Generic points of quadrics and Chow groups

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Abstract

In this article we study the sufficient conditions for the $\overline{k}$-defined element of the Chow group of a smooth variety to be k-rational (defined over k). For 0-cycles this question was addressed earlier. Our methods work for cycles of arbitrary dimension. We show that it is sufficient to check this property over the generic point of a quadric of sufficiently large dimension. Among the applications one should mention the uniform construction of fields with all known u-invariants.

1 Introduction

Let Y be smooth variety defined over the field k. In many situations, one needs to know, if some class of rational equivalence of cycles on $Y|_{\overline{k}}$ is defined over the base field k. In particular, this happens when one computes so-called "generic discrete invariant of a quadric" - see Introduction of [10]. It occurs, that often it is sufficient to check this property not over k, but over some bigger field $k(Q)/k$, where everything could be much simpler. The case of zero cycles modulo some prime l is already quite classical. It is a standard application of the Rost degree formula that if Q is a ν_n-variety, and $dim(Y) < dim(Q) = l^{n-1} - 1$, then the 0-cycle of degree prime to l exists on Y if and only if it exists on $Y|_{k(Q)}$ (see [7],[8]). For the case, where Q is a quadric, this gives: if $dim(Q) \geqslant 2^r - 1 > dim(Y)$, then the existence of zero cycle of odd degree on Y is equivalent to the existence of such cycle on $Y|_{k(Q)}$. If Y is quadric as well, we get the well-known Theorem of D.Hoffmann ([2]). Finally, if one knows more about the quadric Q, not just its dimension, there is stronger result of N.Karpenko-A.Merkurjev (see [3]), saying that the same holds if $dim(Q) - i_1(Q) + 1 > dim(Y)$.

In the current article we address mentioned question for cycles of arbitrary dimension, $l = 2$, Q - quadric. The principal result (Corollary 3.5) says that for class $\overline{y} \in CH^m(Y|_{\overline{k}})/2$, for $m < [dim(Q)+1/2]$, $\overline{y}$ is defined over k if and only if $\overline{y}|_{\overline{k(Q)}}$ is defined over $k(Q)$. If one does not impose any restrictions on Q and Y, the above condition on m can not be improved (Statement 3.7). The stronger result (Theorem 3.1) claims that if $\overline{y}|_{\overline{k(Q)}}$ is defined over $k(Q)$, and $m - [dim(Q)+1/2] < j$, then $S^j(\overline{y})$ is defined over k, where S^* is a Steenrod operation (see [1], [14]). This statement is very useful for the computation of the generic discrete invariant of quadrics. The proof is based on the so-called "symmetric operations" - see [9].

In the end we formulate the Conjecture which is an analogue of the Karpenko-Merkurjev Theorem for cycles of positive dimension.

The methods and results of the current paper serve as a main tool in the uniform construction of fields with all known u-invariants. This construction gives the new values of the u-invariant: $2^r + 1$, $r > 3$ - see [12].

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2 Symmetric operations

Everywhere below we will assume that all our fields have characteristic zero. For a smooth variety X, M.Levine and F.Morel have defined the ring of algebraic cobordisms $\Omega^*(X)$ with a natural surjective map $pr : \Omega^*(X) \to CH^*(X)$. It is an analogue of the complex-oriented cobordisms in topology. In particular, one has the action of the Landweber-Novikov operations there ([6]). In [9, 11] the author constructed some new cohomological operations on Algebraic Cobordisms, so-called, *symmetric operations*. In the questions related to 2-torsion these operations behave in a more subtle way than the Landweber-Novikov operations. This article could serve as a demonstration of this feature. Symmetric operations $\Phi^{t^r} : \Omega^d(X) \to \Omega^{2d+r}(X)$, for $r \geqslant 0$ are defined in the following way. For a smooth morphism $W \to U$, let $\widetilde{\square}(W/U)$ denotes the blow-up of $W \times_U W$ at the diagonal W. For a smooth variety W denote: $\widetilde{\square}(W) := \widetilde{\square}(W/\operatorname{Spec}(k))$. Denote as $\widetilde{C}^2(W)$ and $\widetilde{C}^2(W/U)$ the quotient variety of $\widetilde{\square}(W)$, respectively, $\widetilde{\square}(W/U)$ by the natural $\mathbb{Z}/2$-action. These are smooth varieties. Notice that they have natural line bundle $\mathcal{L}$, which lifted to $\widetilde{\square}$ becomes $\mathcal{O}(1)$ - see [7]. Let $\rho := c_1(\mathcal{L}^{-1}) \in \Omega^1(\widetilde{C}^2)$.

If $[v] \in \Omega^d(X)$ is represented by $v : V \to X$, then v can be decomposed as $V \xrightarrow{g} W \xrightarrow{f} X$, where g is a regular embedding, and f is smooth projective. One gets natural maps:

$$\widetilde{C}^2(V) \overset{\alpha}{\hookrightarrow} \widetilde{C}^2(W) \overset{\beta}{\hookleftarrow} \widetilde{C}^2(W/X) \xrightarrow{\gamma} X.$$

Now, $\Phi^{t^r}([v]) := \gamma_*\beta^*\alpha_*(\rho^r)$. Denote as $\phi^{t^r}([v])$ the composition $pr \circ \Phi^{t^r}([v])$. As was proven in [11, Theorem 2.24], Φ^{t^r} gives a well-defined operation $\Omega^d(X) \to \Omega^{2d+r}(X)$.

It was proven in [9] that the Chow-trace of Φ is the half of the Chow-trace of certain Landweber-Novikov operation.

Proposition 2.1 ([9, Propositions 3.8, 3.9], [11, Proposition 3.14])

(1) $2\phi^{t^r}([v]) = pr((-1)^{r+1}S^{r+d}_{L-N}([v]))$, *for* $r > 0$*;*

(2) $2\phi^{t^0}([v]) = pr(\square - S^d_{L-N}([v]))$,

where S^r_{L-N} *is the Landweber-Novikov operation corresponding to the characteristic number* c_r*.*

The additive properties of ϕ are given by the following:

Proposition 2.2 ([11, Proposition 2.8])

(1) *Operation* ϕ^{t^r} *is additive for* $r > 0$*;*

(2) $\phi^1(x + y) = \phi^1(x) + \phi^1(y) + pr(x \cdot y)$.

Let $[v] \in \Omega^*(X)$ be some cobordism class, and $[u] \in \mathbb{L}$ be class of some smooth projective variety U over k of positive dimension. We will use the standard notation $\eta_2(U)$ for the Rost invariant $\frac{-deg(c_{dim(U)}(-T_U))}{2} \in \mathbf{Z}$ (see [7]).

Proposition 2.3 ([11, Proposition 3.15]) *In the above notations, let* $r = (codim(v) - 2dim(u))$*. Then, for any* $i \geqslant \max(r; 0)$*,*

$$\phi^{t^{i-r}}([v] \cdot [u]) = (-1)^{i-r}\eta_2(U) \cdot (pr \circ S^i_{L-N})([v]).$$

The following proposition describes the behavior of Φ with respect to pull-backs and regular push-forwards. For $q(t) \in CH^*(X)[[t]]$ let us define $\phi^{q(t)} := \sum_{i\geqslant 0} q_i \phi^{t^i}$. For a vector bundle $\mathcal{V}$ denote $c(\mathcal{V})(t) := \prod_i (t+\lambda_i)$, where $\lambda_i \in CH^1$ are the *roots* of $\mathcal{V}$. This is the usual total Chern class of $\mathcal{V}$.

Proposition 2.4 ([11, Propositions 3.1, 3.4]) *Let $f : Y \to X$ be some morphism of smooth quasiprojective varieties, and $q(t) \in CH^*(X)[[t]]$. Then*

(1) $f^*\phi^{q(t)}([v]) = \phi^{f^*q(t)}(f^*[v])$;

(2) *If f is a regular embedding, then* $\phi^{q(t)}(f_*([w])) = f_*(\phi^{f^*q(t)\cdot c(\mathcal{N}_f)(t)}([w]))$, *where $\mathcal{N}_f$ is the normal bundle of the embedding.*

And, consequently, for f - a regular embedding:

(3) $\phi^{q(t)}(f_*([1_Y])\cdot[v]) = \phi^{q(t)\cdot f_*(c(\mathcal{N}_f)(t))}([v])$

3 Results

We say that an element of $\mathrm{CH}^*(Y|_{\overline{F}})/2$ is *defined over F*, if it belongs to the image of the restriction map $ac : CH^*(Y|_F)/2 \to CH^*(Y|_{\overline{F}})/2$.

Theorem 3.1 *Let k be a field of characteristic 0. Let Y be a smooth quasiprojective variety, Q be a smooth projective quadric of dimension n, and $\overline{y} \in CH^m(Y|_{\overline{k}})/2$ be some element. Suppose that $\overline{y}|_{\overline{k(Q)}}$ is defined over $k(Q)$. Then:*

(1) *For all $j > m - [n+1/2]$, $S^j(\overline{y})$ is defined over k;*

(2) *For $j = m - [n+1/2]$, $S^j(\overline{y}) + \overline{z}\cdot\overline{y}$ is defined over k, for some $\overline{z} \in im(ac \circ (\pi_Y)_* \circ (\cdot h^{[n/2]}) : CH^m(Q\times Y)/2 \to CH^j(Y|_{\overline{k}})/2)$.*

Proof: Since $\overline{y}|_{\overline{k(Q)}}$ is defined over $k(Q)$, there exists $x \in CH^m(Q\times Y)/2$ such that $\overline{y}|_{\overline{k(Q)}} = \overline{i^*(x)}\ (mod\, 2)$, where $i : \mathrm{Spec}(k(Q)) \times Y \to Q\times Y$ is the embedding of the generic fiber.

Over $\overline{k}$, quadric Q becomes a cellular variety, with the basis in Chow ring given by the classes $\{h^i, l_i\}_{0\leqslant i\leqslant [n/2]}$ of plane sections and projective subspaces. Hence,

$$\mathrm{CH}^*(Q\times Y|_{\overline{k}}) = \bigoplus_{i=0}^{[n/2]} (\pi_Q^*(h^i)\cdot\pi_Y^*\,\mathrm{CH}^*(Y|_{\overline{k}}) \oplus \pi_Q^*(l_i)\cdot\pi_Y^*\,\mathrm{CH}^*(Y|_{\overline{k}})),$$

and

$$\overline{x} = \sum_{i=0}^{[n/2]} h^i \cdot \overline{x}^i + \sum_{i=0}^{[n/2]} l_i \cdot \overline{x}_i,$$

for certain unique $\overline{x}^i \in \mathrm{CH}^{m-i}(Y|_{\overline{k}})/2$ and $\overline{x}_i \in \mathrm{CH}^{m-n+i}(Y|_{\overline{k}})/2$.

Lemma 3.2 $\overline{x}^0 = \overline{y}$.

Proof: Clearly, $\overline{x}^0|_{\overline{k(Q)}} = \overline{y}|_{\overline{k(Q)}}$. But for any field extension E/F with F algebraically closed, the restriction homomorphism on Chow groups (with any coefficients) is injective by the specialization arguments. □

Proposition 3.3 *Let $x \in \mathrm{CH}^m(Q \times Y)/2$ be some element, and $\overline{x}^i, \overline{x}_i$ be it's coordinates as above. Then*

(1) *For $j > m - [n+1/2]$, $S^j(\overline{x}^0)$ is defined over k.*

(2) *For $j = m - [n+1/2]$, $S^j(\overline{x}^0) + \overline{x}^0 \cdot \overline{x}_{[n/2]}$ is defined over k.*

Proof:
We have the natural map $pr : \Omega^* \to CH^*$ which is surjective by the result of M.Levine-F.Morel (see [6, Theorem 14.1]). Thus, there exists $v \in \Omega^m(Q \times Y)$ such that $pr(v) = x$.

Again, since over $\overline{k}$, quadric Q becomes a cellular variety,

$$\Omega^*(Q \times Y|_{\overline{k}}) = \bigoplus_{\beta \in B} \pi_Q^*(f_\beta) \cdot \pi_Y^* \Omega^*(Y|_{\overline{k}}),$$

where $\{f_\beta\}_{\beta \in B}$ can be any set of elements such that $\{pr(f_\beta)\}_{\beta \in B}$ form a $\mathbb{Z}$-basis of $CH^*(Q|_{\overline{k}})$ - see Section 2 of [13]. In particular, we can take the set $\{h^i, l_i\}_{0 \leqslant i \leqslant [n/2]}$, where h is a (cobordism-) class of a hyperplane section, and l_i is a class of a projective plane of dimension i on Q. Thus,

$$\overline{v} = \sum_{i=0}^{[n/2]} h^i \cdot \overline{v}^i + \sum_{i=0}^{[n/2]} l_i \cdot \overline{v}_i,$$

for certain $\overline{v}^i \in \Omega^{m-i}(Y|_{\overline{k}})$ and $\overline{v}_i \in \Omega^{m-n+i}(Y|_{\overline{k}})$, which satisfy: $pr(\overline{v}^i) = \overline{x}^i$, $pr(\overline{v}_i) = \overline{x}_i$.

Let $n = 2^r - 1 + s$, where $0 \leqslant s < 2^r$. Let us denote $n - s = 2^r - 1$ as d. Let $emb : P \subset Q$ be a (smooth) subquadric of codimension s, and $emb_g : P \subset P_g$ be embedding of codimension g into any (smooth) quadric.

Consider $u_g := (emb_g \times id_Y)_*(emb \times id_Y)^*(v) \in \Omega^{m+g}(P_g \times Y)$. Then

$$\overline{u_g} = \sum_{i=0}^{[n/2]} h^{i+g} \cdot \overline{v}^i + \sum_{i=0}^{[n/2]-s} l_i \cdot \overline{v}_{i+s}.$$

We will obtain the needed cycle $S^j(\overline{x}^0)$ as a linear combination of certain symmetric operations applied to u_g and $(\pi_Y)_*(u_g)$.

For $0 \leqslant g \leqslant [n/2] - s$, consider

$$w_g := ((\pi_Y)_* \circ \phi^{t^{j+d-m-g}} - \phi^{t^{j+2d-m}} \circ (\pi_Y)_*)(u_g),$$

where $\phi^{t^a} : \Omega^b(X) \to CH^{2b+a}(X)$ - symmetric operation defined in [11]. Remind, that operations ϕ^{t^a} are defined only for $a \geqslant 0$. This condition is satisfied in our case, since $(j+d) - (m-d) > (j+d) - (m+g) \geqslant m - [n+1/2] + n - s - m - g = [n/2] - s - g \geqslant 0$.

For $j > m - [n+1/2]$, by Propositions 2.2, the operations we are considering are additive, and $\overline{w_g}$ is equal to

$$\sum_{i=0}^{[n/2]} ((\pi_Y)_* \circ \phi^{t^{j+d-m-g}} - \phi^{t^{j+2d-m}} \circ (\pi_Y)_*)(h^{i+g} \cdot \overline{v}^i) +$$
$$\sum_{i=0}^{[n/2]-s} ((\pi_Y)_* \circ \phi^{t^{j+d-m-g}} - \phi^{t^{j+2d-m}} \circ (\pi_Y)_*)(l_i \cdot \overline{v}_{i+s}).$$

Since for the regular embedding $f : h^{i+g} \subset Q$, the normal Chern polynomial $c(\mathcal{N}_f)(t)$ is $(t+h)^{i+g}$, we have by Proposition 2.4(3),

$$(\pi_Y)_* \circ \phi^{t^{j+d-m-g}}(h^{i+g} \cdot \overline{v}^i) = (\pi_Y)_* \circ \phi^{t^{j+d-m-g} h^{i+g} (t+h)^{i+g}}((\pi_Y)^* \overline{v}^i),$$

which is equal to $2\binom{i+g}{d-i}\phi^{t^{j+2i-m}}(\overline{v}^i)$ and is 0 modulo 2.

By Proposition 2.3, $(\phi^{t^{j+2d-m}} \circ (\pi_Y)_*)(h^{i+g} \cdot \overline{v}^i)$ is equal, modulo 2, to the Chow trace of $(-1)^{j-m+1}\binom{-(d-i+2)}{d-i} S^{i+j}_{L-N}(\overline{v}^i)$, and modulo 2 this is equal to $\binom{-(d-i+2)}{d-i} S^{i+j}(pr(\overline{v}^i))$. By dimensional reasons, the second multiple is zero, if $m - j < 2i$. Otherwise, $2i < [n+1/2]$, and $i < 2^{r-1}$. Since $\binom{-(a+2)}{a}$ is odd only for $a = 2^p - 1$, for some p, we get: modulo 2, the only nontrivial term corresponds to $i = 0$ and is equal to $S^j(pr(\overline{v}^0))$.

Hence, modulo 2,

$$\sum_{i=0}^{[n/2]}((\pi_Y)_*\circ\phi^{t^{j+d-m-g}}-\phi^{t^{j+2d-m}}\circ(\pi_Y)_*)(h^{i+g}\cdot\overline{v}^i)$$

is equal to $S^j(pr(\overline{v}^0))$.

Let now $0\leqslant i\leqslant[n/2]-s$. Again, since for the regular embedding $f: l_i\subset Q$, the normal Chern polynomial $c(\mathcal{N}_f)(t)$ is $(t+h)^{d+g-i+1}(t+2h)^{-1}$, we have by Proposition 2.4(3), that modulo 2,

$$((\pi_Y)_*\circ\phi^{t^{j+d-m-g}}(l_i\cdot\overline{v}_{i+s})=(\pi_Y)_*\phi^{t^{j+d-m-g-1}l_i(t+h)^{d+g-i+1}}((\pi_Y)^*(\overline{v}_{i+s})),$$

which is equal to $\binom{d+g-i+1}{i}\phi^{t^{j-m+2d-2i}}(\overline{v}_{i+s})$. Notice, that $j-m+2d-2i>-[n+1/2]+2d-2i=([n/2]-s-i)+(d-i)\geqslant 0$. Let us denote $\phi^{t^{j-m+2d-2i}}(\overline{v}_{i+s})$ as $\overline{\varepsilon}_i$.

By Proposition 2.3, $\phi^{t^{j+2d-m}}\circ(\pi_Y)_*(l_i\cdot\overline{v}_{i+s})$, for $i>0$, is equal to $(-1)^{j-m+1}\frac{1}{2}\binom{-(i+1)}{i}\cdot pr(S^{j+n-s-i}_{L-N}(\overline{v}_{i+s}))$, and modulo 2 this is equal to $\frac{1}{2}\binom{-(i+1)}{i}\cdot S^{j+n-s-i}(pr(\overline{v}_{i+s}))$, which is zero, since $j+n-s-i>codim(\overline{v}_{i+s})=m-n+i+s$. For $i=0$, the above expression is equal to $\overline{\varepsilon}_0$.

Putting things together, we get:

$$\overline{w_g}=S^j(pr(\overline{v}^0))+\sum_{0<i\leqslant[n/2]-s}\binom{n-s+g-i+1}{i}\overline{\varepsilon}_i.$$

Consider $\alpha:=\sum_{0\leqslant g\leqslant[n/2]-s}\binom{[n/2]-s+1}{g+1}w_g$. Then

$$\overline{\alpha}=\left(\sum_{0\leqslant g\leqslant[n/2]-s}\binom{[n/2]-s+1}{g+1}\right)S^j(pr(\overline{v}^0))+$$
$$\sum_{0<i\leqslant[n/2]-s}\overline{\varepsilon}_i\sum_{0\leqslant g\leqslant[n/2]-s}\binom{n-s+g-i+1}{i}\binom{[n/2]-s+1}{g+1}.$$

Lemma 3.4 (*a*) *The number* $\sum_{0\leqslant g\leqslant[n/2]-s}\binom{[n/2]-s+1}{g+1}$ *is odd.*

(*b*) *The number* $\sum_{0\leqslant g\leqslant[n/2]-s}\binom{n-s+g-i+1}{i}\binom{[n/2]-s+1}{g+1}$ *is even.*

Proof: Let $d=[n/2]-s$, $k=g+1$.
a) The sum is equal to $\sum_{c=1}^{d+1}\binom{d+1}{c}$, which is $2^{d+1}-1$.

b) The sum here is equal to $\sum_{k=1}^{d+1}\binom{n-s+k-i}{i}\binom{d+1}{k}$. Since $\binom{n-s-i}{i}=\binom{2^r-1-i}{i}$ is even, the sum is equal (modulo 2) to $\sum_{k=0}^{d+1}\binom{n-s+k-i}{i}\binom{d+1}{k}$. The latter expression is equal (modulo 2) to $\sum_{k=0}^{d+1}\binom{d+1}{d+1-k}\binom{-(i+1)}{n-s+k-2i}=\binom{d-i}{d+1+n-s-2i}$. Since $(d-i)\geqslant 0$, and $(d+1+n-s-2i)-(d-i)=n-s+1-i=2^r-i>0$, we get 0 (modulo 2). $\square$

It follows from Lemma 3.4 that (modulo 2) $\overline{\alpha}=S^j(pr(\overline{v}^0))=S^j(\overline{x}^0)$. But w_g and thus α are defined over the base field k. Then so is $S^j(\overline{x}^0)$.

For $j=m-[n+1/2]$, $(j+2d-m)$ is always greater than zero, and $(j+d-m-g)$ is greater than zero, except for the case $g=[n/2]-s$. Thus, for $0\leqslant g<[n/2]-s$, $\overline{w}_g$ is given by the same formulas as above, and for $g=[n/2]-s$, we have extra terms $pr(\overline{v}^0\cdot\overline{v}_{[n/2]})+2\cdot$ (*something*). Thus, in this case, modulo 2, $\alpha=S^j(\overline{x}^0)+\overline{x}^0\cdot\overline{x}_{[n/2]}$.

$\square$

By Lemma 3.2, $pr(\overline{v}^0)$ is exactly $\overline{y}$. It remains to take $\overline{z}=pr(\overline{v}_{[n/2]})$. Clearly, $\overline{z}=(\pi_Y)_*\circ pr(h^{[n/2]}\cdot\overline{v})$.

Theorem 3.1 is proven.

$\square$

Remark: If one does not mind moding out also a 2-torsion in the Chow groups, one can get similar result just with the help of the usual Landweber-Novikov operations. Here instead of using Propositions 2.3 and 2.4, one should apply multiplicative properties of the Landweber-Novikov operations. But to obtain the "clean" statement as above, the use of the *symmetric operations* is essential.

Corollary 3.5 *Under the conditions of Theorem 3.1:*

(1) *For $m<[n+1/2]$, $\overline{y}$ is defined over k;*

(2) *For $m=[n+1/2]$, either $\overline{y}$ is defined over k, or $Q|_{k(Y)}$ is completely split.*

Proof: Take $j=0$, and use the fact that $S^0=id$. In (2) observe, that either $\overline{z}$ is zero, or the composition $CH^{[n+1/2]}(Q\times Y)/2\stackrel{\cdot h^{[n/2]}}{\to} CH^n(Q\times Y)/2\stackrel{(\pi_Y)_*}{\to} CH^0(Y)/2\stackrel{ac}{\cong} CH^0(Y|_{\overline{k}})/2=\mathbb{Z}/2$ is onto, and thus $Q|_{k(Y)}$ is completely split.

□

If one does not impose any conditions on the quadric Q, as well as on the relation between codimension of the cycle and the dimension of Y, the boundary in the Corollary 3.5 is optimal.

Let Q be a generic quadric of dimension n (that is, quadric given by the form $\langle a_1, \ldots, a_{n+2}\rangle$ over $F = k(a_1, \ldots, a_{n+2})$), Y be the last grassmannian $G([n/2], Q)$ of Q, and $m = [n+1/2]$. Using the restriction to the field of power series, and the results of Springer, one easily gets: degree of any finite extension E/F which splits Q completely is divisible by $2^{[n+2/2]}$, in other words, the image of $CH_0(G([n/2], Q)) \to CH_0(G([n/2], Q)|_{\overline{F}}) = \mathbb{Z}$ is contained in $2^{[n+2/2]} \cdot \mathbb{Z}$.

Let l_i be fixed projective plane of dimension i on $Q|_{\overline{F}}$. Remind that for $0 \leqslant i < [n+1/2]$, *elementary class* $Z_{[n+1/2]-i} \in CH^{[n+1/2]-i}(G([n/2], Q)|_{\overline{F}})$ is given by the locus of $[n/2]$-dimensional planes on Q intersecting l_i - see [10]. For $i = 0$ and n - even, Z_0 can be chosen as one of the families of middle-dimensional planes. Let z_i denotes $Z_i\ (\,mod\, 2\,) \in CH^i/2$.

Statement 3.6 *If Q is generic, then none of $z_j \in CH^j(G([n/2], Q)|_{\overline{F}})/2$ is defined over F.*

Proof: If n is even, the cycle z_j, $j > 0$ is defined on $G(n/2, Q)$ over F if and only if the cycle z_j is defined on $G(n/2 - 1, P)$ over $F(\sqrt{det(Q)})$, where $P \subset Q|_{F(\sqrt{det(Q)})}$ is any smooth subquadric of codimension 1 - see [10, Definition 5.11]. Since Q was generic, we can take such P to be generic too, and the problem is reduced to the case n - odd.

Cycles of the type $2Z_j \in CH^j(G((n-1/2), Q))$, $1 \leqslant j \leqslant (n+1/2)$ are always defined over F, since they are the Chern classes of tautological bundle - see [10, Theorem 2.5]. On the other hand, $\prod_{1 \leqslant i \leqslant (n+1/2)} Z_i$ is a class of a rational point. And any other product of Z_i's which is a zero-cycle has nesessarily degree divisible by 2 - see [10, Proposition 3.1]. Thus, if z_a would be defined over F, that is, over F would be defined $\lambda Z_a + (something)$, where λ is odd and $(something)$ is a polynomial in Z_b, $b \neq a$, then the cycle $2^{(n-1/2)} \prod_{b \neq a, 1 \leqslant b \leqslant (n+1/2)} Z_b \cdot (\lambda Z_a + (something)) \equiv 2^{(n-1/2)} \lambda \prod_{1 \leqslant b \leqslant (n+1/2)} Z_b\ (\,mod\, 2^{(n+1/2)})$ would be defined over F. Thus on $G((n-1/2), Q)$ there would be a 0-cycle of degree not divisible by $2^{(n+1/2)}$. This contradicts to the fact that Q is generic.

□

The needed example is provided by the following

Statement 3.7 *Let* Q *be generic quadric of dimension* n*,* Y *be* $G([n/2], Q)$*,* $m = [n + 1/2]$*, and* $\overline{y} = z_m \in CH^m(Y|_{\overline{F}})/2$*. Then* $\overline{y}|_{\overline{F(Q)}}$ *is defined over* $F(Q)$*, but* $\overline{y}$ *is not defined over* F*.*

Proof: It follows from the Statement 3.6 that $\overline{y}$ is not defined over F. On the other hand, $Q|_{F(Q)}$ has a rational point, and so not just z_m, but even Z_m is defined over $F(Q)$ by the very definition.

□

Remark: Clearly, in the example above one could as easily take any quadric Q such that $z_{[n+1/2]}(Q)$ is not defined over the base field, in other words, $[n + 1/2] \notin J(Q)$ (see [10, Definition 5.11]).

Moreover, the converse is true as well. As a supplement to Corollary 3.5 one can show the following:

Statement 3.8 *Under the conditions of Theorem 3.1, let* $[n+1/2] \in J(Q)$*. Then, for* $m \leqslant [n + 1/2]$*,* $\overline{y}$ *is defined over* k*.*

We will need some preliminary facts.

Proposition 3.9 *Let* Q *be smooth quadric, and*

$$f : G(Q, [n/2]) \xleftarrow{\alpha} F(Q, 0, [n/2]) \xrightarrow{\beta} Q$$

be the natural correspondence. Suppose $z_{[n+1/2]}$ *is defined.*

Let $t \in \mathrm{CH}_{[n+1/2]}(G(Q, [n/2]))/2$ *be such that* $f_*(t) = 1 \in \mathrm{CH}^0(Q)/2$*. Then* $f_*(t \cdot z_{[n+1/2]}) = l_{[n/2]} \in \mathrm{CH}_{[n/2]}(Q)/2$*.*

Proof: Really, by the definition, $z_{[n+1/2]} = f^*(l_0) = \alpha_*\beta^*(l_0)$. By the projection formula, $f_*(t \cdot z_{[n+1/2]}) = \beta_*\alpha^*(t \cdot z_{[n+1/2]}) = \beta_*\alpha^*\alpha_*(\alpha^*(t) \cdot \beta^*(l_0))$. Again, by the projection formula, $\beta_*(\alpha^*(t) \cdot \beta^*(l_0)) = l_0$. Thus, $\alpha^*(t) \cdot \beta^*(l_0)$ is a zero-cycle of degree 1 on $F(Q, 0, [n/2])$, and $\alpha_*(\alpha^*(t) \cdot \beta^*(l_0))$ is a zero cycle of degree 1 on $G(Q, [n/2])$. Proposition follows. □

Let $x \in \mathrm{CH}^m(Y \times Q)/2$ be some element. Then

$$\overline{x} = \sum_{i=0}^{[n/2]} (\overline{x}^i \cdot h^i + \overline{x}_i \cdot l_i).$$

Statement 3.10 *Suppose that $z_{[n+1/2]}(Q)$ is defined. Then for any $x \in \mathrm{CH}^{[n+1/2]}(Y \times Q)/2$, there exists $u \in \mathrm{CH}^{[n+1/2]}(Y \times Q)/2$ such that $\overline{u}^0 = \overline{x}^0$, and $\overline{u}_{[n/2]} = 0$.*

Proof: If $\overline{x}_{[n/2]} = 0$, there is nothing to prove. Otherwise, the class $l_{[n/2]} \in \mathrm{CH}_{[n/2]}(Q|_{k(Y)})/2$ is defined. Indeed, let

$$\rho_X : \mathrm{CH}^*(Y \times X)/2 \twoheadrightarrow \mathrm{CH}^*(X|_{k(Y)})/2$$

be the natural restriction. Then $\rho_Q(\overline{v}) = l_{[n/2]}$ plus $\lambda \cdot h^{[n/2]}$, if n is even (notice, that $\overline{x}_{[n/2]} \in \mathrm{CH}^0$). Anyway, this implies that over $k(Y)$ the variety $G(Q, [n/2])$ has a zero-cycle of degree 1, and thus, a rational point. Let $s \in \mathrm{CH}_{\dim(Y)}(G(Q, [n/2]) \times Y)/2$ be arbitrary lifting of the class of a point on $G(Q, [n/2])|_{k(Y)}$ with respect to $\rho_{G(Q,[n/2])}$. Let

$$f : G(Q, [n/2]) \stackrel{\alpha}{\leftarrow} F(Q, 0, [n/2]) \stackrel{\beta}{\rightarrow} Q$$

be the natural correspondence. Consider the element $u' := (f \times id)_*(s) \in \mathrm{CH}^{[n+1/2]}(Q \times Y)/2$. Proposition 3.9 implies that the (defined over k) class

$$u" := \pi_Y^*(\pi_Y)_*((h^{[n/2]} \times 1_Y) \cdot (f \times id)_*(s \cdot z_{[n+1/2]}(Q)))$$

satisfy: $\overline{u"}^0 = \overline{u'}^0$, and (evidently) $\overline{u"}_{[n/2]} = 0$. Since $\overline{u'}_{[n/2]} = 1 = \overline{x}_{[n/2]}$, it remains to take: $u := x - u' + u"$. □

Proof of Statement 3.8: If $m < [n+1/2]$, the statement follows from Corollary 3.5. For $m = [n+1/2]$, let $y' \in \mathrm{CH}^m(Y|_{k(Q)})/2$ be such element that $y'|_{\overline{k(Q)}} = \overline{y}|_{\overline{k(Q)}}$. Let us lift y' via surjection $\mathrm{CH}^*(Y \times Q)/2 \twoheadrightarrow \mathrm{CH}^*(Y|_{k(Q)})/2$ to some element x. Then it follows from the Statement 3.10 that x can be chosen in such a way that $\overline{x}_{[n/2]} = 0$. It remains to apply Proposition 3.3. □

The Statement 3.8 extends Corollary 3.5 in the direction of the following conjecture, which serves as an analogue of the Karpenko-Merkurjev Theorem for the cycles of positive dimension.

To introduce the Conjecture we will need first to define some objects.

Let $0 \leqslant i \leqslant [n/2]$, and $G(Q,i)$ be the Grassmannian of i-dimensional projective subspaces on Q. For the standard correspondence

$$f_i : G(Q,i) \stackrel{\alpha_i}{\leftarrow} F(Q,0,i) \stackrel{\beta_i}{\rightarrow} Q,$$

denote as $z_{n-i}^{\boxed{i-[n/2]}}$ the class $(f_i)^*(l_0) \in \mathrm{CH}^{n-i}(G(Q,i)|_{\overline{k}})/2$. In particular, in this notations, $z_{[n+1/2]}^{\boxed{0}}$ will be our class $z_{[n+1/2]}$. Notice also, that $z_n^{\boxed{-[n/2]}}$ is the class of a point on $Q|_{\overline{k}}$, and $z_{n-1}^{\boxed{1-[n/2]}}$ is defined if and only if Q posses the Rost projector. For $i < j$, $z_{n-i}^{\boxed{i-[n/2]}}$ is defined over $k \Rightarrow z_{n-j}^{\boxed{j-[n/2]}}$ is defined over k (see [12]).

Conjecture 3.11 *Suppose, the class* $z_{n-i}^{\boxed{i-[n/2]}}$ *is defined over* k*. Then for all* $m \leqslant n-i$,

$$\overline{y}|_{\overline{k(Q)}} \text{ is defined over } k(Q) \ \Leftrightarrow\ \overline{y} \text{ is defined over } k.$$

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