# Hyperbolic geometry for 3d gravity

## 5. AdS 3-manifolds

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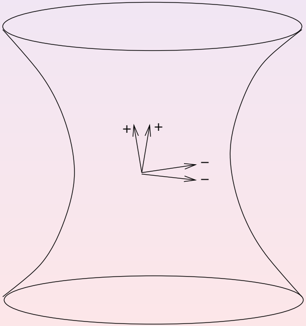
# The AdS space

$$AdS^3 = \{x \in \mathbb{R}^{2,2} \mid -x_0^2 - x_1^2 + x_2^2 + x_3^2 = -1\} .$$

$AdS^3$ is a Lorentz space with constant curvature $-1$. It has a projective model (as for $H^2$), interior of a quadric $Q$.
$Isom_0(AdS^3) = PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R})$ : $Q$ is ruled by two families of lines, preserved by $Isom_0(AdS^3)$.

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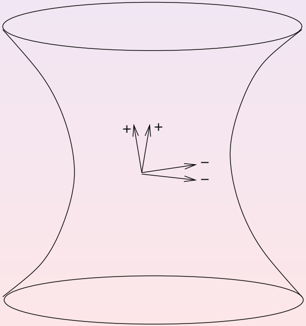


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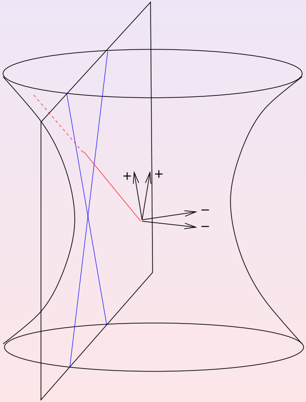


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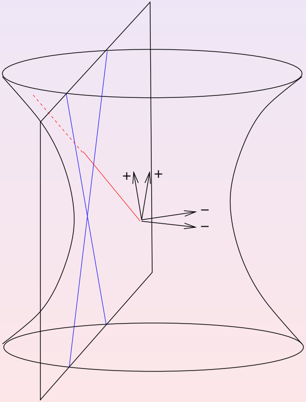


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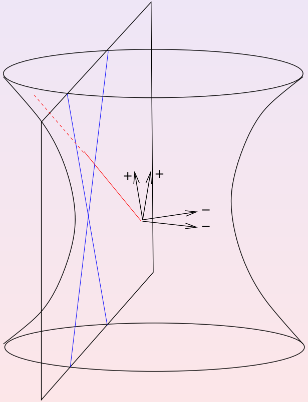


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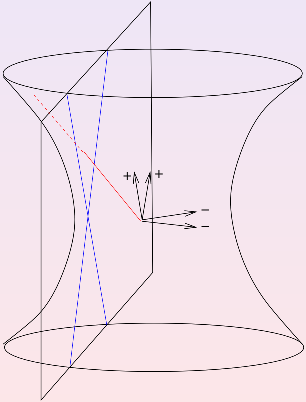


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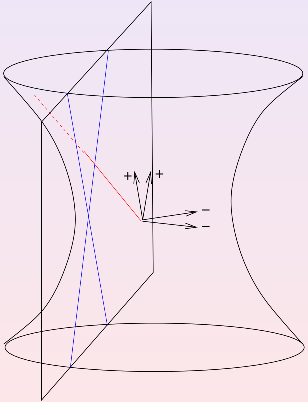


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# Fuchsian AdS manifolds

Simplest examples – analogs of Fuchsian hyperbolic manifolds.

Start with a closed hyperbolic surface $(S, g)$, consider the Lorentz manifold :

$$M = (S \times (-\pi/2, \pi/2), -dt^2 + \cos(t)^2 g) \ .$$

$M$ has constant curvature $-1$, $t = 0$ is a Cauchy surface.

$M = \Omega/\Gamma$, where $\Omega \subset AdS^3$ is the future cone of a point, and $\Gamma \simeq \pi_1(S)$ acts on a totally geodesic surface in $\Omega$, isometric to $H^2$.

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# GHMC AdS manifolds

An AdS 3-mfld is GHMC if :

- it is globally hyperbolic
- it contains a closed (oriented) space-like surface $S$ of genus $\geq 2$,
- it is maximal.

General idea : GHMC AdS mflds are very similar to quasifuchsian hyperbolic mflds.

Thm (Mess, 1990) : let $M$ be a GHMC AdS mfld. Then $M = \Omega/\rho(\pi_1(S))$, where $\Omega \subset AdS^3$ is convex and $\rho : \pi_1(S) \to Isom_0(AdS^3)$. $\rho = (\rho_l, \rho_r) : \pi_1(S) \to PSL(2, \mathbb{R}) \times PSL(2, \mathbb{R})$, and $\rho_l, \rho_r \in \mathcal{T}_S$. Any $(\rho_l, \rho_r) \in \mathcal{T}_S$ can be uniquely obtained.

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Applications to quantization ? $\mathcal{T}$ appears to be easier to quantize (Fock, $\cdots$).

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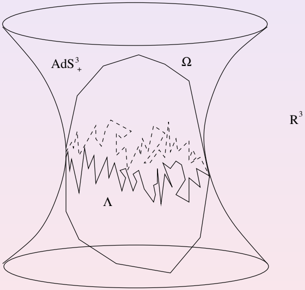
# The convex core

The limit set $\Lambda$ of $M$ can be defined (almost) as for quasifuchsian manifolds.




- It is still a Jordan curve, $C^\alpha$. $CC(M) = CH(\Lambda)/\rho(\pi_1(S))$ is the smallest convex subset of $M$ containing a space-like surface.
- Its boundary has two components, each is a convex, ruled space-like surface, with hyperbolic induced metric $h_\pm$, bent along a measured lamination $\lambda_\pm$.

Conjecture (Mess 1990) : the maps $(h_+, h_-) : \mathcal{GH} \to \mathcal{T} \times \mathcal{T}$ and $(\lambda_+, \lambda_-) : \mathcal{GH} \to \mathcal{ML} \times \mathcal{ML}$ are homeomorphisms. ? ? ? ?

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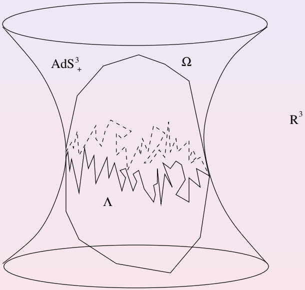


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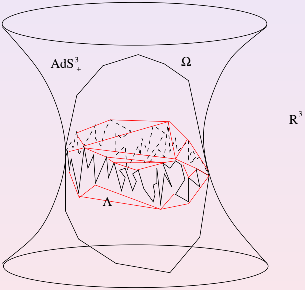


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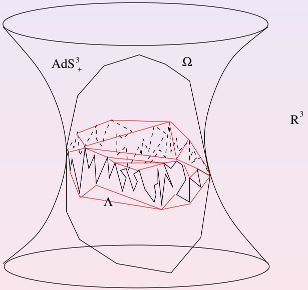


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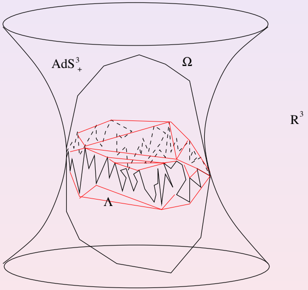


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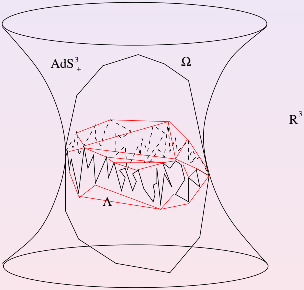


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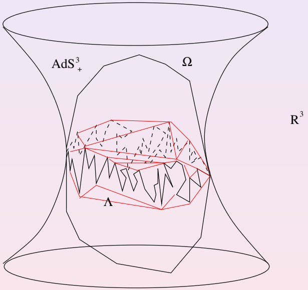


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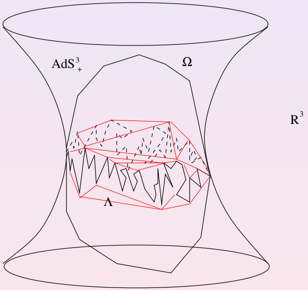


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# A proof of Thurston's Earthquake Theorem (Mess)

GHMC AdS mflds provide a direct proof of the Earthquake Theorem.

Thm (Mess 1990) : $\rho_l = E_l(\lambda_+)(h_+)$, and similarly for $\rho_-, h_-$.
Cor : given $\rho_l = E_l(\lambda_+)^{-1} \circ E_r(\lambda_+)(\rho_r)$
$= E_r(\lambda_+)^2(\rho_r) = E_r(2\lambda_+)(\rho_r)$.

Given $\rho_l, \rho_r \in \mathcal{T}$, they define a unique GHMC AdS mfld $M$, then $\rho_l = E_r(2\lambda_+)(\rho_r)$. The uniqueness also follows from this construction.

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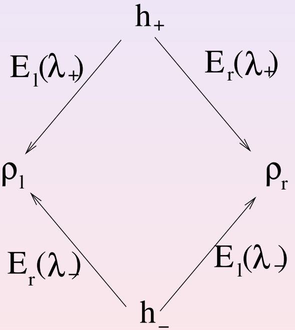


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Cor : given $\rho_l = E_l(\lambda_+)^{-1} \circ E_r(\lambda_+)(\rho_r)$ $= E_r(\lambda_+)^2(\rho_r) = E_r(2\lambda_+)(\rho_r)$.

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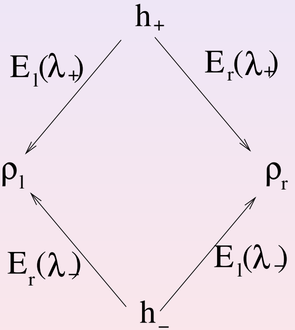


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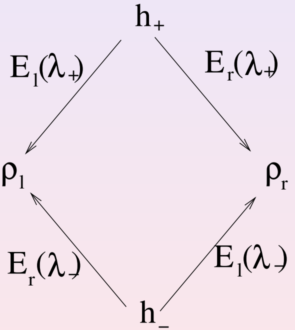


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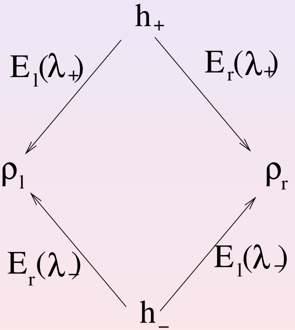


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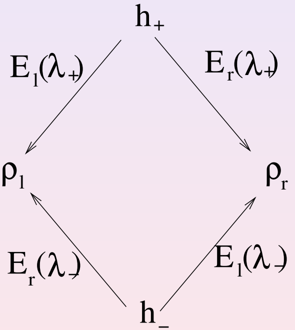


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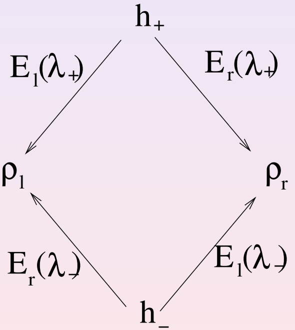


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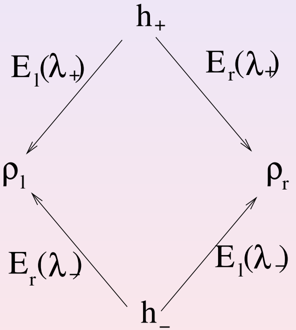


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Let $S$ be a surface with a metric $g$ and a bilinear symmetric form $h$. Then :

1. $tr_{[g]}(h) = 0$ iff $h = Re(q)$ for a quadratic differential $q$.
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For fixed $g$, set $g' = e^{2u}g$. Then $K' = e^{-2u}(-\Delta u + K)$, while $det_{g'} h = e^{-4u} det_g h$. So condition (3) for $g'$ is :

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Sols correspond to critical points of :

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# Maximal surfaces

Considering maximal surfaces yields another interesting parametrization of $\mathcal{GH}$.

Thm : any GHMC AdS manifold contains a unique closed space-like maximal surface.

Conversely, the maximal surfaces in AdS constructed in the previous slide all "extend" to a GHMC AdS manifold.

Recall that QHD for $c \simeq T^*_c\mathcal{T}$.

Thm (Krasnov, S. ; Fock, Taubes, etc) : the map $([I], II) : \mathcal{GH} \to T^*\mathcal{T}$ is a homeomorphism.

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## Particles

Def : "particles" are cone singularities along time-like lines (cf "hinges" in Ruth Williams' course). The angle is less than $2\pi$. Two cases :

- angles $< \pi$ : the mathematical theory works well but collisions between particles are (almost) forbidden.
- angles $\in (\pi, 2\pi)$ : collisions are possible but global descriptions are more complicated.

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# Teichmüller space with marked points

Now $S$ is a closed surface of genus $g \geq 2$ with some marked points $x_1, \cdots, x_n$. $\mathcal{T}_{g,n}$ is the space of complex structures on $S$, up to isotopies fixing the $x_i$.

Thm : any $h \in \mathcal{T}_{g,n}$ is compatible with a unique complete hyperbolic metric with cusps at the $x_i$.

Thm (Troyanov, '90) : let $c \in \mathcal{T}_{g,n}$, and let $\theta_1, \cdots, \theta_n \in (0, 2\pi)$. There is a unique hyperbolic metric $h$ compatible with $c$, with cone singularities at the $x_i$ of angle $\theta_i$.

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# The Mess parametrization with particles

Unfortunately, for $M$ GHMC AdS with particles, the holonomy is rather bad : no action on a "nice" space, etc. But hyperbolic metrics can be used (Krasnov, S.). Let $S \subset M$ be a closed space-like surface, orthogonal to the particles, with $|k_i| < 1$. Let $I^{\#}_{\pm}(x, y) = I((E \pm JB)x, (E \pm JB)y)$. Then

- $I^{\#}_{\pm}$ are hyperbolic metrics on $S$,
- they do not depend on the choice of $S$,
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- with particles, they have cone sings of prescribed angle.

Thm (Bonsante, S.) : the map $(I^{\#}_{+}, I^{\#}_{-}) : \mathcal{GH} \to \mathcal{T}_{g,n} \times \mathcal{T}_{g,n}$ is a homeomorpism.

Mathematical side : an extension of the earthquake theorem to hyperbolic surfaces with cone singularities. There is also a parametrization of $\mathcal{GH}$ by $T^*\mathcal{T}_{g,n}$ using maximal surfaces (Krasnov, S.).

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Unfortunately, for $M$ GHMC AdS with particles, the holonomy is rather bad : no action on a "nice" space, etc. But hyperbolic metrics can be used (Krasnov, S.). Let $S\subset M$ be a closed space-like surface, orthogonal to the particles, with $|k_i|<1$. Let $I^\#_\pm(x,y)=I((E\pm JB)x,(E\pm JB)y)$. Then

- $I^\#_\pm$ are hyperbolic metrics on $S$,
- they do not depend on the choice of $S$,
- when no particle is present, they correspond to $\rho_l,\rho_r$.
- with particles, they have cone sings of prescribed angle.

Thm (Bonsante, S.) : the map $(I^\#_+,I^\#_-):\mathcal{GH}\to\mathcal{T}_{g,n}\times\mathcal{T}_{g,n}$ is a homeomorpism.

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# Multi Black holes

Simplest example ("non-rotating") : start from a complete hyperbolic surface $(S, g)$ with ends of infinite area (not cusps), consider again

$$M = (S \times (-\pi/2, \pi/2), -dt^2 + \cos(t)^2 g) \ .$$

Not globally hyperbolic, the infinite ends do not "see" what happens in the part with topology, or in the other infinite ends (wormhole). $M = \Omega/\Gamma$, where $\Omega \subset AdS^3$ and $\Gamma \simeq \pi_1(S)$ is a free group in $PSL(2, \mathbb{R}) \times PSL(2, \mathbb{R})$. This example can be deformed ("rotating" case). The space of MBH of given topology is parametrized by two copies of the Teichmüller space of hyperbolic metrics with geodesic boundary components (Barbot).

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