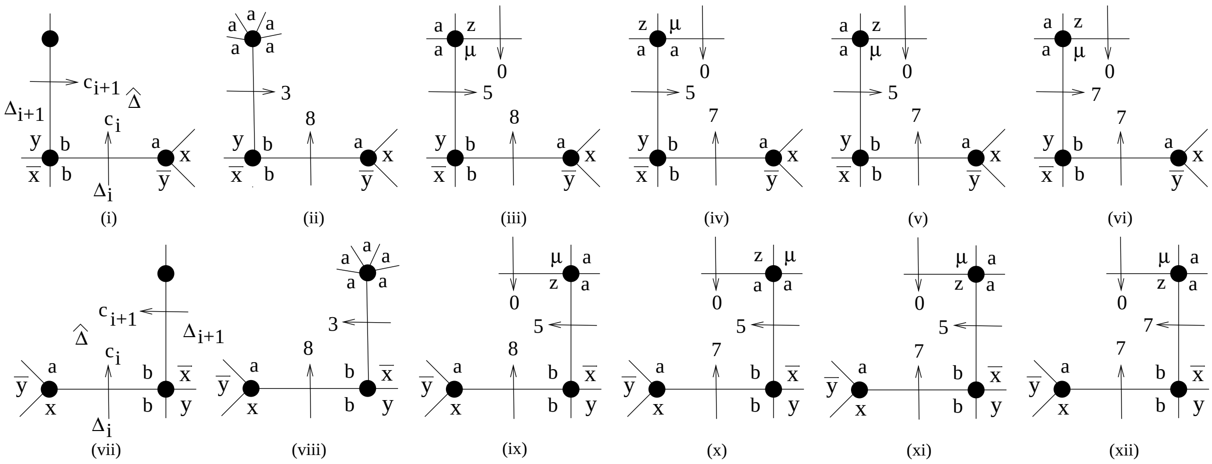
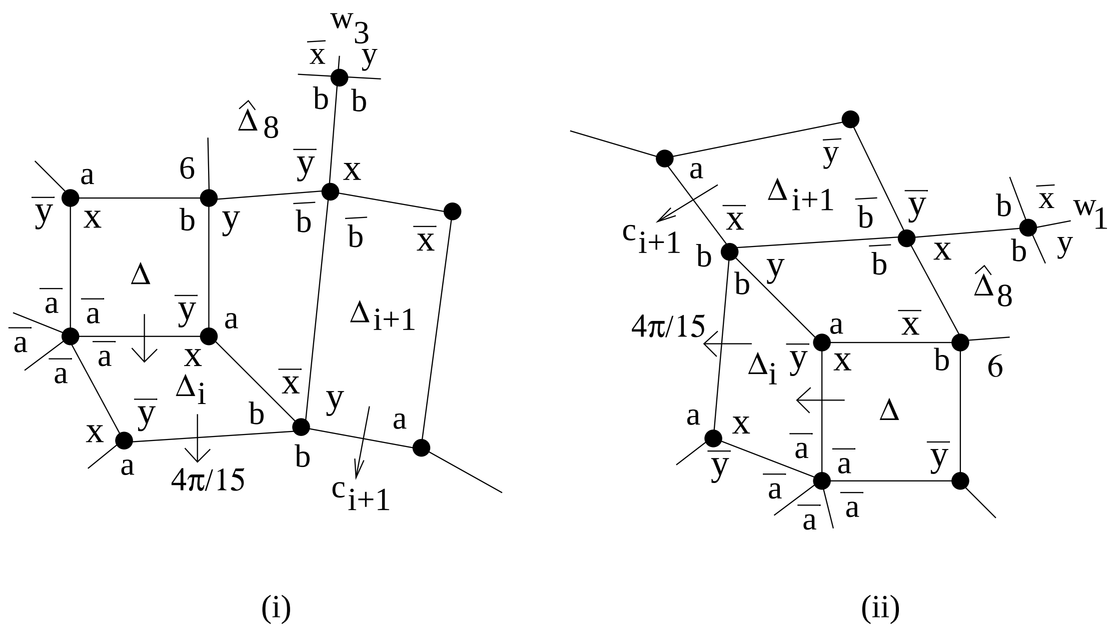


Figure 5.10



w3
x̄
y
b
b
$\hat{\Delta}_8$
a
6
ȳ
x
ȳ
x
b
y
b̄
b̄
x̄
Δ
ā
ā
ȳ
a
ā
ā
x
ā
Δ_{i+1}
Δ_i
x̄
y
x
ȳ
b
a
a
b
4π/15
c_{i+1}
(i)
a
ȳ
Δ_{i+1}
x̄
b̄
ȳ
b
x̄
w1
c_{i+1}
b
y
b̄
x
b
y
b
$\hat{\Delta}_8$
4π/15
a
x̄
Δ_i
ȳ
x
b
6
a
x
Δ
ā
ā
ȳ
ȳ
ā
ā
ā
(ii)

Figure 5.11

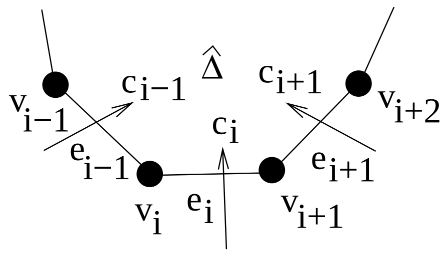


Figure 6.1

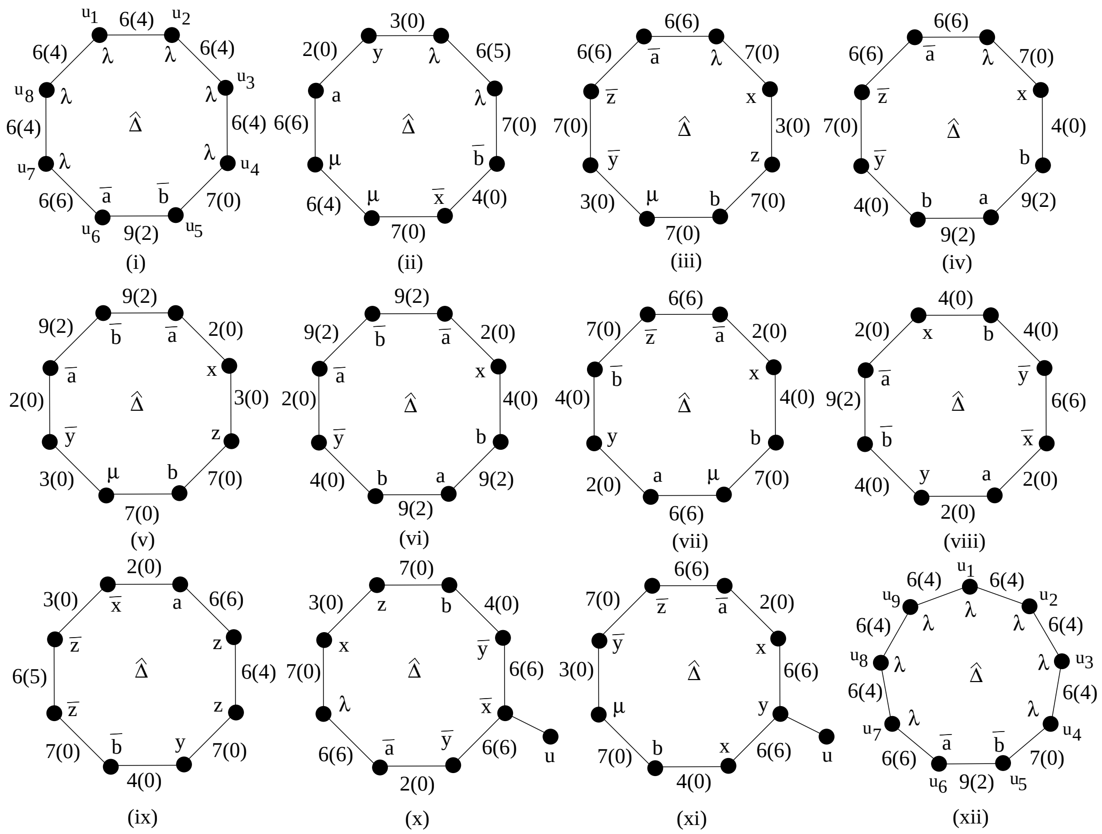


Figure 6.2

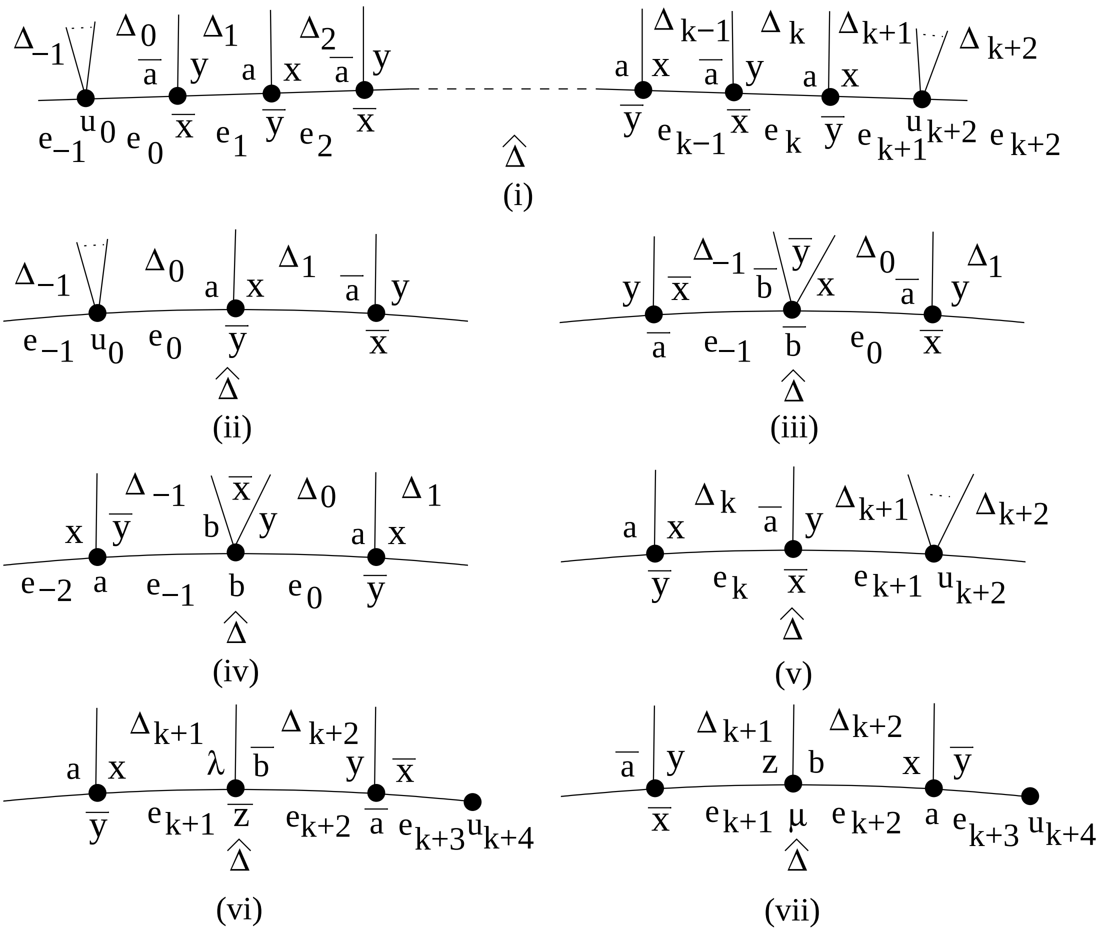
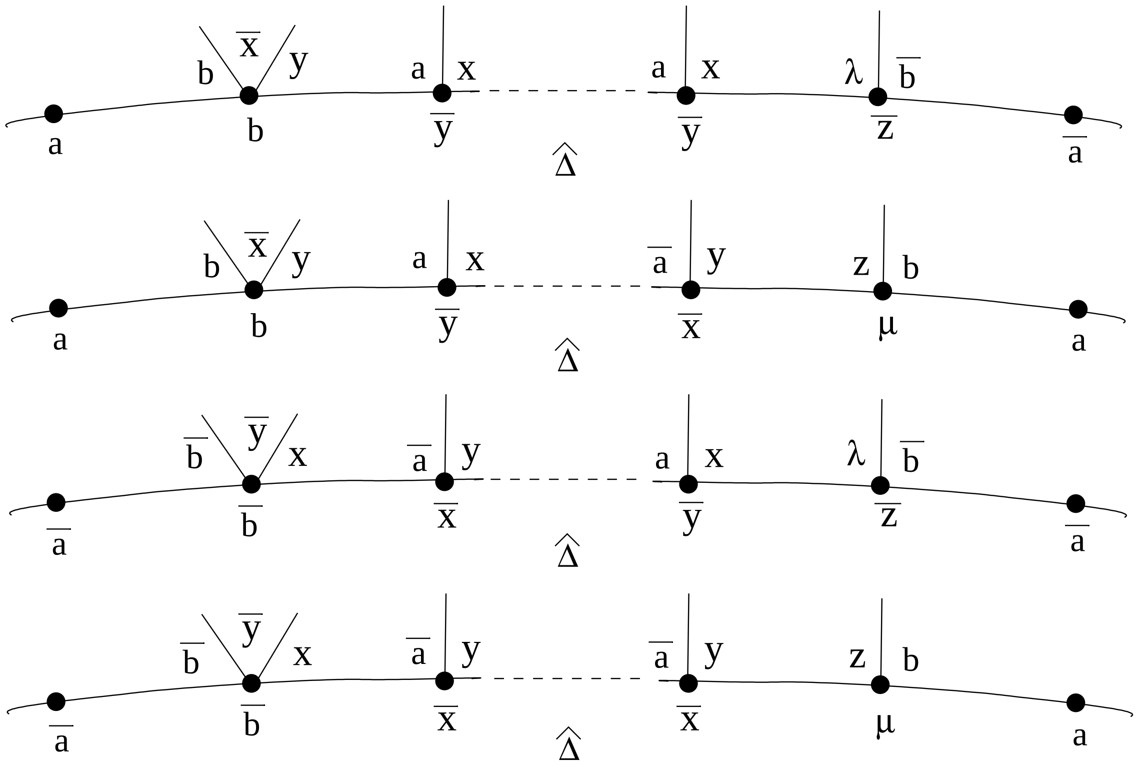


Figure 7.1



b $\bar{x}$ y a x a x λ $\bar{b}$
a b $\bar{y}$ $\bar{y}$ $\bar{z}$ $\bar{a}$
$\hat{\Delta}$
b $\bar{x}$ y a x $\bar{a}$ y z b
a b $\bar{y}$ $\bar{x}$ μ a
$\hat{\Delta}$
$\bar{b}$ $\bar{y}$ x $\bar{a}$ y a x λ $\bar{b}$
$\bar{a}$ $\bar{b}$ $\bar{x}$ $\bar{y}$ $\bar{z}$ $\bar{a}$
$\hat{\Delta}$
$\bar{b}$ $\bar{y}$ x $\bar{a}$ y $\bar{a}$ y z b
$\bar{a}$ $\bar{b}$ $\bar{x}$ $\bar{x}$ μ a
$\hat{\Delta}$

Figure 7.2

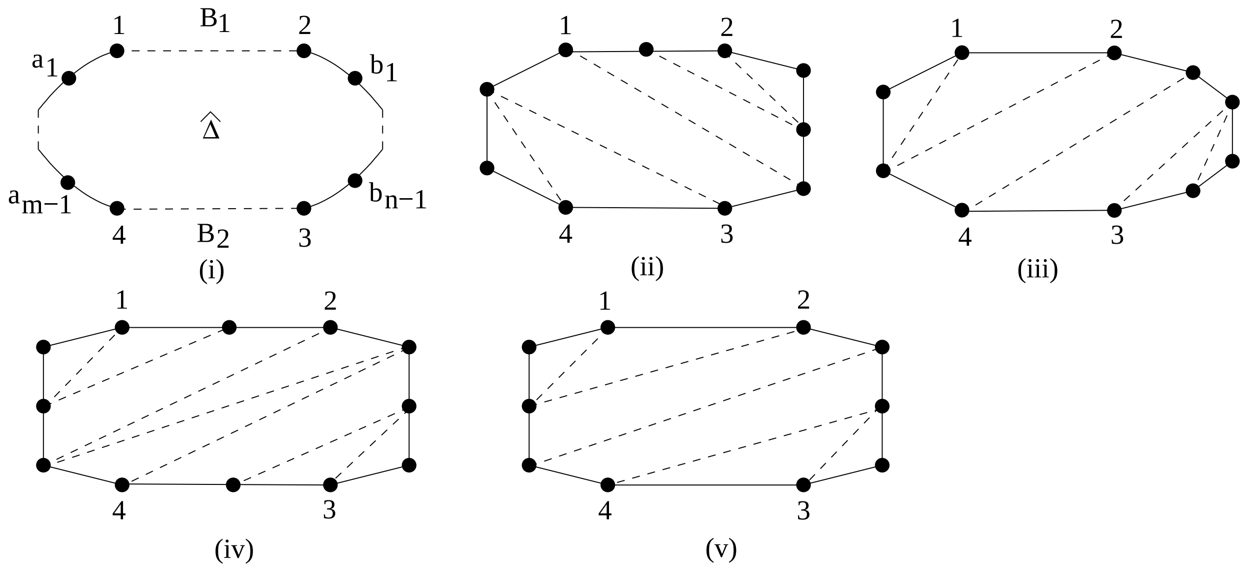

1
B_1
2
a_1
b_1
$\widehat{\Delta}$
a_{m-1}
b_{n-1}
4
B_2
3
(i)
1
2
4
3
(ii)
1
2
4
3
(iii)
1
2
4
3
(iv)
1
2
4
3
(v)

Figure 7.3

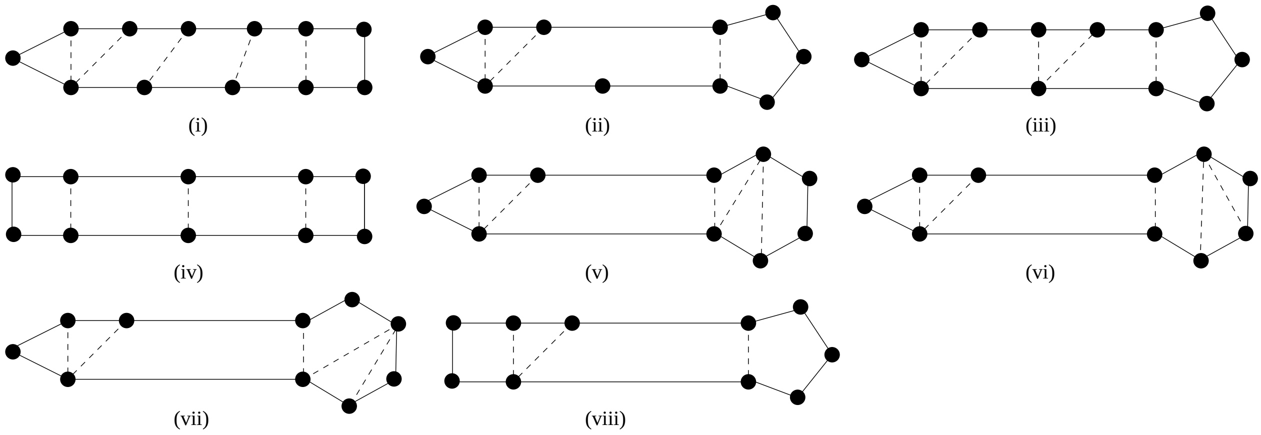


Figure 7.4

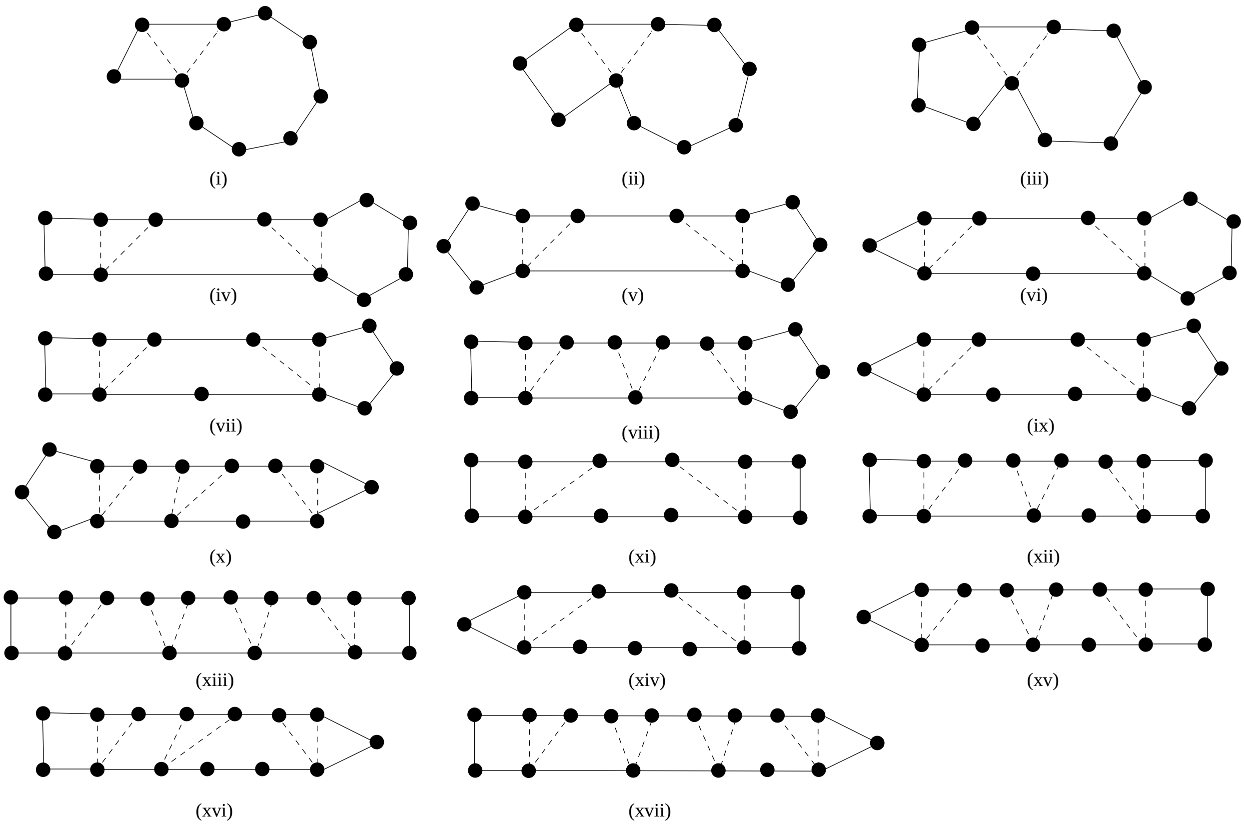
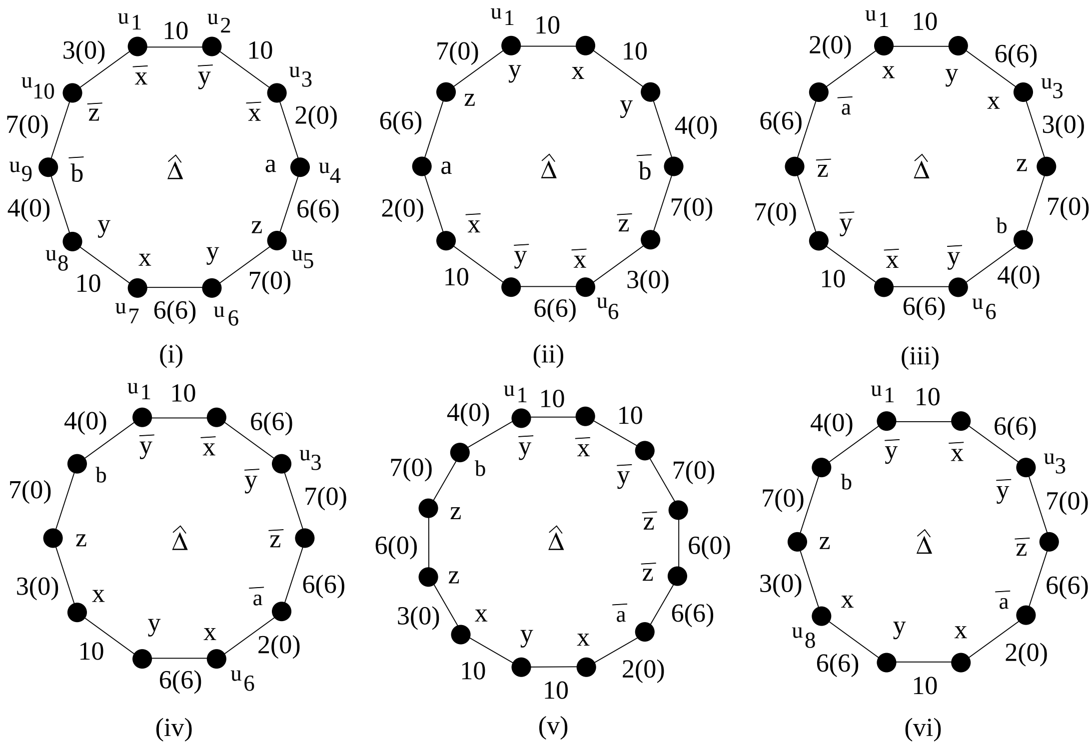


Figure 7.5



u1 10 u2 10 u3 2(0) u4 6(6) u5 7(0) u6 6(6) u7 10 u8 4(0) u9 7(0) u10 3(0)
x̄ ȳ x̄ a z y x y b̄ z̄
Δ̂
(i)
u1 10 10 4(0) 7(0) 3(0) u6 6(6) 10 2(0) 6(6) 7(0)
y x y b̄ z̄ x̄ ȳ x̄ a z
Δ̂
(ii)
u1 10 6(6) u3 3(0) 7(0) 4(0) u6 6(6) 10 7(0) 6(6) 2(0)
x y x z b ȳ x̄ ȳ z̄ ā
Δ̂
(iii)
u1 10 6(6) u3 7(0) 6(6) 2(0) u6 6(6) 10 3(0) 7(0) 4(0)
ȳ x̄ ȳ z̄ ā x y x z b
Δ̂
(iv)
u1 10 10 7(0) 6(0) 6(6) 2(0) 10 10 3(0) 6(0) 7(0) 4(0)
ȳ x̄ ȳ z̄ z̄ ā x y x z z b
Δ̂
(v)
u1 10 6(6) u3 7(0) 6(6) 2(0) 10 6(6) u8 3(0) 7(0) 4(0)
ȳ x̄ ȳ z̄ ā x y x z b
Δ̂
(vi)

Figure 7.6

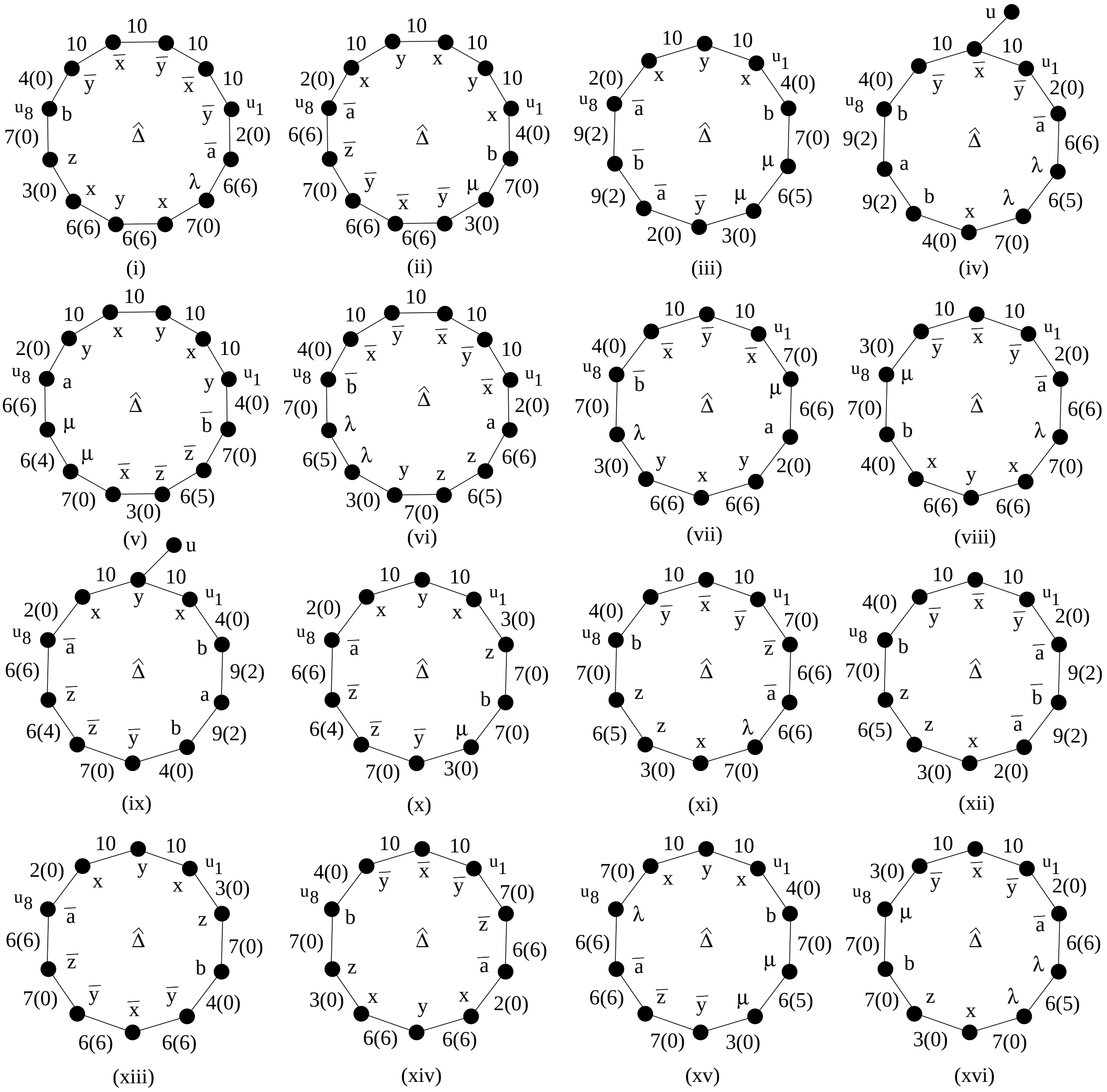


Figure 7.7

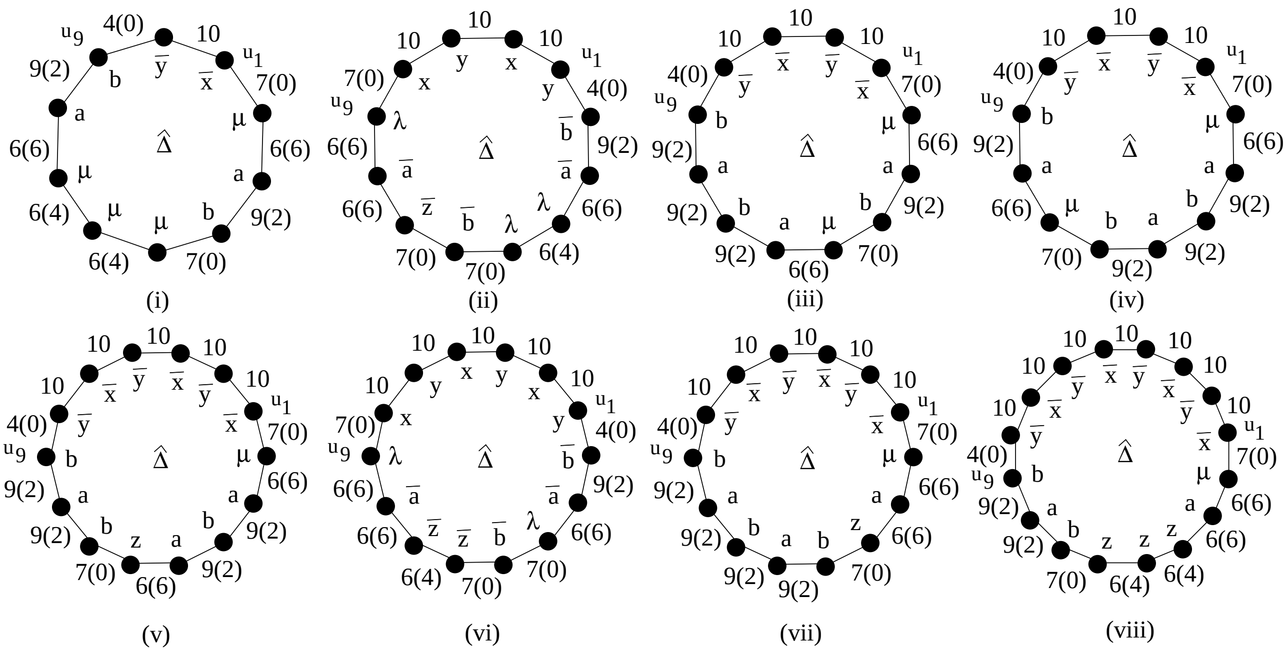


Figure 7.8

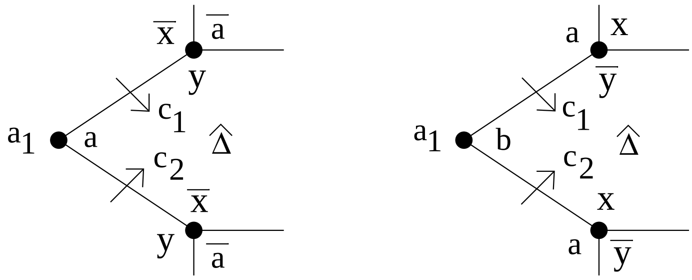


Figure 7.9

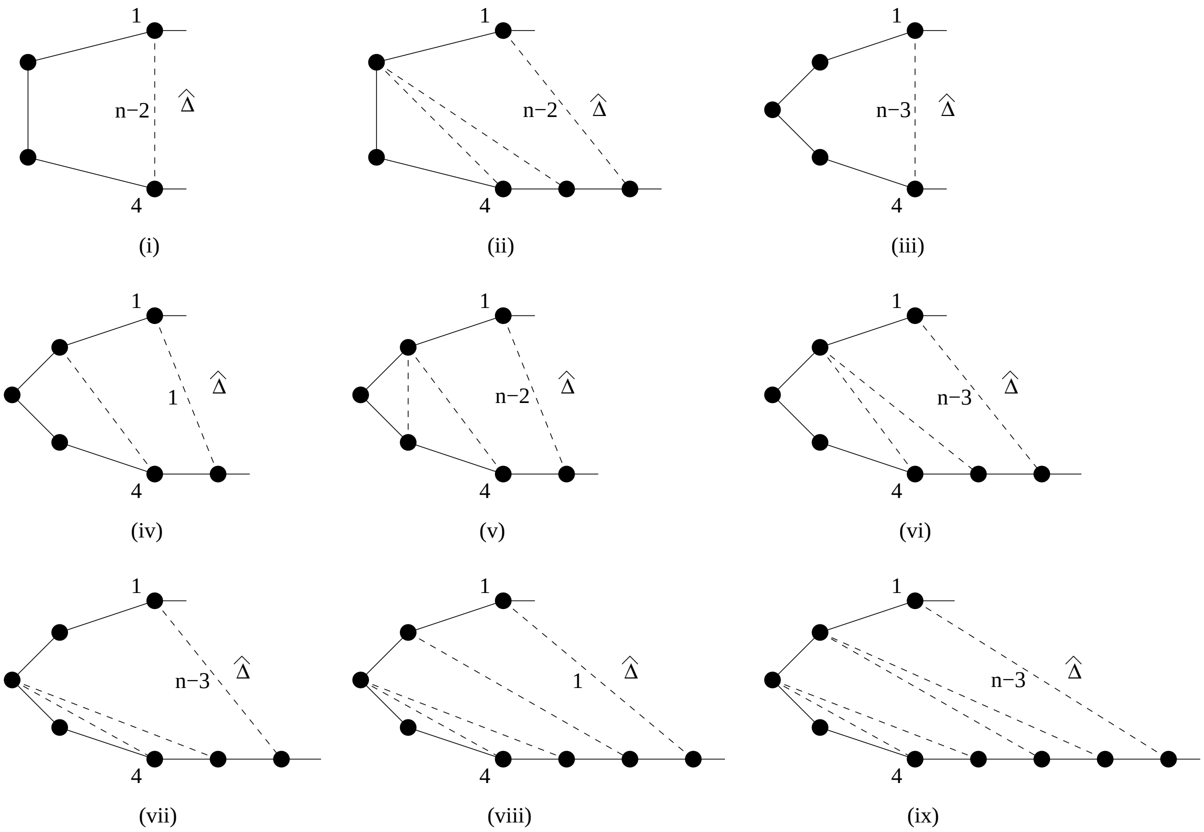

1
n−2
$\widehat{\Delta}$
4
(i)
1
n−2
$\widehat{\Delta}$
4
(ii)
1
n−3
$\widehat{\Delta}$
4
(iii)
1
1
$\widehat{\Delta}$
4
(iv)
1
n−2
$\widehat{\Delta}$
4
(v)
1
n−3
$\widehat{\Delta}$
4
(vi)
1
n−3
$\widehat{\Delta}$
4
(vii)
1
1
$\widehat{\Delta}$
4
(viii)
1
n−3
$\widehat{\Delta}$
4
(ix)

Figure 7.10

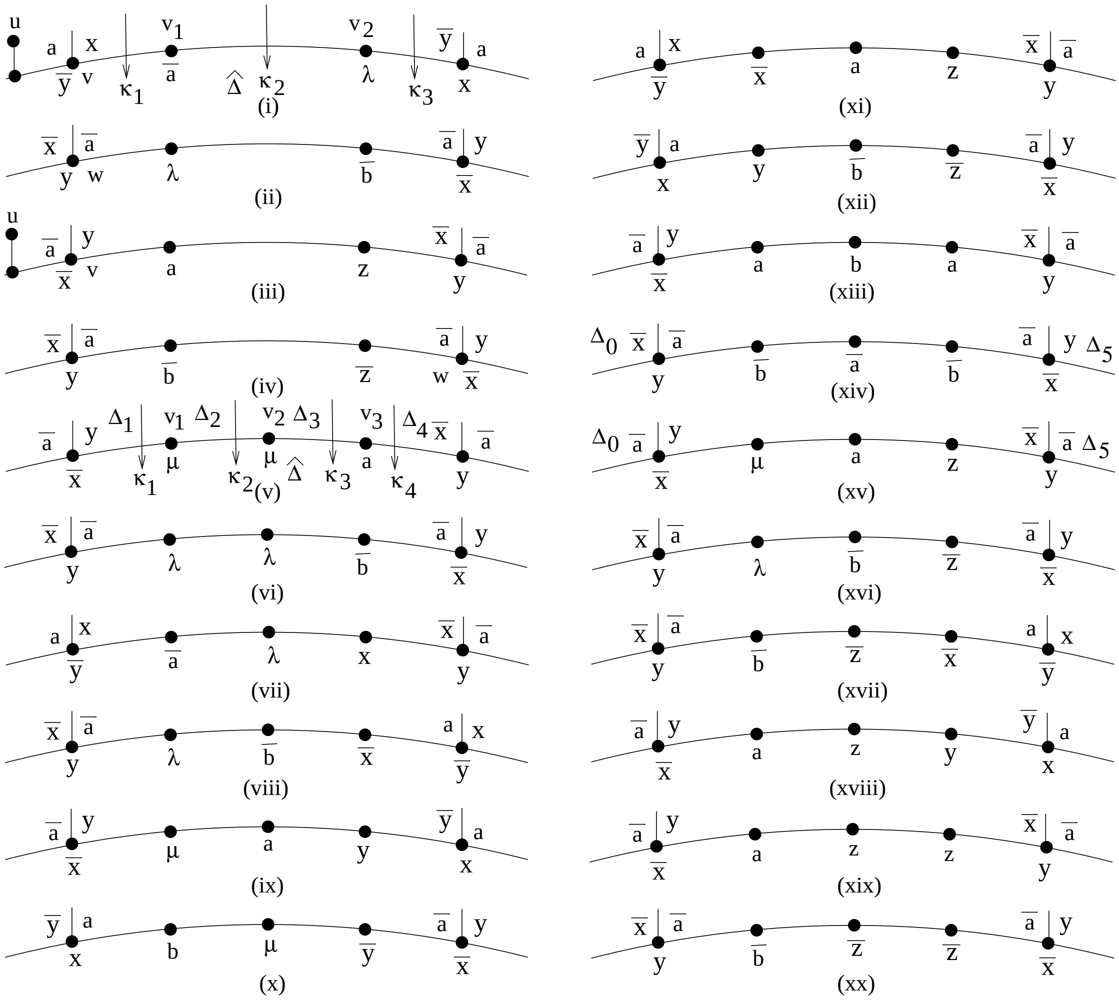

(i)
(ii)
(iii)
(iv)
(v)
(vi)
(vii)
(viii)
(ix)
(x)
(xi)
(xii)
(xiii)
(xiv)
(xv)
(xvi)
(xvii)
(xviii)
(xix)
(xx)

Figure 7.11

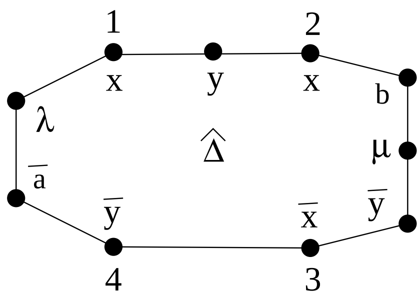


Figure 7.12

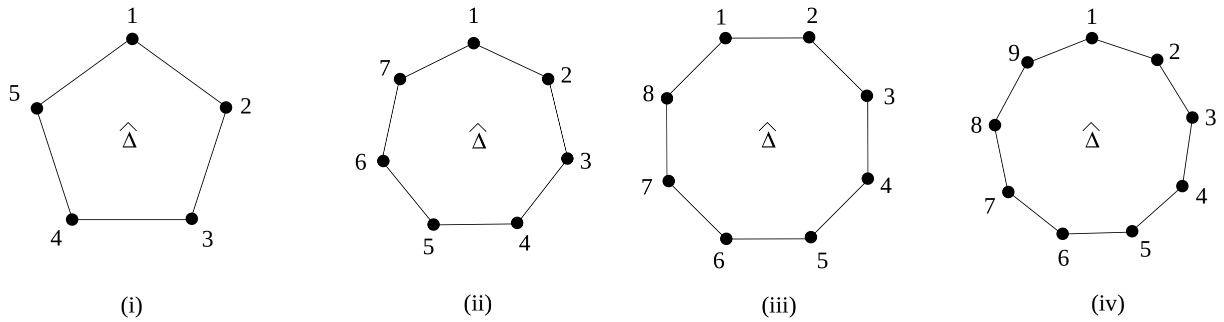
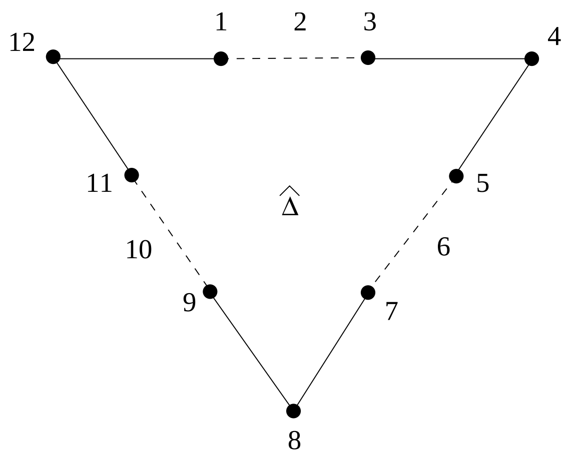
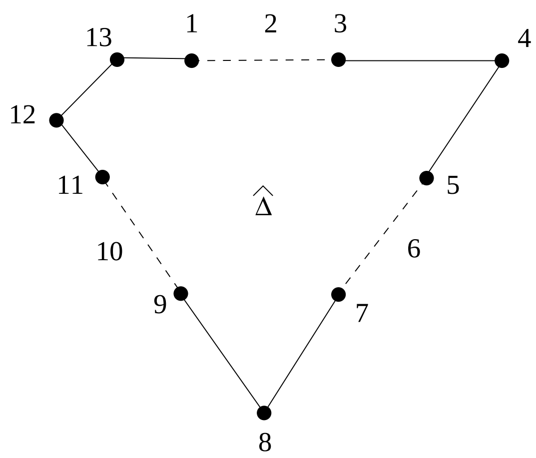

(i) 1 2 3 4 5 $\widehat{\Delta}$
(ii) 1 2 3 4 5 6 7 $\widehat{\Delta}$
(iii) 1 2 3 4 5 6 7 8 $\widehat{\Delta}$
(iv) 1 2 3 4 5 6 7 8 9 $\widehat{\Delta}$

Figure A.1



(i)



(ii)

Figure A.2

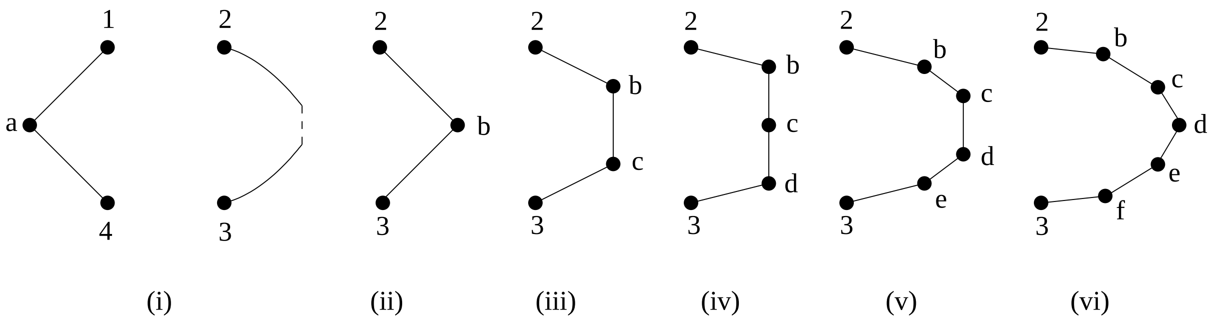


Figure A.3

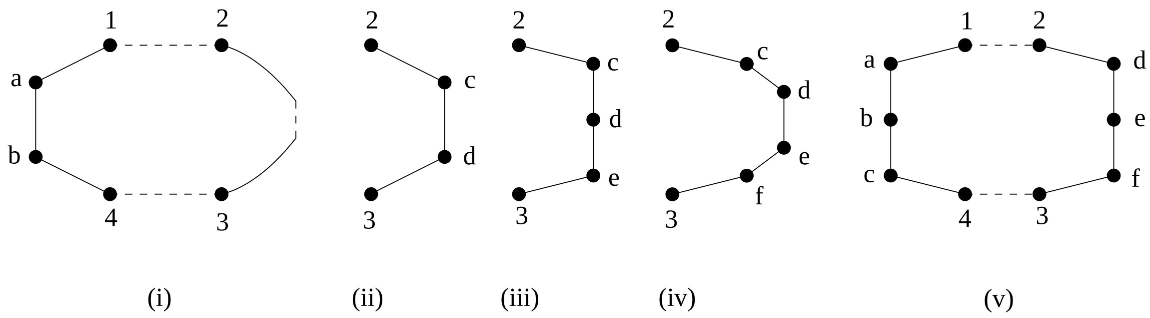
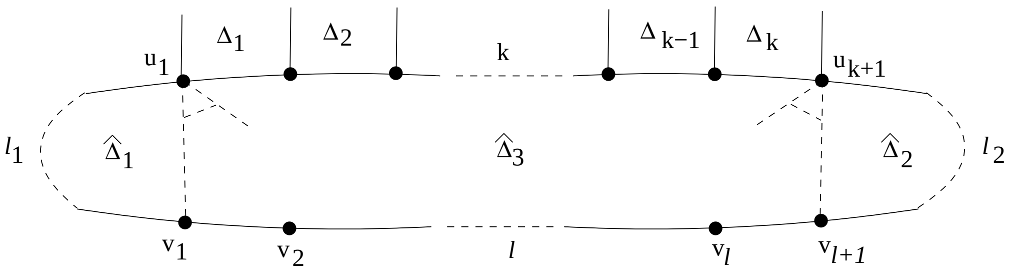


Figure A.4



Type B region $\widehat{\Delta}$

Figure A.5

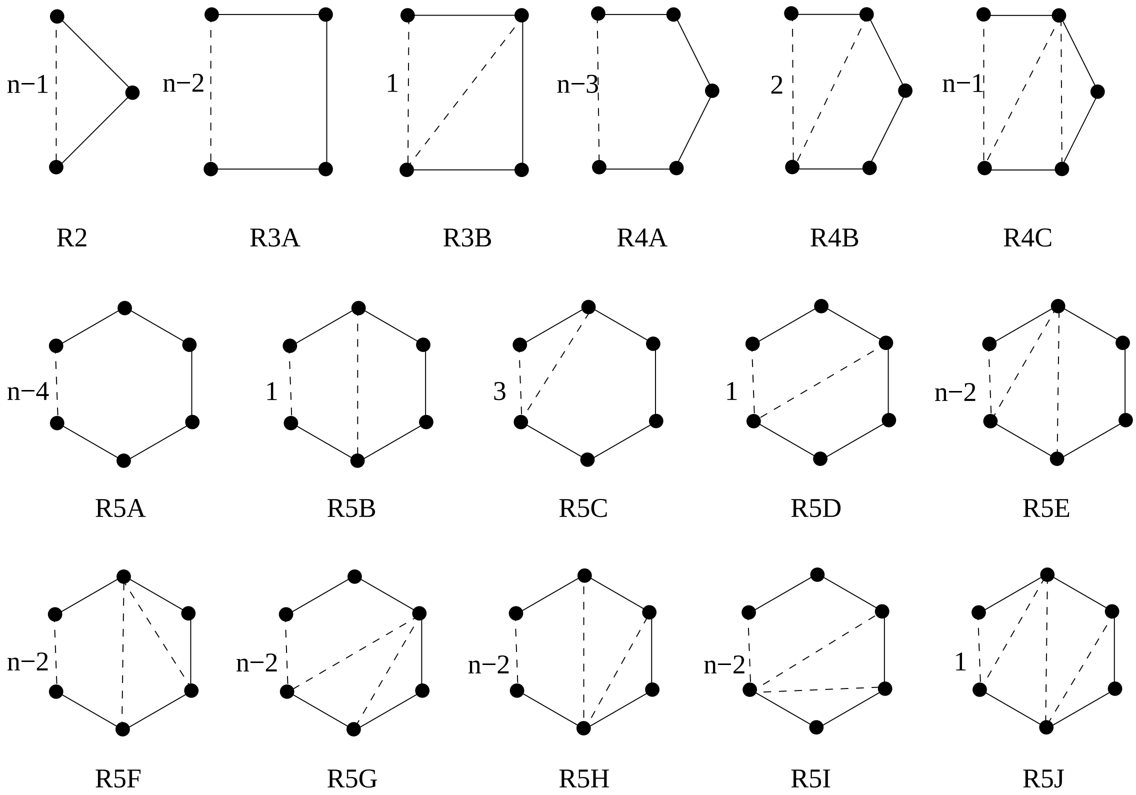

n−1
n−2
1
n−3
2
n−1
R2
R3A
R3B
R4A
R4B
R4C
n−4
1
3
1
n−2
R5A
R5B
R5C
R5D
R5E
n−2
n−2
n−2
n−2
1
R5F
R5G
R5H
R5I
R5J

Figure A.6

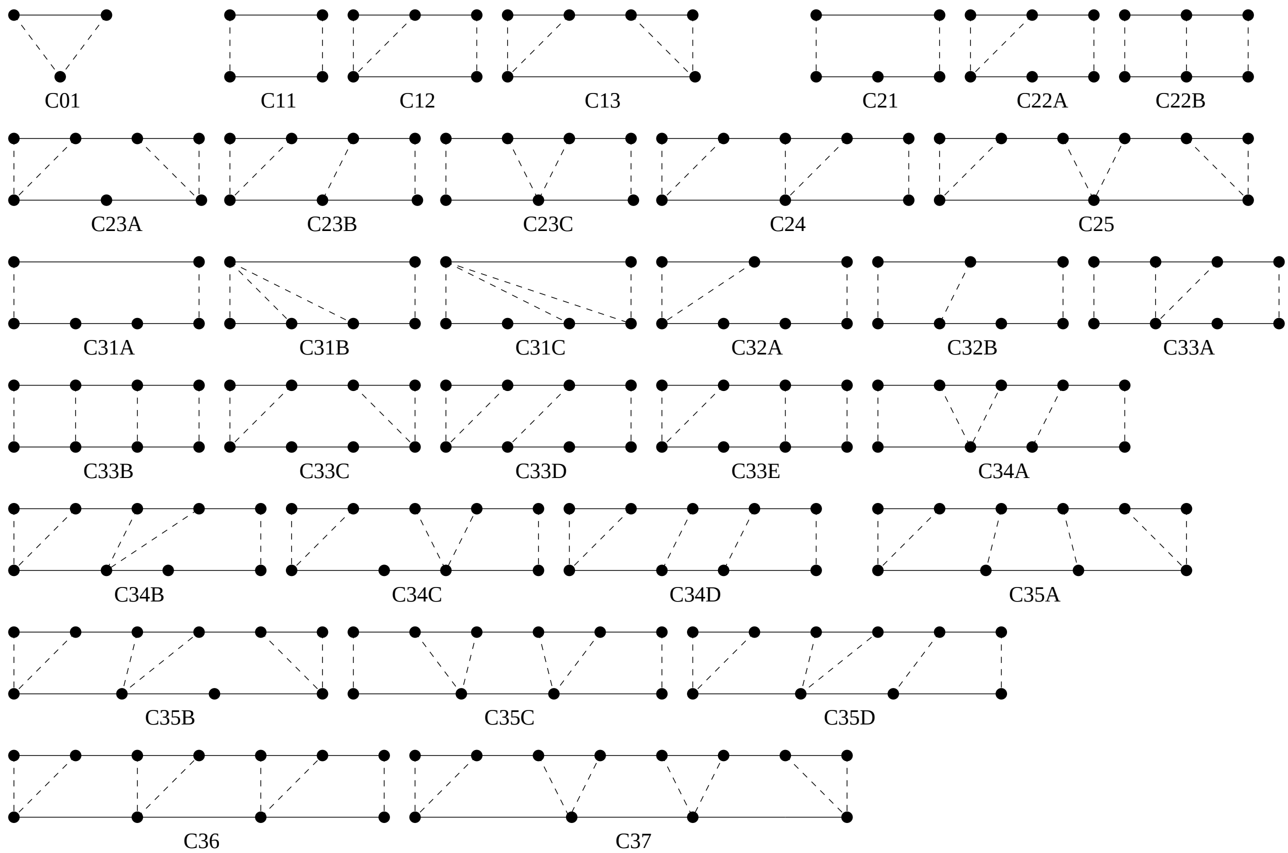

C01
C11
C12
C13
C21
C22A
C22B
C23A
C23B
C23C
C24
C25
C31A
C31B
C31C
C32A
C32B
C33A
C33B
C33C
C33D
C33E
C34A
C34B
C34C
C34D
C35A
C35B
C35C
C35D
C36
C37

Figure A.7

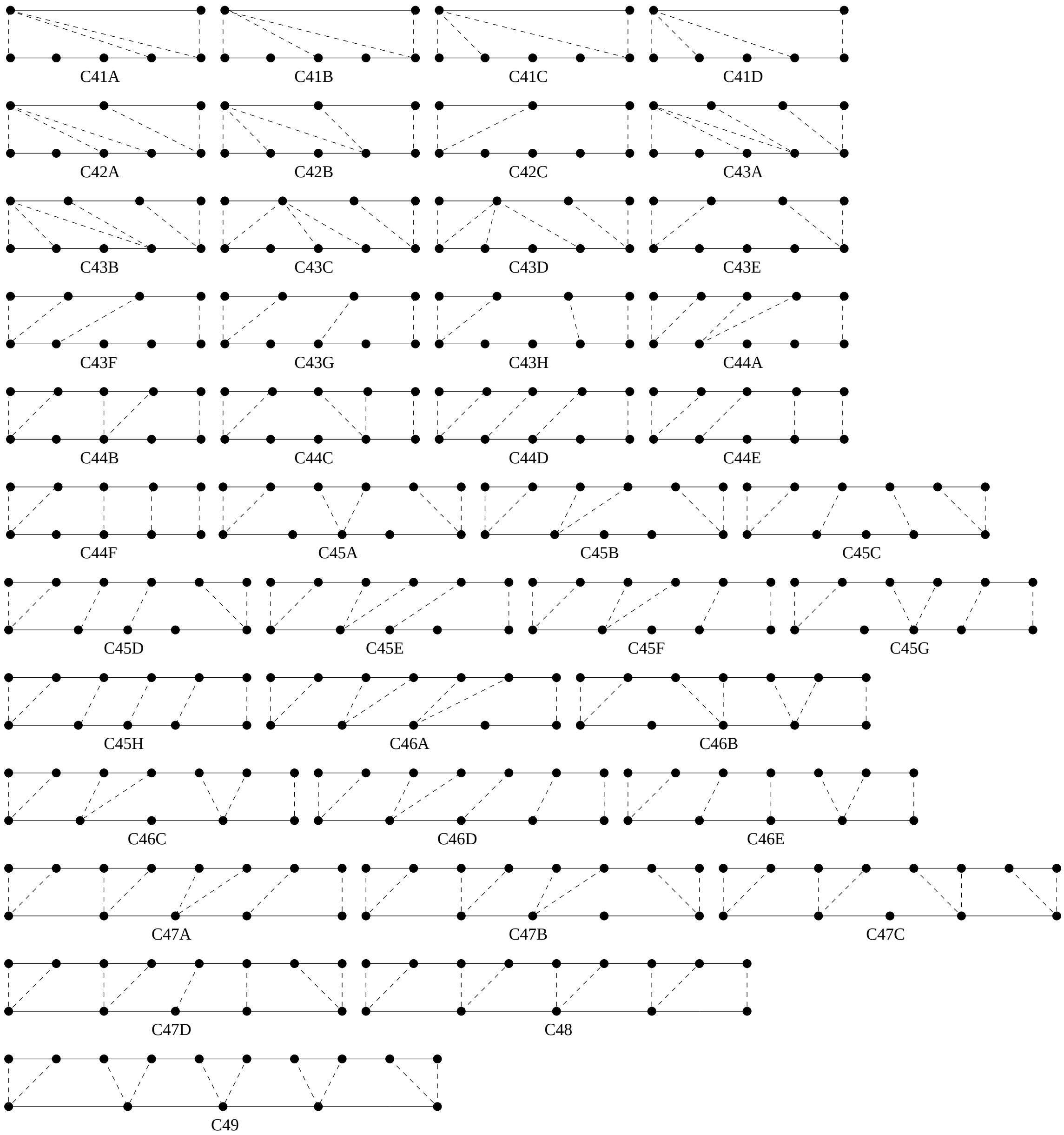

C41A
C41B
C41C
C41D
C42A
C42B
C42C
C43A
C43B
C43C
C43D
C43E
C43F
C43G
C43H
C44A
C44B
C44C
C44D
C44E
C44F
C45A
C45B
C45C
C45D
C45E
C45F
C45G
C45H
C46A
C46B
C46C
C46D
C46E
C47A
C47B
C47C
C47D
C48
C49

Figure A.8

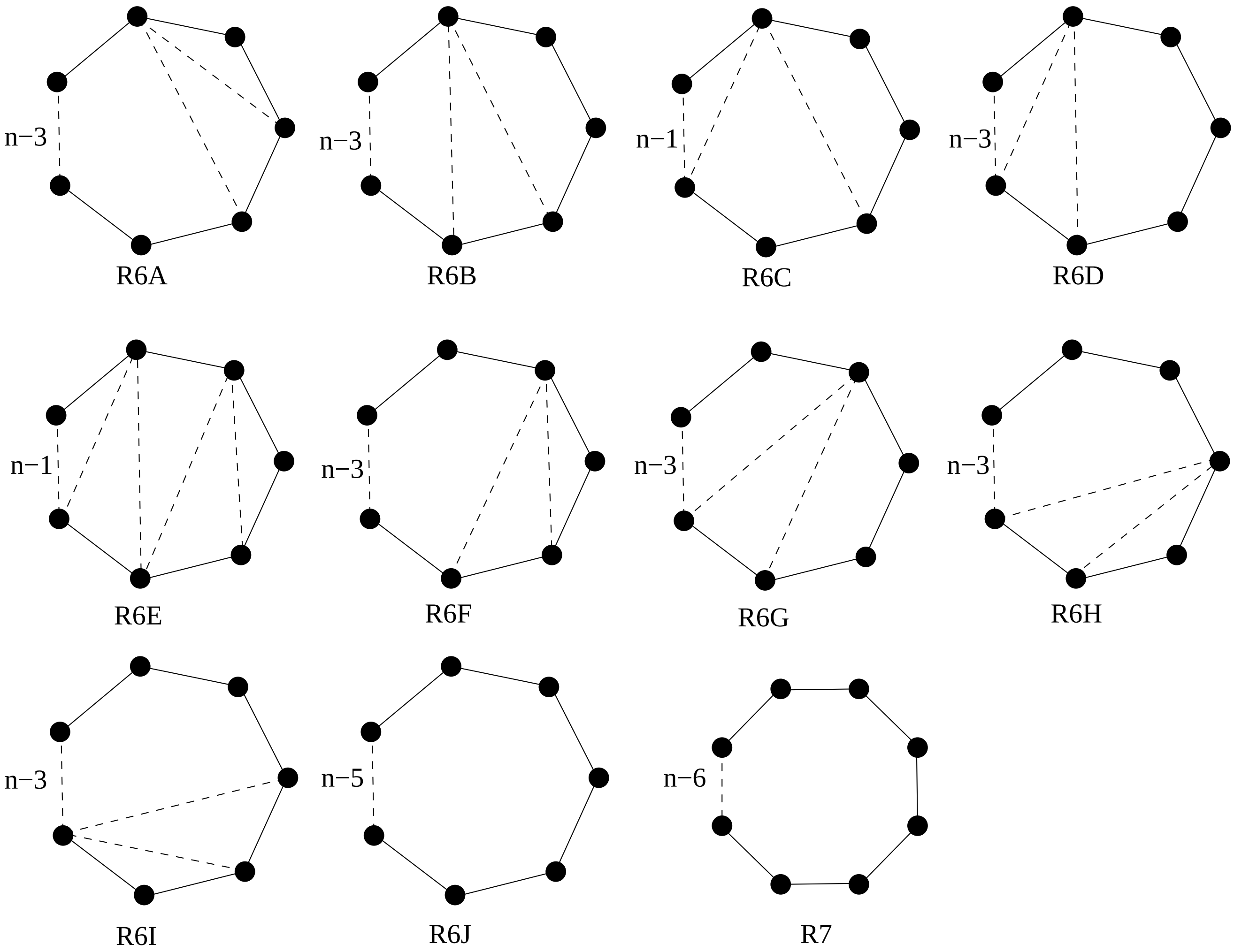

n−3
R6A
n−3
R6B
n−1
R6C
n−3
R6D
n−1
R6E
n−3
R6F
n−3
R6G
n−3
R6H
n−3
R6I
n−5
R6J
n−6
R7

Figure A.9