The solution of equations of length seven over torsion-free groups - a University of Nottingham Mathematics Preprint 2015

Mairaj Bibi and Martin Edjvet
School of Mathematical Sciences
The University of Nottingham
University Park, Nottingham NG7 2RD, UK

1 Introduction

Let A be a non-trivial group and let $s(t) = 1$ be an element in the free product $A* < t >$ where t is distinct from A. That is, $s(t) = a_1 t^{k_1} a_2 t^{k_2} ... a_n t^{k_n}$ where $a_i \in A$, $k_i = \pm 1$ and $a_i \neq 1$ if $k_i = -k_{i-1}$ $1 \leq i \leq n-1$ (subscript mod n). The integer n is called *length* of the equation. The equation $s(t) = 1$ is solvable over A if and only if the natural map $a \mapsto a$ from A to $< A, t | s(t) >$ is injective. It was conjectured by Levin [15] that any equation over a torsion free group is solvable. Prishchepov [17] and, with a different proof, Ivanov and Klyachko [11] showed that the conjecture is true for equations of length at most six. Our aim is to prove the following theorem.

Theorem 1.1. *Any equation of length seven is solvable over any torsion free group.*

There are (up to cyclic permutation and inversion) nine distinct equations of length seven:

1. $a_1 t a_2 t a_3 t a_4 t a_5 t a_6 t a_7 t = 1$

2. $a_1 t a_2 t a_3 t a_4 t a_5 t a_6 t a_7 t^{-1} = 1$

3. $a_1 t a_2 t a_3 t a_4 t a_5 t a_6 t^{-1} a_7 t^{-1} = 1$

4. $a_1 t a_2 t a_3 t a_4 t a_5 t^{-1} a_6 t a_7 t^{-1} = 1$

5. $a_1 t a_2 t a_3 t a_4 t^{-1} a_5 t a_6 t a_7 t^{-1} = 1$

6. $a_1 t a_2 t a_3 t a_4 t a_5 t^{-1} a_6 t^{-1} a_7 t^{-1} = 1$

7. $a_1 t a_2 t a_3 t a_4 t^{-1} a_5 t a_6 t^{-1} a_7 t^{-1} = 1$

8. $a_1 t a_2 t a_3 t^{-1} a_4 t a_5 t a_6 t^{-1} a_7 t^{-1} = 1$

9. $a_1ta_2ta_3t^{-1}a_4ta_5t^{-1}a_6ta_7t^{-1} = 1$

The exponent sum of t in equations 6, 7, 8, 9 is equal to 1, which is solvable by Klyachko [14] while equations 2 and 3 are solvable by Stalling [18] and equation 1 by Levin [15]. Equation 4 is solvable by putting $n = 2$, $m = 4$ in [3, p. 493]. This leaves equation 5 which we rewrite as

$$s(t) = atbtctdt^{-1}etftgt^{-1} = 1.$$

2 Method of proof

The definitions of this section are taken from [1].

A *relative group presentation* is a presentation of the form $\mathcal{P} =< A, t|\mathbf{r} >$ where A is a group, t is disjoint from A. Denoting the free group on t by $< t >$, $\mathbf{r}$ is a set of cyclically reduced words in the free product $A* < t >$. The group defined by $\mathcal{P}$ is $A(\mathcal{P}) = \dfrac{A* < t >}{N}$, where N is the normal closure of $\mathbf{r}$ in $A* < t >$.

A *picture* $\mathbb{P}$ is a finite collection of pairwise disjoint discs $\{D_1, \ldots, D_r\}$ in the interior of a disc D^2, together with a finite collection of pairwise disjoint simple arcs $\{\sigma_1, \ldots, \sigma_s\}$ embedded in the closure of D^2 - $\bigcup\{D_1, \ldots, D_r\}$. The *boundary* of $\mathbb{P}$ is the circle ∂D^2, denoted by $\partial\mathbb{P}$. For $j \in \{1, \ldots, r\}$, the *corners* of D_j are closures of the connected components of $\partial D_j - \bigcup\{\sigma_1, \ldots, \sigma_s\}$, where ∂D_j is the boundary of D_j.

The *regions* of $\mathbb{P}$ are the closures of the connected components of D^2 -$(\bigcup\{D_1, \ldots, D_r\} \cup \bigcup\{\sigma_1, \ldots, \sigma_s\})$. An *inner region* of $\mathbb{P}$ is a simply connected region of $\mathbb{P}$ that does not meet $\partial\mathbb{P}$.

The picture $\mathbb{P}$ is *nontrivial* if $r \geq 1$, is *connected* if $\bigcup\{D_1, \ldots, D_r\} \cup \bigcup\{\sigma_1, \ldots, \sigma_s\}$ is connected, and is *spherical* if it is non-trivial and $\bigcup\{\sigma_1, \ldots, \sigma_s\} \cap \partial\mathbb{P} = \emptyset$.

Suppose that the picture $\mathbb{P}$ is labelled in the following sense.

Each arc is given a normal orientation indicated by a short arrow meeting the arc transversely, and labelled by t or t^{-1}.

Each corner is oriented anticlockwise (with respect to D^2) and labelled by an element of A.

If κ is a corner of a disc D_j of $\mathbb{P}$, then $W(\kappa)$ will be the word obtained by reading in anticlockwise order the labels on the arcs and corners meeting ∂D beginning with the label on the arc at the terminal point of the anticlockwise oriented corner κ. If we cross an arc labelled t in the direction of its normal orientation we read t, otherwise t^{-1}.

We say that $\mathbb{P}$ is a *picture* over the presentation $\mathcal{P}$ if the above labelling satisfies the following two conditions:

1. For each corner κ of $\mathbb{P}$, $W(\kappa) \in \mathbf{r}^*$, where $\mathbf{r}^*$ is the set of all cyclic permutations of $\mathbf{r} \cup \mathbf{r}^{-1}$ which begin with t or t^{-1}.

2. If $a_1, a_2..., a_n$ is the sequence of corner labels encountered in a clockwise traversal of the boundary of an inner region of $\mathbb{P}$, then $a_1 a_2 ... a_n = 1$ in A.

A *dipole* in a picture $\mathbb{P}$ over $\mathcal{P}$ consists of a pair of corners κ, κ' of $\mathbb{P}$ together with an arc σ joining the head of one corner to the tail of the other such that the corners κ, κ' lie in the same region and such that $W(\kappa) = \overline{W(\kappa')}$, where the operator - is defined as follow. Let $R = Sa \in \mathbf{r}$, where $a \in A$ and S begins and ends with t or t^{-1}. Then $\bar{R} = S^{-1}a^{-1}$. The picture $\mathbb{P}$ is *reduced* if it does not contain a dipole. A relative presentation $\mathcal{P}$ is *aspherical* if every connected spherical picture over $\mathcal{P}$ contains a dipole (that is, fails to be reduced).
If the relative presentation $\mathcal{P}$ is orientable (that is no element of $\mathbf{r}$ is a cyclic permutation of its inverse) and aspherical then the natural homomorphism $A \longrightarrow \dfrac{A * < t >}{N}$ is injective. In our case $\mathbf{r}$ consists of a single element $s(t) = atbtctdt^{-1}etftgt^{-1}$. Clearly $\mathcal{P} =< A, t | s(t) >$ is orientable. If we show that $\mathcal{P}$ is aspherical then $s(t) = 1$ is solvable over A. We will use two approaches to asphericity: weight test [1] and curvature distribution method [5]. Moreover,we work with *relative diagrams* [9] which are the dual of pictures.
The star graph [1] Γ for the relative presentation $\mathcal{P} =< A, t | atbtctdt^{-1}etftgt^{-1} = 1 >$ is the directed graph shown in Figure 2.1.
A non-empty cyclically reduced cycle (closed path) in the star graph Γ is called *admissible* if it has a label which is trivial in A. Each vertex label of a relative diagram gives an admissible cycle in Γ.
A *weight function* θ on the star graph Γ is a real valued function on the edges of Γ. If e is an edge of Γ, then $\theta(e) = \theta(e^{-1})$. The weight of a closed cycle is the sum of the weights of the constituent edges.
A weight function θ on Γ is called *aspherical* if it satisfies the following conditions:
(**W1**) Let $R \in \mathbf{r}$ be cyclically reduced word, say $R = t_1^{\varepsilon_1} a_1 \dots t_n^{\varepsilon_n} a_n$, where $\varepsilon_i = \pm 1$ and $a_i \in A$. Then

$$\sum_{i=1}^{n} (1 - (\theta(t_i^{\varepsilon_i} a_i \dots t_n^{\varepsilon_n} a_n t_1^{\varepsilon_1} a_1 \dots t_{i-1}^{\varepsilon_{i-1}} a_{i-1})) \geq 2.$$

(**W2**) Each admissible cycle in Γ has weight at least 2.
(**W3**) Each edge of Γ has a non-negative weight.

If Γ admits an aspherical weight function then the relative presentation $\mathcal{P}$ is *aspherical* [1].
Let K be a reduced spherical diagram over $\mathcal{P}$. Given an angle function α on the corners of K we define the curvature function c on a vertex v of a diagram K by

$$c(v) = 2\pi - \sum_{\kappa \in v} \alpha(\kappa), \text{where } \kappa \text{ is a corner of the vertex } v$$

and on regions Δ of K by :

$$c(\Delta) = 2\pi - \sum_{\kappa \in \Delta} (\pi - \alpha(\kappa)).$$

For a vertex v with degree d, allocate an angle $\frac{2\pi}{d}$ to each corner κ of v. The curvature of each vertex v will be then $2\pi - \frac{2\pi}{d}d = 0$, while the curvature of the region Δ of degree n will be

$$\begin{aligned} & 2\pi - \left[(\pi - \frac{2\pi}{d_1}) + ... + (\pi - \frac{2\pi}{d_n})\right] \text{ where } d_i \text{ is the degree of } v_i \\ = & \ \pi \left[2 - n + \sum_{i=1}^{n} \frac{2}{d_n}\right]. \end{aligned}$$

The curvature of K , $c(K)$, is the sum of curvatures of all vertices and regions of K. It is well known that $c(K) = 4\pi$. Here, the curvature of each vertex is 0, so $c(K)$ is the sum of curvature of all regions of K. Suppose that $c(\Delta) > 0$. The curvature distribution method is based on distributing the positive curvature $c(\Delta)$ to neighbouring regions of Δ, say $\hat{\Delta}_i$, with negative curvature. Let $c^*(\hat{\Delta}_i)$ be the total curvature obtained after adding to $c(\hat{\Delta}_i)$ all curvature that it can possibly receive from positively curved neighbouring regions. Then $c(K) \leq \sum c^*(\hat{\Delta})$. Thus if we can show that $c^*(\hat{\Delta}_i) \leq 0$ for each $\hat{\Delta}_i$ this will contradict the fact that $c(K) = 4\pi$ and it can be deduced that $\mathcal{P}$ is aspherical.

3 Preliminary results

Recall that $\mathcal{P} =< A, t|s(t) >$ where A is torsion-free, $s(t) = atbtctdt^{-1}etftgt^{-1}$ and it can be assumed without any loss that $b = 1$.

Lemma 3.1. *The equation $s(t) = 1$ is solvable over A if one of the following sets of conditions holds: (i) $a = e$ and $d = g$ (ii) $a = e$ and $f = b$ (iii) $d = g$ and $f = c$ (iv) $a = f^{-1} = e$ and $b = c$.*

Proof. If (i) holds then adding the generator $x = tdt^{-1}at$ yields $\mathcal{P} =< A, x|r(x) >$ where $r(x) = xfx^2cxfxa^{-1}x^{-1}f^{-1}x^{-1}c^{-1}d^{-1}c$; if (ii) holds then $x = t^{-1}at^2$ and $r(x) = xdxgxcdxgxca^{-1}c^{-1}x^{-1}g^{-1}x^{-1}d^{-1}$; if (iii) holds then $x = tctdt^{-1}$ and $r(x) = xexac^{-1}xexaxa^{-1}x^{-1}e^{-1}x^{-1}d^{-1}$; and if (iv) holds then $x = t^{-1}ata^{-1}t$ and $r(x) = xgx^3dxgx^{-1}g^{-1}x^{-1}d^{-1}ad$. In each case the equation is solvable [18].

□

Lemma 3.2. *The presentation $\mathcal{P} =< A, t|s(t) >$ is aspherical if one of the following sets of conditions holds: (i) $a = e^{-1}$, $g = d^{-1}$ (ii) $a = e^{-1}$, $g = d$; (iii) $a = e^{-1}$ and $b = c = f$.*

Proof. (i)

$$\begin{aligned} \mathcal{P} &= < A, t|at^2ctdt^{-1}a^{-1}tftd^{-1}t^{-1} > \\ &= < A, t, x|xtctdx^{-1}td^{-1}f = 1 = x^{-1}td^{-1}t^{-1}at > . \end{aligned}$$

The star graph Γ for the new presentation is given by Figure 3.1(i), where (from the first relator) $\chi \longleftrightarrow 1, \gamma \longleftrightarrow c, \delta \longleftrightarrow d, \xi \longleftrightarrow 1, \alpha \longleftrightarrow d^{-1}f$; and (from the second relator) $\tau \longleftrightarrow 1, \kappa \longleftrightarrow d, \epsilon \longleftrightarrow a, \beta \longleftrightarrow 1$. Assign the following weights to edges of Γ: $\theta(\alpha) = \theta(\kappa) = \theta(\chi) = \theta(\epsilon) = 0$ and $\theta(\beta) = \theta(\gamma) = \theta(\delta) = \theta(\xi) = \theta(\tau) = 1$. Then θ satisfies conditions **W1** and **W3** for asphericity given in Section 2. Moreover each cycle of weight less than 2 in Γ has label a^m or d^m where $m \in \mathbb{Z} \setminus \{0\}$. Since A is torsion-free it follows that **W2** is satisfied and θ is an aspherical weight function.
(ii)

$$\begin{aligned} \mathcal{P} &= < A, t|at^2ctdt^{-1}a^{-1}tftdt^{-1} > \\ &= < A, t, x|xtctdx^{-1}tdf = 1 = x^{-1}tdt^{-1}at > . \end{aligned}$$

The star graph Γ for the new presentation is given by Figure 3.1(ii), where $\chi \longleftrightarrow 1, \gamma \longleftrightarrow c, \delta \longleftrightarrow d, \xi \longleftrightarrow 1, \alpha \longleftrightarrow df$; and $\tau \longleftrightarrow 1, \kappa \longleftrightarrow d, \epsilon \longleftrightarrow a, \beta \longleftrightarrow 1$. Assign the following weights to edges of Γ: $\theta(\alpha) = \theta(\epsilon) = \theta(\kappa) = \theta(\chi) = 0$ and $\theta(\beta) = \theta(\gamma) = \theta(\delta) = \theta(\xi) = \theta(\tau) = 1$. Then θ is an aspherical weight function.
(iii)

$$\begin{aligned} \mathcal{P} &= < A, t|at^3dt^{-1}a^{-1}t^2gt^{-1} > \\ &= < A, t, x|xtdtx^{-1}tg = 1 = x^{-1}t^{-1}at^2 > . \end{aligned}$$

The star graph Γ for the new presentation is given by Figure 3.1(iii), where $\chi \longleftrightarrow 1, \delta \longleftrightarrow d, \beta \longleftrightarrow 1, \xi \longleftrightarrow 1, \gamma \longleftrightarrow g$; and $\alpha \longleftrightarrow 1, \epsilon \longleftrightarrow a, \kappa \longleftrightarrow 1, \tau \longleftrightarrow 1$.

Assign the following weights to edges of Γ: $\theta(\alpha) = \theta(\gamma) = \theta(\epsilon) = \theta(\chi) = 0$ and $\theta(\beta) = \theta(\kappa) = \theta(\delta) = \theta(\xi) = \theta(\tau) = 1$. Then θ is an aspherical weight function.

□

Lemma 3.3. *The presentation $\mathcal{P} =< A, t|s(t) >$ is aspherical if one of the following conditions holds: (i) $a^{\pm 1} = e^2$, $g = c = d$ and $b = f$; (ii) $a^2 = e^{\pm 1}$, $d = g = c$ and $f = b$; (iii) $a = e^{-1} = f = c$; (iv) $a = e^{-1}$ and $g^{-1} = f = c$; (v) $a = e^{-1}$, $g = f^{-1}$ and $b = c$; (vi) $a^{\pm 1} \neq e^2$, $a^2 \neq e^{\pm 1}$, $a \neq e^{\pm 1}$, $g = c = d$ and $b = f$.*

Proof. (i)

$$\begin{aligned}\mathcal{P} &= < A, t|at^2ctct^{-1}et^2ct^{-1} > \\ &= < A, t, x|ax^2ex = 1 = x^{-1}t^2ct^{-1} > .\end{aligned}$$

The star graph Γ for the new presentation $\mathcal{P}$ is given by Figure 3.2(i), where $\alpha \longleftrightarrow a, \beta \longleftrightarrow 1, \gamma \longleftrightarrow e$; and $\epsilon \longleftrightarrow 1, \xi \longleftrightarrow 1, \tau \longleftrightarrow c, \delta \longleftrightarrow 1$. Assign the following weights to edges of Γ: $\theta(\alpha) = \theta(\beta) = \theta(\xi) = \theta(\tau) = 0$ and $\theta(\gamma) = \theta(\delta) = \theta(\varepsilon) = 1$. Then, since $a^{\pm 1} = e^2$, θ is an aspherical weight function.

(ii) As in (i), the presentation $\mathcal{P} =< A, t, x|ax^2ex = 1 = x^{-1}t^2ct^{-1} >$. Also the star graph Γ for $\mathcal{P}$ is shown in Figure 3.2(i). Assign the following weight function θ to the edges of Γ: $\theta(\gamma) = \theta(\beta) = \theta(\xi) = \theta(\tau) = 0$ and $\theta(\alpha) = \theta(\delta) = \theta(\varepsilon) = 1$. Then, since $a^2 = e^{\pm 1}$, θ is an aspherical weight function.

(iii)

$$\begin{aligned}\mathcal{P} &= < A, t|ct^2ctdt^{-1}c^{-1}tctgt^{-1} > \\ &= < A, t, x|xt^2xdx^{-1}txg = 1 = x^{-1}t^{-1}ct > .\end{aligned}$$

The star graph Γ for the new presentation is given by Figure 3.2(ii), where $\chi \longleftrightarrow 1, \kappa \longleftrightarrow 1, \alpha \longleftrightarrow 1, \delta \longleftrightarrow d, \xi \longleftrightarrow 1, \beta \longleftrightarrow 1, \eta \longleftrightarrow g$; and $\gamma \longleftrightarrow 1, \epsilon \longleftrightarrow c, , \tau \longleftrightarrow 1$. Assign the following weights to edges of Γ: $\theta(\delta) = \theta(\varepsilon) = \theta(\tau) = \theta(\xi) = 0$ and $\theta(\alpha) = \theta(\beta) = \theta(\gamma) = \theta(\chi) = \theta(\eta) = \theta(\kappa) = 1$. Then θ is an aspherical weight function.

(iv)

$$\begin{aligned}\mathcal{P} &= < A, t|at^2ctdt^{-1}a^{-1}tctc^{-1}t^{-1} > \\ &= < A, t, x|x^{-1}tctdxt = 1 = x^{-1}t^{-1}a^{-1}tct > .\end{aligned}$$

The star graph Γ for the new presentation is given by Figure 3.2(iii), where $\xi \longleftrightarrow 1, \delta \longleftrightarrow c, \gamma \longleftrightarrow d, \chi \longleftrightarrow 1, \beta \longleftrightarrow 1$; and $\alpha \longleftrightarrow 1, \epsilon \longleftrightarrow a, \kappa \longleftrightarrow c, \tau \longleftrightarrow 1$. Assign the following weights to edges of Γ: $\theta(\alpha) = \theta(\gamma) = \theta(\varepsilon) = \theta(\chi) = 0$ and

$\theta(\beta) = \theta(\kappa) = \theta(\delta) = \theta(\xi) = \theta(\tau) = 1$. Then θ is an aspherical weight function.

(v)

$$\begin{aligned}\mathcal{P} &= < A, t | at^3dt^{-1}a^{-1}tftf^{-1}t^{-1} > \\ &= < A, t, x | t^2dxtx^{-1} = 1 = x^{-1}t^{-1}a^{-1}tft > .\end{aligned}$$

The star graph Γ for the new presentation is given by Figure 3.2(iv), where $\kappa \longleftrightarrow 1, \alpha \longleftrightarrow d, \chi \longleftrightarrow 1, \beta \longleftrightarrow 1, \xi \longleftrightarrow 1$; and $\gamma \longleftrightarrow 1, \epsilon \longleftrightarrow a, \delta \longleftrightarrow f, \tau \longleftrightarrow 1$. Assign the following weights to edges of Γ: $\theta(\alpha) = \theta(\gamma) = \theta(\epsilon) = \theta(\chi) = 0$ and $\theta(\beta) = \theta(\kappa) = \theta(\delta) = \theta(\xi) = \theta(\tau) = 1$. Then θ is an aspherical weight function.

(vi) As in (i), the presentation $\mathcal{P} =< A, t, x | ax^2ex = 1 = x^{-1}t^2ct^{-1} >$. Also the star graph Γ for $\mathcal{P}$ is shown in Figure 3.2(i). Assign the following weight function θ to the edges of Γ: $\theta(\beta) = \theta(\xi) = \theta(\tau) = 0$, $\theta(\alpha) = \theta(\gamma) = \frac{1}{2}$ and $\theta(\delta) = \theta(\varepsilon) = 1$. Then θ is an aspherical weight function.

□

4 Proof of Theorem 1.1

Using the star graph Γ and the fact that A is a torsion free group, the possible labels of vertices of degree 2 for $s(t)$ are (up to inversion and cyclic permutation)

$$S = \{ae, ae^{-1}, dg, dg^{-1}, fc^{-1}, fb^{-1}, cb^{-1}\}.$$

We will classify the cases according to the number N of members of S that are admissible. These possibilities can be reduced as follows: we can work modulo *equivalence*, that is, modulo inversion, cyclic permutation, $t \leftrightarrow t^{-1}$ and $(a, b, c, d, e, f, g) \leftrightarrow (d^{-1}, c^{-1}, b^{-1}, a^{-1}, g^{-1}, f^{-1}, e^{-1}$ (for example ae, gd^{-1} is equivalent to ae^{-1}, gd). Since $c(\Delta) \leq c(3, 3, 3, 3, 3, 3, 3) = -\frac{\pi}{3}$, it can be assumed that $N > 0$; it follows from Lemma 3.1(i) and Lemma 3.2(i)-(ii) and A torsion-free that at most one of ae, ae^{-1}, dg, dg^{-1} is admissible; if any two of fb^{-1}, fc^{-1}, cb^{-1} are admissible then so is the third; it follows from Lemmas 3.1 and 3.2 that $N \leq 3$. A routine check now shows that there are the following 10 cases to be considered: Case 1: $a = e$; Case 2: $a = e^{-1}$; Case 3: $b = f$; Case 4: $b = c$; Case 5 : $a = e^{-1}$, $f = c$; Case 6: $a = e^{-1}$, $f = b$; Case 7: $a = e^{-1}$, $c = b$; Case 8: $d = g$, $f = b$; Case 9: $a = e$, $c = b$; Case 10 : $f = c$, $f = b$, $b = c$

Remark 4.1. *Since $c(2, 3, 3, 3, 3, 3, 3) = 0$, a positively curved region Δ must include at least two vertices of degree 2.*

Recall that we suppose by way of contradiction that K is a reduced spherical *diagram* over $\mathcal{P}$. The diagram K will have a *distinguished* vertex v_0 (corresponding to the annular region of the spherical picture dual to K) and, in general, $l(v_0)$ yields a cycle in the star graph Γ which is not necessarily admissible, that is, $l(v_0)$ need not equal 1 in A. Assume until otherwise stated (near the end of Section 4) that $l(v_0) = 1$ in A. Then the product of the corner labels read anti-clockwise around *any* vertex of K yields an admissible cycle in Γ. We will consider each of the ten cases in turn. It will be shown that if $\hat{\Delta}_1$ receives positive curvature then either $c^*(\hat{\Delta}_1) \leq 0$ or $c^*(\hat{\Delta}_1)$ is distributed to another region $\hat{\Delta}_2$ such that $c^*(\hat{\Delta}_2) \leq 0$. The comments made at the end of Section 2 show that this suffices to prove the theorem. Throughout what follows we assume $c(\Delta) > 0$.

4.1 Case 1: $a = e$

By Remark 4.1, we must have $d(v_a) = d(v_e) = 2$ in Δ as shown in Figure 4.1(i). Suppose that $d(v_b) = d(v_c) = d(v_d) = d(v_f) = d(v_g) = 3$. Then $l(v_g) \in \{g^2d^{-1}, gd^{-2}\}$. But gd^{-2} forces $d(v_f) > 3$, so let $l(v_g) = g^2d^{-1}$ which forces $l(v_f) = b^{-1}fa$ and $l(v_d) = dg^{-2}$. Now $l(v_b) \in f^{-1}b\{a^{\pm 1}, e^{\pm 1}\}$. The label $f^{-1}b\{a, e\}$ implies $a^2 = 1$ and $f^{-1}b\{a^{-1}, e^{-1}\}$ forces $d(v_c) > 3$. Therefore there are the following five cases to consider: (i) $d(v_b) > 3$ only; (ii) $d(v_c) > 3$ only; (iii) $d(v_d) > 3$ only; (iv) $d(v_f) > 3$ only; and (v) $d(v_g) > 3$ only.

In (i)-(iii) $l(v_g) = g^2d^{-1}$ and $l(v_f) = b^{-1}fa$ as in Figure 4.1(ii). Add $c(\Delta) \leq \frac{\pi}{6}$ to $c(\hat{\Delta})$ as shown. If $d(v_e) > 2$ in $\hat{\Delta}$ then $c(\hat{\Delta}) \leq -\frac{\pi}{3}$ and if $d(v_e) = 2$ as shown then this forces $d(v_f) > 3$ and $c(\hat{\Delta}) \leq -\frac{\pi}{6}$.

In cases (iv) and (v) $d(v_b) = 3$ implies $l(v_b) = f^{-1}b\{a^{\pm 1}, e^{\pm 1}\}$. Suppose firstly that $l(v_b) = f^{-1}ba$. Then $l(v_c) = gc\{b^{-1}, f^{-1}\}$. But gcb^{-1} forces $d(v_d) > 3$ so $l(v_c) = gcf^{-1}$ which forces $l(v_d) = dg^{-2}$. Add $c(\Delta) \leq \frac{\pi}{6}$ to $c(\hat{\Delta})$ as shown in Figure 4.1(iii). Observe that $d(v_b) > 3$ in $\hat{\Delta}$ and $c(\hat{\Delta}) \leq -\frac{\pi}{6}$. Now suppose that $l(v_b) = f^{-1}ba^{-1}$. Then $l(v_c) \in b^{-1}c\{a^{\pm 1}, e^{\pm 1}\}$. But $b^{-1}c\{a^{\pm 1}, e^{-1}\}$ forces $d(v_d) > 3$ so $l(v_c) = b^{-1}ce$. Add $c(\Delta) \leq \frac{\pi}{6}$ to $c(\hat{\Delta})$ as shown in Figure 4.1(iv). Observe that $d(v_g) > 3$ in $\hat{\Delta}$ and so $c(\hat{\Delta}) \leq -\frac{\pi}{6}$. Suppose that $l(v_b) = f^{-1}be$. Then $l(v_c) = dc\{f^{-1}, b^{-1}\}$. But dcb^{-1} forces $d(v_d) > 3$ so $l(v_c) = dcf^{-1}$. Add $c(\Delta) \leq \frac{\pi}{6}$ to $c(\hat{\Delta})$ as shown in Figure 4.1(v). Observe that $d(v_f) > 3$ in $\hat{\Delta}$ and so $c(\hat{\Delta}) \leq -\frac{\pi}{6}$. Finally suppose that $l(v_b) = f^{-1}be^{-1}$. Then $l(v_c) = f^{-1}c\{e^{\pm 1}, a^{\pm 1}\}$. But $f^{-1}c\{e^{-1}, a^{\pm 1}\}$ forces $d(v_d) > 3$ so $l(v_c) = f^{-1}ce$. Add $c(\Delta) \leq \frac{\pi}{6}$ to $c(\hat{\Delta})$ as shown in Figure 4.1(vi). Observe that $d(v_d) > 3$ in $\hat{\Delta}$ and so $c(\hat{\Delta}) \leq -\frac{\pi}{6}$. This completes the distribution of curvature. Checking shows that

$\hat{\Delta}$ cannot receive positive curvature across more than one edge and it follows that $c^*(\hat{\Delta}) \leq 0$.

4.2 Case 2: $a = e^{-1}$

In this case $d(v_a) = d(v_e) = 2$ in Δ as shown in Figure 4.2(i). Observe that if $d(v_g) = 3$ then $l(v_g) = gf\{b^{-1}, c^{-1}\}$. The label gfb^{-1} implies $l(v_f) = gfa^{-1}w$ and gfc^{-1} implies $l(v_f) = gfb^{-1}w$ therefore $d(v_f)$ and $d(v_g)$ cannot both be 3. This gives the following two cases: (i) $d(v_g) > 3$ only; (ii) $d(v_f) > 3$ only

Consider case (i). Here $l(v_d) = db\{f^{-1}, c^{-1}\}$ and $l(v_b) = db\{f^{-1}, c^{-1}\}$. If $l(v_d) = dbf^{-1}$ and $l(v_b) = dbf^{-1}$ or dbc^{-1} then $d(v_c) > 3$. If $l(v_d) = dbc^{-1}$ and $l(v_b) = dbf^{-1}$ then $l(v_c) = cb^{-1}g^{-1}$ which implies $d = g$, a contradiction. This leaves $l(v_d) = l(v_b) = dbc^{-1}$ which forces $l(v_c) = cb^{-1}d^{-1}$. Now $l(v_f) = gf\{b^{-1}, c^{-1}\}$. Suppose that $l(v_f) = gfb^{-1}$. Add $c(\Delta) \leq \frac{\pi}{6}$ to $c(\hat{\Delta})$ as shown in Figure 4.2(ii). Observe that $d(v_{a^{-1}})$ and $d(v_{g^{-1}})$ cannot both be 3 in $\hat{\Delta}$ so $c(\hat{\Delta}) \leq -\frac{\pi}{3}$. Now suppose that $l(v_f) = gfc^{-1}$. Then add $c(\Delta) \leq \frac{\pi}{6}$ to $c(\hat{\Delta})$ as shown in Figure 4.2(iii). If $d(v_{a^{-1}}) > 2$ in $\hat{\Delta}$ then $c(\hat{\Delta}) \leq -\frac{\pi}{6}$; or if $d(v_{a^{-1}}) = 2$ as shown then $d(v_{f^{-1}})$ and $d(v_{g^{-1}})$ cannot both be 3 so again $c(\hat{\Delta}) \leq -\frac{\pi}{6}$. Now consider case (ii), that is, $d(v_f) > 3$. It follows from case (i) that $l(v_b) = l(v_d) = bc^{-1}d$ and $l(v_c) = cb^{-1}d^{-1}$. Now $l(v_g) = gf\{b^{-1}, c^{-1}\}$. If $l(v_g) = gfb^{-1}$ then add $c(\Delta) \leq \frac{\pi}{6}$ to $c(\hat{\Delta})$ as shown in Figure 4.2(iv). Note that here $c\hat{\Delta}) \leq -\frac{\pi}{6}$. If $l(v_g) = gfc^{-1}$ then add $c(\Delta) \leq \frac{\pi}{6}$ to $c(\hat{\Delta})$ as shown in Figure 4.2(v). If $d(v_{e^{-1}}) > 2$ in $\hat{\Delta}$ then $c(\hat{\Delta}) \leq -\frac{\pi}{6}$; or if $d(v_{e^{-1}}) = 2$ as shown then this forces $d(v_{d^{-1}}) > 3$. Now $d(v_{f^{-1}})$ and $d(v_{g^{-1}})$ cannot both be 3 so again $c(\hat{\Delta}) \leq -\frac{\pi}{6}$. This completes the distribution of curvature.

If $\hat{\Delta}$ receives positive curvature across at most one edge then $c^*(\hat{\Delta}) \leq 0$ so it remains to check when $\hat{\Delta}$ receives positive curvature across more than one edge. From the above we see that either $c^*(\hat{\Delta}) \leq 0$ or $\hat{\Delta}$ receives positive curvature across the (b^{-1}, c^{-1}) and (c^{-1}, d^{-1}) edges as shown in Figure 4.2(vi). Now $d(v_{f^{-1}})$ and $d(v_{g^{-1}})$ cannot both be 3. If $d(v_{f^{-1}}) = 3$ then $l(v_{f^{-1}}) = f^{-1}g^{-1}c$. Add $c^*(\hat{\Delta}) \leq c(\hat{\Delta}) + \frac{\pi}{3} \leq \frac{\pi}{6}$ to $c(\hat{\Delta}_1)$ as shown in Figure 4.2(vii). If $d(v_a) > 2$ then $c(\hat{\Delta}_1) \leq -\frac{\pi}{6}$; if $d(v_a) = 2$ as shown then $d(v_b) > 3$ and again $c(\hat{\Delta}_1) \leq -\frac{\pi}{6}$. If $d(v_{g^{-1}}) = 3$ in $\hat{\Delta}$, then $l(v_{g^{-1}}) = f^{-1}g^{-1}c$. Add $c^*(\hat{\Delta}) \leq c(\hat{\Delta}) + \frac{\pi}{3} \leq \frac{\pi}{6}$ to $c(\hat{\Delta}_1)$, where $\hat{\Delta}_1$ is shown in Figure 4.2(viii). Similarly $c^*(\hat{\Delta}_1) \leq -\frac{\pi}{6}$. If $\hat{\Delta}_1$ receive positive curvature across exactly one edge then $c^*(\hat{\Delta}_1) \leq 0$. Otherwise $\hat{\Delta}_1$ receives positive curvature across the (b, c) and (c, d) edges as shown in Figure 4.2(ix). Repeat the above argument for $\hat{\Delta}_1/\hat{\Delta}_2$. Since positive curvature is distributed across the same pairs of edges each time and since the region that receives positive curvature is negatively curved it follows that this procedure must terminate at a region $\hat{\Delta}_k$ where $c^*(\hat{\Delta}_k) \leq 0$.

4.3 Case 3: $b = f$

In this case $d(v_b) = d(v_f) = 2$ in Δ as shown in Figure 4.3(i). There are four cases to consider: (i) $d(v_c) = d(v_d) = d(v_e) = 3$; (ii) $d(v_c) > 3$ only; (iii) $d(v_d) > 3$ only; and (iv) $d(v_e) > 3$ only.

In (i) $l(v_e) \in \{ea^{-2}, e^2a^{-1}\}$. Let $l(v_e) = ea^{-2}$. Now $l(v_d) = d^2g^{-1}$ implies $d(v_c) > 3$ and so $l(v_d) = dg^{-2}$ forcing $l(v_c) = g^{-1}cf^{-1}$. Add $c(\Delta) \leq \frac{\pi}{3}$ to $c(\hat{\Delta})$ as shown in Figure 4.3(ii). Observe that $d(v_{f^{-1}}) > 3$ in $\hat{\Delta}$ and so $c(\hat{\Delta}) \leq -\frac{\pi}{2}$.

If on the other hand $l(v_e) = e^2a^{-1}$ then $l(v_d) = df\{b^{-1}, c^{-1}\}$. But dfb^{-1} implies $d = 1$ so $l(v_d) = dfc^{-1}$ which implies $l(v_c) = cb^{-1}g^{-1}$. Add $c(\Delta) \leq \frac{\pi}{6}$ to $c(\hat{\Delta})$ as shown in Figure 4.3(iii). Note that $d(v_d)$ and $d(v_g)$ are both greater than 3 in $\hat{\Delta}$ so $c(\hat{\Delta}) \leq -\frac{\pi}{3}$.

In cases (ii)-(iv) $d(v_a) = d(v_g) = 3$. It follows that Δ is given by Figure 4.3(iv). Add $c(\Delta) \leq \frac{\pi}{6}$ to $c(\hat{\Delta})$ as shown. If $d(v_f) > 2$ in $\hat{\Delta}$ then $c(\hat{\Delta}) \leq -\frac{\pi}{3}$ and if $d(v_f) = 2$ then this forces $d(v_g) > 3$ and $c(\hat{\Delta}) \leq -\frac{\pi}{6}$. This completes the initial distribution of curvature. If $\hat{\Delta}$ receive positive curvature across exactly one edge then $c^*(\hat{\Delta}) \leq 0$ and if $\hat{\Delta}$ receive positive curvature across more than one edge then it is $\hat{\Delta}$ of Figure 4.3(ii) and $c^*(\hat{\Delta}) \leq -\frac{\pi}{2} + \frac{\pi}{3} + \frac{\pi}{6}$.

4.4 Case 4: $b = c$

Since $d(v_b)$ and $d(v_c)$ cannot both be 3 it follows that $c(\Delta) \leq 0$ in this case.

4.5 Case 5: $a = e^{-1}$ and $f = c$

If $l(v_a) = ae$ and $d(v_g) = 3$ then $l(v_g) = gf\{b^{-1}, c^{-1}\}$ which implies $g = 1$ or $g = f^{-1}$ and the presentation $\mathcal{P}$ is aspherical by Lemma 3.3(iv); if $l(v_c) = cf^{-1}$ and $d(v_b) = 3$ then $l(v_b) = be^{-1}\{c^{-1}, f^{-1}\}$ and $\mathcal{P}$ is aspherical by Lemma 3.3(iii); if $l(v_e) = ea$ and $d(v_f) = 3$ then $l(v_f) = gf\{b^{-1}, c^{-1}\}$ and again either $g = 1$ or $\mathcal{P}$ is aspherical by Lemma 3.3(iv); if $l(v_f) = fc^{-1}$ and $d(v_e) = 3$ then $l(v_e) = eb^{-1}\{c, f\}$ and $\mathcal{P}$ is aspherical by Lemma 3.3(iii); and if both $l(v_c) = cf^{-1}$ and $l(v_e) = ea$ then $l(v_d) = g^{-1}dbw$ forcing $d(v_d) \geq 4$. It follows that the only case that remains to be considered is $d(v_a) = d(v_c) = d(v_f) = 2$. Since this forces each of $d(v_b)$, $d(v_e)$, $d(v_g)$ to be greater than 3, it follows that $d(v_d) = 3$ and $l(v_d) \in \{d^2g^{-1}, dg^{-2}\}$. Suppose that $l(v_d) = d^2g^{-1}$ as shown in Figure 4.4(i). Then $d(v_g) = d(v_b) = 4$ or $d(v_g) = d(v_e) = 4$ cannot occur. Add $c(\Delta) \leq \frac{\pi}{15}$ to $c(\hat{\Delta})$ as shown. Observe that $d(v_c) > 3$ and $d(v_e) > 3$ in $\hat{\Delta}$. If at least

one of v_a or v_f in $\hat{\Delta}$ has degree greater than 2 then $c(\hat{\Delta}) \le -\frac{\pi}{3}$. If $d(v_a) = d(v_f) = 2$ in $\hat{\Delta}$ as shown then this forces $d(v_g) > 3$ and so $c(\hat{\Delta}) \le -\frac{\pi}{6}$.
Now suppose that $l(v_d) = dg^{-2}$. Then again $d(v_g) = d(v_b) = 4$ or $d(v_g) = d(v_e) = 4$ cannot occur. Add $c(\Delta) \le \frac{\pi}{15}$ to $c(\hat{\Delta})$ as shown in Figure 4.4(ii). Observe that $d(v_{e^{-1}}) > 3$ and $d(v_{a^{-1}}) > 3$ in $\hat{\Delta}$ otherwise $d(v_e) > 4$ in Δ and $c(\Delta) < 0$. If $d(v_{c^{-1}}) > 2$ in $\hat{\Delta}$ then $c(\hat{\Delta}) \le -\frac{\pi}{3}$. If $d(v_{c^{-1}}) = 2$ then this forces $d(v_{b^{-1}}) > 3$ otherwise $d(v_e)$ and $d(v_b)$ would be greater than 4 in Δ and $c(\Delta) < 0$. So $c(\hat{\Delta}) \le -\frac{\pi}{6}$.
This completes the distribution of curvature. Since $\hat{\Delta}$ receive positive curvature across exactly one edge $c^*(\hat{\Delta}) \le 0$.

4.6 Case 6: $a = e^{-1}$ and $f = b$

There are the following four cases to consider: (i) $d(v_a) = d(v_e) = 2$; (ii) $d(v_b) = d(v_f) = 2$; (iii) $d(v_a) = d(v_f) = 2$; and (iv) $d(v_b) = d(v_e) = 2$.
(i) It follows from Case 2 that $d(v_f)$ and $d(v_g)$ cannot both be 3. Now $l(v_b) = db\{f^{-1}, c^{-1}\} = l(v_d)$. But dbf^{-1} implies $d = 1$. Therefore $l(v_b) = dbc^{-1} = l(v_d)$ which implies $l(v_c) = cb^{-1}d^{-1}$ as shown in Figure 4.5(i). But then $l(v_f)$ or $l(v_g)$ is $gf\{b^{-1}, c^{-1}\}$ which implies $g = 1$ or $g = d$. Therefore $c(\Delta) \le 0$.
(ii) Observe that $d(v_a)$ and $d(v_e)$ are greater than 3 in Δ as shown in Figure 4.5(ii) and so $c(\Delta) \le 0$ in this case.
(iii) Observe that $d(v_e) > 3$ in Δ as shown in Figure 4.5(iii). Now $l(v_b) = dbc^{-1}$ since dbf^{-1} implies $d = 1$. But then $l(v_g) = gfc^{-1}$ implies $d = g$. Therefore $c(\Delta) \le 0$.
(iv) Observe that $d(v_a) > 3$ in Δ as shown in Figure 4.5(iv). Now $l(v_f) = gfc^{-1}$ since gfb^{-1} implies $g = 1$ and this forces $l(v_g) \in \{gd^{-2}, g^2d^{-1}\}$. Now $l(v_d) = db\{c^{-1}, f^{-1}\}$ which implies $d = 1$ or $d = c$ and A is cyclic. Therefore $c(\Delta) \le 0$.

4.7 Case 7: $a = e^{-1}$ and $c = b$

There are five cases to consider: (i) $d(v_a) = d(v_e) = 2$; (ii) $d(v_a) = d(v_c) = 2$; (iii) $d(v_b) = d(v_e) = 2$; (iv) $d(v_c) = d(v_e) = 2$; and (v) $d(v_a) = d(v_c) = d(v_e) = 2$.
(i) If $d(v_f)$ or $d(v_g)$ is 3 in Δ as shown in Figure 4.6(i) then $l(v_f)$ or $l(v_g) = gf\{b^{-1}, c^{-1}\}$ which implies $g = f^{-1}$ and the presentation $\mathcal{P}$ is aspherical by Lemma 3.3(v).
(ii) Observe that $d(v_b) > 3$ in Δ as shown in Figure 4.6(ii). Now $l(v_d) = c^{-1}df$ which forces $d(v_e) > 3$ and so $c(\Delta) \le 0$.
(iii) It follows from (i) that $d(v_f) > 3$ in Δ. If $d(v_c) = 3$ then $l(v_c) = d^{-1}cf^{-1}$ as shown in Figure 4.6(iii) since $d^{-1}cb^{-1}$ implies $d = 1$. But this forces $d(v_d) > 3$ and so $c(\Delta) \le 0$.

(iv) Observe that $d(v_d)$ and $d(v_f)$ are greater than 3 in Δ as shown in Figure 4.6(iv) and so $c(\Delta) \leq 0$.
(v) Observe that $d(v_b) > 3$ and $d(v_d) > 3$ in Δ as shown in Figure 4.6(v). It follows from (i) that $d(v_f)$ and $d(v_g)$ are greater than 3 and so $c(\Delta) \leq 0$.

4.8 Case 8: $d = g$ and $f = b$

There are 7 cases to consider: (i) $d(v_d) = d(v_g) = 2$; (ii) $d(v_b) = d(v_f) = 2$; (iii) $d(v_b) = d(v_g) = 2$; (iv) $d(v_b) = d(v_d) = 2$; (v) $d(v_d) = d(v_f) = 2$; (vi) $d(v_b) = d(v_d) = d(v_g) = 2$; and (vii) $d(v_b) = d(v_d) = d(v_f) = 2$.
(i) If $d(v_e) = 3$ then $l(v_e) \in \{a^{-1}e^2, a^{-2}e\}$. The label $a^{-2}e$ forces $d(v_f) > 3$ and $a^{-1}e^2$ and $d(v_f) = 3$ forces $l(v_f) = dfc^{-1}$ which implies $d = c = g$ and the presentation $\mathcal{P}$ is aspherical by Lemma 3.3(i). So $d(v_e)$ and $d(v_f)$ cannot both be 3 as shown in Figure 4.7(i). If $d(v_a) = 3$ then $l(v_a) \in \{ae^{-2}, e^{-1}a^2\}$. But ae^{-2} forces $d(v_b) > 3$ so $l(v_a) = e^{-1}a^2$ which implies $l(v_b) = gb\{c^{-1}, f^{-1}\}$ which implies $g = 1$ or $g = c$ and the presentation $\mathcal{P}$ is aspherical by Lemma 3.3(ii).
(ii) If $d(v_e) = d(v_c) = 3$ then $l(v_c) = g^{-1}c\{b^{-1}, f^{-1}\}$ which implies $g = c = d$ and $l(v_e) \in \{e^2a^{-1}, ea^{-2}\}$ which implies $a = e^2$ or $e = a^2$ and the presentation $\mathcal{P}$ is aspherical by Lemma 3.3(i) or (ii). If $d(v_a) = 3$ then $l(v_a) \in \{ae^{-2}, a^2e^{-1}\}$. But $l(v_a) = ae^{-2}$ forces $d(v_g) > 3$, so $l(v_a) = a^2e^{-1}$ which forces $l(v_g) = gbc^{-1}$ as shown in Figure 4.7(ii) which implies $g = c$ and again the presentation $\mathcal{P}$ is aspherical by Lemma 3.3(ii).
(iii) If $d(v_a) = d(v_c) = 3$ then $l(v_a) = ae^{-2}$ and $l(v_c) = g^{-1}c\{b^{-1}, f^{-1}\}$ which implies $g = c = d$ and the presentation $\mathcal{P}$ is aspherical by Lemma 3.3(i). First consider $d(v_a) > 3$ in Δ as shown in Figure 4.7(iii). Then $l(v_c) = g^{-1}c\{b^{-1}, f^{-1}\}$. But $l(v_c) = g^{-1}cf^{-1}$ forces $d(v_d) > 3$, so $l(v_c) = g^{-1}cb^{-1}$ which implies $l(v_d) = c^{-1}d\{b, f\}$. But then $l(v_e) \in \{a^2e, ae^2\}$ or $l(v_e) = e^2a^{\pm 1}$ and the presentation $\mathcal{P}$ is aspherical. Now consider $d(v_c) > 3$ in Δ as shown in Figure 4.7(iv). Then $l(v_a) = ae^{-2}$ and $l(v_f) = fc^{-1}\{d^{-1}, g^{-1}\}$ since $fc^{-1}\{d, g\}$ implies $d = c = g$ and the presentation is aspherical by Lemma 3.3(i). But then $l(v_e) = ec^{-1}\{b, f\}$ or $l(v_e) = ef^{-1}\{b, c\}$ which implies $e = 1$ or A cyclic and so $c(\Delta) \leq 0$.
(iv) If $d(v_c) = d(v_e) = 3$ in Δ in Figure 4.7(v) then $l(v_c) = g^{-1}cf^{-1}$ which implies $g = c = d$ and $l(v_e) \in \{a^{-1}e^2, a^{-2}e^1\}$ which implies $a = e^2$ or $e = a^2$ and the presentation $\mathcal{P}$ is aspherical by Lemma 3.3(i) or 3.3(ii). If $d(v_a) = 3$ then $l(v_a) \in \{ae^{-2}, a^2e^{-1}\}$. But ae^{-2} forces $d(v_g) > 3$, so $l(v_a) = a^2e^{-1}$ which forces $l(v_g) = gb\{c^{-1}, f^{-1}\}$ which implies $g = 1$ or $g = c$ and again the presentation $\mathcal{P}$ is aspherical by Lemma 3.3(ii).
(v) If $d(v_e) = d(v_g) = 3$ then $l(v_e) = ea^{-2}$ and $l(v_g) = c^{-1}g\{b, f\}$ which implies $g = c$

and the presentation $\mathcal{P}$ is aspherical by Lemma 3.3(ii). First consider $d(v_e) > 3$ in Δ as shown in Figure 4.7(vi). Then $l(v_g) = c^{-1}g\{b,f\}$ which forces $l(v_a) = a^2e^{\pm 1}$ or $l(v_a) \in \{a^2e, ae^2\}$ and the presentation $\mathcal{P}$ is aspherical by Lemma 3.3(i) or 3.3(ii). Now consider $d(v_g) > 3$ in Δ as shown in Figure 4.7(vii). Then $l(v_e) = ea^{-2}$ and $l(v_c) = cf^{-1}\{d,g\}$ since $cf^{-1}\{d^{-1},g^{-1}\}$ implies $c = d = g$ and the presentation $\mathcal{P}$ is aspherical by Lemma 3.3(ii). But this forces $l(v_b) = be\{c^{-1},f^{-1}\}$ which implies $e = 1$ or $e = c$ and A is cyclic or $l(v_b) = ba\{c^{-1},f^{-1}\}$ which implies $a = 1$ or $a = c$ and A is cyclic. So $c(\Delta) \leq 0$.

(vi) If $d(v_c) = 3$ then $l(v_c) = g^{-1}cf^{-1}$ which implies $g = c$ and the presentation $\mathcal{P}$ is aspherical by Lemma 3.3(vi). It can be assumed that $d(v_c) > 3$. If $d(v_e) = d(v_f) = 3$ then $l(v_e) \in \{ea^{-2}, e^2a^{-1}\}$. But ea^{-2} forces $d(v_f) > 3$, so $l(v_e) = e^2a^{-1}$ which forces $l(v_f) = dfc^{-1}$ which implies $d = c$ and again the presentation $\mathcal{P}$ is aspherical by Lemma 3.3(i).

Let $d(v_a) = d(v_f) = 3$ in Δ. Then $l(v_a) = ae^{-2}$ and as before $l(v_f) = fc^{-1}\{d^{-1},g^{-1}\}$. First let $l(v_f) = fc^{-1}d^{-1}$. Then $d(v_c)$ and $d(v_e)$ are greater than 4. If $d(v_e) = 5$ then $l(v_e) = a^{-1}ec^{-1}\{d^{\pm 1}b, d^{\pm 1}c, d^{\pm 1}f, g^{\pm 1}b, g^{\pm 1}c, g^{\pm 1}f, ba^{-1}, fa^{-1}, be^{\pm 1}, fe^{\pm 1}\}$ and A cyclic. Therefore $d(v_e) > 5$. Add $c(\Delta) \leq c(2,2,2,3,3,5,6) = \frac{\pi}{15}$ to $c(\hat{\Delta})$ as shown in Figure 4.7(viii). If one of $v_{d^{-1}}$ or $v_{b^{-1}}$ has degree greater than 2 then $c(\hat{\Delta}) \leq -\frac{4\pi}{15}$. If $d(v_{b^{-1}}) = d(v_{d^{-1}}) = 2$ then this forces $d(v_{c^{-1}}) > 3$ and $c(\hat{\Delta}) \leq -\frac{\pi}{10}$. Now let $l(v_f) = fc^{-1}g^{-1}$. Then again $d(v_c)$ and $d(v_e)$ are greater than 4. If $d(v_e) = 5$ then the only valid labeling for v_e is $a^{-1}ef^{-1}be$. But then we can increase the number of vertices of degree 2 as shown in Figure 4.7(ix) and so $d(v_e) > 5$. Add $c(\Delta) \leq \frac{\pi}{15}$ to $c(\hat{\Delta})$ as shown in Figure 4.7(x). Observe that $d(v_{a^{-1}}) > 3$ in $\hat{\Delta}$ If one of $d(v_{b^{-1}})$ and $d(v_{d^{-1}})$ is greater than 2 then $c(\hat{\Delta}) \leq -\frac{\pi}{2}$. If $d(v_{b^{-1}}) = d(v_{d^{-1}}) = 2$ as shown then this forces $d(v_{c^{-1}}) > 3$ and so $c(\hat{\Delta}) \leq -\frac{\pi}{3}$.

Let $d(v_a) = d(v_e) = 3$ in Δ. Then $l(v_a) = ae^{-2}$ which implies $l(v_e) = a^{-1}e^2$. Add $c(\Delta) \leq \frac{\pi}{3}$ to $c(\hat{\Delta})$ as shown in Figure 4.7(xi). Now $d(v_{d^{-1}}) > 3$ in $\hat{\Delta}$. If $d(v_{b^{-1}}) > 2$ then $c(\hat{\Delta}) \leq -\frac{\pi}{3}$. If $d(v_{b^{-1}}) = 2$ as shown then this forces $d(v_{c^{-1}}) > 3$. If $d(v_{a^{-1}}) > 3$ in $\hat{\Delta}$ then $c(\hat{\Delta}) \leq -\frac{\pi}{3}$. So assume that $d(v_{a^{-1}}) = 3$ which implies $l(v_{a^{-1}}) = a^{-1}e^2$. But this forces $d(v_c) > 4$ in Δ otherwise $l(v_c) = dg^{-1}cf^{-1}$ which implies $c = 1$. In this case add $c(\Delta) \leq c(2,2,2,3,3,4,5) = \frac{7\pi}{30}$ to $c(\hat{\Delta}) \leq c(2,2,3,3,4,4,5) = -\frac{4\pi}{15}$.

If $d(v_e) = 3$ only then $l(v_e) \in \{ea^{-2}, e^2a^{-1}\}$. Suppose that $l(v_e) = ea^{-2}$. Then $d(v_a)$ and $d(v_f)$ are greater than 4 and $c(\Delta) < 0$. Now suppose that $l(v_e) = e^2a^{-1}$. Then $d(v_a) > 5$ and so $c(\Delta) \leq 0$ in this case.

If $d(v_f) = 3$ only then as before $l(v_f) = fc^{-1}\{d^{-1},g^{-1}\}$. If $l(v_f) = fc^{-1}d^{-1}$ as shown in Figure 4.7(xii) then $d(v_c) > 4$. Now $l(v_e) = a^{-1}ec^{-1}\{b, f\}$. Add $c(\Delta) \leq \frac{\pi}{15}$ to $c(\hat{\Delta})$. Observe that $d(v_{b^{-1}}) > 2$ in $\hat{\Delta}$ and so $c(\hat{\Delta}) \leq -\frac{\pi}{10}$. Now let $l(v_f) = fc^{-1}g^{-1}$ as shown in Figure 4.7(xiii). Then again $d(v_c) > 4$ and $l(v_e) = a^{-1}ef^{-1}\{b, c\}$. If $l(v_e) = a^{-1}ef^{-1}b$ then $a = e$, contrary to our assumptions, so $l(v_e) = a^{-1}ef^{-1}c$. Add $c(\Delta) \leq \frac{\pi}{15}$ to $c(\hat{\Delta})$. Observe that $d(v_{b^{-1}}) > 2$ in $\hat{\Delta}$ and so $c(\hat{\Delta}) \leq c(2,2,3,3,3,4,5) = -\frac{\pi}{10}$.

If $d(v_a) = 3$ only then $l(v_a) = ae^{-2}$. Add $c(\Delta) \leq \frac{\pi}{6}$ to $c(\hat{\Delta})$ as shown in Figure 4.7(xiv). If $d(v_{d^{-1}}) = 3$ in $\hat{\Delta}$ then $l(v_{d^{-1}}) = f^{-1}d^{-1}\{b, c\}$ which implies $d = 1$ or $d = c$ and the presentation $\mathcal{P}$ is aspherical by Lemma 3.3(i). If $d(v_{b^{-1}}) > 2$ then $c(\hat{\Delta}) \leq -\frac{\pi}{3}$ so assume $d(v_{b^{-1}}) = 2$ in $\hat{\Delta}$ as shown which forces $d(v_{c^{-1}}) > 3$. It follows that $c(\hat{\Delta}) \leq c(2,2,3,3,4,4,4) = -\frac{\pi}{6}$.

(vii) If $l(v_c) = g^{-1}cf^{-1}$ or $l(v_g) = c^{-1}g\{b, f\}$ then $g = c$ and the presentation $\mathcal{P}$ is aspherical by Lemma 3.2(vi) so assume $d(v_c) > 3$ and $d(v_g) > 3$. First consider $d(v_a) = d(v_e) = 3$. Then $l(v_e) = ea^{-2}$ and $l(v_a) = a^2e^{-1}$ as shown in Figure 4.7(xv). Now $d(v_c)$ and $d(v_g)$ cannot both be 4. Add $c(\Delta) \leq \frac{7\pi}{30}$ to $c(\hat{\Delta})$ as shown. If $d(v_{b^{-1}}) = 3$ in $\hat{\Delta}$ then $l(v_{b^{-1}}) = b^{-1}g^{-1}\{c, f\}$ which implies $g = 1$ or $g = c$ and the presentation $\mathcal{P}$ is aspherical by Lemma 3.2. So $d(v_{b^{-1}}) > 3$. If $d(v_{d^{-1}}) > 2$ then $c(\hat{\Delta}) \leq -\frac{\pi}{3}$ so let $d(v_{d^{-1}}) = 2$ as shown. If both of $v_{c^{-1}}$ and $v_{e^{-1}}$ have degree greater than 3 then $c(\hat{\Delta}) \leq -\frac{\pi}{3}$. If $d(v_{e^{-1}}) = 3$ and $d(v_{c^{-1}}) > 3$ then $l(v_{e^{-1}}) = e^{-1}a^2$ forces $d(v_{f^{-1}}) > 4$. So $c(\hat{\Delta}) \leq -\frac{4\pi}{15}$. If $d(v_{c^{-1}}) = 3$ and $d(v_{e^{-1}}) \geq 3$ then $l(v_{c^{-1}}) = fc^{-1}\{d^{-1}, g^{-1}\}$ forces $d(v_{b^{-1}})$ and $d(v_{f^{-1}})$ to be greater than 4 in $\hat{\Delta}$ and $d(v_g) > 4$ in Δ. So add $c(\Delta) \leq c(2,2,2,3,3,5,5) = \frac{2\pi}{15}$ to $c(\hat{\Delta}) \leq -\frac{\pi}{5}$.

Now consider $d(v_a) = 3$ only then $l(v_a) \in \{ae^{-2}, a^2e^{-1}\}$. If $l(v_a) = ae^{-2}$ then $d(v_e) > 4$ otherwise $l(v_e) = a^{-1}ea^{-1}\{e^{\pm 1}, a^{-1}\}$ which implies $|e| \leq 6$. Now $l(v_g) = c^{-1}gd^{-1}\{b, f\}$ which implies $c = 1$, and so $c(\Delta) \leq 0$. If $l(v_a) = a^2e^{-1}$ then again $d(v_e) > 4$. Now $l(v_c) = g^{-1}cf^{-1}\{g^{-1}, d^{\pm 1}\}$. But $l(v_c) = g^{-1}cf^{-1}d$ implies $c = 1$ so $l(v_c) = g^{-1}cf^{-1}\{g^{-1}, d^{-1}\}$. and $l(v_g) = c^{-1}gb\{a^{\pm 1}, e^{\pm 1}\}$ which implies $g = a^{\pm 1}$ or $g = e^{\pm 1}$ and A is cyclic. Therefore $c(\Delta) \leq 0$ in this case. Finally consider $d(v_e) = 3$ only then $l(v_e) = ea^{-2}$. Add $c(\Delta) \leq \frac{\pi}{6}$ to $c(\hat{\Delta})$ as shown in Figure 4.7(xvi). Observe that $d(v_{b^{-1}}) > 3$ in $\hat{\Delta}$. If $d(v_{d^{-1}}) > 2$ then $c(\hat{\Delta}) \leq -\frac{\pi}{3}$. If $d(v_{d^{-1}}) = 2$ and at least one of $v_{c^{-1}}$ and $v_{e^{-1}}$ has degree greater than 3 then $c(\hat{\Delta}) \leq -\frac{\pi}{6}$. If $d(v_{c^{-1}}) = d(v_{e^{-1}}) = 3$ then $l(v_{e^{-1}}) = e^{-1}a^2$ which forces $l(v_c) = g^{-1}cf^{-1}bw$ and $d(v_c) > 4$ in Δ. So add $c(\Delta) \leq c(2,2,2,3,4,4,5) = \frac{\pi}{15}$ to $c(\hat{\Delta}) \leq c(2,2,3,3,3,4,5) = -\frac{\pi}{10}$. This complete the distribution of curvature. Observe that $\hat{\Delta}$ cannot receive positive curvature across more than one edge and so $c^*(\hat{\Delta}) \leq 0$

4.9 Case 9: $a = e$ and $b = c$

There are five cases to consider: (i) $d(v_a) = d(v_e) = 2$; (ii) $d(v_a) = d(v_c) = 2$; (iii) $d(v_c) = d(v_e) = 2$; (iv) $d(v_b) = d(v_e) = 2$; and (v) $d(v_a) = d(v_c) = d(v_e) = 2$.
(i) If $d(v_g) = 3$ in Δ as shown in Figure 4.8(i) then $l(v_g) \in \{g^2d^{-1}, gd^{-2}\}$. If $l(v_g) = gd^{-2}$ then $l(v_f) = b^{-1}fc^{-1}w$ and $d(v_f) > 3$. If $l(v_g) = g^2d^{-1}$ then $d(v_f) = 3$ implies $l(v_f) = b^{-1}fa$ which implies $a = f^{-1}$ and the equation $s(t)$ is solvable by Lemma 3.1(iv). If $d(v_b) = d(v_c) = d(v_d) = 3$ then $l(v_b) = f^{-1}b\{a^{\pm1}, e^{\pm1}\}$. But $f^{-1}b\{a^{-1}, e^{-1}\}$ implies $f = a^{-1}$ and the result follows by Lemma 3.1(iv) so $l(v_b) = f^{-1}b\{a, e\}$. If $l(v_b) = f^{-1}ba$ then $l(v_c) = gc\{b^{-1}, f^{-1}\}$. If $l(v_c) = gcb^{-1}$ then $g = 1$ so $l(v_c) = gcf^{-1}$. But then $l(v_d) = dg^{-2}$ and A is cyclic. If $l(v_b) = f^{-1}be$ then $l(v_c) = dcf^{-1}$ since dcb^{-1} implies $d = 1$. But then $l(v_d) = dg^{-2}$ which implies A is cyclic. So $c(\Delta) \leq 0$.
(ii) Observe that $d(v_b) > 3$ in Δ as shown in Figure 4.8(ii). Now $l(v_d) = c^{-1}df$ as shown since $c^{-1}db$ implies $d = 1$. But this forces $d(v_e) > 3$ and so $c(\Delta) \leq 0$.
(iii) In this case $d(v_d) > 3$ in Δ as shown in Figure 4.8(iii). Now $l(v_b) = ba^{-1}\{c^{-1}, f^{-1}\}$ which implies $a = 1$ or $a = f^{-1}$ and the equation is solvable by Lemma 3.1(iv).
(iv) If $d(v_a) = 3$ then $l(v_a) = ab^{-1}\{c, f\}$ which implies $a = 1$ or $a = f^{-1}$ and the equation is solvable by Lemma 3.1(iv). Therefore $d(v_a) > 3$ as shown in Figure 4.8(iv). Now $l(v_c) = d^{-1}cf^{-1}$ since $d^{-1}cb^{-1}$ implies $d = 1$ which forces $l(v_d) = dg^{-2}$ as shown. But then $l(v_f) = b^{-1}f\{a^{\pm1}, e^{\pm1}\}$ which implies $f = a^{\pm1}$ and A is cyclic so $c(\Delta) \leq 0$.
(v) Observe that $d(v_d) > 3$ as shown in Figure 4.8(v). If $d(v_b) = 3$ then $l(v_b) = f^{-1}ba^{-1}$ which implies $a = f^{-1}$ and the equation is solvable by Lemma 3.1(iv). Therefore $d(v_b) > 3$. As in subcase (i) $d(v_g)$ and $d(v_f)$ cannot be both be 3. First consider $d(v_g) = 3$ as shown in Figure 4.8(vi). Then $l(v_g) \in \{gd^{-2}, g^2d^{-1}\}$. If $l(v_g) = gd^{-2}$ then $d(v_f) > 4$ otherwise $l(v_f) = b^{-1}fc^{-1}f$ which implies $f^2 = 1$. But then $d(v_b) = d(v_d) = 4$ cannot occur. Therefore $c(\Delta) \leq 0$ in this case. If $l(v_g) = g^2d^{-1}$ then $d(v_d) = 4$ implies $l(v_d) = c^{-1}dg^{-1}f$. But then $d(v_b)$ and $d(v_f)$ are greater than 4 and $c(\Delta) \leq 0$. For otherwise $l(v_b) = f^{-1}ba^{-1}\{a^{-1}, e^{\pm1}\}$ or $l(v_f) = b^{-1}fa\{a, e^{\pm1}\}$ which implies $f = 1$ or $f = a^{-2}$ and A is cyclic. Therefore $d(v_d) \geq 5$ and so add $c(\Delta) \leq \frac{\pi}{15}$ to $c(\hat{\Delta})$ as shown and observe that $c(\hat{\Delta}) \leq c(2, 2, 3, 3, 3, 4, 5) = -\frac{\pi}{10}$.
Now consider $d(v_f) = 3$ as shown in Figure 4.8(vii). Then $l(v_f) = b^{-1}f\{a^{\pm1}, e^{\pm1}\}$. But $b^{-1}f\{a, e\}$ implies $f = a^{-1}$ or $f = e^{-1}$ and the equation is solvable by Lemma 3.1(iv). So $l(v_f) = b^{-1}f\{a^{-1}, e^{-1}\}$ forcing $d(v_b) > 4$. Add $c(\Delta) \leq \frac{\pi}{15}$ to $c(\hat{\Delta})$ as shown and observe that $c(\hat{\Delta}) \leq -\frac{\pi}{10}$.
This completes the initial distribution of curvature. If $\hat{\Delta}$ receive positive curvature across exactly one edge then $c^*(\hat{\Delta}) \leq 0$. Moreover $\hat{\Delta}$ cannot receives positive curvature

across more than one edge since positive curvature is received across (a^{-1}, b^{-1}, c^{-1}) each time.

4.10 Case 10 : $f = c, f = b$ and $b = c$

In this case $s(t) = t^3dt^{-1}et^2gt^{-1}a$ which is solvable by Theorem 1 in [13].

Now assume that $l(v_0) \neq 1$ in A and let $d(v_0) = k_0$. A region of K is *interior* if it does not involve v_0; otherwise it is said to be a *boundary* region. For each interior region Δ such that $c(\Delta) > 0$ distribute $c(\Delta)$ to the region $\hat{\Delta}$ exactly as before as shown in Figures 4.1-4.8 except possibly Figure 4.2 of Case 2. As described above there may be a sequence of distributions that termin ated at a region $\hat{\Delta}_k$ for which $c^*(\hat{\Delta}_k) \leq 0$. The difference here is if there is a $\hat{\Delta}_j$ where $1 \leq j \leq k$ which is a boundary region then the distribution terminates at the first such region encou ntered and the $2.\frac{\pi}{6}$ remains with $\hat{\Delta}_j$. We have shown that if $\hat{\Delta}$ is always interior then $c^*(\hat{\Delta}) \leq 0$ and so it follows that $c(K) \leq \sum c^*(\hat{\Delta})$ where the sum is taken over all th e boundary regions of K.

Let $\hat{\Delta}$ be a boundary region of K. Checking Figures Cases 1,5,8 and 9 shows that $\hat{\Delta}$ remains uniquely associated with Δ and so the maximum curvature $\hat{\Delta}$ can receive is $\frac{\pi}{3}$; for Case 3 the maximum $\hat{\Delta}$ can received is $\frac{\pi}{3} + \frac{\pi}{6}$ (see Figure 4.3(ii)); and for Case 2 the maximum $\hat{\Delta}$ can receive is $2.\frac{\pi}{6}$ (see Figure 4.2). Suppose that $\hat{\Delta}$ has at most one interior ($\neq v_0$) vertex of degree 2. Then $c^*(\hat{\Delta}) \leq c(k_0, 2, 3, 3, 3, 3, 3) + (\frac{\pi}{3} + \frac{\pi}{6}) = \frac{2\pi}{k_0} - \frac{\pi}{6}$. (Note that this clearly holds if $k_0 \geq 3$; and if $k_0 = 2$ then at most one vertex of $\hat{\Delta}$ can coincide with v_0.) Now let $\hat{\Delta}$ have exactly two interior vertices of degree 2. Then $\hat{\Delta}$ cannot be the region $\hat{\Delta}$ of Figure 4.3(ii)-(iv) and so $c^*(\hat{\Delta}) \leq c(k_0, 2, 2, 3, 3, 3, 3) + \frac{\pi}{3} = \frac{2\pi}{k_0}$. Finally suppose that $\hat{\Delta}$ has the maximum of three interior vertices of degree 2. Then either $c^*(\hat{\Delta}) = c(\hat{\Delta}) \leq c(k_0, 2, 2, 2, 3, 3, 3,) = \frac{2\pi}{k_0}$ or $\hat{\Delta}$ is given by Figure 4.7(viii), in which case $c^*(\hat{\Delta}) \leq c(k_0, 2, 2, 2, 3, 5, 6) + \frac{\pi}{15} = \frac{2\pi}{k_0} - \frac{3\pi}{5}$. In conclusion it follows that $c^*(\hat{\Delta}) \leq \frac{2\pi}{k_0}$ for any given boundary region $\hat{\Delta}$ and so $c(K) \leq k_0.\left(\frac{2\pi}{k_0}\right) < 4\pi$, our final contradiction.

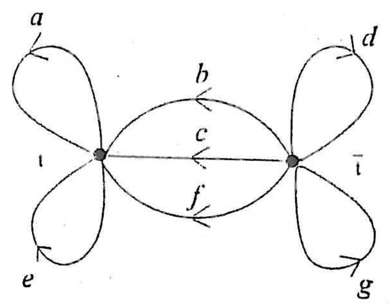

Figure 2.1: Star graph Γ

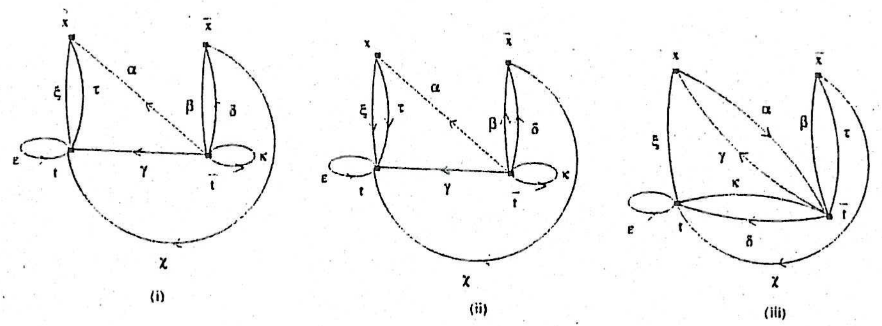

Figure 3.1: New star graphs

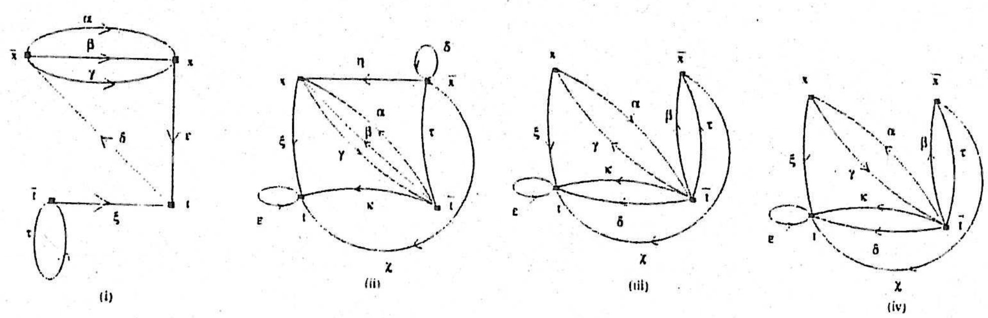

Figure 3.2: New star graphs

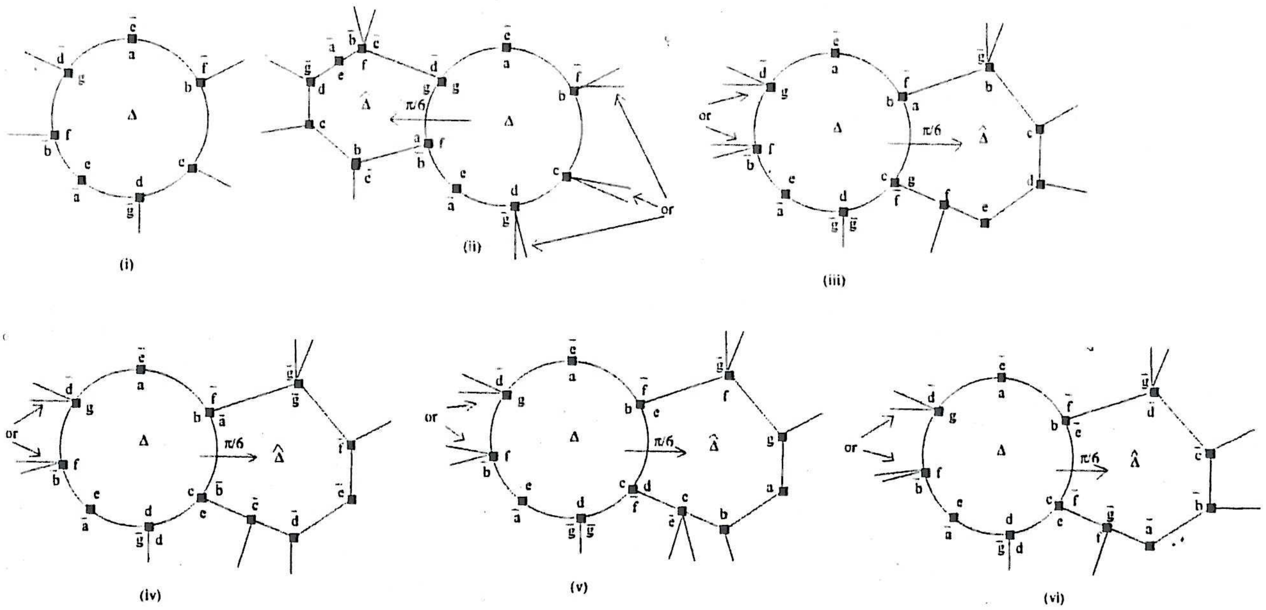

Figure 4.1: Case 1: $a = e$

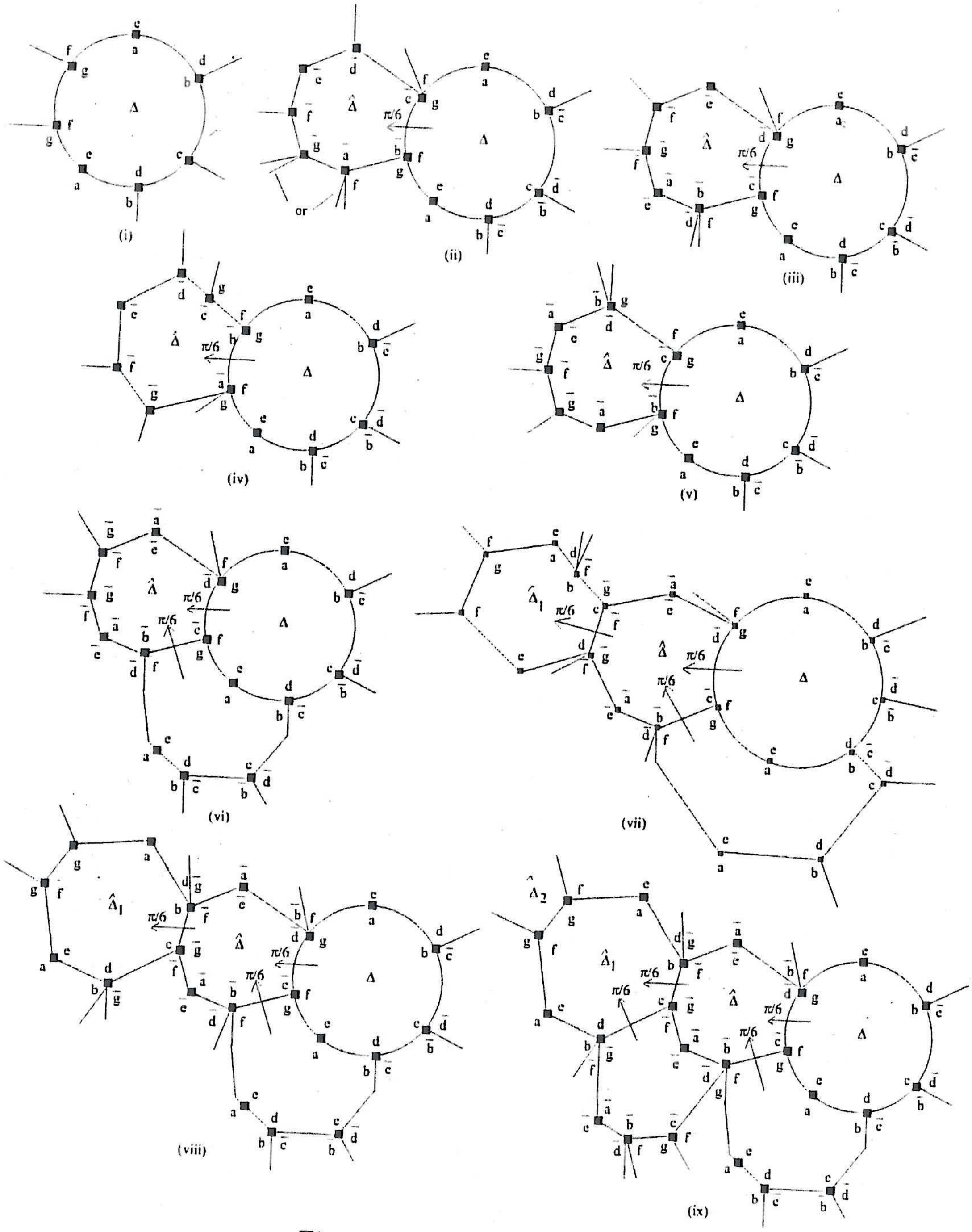

Figure 4.2: Case 2: $a = e^{-1}$

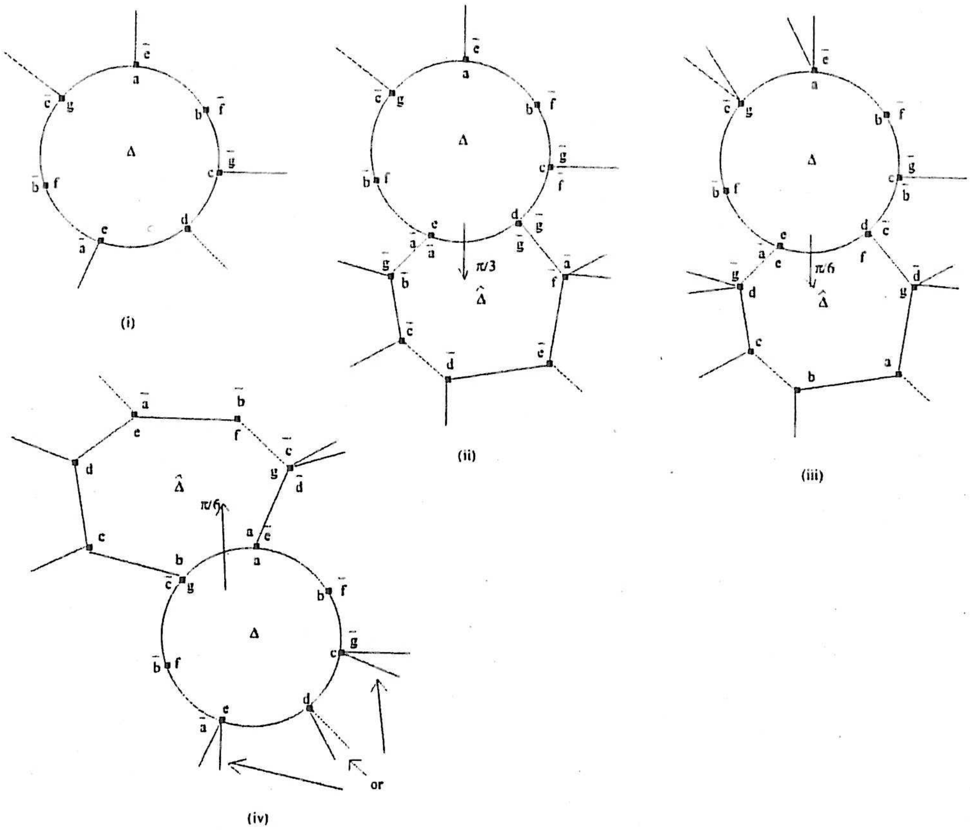

Figure 4.3: Case 3: $b = f$

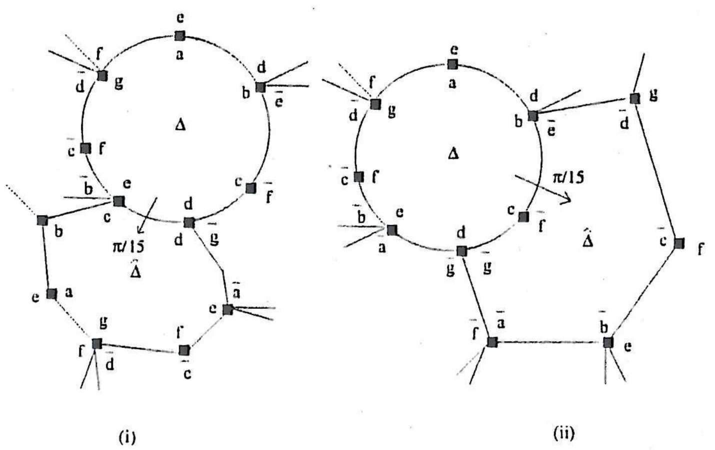

Figure 4.4: Case 5: $a = e^{-1}$ and $f = c$

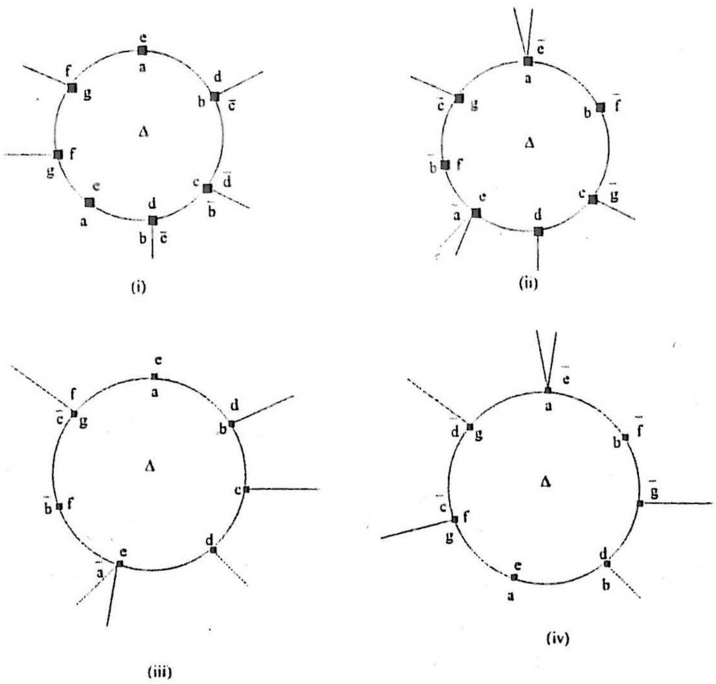

Figure 4.5: Case 6: $a = e^{-1}$ and $f = b$

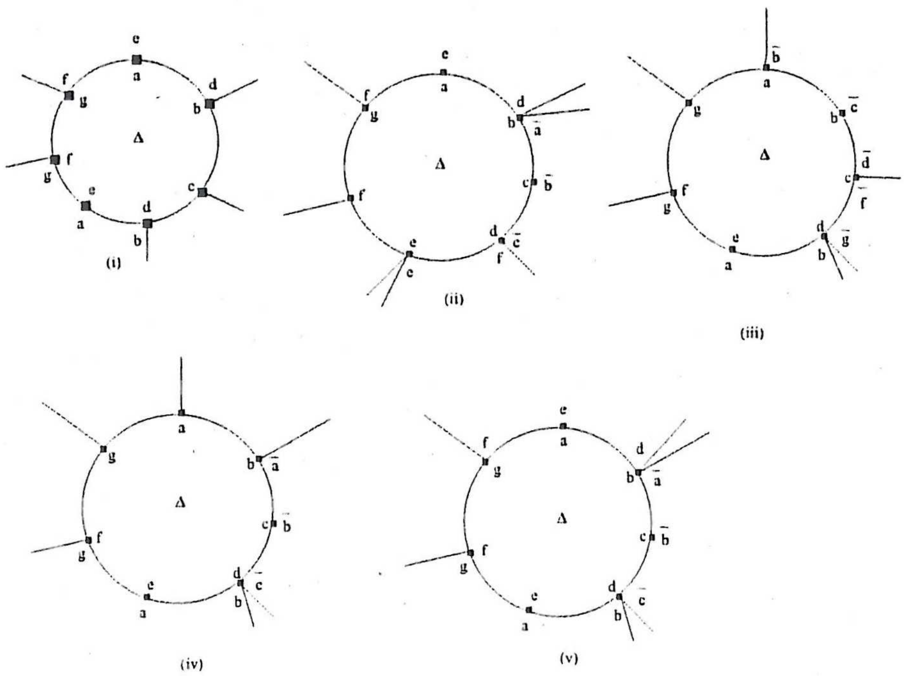

Figure 4.6: Case 7: $a = e^{-1}$ and $b = c$

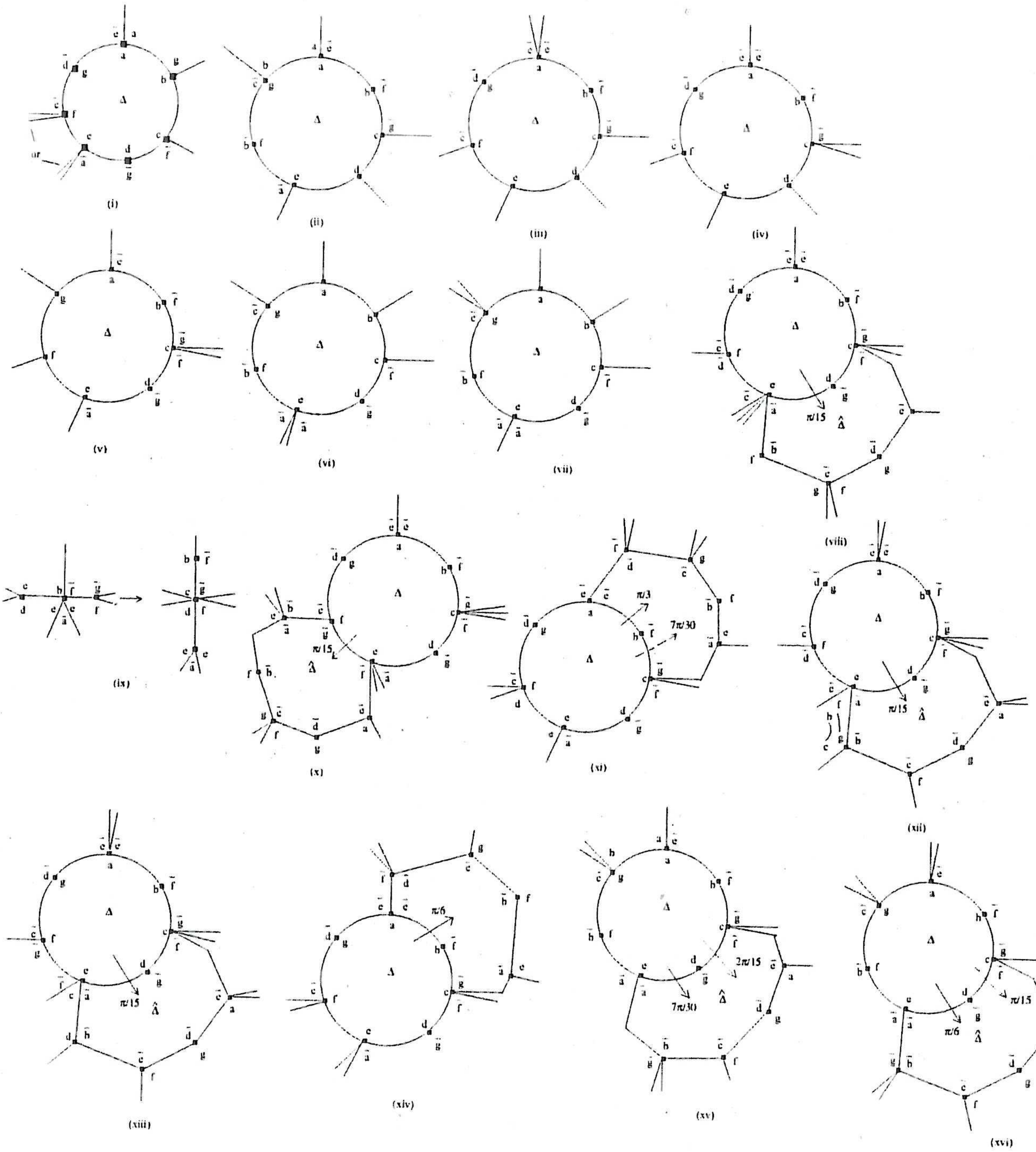

Figure 4.7: Case 8: $d = g$, $b = f$

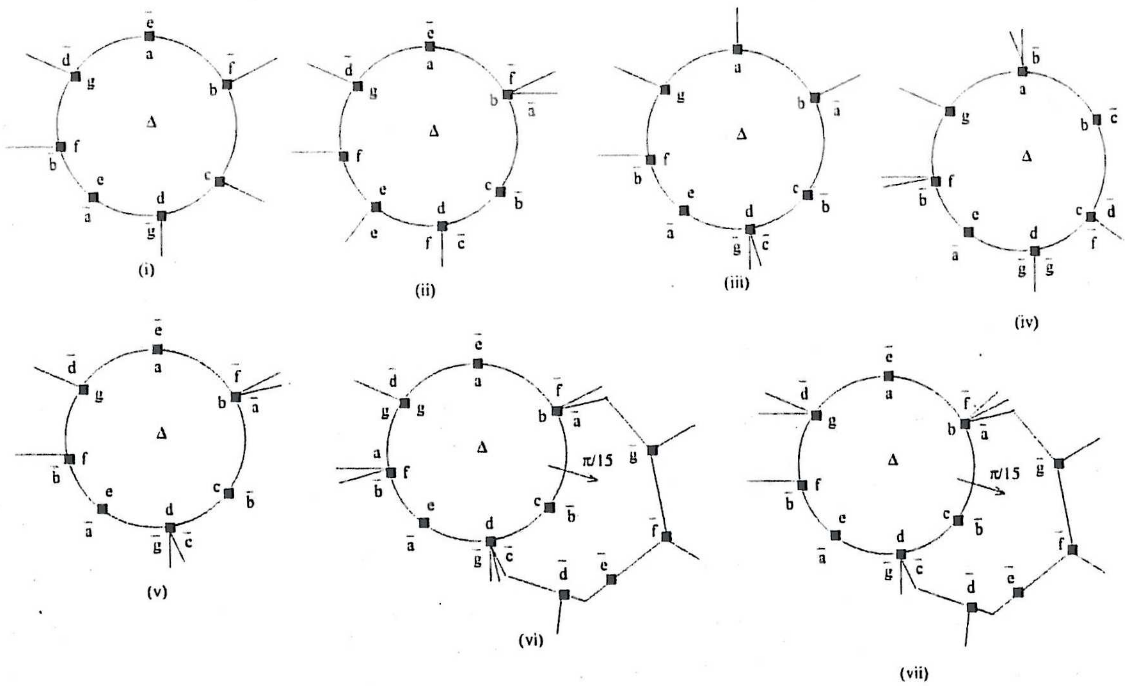

Figure 4.8: Case 9: $a = e$ and $b = c$

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