Hyperbolic geometry for 3d gravity

2. Hyperbolic surfaces

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Hyperbolic surfaces

A hyperbolic surface is locally modeled on the hyperbolic plane.
Equivalently : a surface with a Riemannian metric of curvature -1.
H^2 is the only 1-connected complete hyperbolic surface. If S is closed and hyperbolic, its universal cover is H^2. Proof : complete, simply connected, hence H^2.
So $\pi_1(S) \subset PSL(2, R)$.

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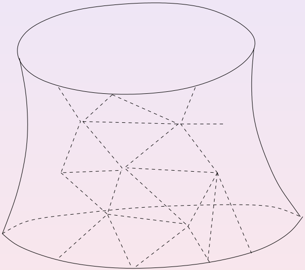
The Gauss-Bonnet formula

Consider a surface S, perhaps with boundary, with a triangulation T. Let $\chi(S,t) = \#(\mathit{vertices}) - \#(\mathit{edges}) + \#(\mathit{faces})$.

Thm : $\chi(S,T)$ does not depend on T, i.e. $\chi(S)$. $\chi(S)$ = Euler characteristic of S.

Gauss-Bonnet thm : on a hyperbolic surface, the sum of the exterior angles of a polygonal region is $2\pi\chi + A$.



Disk : $\chi = 1$. Annulus, torus : $\chi = 0$. Sphere : $\chi = 2$.

Application : the area of an ideal triangle (all vertices at infinity) is π.

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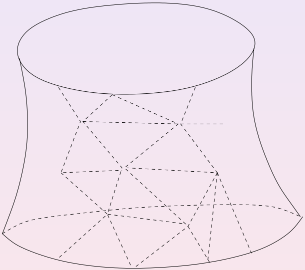

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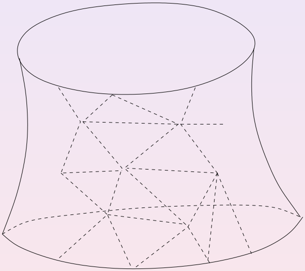

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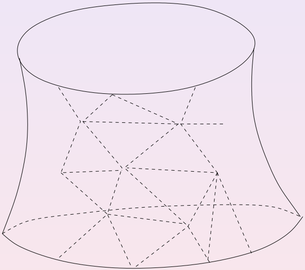

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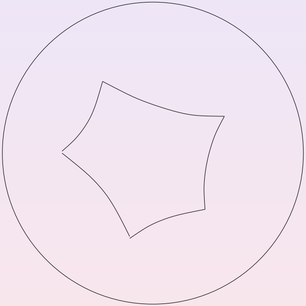

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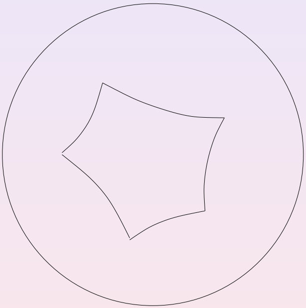

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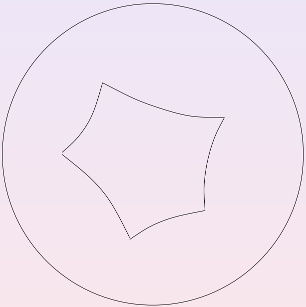

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Closed geodesics

Thm : any closed curve on a hyperbolic surface S is homotopic to a unique closed curve.

Proof of existence : take shortest curve in the homotopy class, has to be geodesic.
Proof of uniqueness : based on Gauss-Bonnet.

- Two homotopic geodesic can not intersect ($\chi = 1$ for a disk).
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Therefore, $\pi_1(S)$ is the set of closed geodesics (perhaps with self-intersections).

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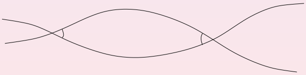

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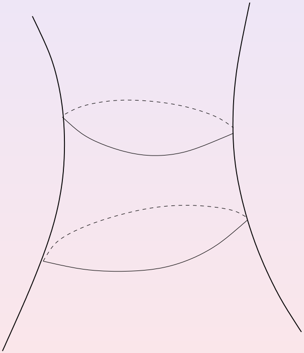

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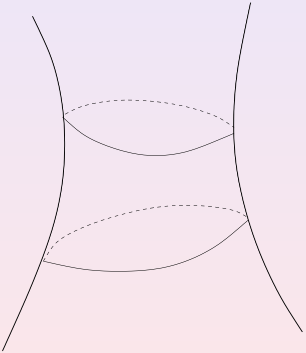

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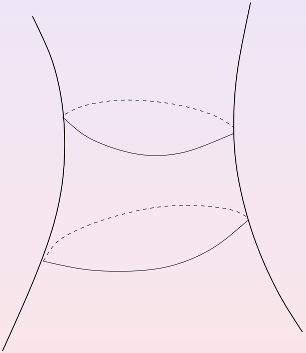

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Surfaces with cusps

Beyond closed hyperbolic surfaces : surfaces with cusps.

Start with a hyperbolic polygon, with some vertices at infinity. Its area is bounded (Gauss-Bonnet). Glue to another copy of itself, no singularity occurs on the (geodesic) edges.

Result : finite area hyperbolic surface.

Same is possible with infinite area surfaces. The "cusps" at infinity are all the same (isometric neighborhoods). The corresponding representations have "parabolic" elements (fixing a point at infinity).

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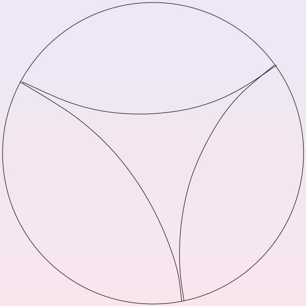

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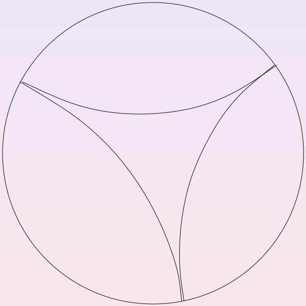

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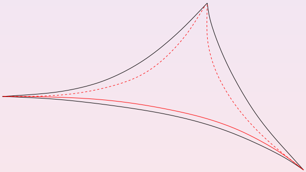

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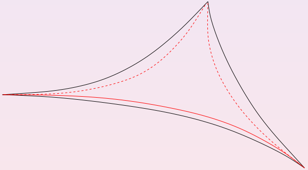

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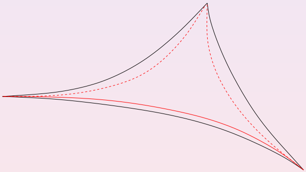

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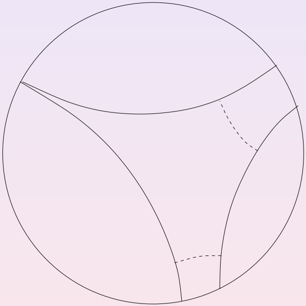

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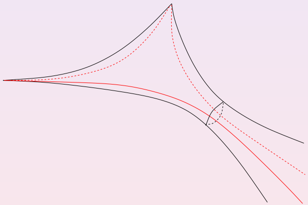

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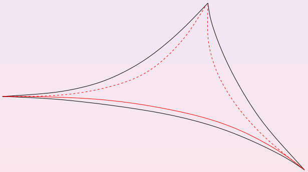

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Hyperbolic surfaces as representations

Each hyperbolic metric on a surface S defines an isometric action of $\pi_1(S)$ on H^2, therefore a representation $\rho : \pi_1(S) \to PSL(2,\mathbb{R})$. So $\mathcal{T}$ embeds in the *representation space* $Hom_{irr}(\pi_1(S), PSL(2,\mathbb{R}))/PSL(2,\mathbb{R})$.

This space is not connected : the connected components are characterized by a topological invariant, the Euler number.

$\mathcal{T}$ corresponds to the connected component with maximal Euler number.

There are several "elementary" ways to understand $\mathcal{T}$:

- Fenchel-Nielsen coordinates, based on the decomposition into "pairs of pants",
- shear coordinates (for surfaces with a cusp),
- complex analytic viewpoint.

Each is important for 3-dim geometry, in different ways. We will concentrate first on Fenchel-Nielsen coordinates.

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Each hyperbolic metric on a surface S defines an isometric action of $\pi_1(S)$ on H^2, therefore a representation $\rho : \pi_1(S) \to PSL(2, \mathbb{R})$. So $\mathcal{T}$ embeds in the *representation space* $Hom_{irr}(\pi_1(S), PSL(2, \mathbb{R}))/PSL(2, \mathbb{R})$. This space is not connected : the connected components are characterized by a topological invariant, the Euler number.
$\mathcal{T}$ corresponds to the connected component with maximal Euler number.
There are several "elementary" ways to understand $\mathcal{T}$:

- Fenchel-Nielsen coordinates, based on the decomposition into "pairs of pants",
- shear coordinates (for surfaces with a cusp),
- complex analytic viewpoint.

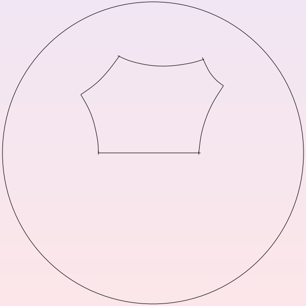
Each is important for 3-dim geometry, in different ways. We will concentrate first on Fenchel-Nielsen coordinates.

Right-angled hexagons

The building block used here are hyperbolic pairs of pants, themselves built from two copies of a hexagon with right angles.

Lemma : $\forall l_1, l_2, l_3 \in \mathbb{R}_{>0}, \exists!$ hyperbolic hexagon with right angles with alternate lengths l_1, l_2, l_3.

Proof : start from one edge, length d, add two orthogonal edges, length l_1, l_2, then two more orthogonal edges, if $d \geq d_0$, they are connected by a unique orthogonal edge of length $l_3(d)$. $l_3(d)$ is an increasing function going from 0 (for d_0) to infinity. qed.



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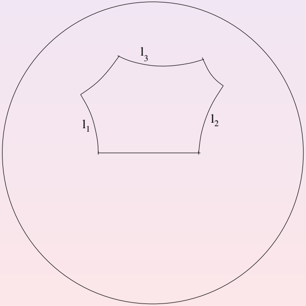

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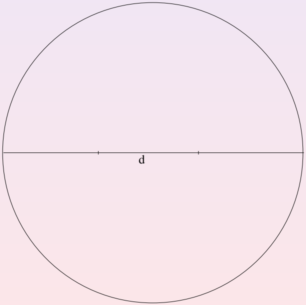

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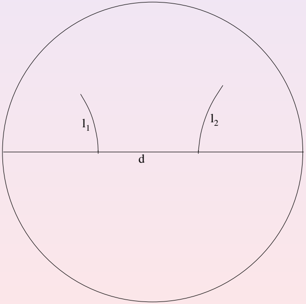

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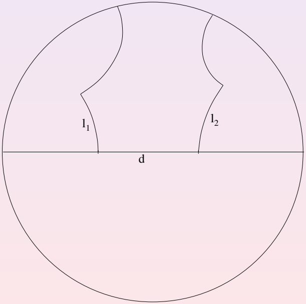

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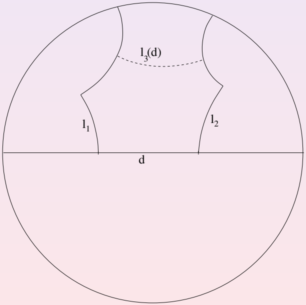

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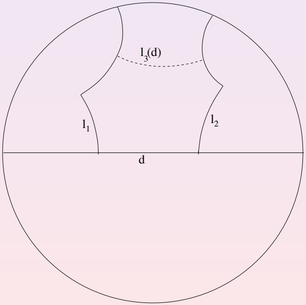

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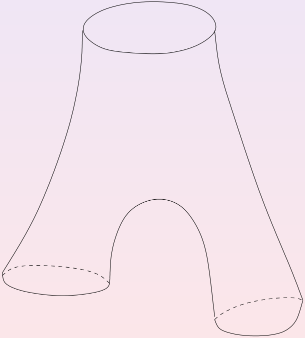

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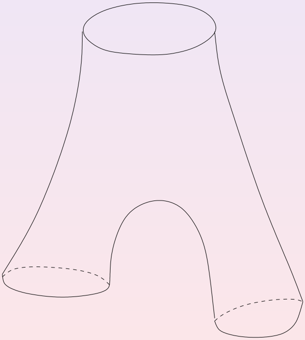

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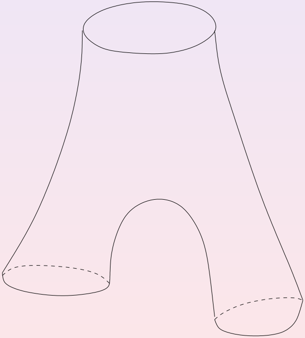

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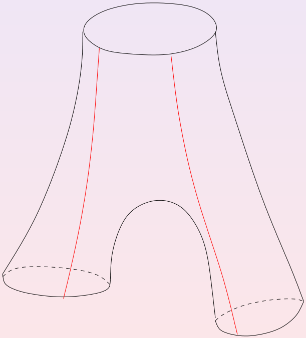

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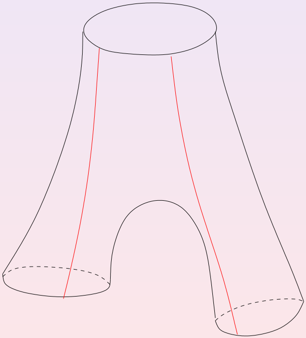

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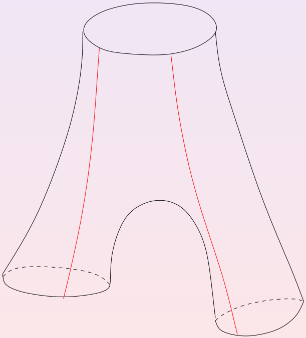

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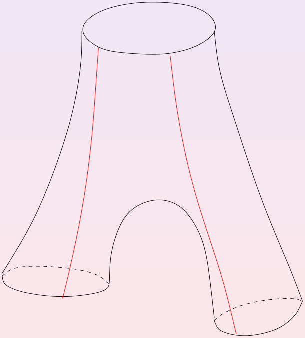

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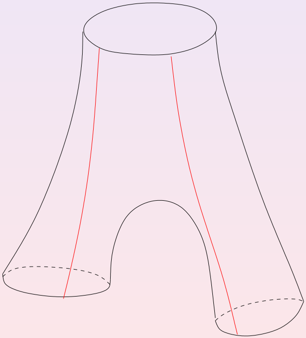

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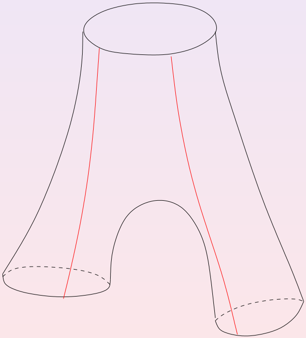

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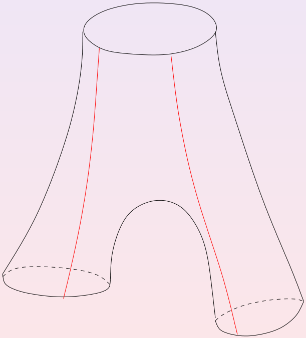

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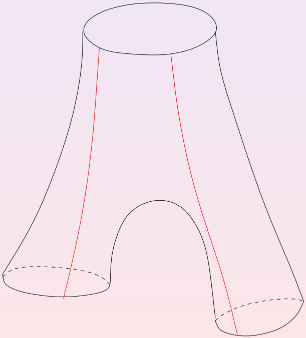

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Fenchel-Nielsen coordinates

Consider a surface S of genus ≥ 2, with a *pant decomposition* : a family of n disjoint simple closed curves with complement pairs of pants.

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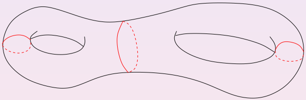
Moreover, the *gluing angles* define $\theta_1, \cdots, \theta_n$. The l_i, θ_i describe the metric up to diffeo.

Thm : the $\theta_i \in \mathbb{R}$ are well-defined on $\mathcal{T}_S$.

The (l_i, θ_i) are global coordinates on $\mathcal{T}_S$.

Cor : $\mathcal{T}_S$ is homeomorphic to a ball of dimension $6g - 6$.



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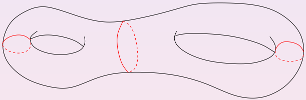

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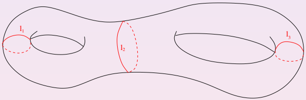

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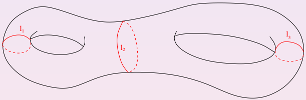

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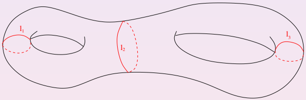

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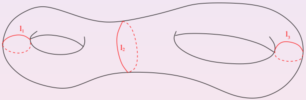

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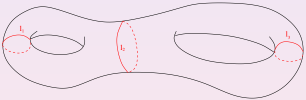

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