Quadratic forms with absolutely maximal splitting

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ABSTRACT. Let F be a field and ϕ be a quadratic form over F. The higher Witt indices of ϕ are defined recursively by the rule $i_{k+1}(\phi) = i_k((\phi_{an})_{F(\phi_{an})})$, where $i_0(\phi) = i_W(\phi)$ is the usual Witt index of the form ϕ. We say that anisotropic form ϕ has *absolutely maximal splitting* if $i_1(\phi) > i_k(\phi)$ for all $k > 1$.

One of the main results of this paper claims that for all anisotropic forms ϕ satisfying the condition $2^{n-1} + 2^{n-3} < \dim\phi \le 2^n$, the following three conditions are equivalent: (i) the kernel of the natural homomorphism $H^n(F, \mathbb{Z}/2\mathbb{Z}) \to H^n(F(\phi), \mathbb{Z}/2\mathbb{Z})$ is nontrivial, (ii) ϕ has absolutely maximal splitting, (iii) ϕ has maximal splitting (i.e., $i_1(\phi) = \dim\phi - 2^{n-1}$). Moreover, we show that if we assume additionally that $\dim\phi \ge 2^n - 7$, then these three conditions hold if and only if ϕ is an anisotropic n-fold Pfister neighbor. In our proof we use the technique developed by V. Voevodsky in his proof of Milnor's conjecture.

1. Introduction

Let F be a field of characteristic $\neq 2$ and let $H^n(F)$ be the Galois cohomology group of F with $\mathbf{Z}/2\mathbf{Z}$-coefficients. For a given extension L/F, we denote by $H^n(L/F)$ the kernel of the natural homomorphism $H^n(F) \to H^n(L)$. Now, let ϕ be a quadratic form over F. An important part of the algebraic theory of quadratic forms deals with the behavior of the groups $H^n(F)$ under the field extension $F(\phi)/F$. Of particular interest is the group

$$H^n(F(\phi)/F) = \ker(H^n(F) \to H^n(F(\phi))).$$

The computation of this group is connected to Milnor's conjecture and plays an important role in K-theory and in the theory of quadratic forms.

The first nontrivial result in this direction is due to J. K. Arason. In [**1**], he computed the group $H^n(F(\phi)/F)$ for the case $n \le 3$. The case $n = 4$ was completely studied by Kahn, Rost and Sujatha ([**13**]). In the cases where $n \ge 5$, there are only partial results depending on Milnor's conjecture: the group $H^n(F(\phi)/F)$ was computed for all Pfister neighbors ([**26**]) and for all 4-dimensional forms ([**33**]). All known results make natural the following conjecture.

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Conjecture 1.1. *Let F be a field, n be a positive integer, and let ϕ be an F-form of dimension $> 2^{n-1}$. Then the following conditions are equivalent:*

(1) *the group $H^n(F(\phi)/F) = \ker(H^n(F) \to H^n(F(\phi)))$ is nonzero,*
(2) *the form ϕ is an anisotropic n-fold Pfister neighbor.*

Moreover, if these conditions hold, then the group $H^n(F(\phi)/F)$ is isomorphic to $\mathbf{Z}/2\mathbf{Z}$ and is generated by $e^n(\pi)$, where π is the n-fold Pfister form associated with ϕ.

The proof of the implication (2)⇒(1) follows from the fact that the Pfister quadric is isotropic if and only if the corresponding *pure symbol* $\alpha \in \mathrm{K}_n^M(k)/2$ is zero, and the fact that the *norm-residue* homomorphism is injective on α (see [**36**]). The implication (1)⇒(2) seems much more difficult. In this paper, we give only a partial answer to the conjecture.

1.1. Forms with maximal splitting. It turns out that Conjecture 1.1 is closely related to a conjecture concerning so-called forms with maximal splitting. Let us recall some basic definitions and results. By $i_W(\phi)$ we denote the Witt index of ϕ. For an anisotropic quadratic form ϕ, the *first higher Witt index of ϕ* is defined as follows: $i_1(\phi) = i_W(\phi_{F(\phi)})$. Since $\phi_{F(\phi)}$ is isotropic, we obviously have $i_1(\phi) \geq 1$. In [**4**] Hoffmann proved the following

Theorem 1.2. *Let ϕ be an anisotropic quadratic form. Let n be such that $2^{n-1} < \dim\phi \leq 2^n$ and m be such that $\dim\phi = 2^{n-1} + m$. Then*

- *$i_1(\phi) \leq m$,*
- *if ϕ is a Pfister neighbor, then $i_1(\phi) = m$.*

This theorem gives rise to the following

Definition 1.3 (see [**4**]). Let ϕ be an anisotropic quadratic form. Let us write $\dim\phi$ in the form $\dim\phi = 2^{n-1} + m$, where $0 < m \leq 2^{n-1}$. We say that ϕ has maximal splitting if $i_1(\phi) = m$.

Our interest in forms with maximal splitting is motivated (in particular) by the following observation (which depends on the Milnor conjecture, see Proposition 7.5, and requires $char(F) = 0$): *Let ϕ and n be as in Conjecture 1.1. If $H^n(F(\phi)/F) \neq 0$, then ϕ has maximal splitting and $H^n(F(\phi)/F) \simeq \mathbf{Z}/2\mathbf{Z}$.* Therefore, the problem of classification of forms with maximal splitting is closely related to Conjecture 1.1. On the other hand, there are many other problems depending on the classification of forms with maximal splitting.

Let us explain some known results concerning this classification. By Theorem 1.2, all Pfister neighbors and all forms of dimension $2^n + 1$ have maximal splitting. By [**5**], these examples present an exhaustive list of forms with maximal splitting of dimension ≤ 9. The case $\dim\phi = 10$ is much more complicated. In [**9**], it was proved that a 10-dimensional form ϕ has maximal splitting only in the following cases:

- ϕ is a Pfister neighbor,
- ϕ can be written in the form $\phi = \langle\langle a\rangle\rangle\, q$, where q is a 5-dimensional form.

The structure of quadratic forms with maximal splitting of dimensions 11, 12, 13, 14, 15, and 16 is very simple: they are Pfister neighbors (see [**5**] or [**7**]). Since $17 = 2^4 + 1$, it follows that any 17-dimensional form has maximal splitting. The previous discussion shows that we have a complete classification of forms with maximal splitting of dimensions ≤ 17.

Conjecture 1.1 together with our previous discussion make the following problem natural:

PROBLEM 1.4. *Find the condition on the positive integer d such that each d-dimensional form ϕ with maximal splitting is necessarily a Pfister neighbor.*

The following example is due to Hoffmann. Let F be the field of rational functions $k(x_1, \dots, x_{n-3}, y_1, \dots, y_5)$ and let

$$q = \langle\langle x_1, \dots, x_{n-3} \rangle\rangle \otimes \langle y_1, y_2, y_3, y_4, y_5 \rangle \,.$$

Then q has maximal splitting and is not a Pfister neighbor. We obviously have $\dim q = 2^{n-1} + 2^{n-3}$. This example gives rise to the following

PROPOSITION 1.5. *Let d be an integer satisfying $2^{n-1} \leq d \leq 2^{n-1} + 2^{n-3}$ for some $n \geq 4$. Then there exists a field F and a d-dimensional F-from ϕ with maximal splitting which is not a Pfister neighbor.*

For the proof, we can define ϕ as an arbitrary d-dimensional subform of the form q constructed above. We remind that if $\phi \subset q$ is a subform, where $2^{n-1} \leq \dim \phi \leq \dim q \leq 2^n$, and q has maximal splitting , then ϕ also has maximal splitting (see [**4**], Prop.4).

Let us return to Problem 1.4. By Proposition 1.5, it suffices to study Problem 1.4 only in the case where $2^{n-1} + 2^{n-3} < d \leq 2^n$. Here, we state the following

CONJECTURE 1.6. *Let $n \geq 3$ and F be an arbitrary field. For any anisotropic quadratic F-form with maximal splitting the condition $2^{n-1} + 2^{n-3} < \dim \phi \leq 2^n$ implies that ϕ is a Pfister neighbor.*

This conjecture is true in the cases $n = 3$ and $n = 4$ (see [**5**], [**7**]). At the time we cannot prove Conjecture 1.6 in the case $n \geq 5$. However, we prove the following partial case of the conjecture.

THEOREM 1.7. *Let $n \geq 5$ and q be an anisotropic form such that $2^n - 7 \leq \dim q \leq 2^n$. Then the following conditions are equivalent:*

(i) *q has maximal splitting,*
(ii) *q is a Pfister neighbor.*

Moreover, we show that for any form ϕ satisfying the condition $2^{n-1} + 2^{n-3} < \dim \phi \leq 2^n$, Conjectures 1.1 and 1.6 are equivalent for all fields of characteristic zero (here we use the Milnor conjecture). The equivalence of the conjectures follows readily from the following theorem.

THEOREM 1.8. ((*M*), see the end of Section 1) *Let n be an integer ≥ 4 and F be a field of characteristic 0. Let ϕ be an anisotropic form such that $2^{n-1} + 2^{n-3} < \dim \phi \leq 2^n$. Then the following conditions are equivalent:*

(1) *ϕ has maximal splitting,*
(2) *$H^n(F(\phi)/F) \neq 0$.*

On the other hand, Theorems 1.7 and 1.8 give rise to the proof of the following partial case of Conjecture 1.1.

COROLLARY 1.9. ((*M*), see the end of Section 1) *Let F be a field of characteristic zero and let $n \geq 5$. Then for any F-form ϕ satisfying the condition $\dim\phi \geq 2^n - 7$, the following conditions are equivalent:*

(1) *the group $H^n(F(\phi)/F) = \ker(H^n(F) \to H^n(F(\phi)))$ is nonzero,*
(2) *the form ϕ is an anisotropic n-fold Pfister neighbor.*

1.2. Plan of works. In section 3, we prove Theorem 1.7. Our proof is based on the following ideas of Bruno Kahn ([**11**]): First of all, we recall some result of M. Knebusch: *let q be an anisotropic F-form. If the $F(q)$-form $(q_{F(q)})_{an}$ is defined over F, then q is a Pfister neighbor.* Now, let q be an F-form satisfying the hypotheses of Theorem 1.7 (in particular, dim $q > 16$). Let us consider the $F(q)$-form $\phi = (q_{F(q)}))_{an}$. By the definition of forms with maximal splitting, we obviously have $\dim\phi \leq 7$. By the construction, the form ϕ belongs to the image of the homomorphism $W(F) \to W(F(q))$. This implies that ϕ belongs to the unramified part $W_{nr}(F(q))$ of the Witt group $W(F(q))$. Using some deep results concerning the group $W_{nr}(F(q))$, we prove that all forms of dimension ≤ 7 belonging to $W_{nr}(F(q))$ are necessarily defined over F (provided that $\dim q > 16$). In particular, this implies that $\phi = (q_{F(q)})_{an}$ is defined over F. Then Knebusch's theorem says that q is a Pfister neighbor. This completes the proof of Theorem 1.7.

To prove Theorem 1.8, we need the Milnor conjecture. The implication (1)⇒(2) is the most difficult part of the theorem. To explain the plan, we introduce the notion of "forms with absolutely maximal splitting". First, we recall that for any F-form ϕ, we can define the higher Witt indices by the following recursive rule: $i_{s+1}(\phi) = i_s((\phi_{an})_{F(\phi_{an})})$.

DEFINITION 1.10. Let ϕ be an anisotropic quadratic form. We say that ϕ has absolutely maximal splitting, or that ϕ is an AMS-form, if $i_1(\phi) > i_r(\phi)$ for all $r > 1$.

Such terminology is justified by the fact that at least in the case $char(F) = 0$, AMS implies maximal splitting (see Theorem 7.1). It is not difficult to show, that if the form ϕ has maximal splitting and satisfies the condition $2^{n-1} + 2^{n-3} < \dim\phi \leq 2^n$, then ϕ is an AMS-form (see Lemma 4.1). This shows, that the form ϕ satisfying the condition (1) of Theorem 1.8, is necessarily an AMS-form. Therefore, it suffices to prove the following theorem.

THEOREM 1.11. ((*M*), see the end of Section 1) *Let F be a field of characteristic zero. Let ϕ be an AMS-form satisfying the condition $2^{n-1} < \dim\phi \leq 2^n$. Then $H^n(F(\phi)/F) \neq 0$.*

To prove this theorem, we study the motive of the projective quadric Q corresponding to a subform q of ϕ of codimension $i_1(\phi) - 1$. It is well known that the function fields of the forms ϕ and q are stably equivalent. Hence, it suffices to prove that $H^n(F(q)/F) \neq 0$. In §5, we show that the motive $M(Q)$ of the quadric Q has some specific endomorphism $\omega : M(Q) \to M(Q)$ which we call the *Rost projector*. Let us give the definition of the latter. First, we recall that the set of endomorphisms $M(Q) \to M(Q)$ is defined as $\mathrm{CH}^d(Q \times Q)$, where $d = \dim Q$. We say that $\omega \in \mathrm{End}(M(Q))$ is a Rost projector, if ω is an idempotent ($\omega \circ \omega = \omega$), and the identity $\omega_{\bar{F}} = pt \times Q_{\bar{F}} + Q_{\bar{F}} \times pt$ holds over the algebraic closure $\bar{F}$ of F. The existence of the *Rost projector* means that $M(Q)$ contains a direct summand N such that $N_{\overline{k}}$ is isomorphic to the direct sum of two so-called *Tate-motives*

$\mathbb{Z} \oplus \mathbb{Z}(d)[2d]$ (since the mutually orthogonal projectors $Q_{\bar{F}} \times pt$ and $pt \times Q_{\bar{F}}$ give direct summands isomorphic to $\mathbb{Z}$ and $\mathbb{Z}(d)[2d]$, respectively). The final step in the proof of theorem 1.8 is based on the following theorem.

THEOREM 1.12. ((*M*), see the end of Section 1) *Let F be a field of characteristic zero. Let Q be the projective quadric corresponding to an anisotropic F-form q. Assume that Q admits a Rost projector. Then* $\dim q = 2^{m-1} + 1$ *for suitable m. Moreover, $H^m(F(q)/F) \neq 0$.*

Now, it is very easy to complete the proof of the implication (1)⇒(2) of Theorem 1.8. Since $2^{m-1} < \dim q \leq 2^m$, $2^{n-1} < \dim \phi \leq 2^n$, and the extensions $F(\phi)/F$ and $F(q)/F$ are stably equivalent, it follows that $n = m$ (this follows easily from Hoffmann's theorem [**4**]). Therefore, $H^n(F(\phi)/F) = H^m(F(q)/F) \neq 0$. This completes the proof of the implication (1)⇒(2).

The proof of Theorem 1.12 is given in section 6. It is based on the technique developed by V.Voevodsky for the proof of Milnor's conjecture (see [**36**]). All needed results of Voevodsky's preprints are collected in Appendix A. Aside from the Appendix we also use the main results of [**26**] (in Theorem 7.3). Here we should point out that these results can be obtained from those of the Appendix in a rather simple way (the recipe is given, for example, in [**14**, Remark 3.3.]). All the major statements of the current paper which are using the abovementioned unpublished results are marked with (*M*) with the reference to this page.

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2. Notation and background

In this article we use the standard quadratic form terminology from [**19**],[**30**]. We use the notation $\langle\langle a_1, \ldots, a_n \rangle\rangle$ for the *Pfister form* $\langle 1, -a_1 \rangle \otimes \cdots \otimes \langle 1, -a_n \rangle$. Under $GP_n(F)$ we mean the set of forms over F which are similar to n-fold Pfister forms. The n-fold Pfister forms provide a system of generators for the abelian group $I^n(F)$. We recall that the Arason–Pfister Hauptsatz (APH in what follows) states that: *every quadratic form over F of dimension $< 2^n$ which lies in $I^n(F)$ is necessarily hyperbolic; if $\phi \in I^n(F)$ and* $\dim \phi = 2^n$, *then the form ϕ is necessarily similar to a Pfister form.* We use the notation e^n for the *generalized Arason invariant* [1]

$$I^n(F)/I^{n+1}(F) \to H^n(F), \text{ where } \langle\langle a_1, \ldots, a_n \rangle\rangle \mapsto (a_1, \ldots, a_n).$$

The following statements describe the relationship between the Witt ring $W(F)$ and the cohomology $H^n(F)$. They will be used extensively in the next two sections.

THEOREM 2.1. ([**24**],[**1**],[**10**],[**32**],[**21**],[**23**],[**27**],Rost-unpublished)
For $n \leq 4$, we have canonical isomorphisms $e^n : I^n(F)/I^{n+1}(F) \to H^n(F)$

[1]The existence of e^n was proven by Arason for $n \leq 3$, and by Jacob-Rost/Szyjevski for $n = 4$.

THEOREM 2.2. (Arason[**1**], Kahn-Rost-Sujatha[**13**], Merkurjev[**22**])
Let $0 \le m \le n \le 4$ and π be an m-fold Pfister form over F. Then $H^n(F(\pi)/F) = e^m(\pi)H^{n-m}(F)$.

THEOREM 2.3. (Arason[**1**], Kahn-Rost-Sujatha[**13**])
Let $n \le 4$ and ρ be a form over F of dimension $> 2^n$. Then $H^n(F(\rho)/F) = 0$.

The following statement is an evident corollary of the theorems above.

COROLLARY 2.4. *Let ρ be a form over F and n be a positive integer ≤ 5. Let ξ be a form over F such that $\xi_{F(\rho)} \in I^n(F(\rho))$. Then*

- *if ρ is a Pfister neighbour of a Pfister form π, then $\xi \in \pi W(F) + I^n(F)$,*
- *if $\dim \rho > 2^{n-1}$, then $\xi \in I^n(F)$.*

REMARK 2.5. Actually, the restriction on the integer n here is unnecessary (at least in characteristic 0) - see Theorem 7.3.

In section 5 we use the notation $\mathbb{Z}$ for the trivial *Tate-motive* (which is just the motive of a point $M(\mathrm{Spec}(k))$), and $\mathbb{Z}(m)[2m]$ for the tensor power $\mathbb{Z}(1)[2]^{\otimes m}$ of the *Tate-motive* $\mathbb{Z}(1)[2]$, where the latter is defined as a complementary direct summand to $\mathbb{Z}$ in $M(\mathbb{P}^1)$ ($M(\mathbb{P}^1) = \mathbb{Z} \oplus \mathbb{Z}(1)[2]$). For this reason, we use the notation $\mathbf{Z}$ for all groups and rings $\mathbb{Z}$ throughout the text. In section 5 we work in the classical Chow-motivic category of Grothendieck (see [**3**],[**20**],[**31**],[**29**]). We remind that in this category the group $\mathrm{Hom}(M(P), M(Q))$ is naturally identified with $\mathrm{CH}^{\dim(Q)}(P \times Q)$, for any smooth connected projective varieties P and Q over k.

In this connection we should mention that the motive of a completely split quadric P (of dimension d) is a direct sum of *Tate-motives*:

$M(P) = \oplus_{0 \le i \le d}\mathbb{Z}(i)[2i]$, if d is odd;
$M(P) = (\oplus_{0 \le i \le d}\mathbb{Z}(i)[2i]) \oplus \mathbb{Z}(d/2)[d]$, if d is even.

The corresponding mutually orthogonal projectors in $\mathrm{End}(M(P))$ are given by $h^i \times l_i$, and $l_i \times h^i$, where $0 \le i < d/2$, and $h^i \subset P$ is a plane section of codimension i, and $l_i \subset P$ is a projective subspace of dimension i (in the case d even, we also have $l^1_{d/2} \times l^2_{d/2}$ and $l^2_{d/2} \times l^1_{d/2}$, where $l^1_{d/2}, l^2_{d/2} \subset P$ are the projective subspaces of half the dimension from the two different families).

In section 6 we work in the bigger triangulated category of motives $DM^{eff}_{-}(k)$ constructed by V.Voevodsky (see [**34**]). This category contains the category of Chow-motives as a full additive subcategory closed with respect to direct summands. All the necessary facts and references are given in the Appendix.

3. Descent problem and forms with maximal splitting

The main goal of this section is to prove Theorem 1.7. It should be noticed that in all cases except for $\dim q = 2^n - 7$ this theorem was proved earlier:

- if $\dim q = 2^n$ or $2^n - 1$, the theorem was proved by M. Knebusch and A. Wadsworth (independently);
- if $\dim q = 2^n - 2$ or $2^n - 3$, the theorem was proved by D. Hoffmann [**4**];
- if $\dim q = 2^n - 4$ or $2^n - 5$, the theorem was proved by B. Kahn [**11**, remark after Th.4] (see also more elementary proofs in [**5**] or [**7**]);
- In the case $\dim q = 2^n - 6$, the theorem follow easily from a result of A. Laghribi [**18**].

To prove the theorem in the case $\dim q = 2^n - 7$ we use the same method as in the paper of Bruno Kahn [**11**]. Namely, we reduce Theorem 1.7 to the study of *a descent problem for quadratic forms* (see Proposition 3.8 and Theorem 3.9). As in the paper of B. Kahn, we work modulo a suitable power $I^n(F)$ of the fundamental ideal $I(F)$.

We start with the following notation.

DEFINITION 3.1. Let ψ be a form over F and $n \geq 0$ be an integer. We define $\dim_n \psi$ as follows:

$$\dim_n \psi = \min\{\dim \phi \,|\, \phi \equiv \psi \pmod{I^n(F)}\}$$

LEMMA 3.2. *Let ψ be a form over F and L/F be some field extension. Then $\dim_n \psi_L \leq \dim_n \psi$. If L/F is unirational, then $\dim_n \psi_L = \dim_n \psi$.*

PROOF. The inequality $\dim_n \psi_L \leq \dim_n \psi$ is obvious. If L/F is unirational, the identity $\dim_n \psi_L = \dim_n \psi$ follows easily from standard specialization arguments. □

COROLLARY 3.3. *Let ψ and $q_0 \subset q$ be forms over F. Then $\dim_n \psi_{F(q_0)} \leq \dim_n \psi_{F(q)}$.*

PROOF. Since $q_0 \subset q$, it follows that $q_{F(q_0)}$ is isotropic and hence the extension $F(q, q_0)/F(q_0)$ is purely transcendental. By Lemma 3.2, we have $\dim_n \psi_{F(q_0)} = \dim_n \psi_{F(q,q_0)} \leq \dim_n \psi_{F(q)}$. □

Now, we recall an evident consequence of Merkurev's index reduction formula: *if A is a central simple algebra of index 2^n and q is a form of dimension $> 2n+2$, then* $\operatorname{ind} A_{F(q)} = \operatorname{ind} A$. The following lemma is an obvious generalization of this statement.

LEMMA 3.4. *Let A be a central simple F-algebra of index 2^n and q be a quadratic form over F. Let $F_0 = F, F_1, \dots, F_h$ be the generic splitting tower of q. Let $i \geq 1$ be an integer such that* $\dim((q_{F_{i-1}})_{an}) > 2n+2$. *Then* $\operatorname{ind} A_{F_i} = \operatorname{ind} A$.

LEMMA 3.5. *Let A be a central simple F-algebra of index 2^n and q be a quadratic form of dimension $> 2n+4$. Then there exists a unirational extension E/F and a 3-dimensional form $q_0 \subset q_E$ such that* $\operatorname{ind}(A_E \otimes C_0(q_0)) = 2^{n+1}$

PROOF. Let $\tilde{F} = F(X,Y,Z)$, $\tilde{q} = q_{\tilde{F}} \perp -X\langle\langle Y,Z\rangle\rangle$, and $\tilde{A} = A_{\tilde{F}} \otimes (Y,Z)$. Clearly, $\operatorname{ind} \tilde{A} = 2 \operatorname{ind} A = 2^{n+1}$. Let $\tilde{F}_0 = \tilde{F}, \tilde{F}_1, \dots, \tilde{F}_h$ be the generic splitting tower for $\tilde{q}$. Let $\tilde{q}_i = (\tilde{q}_{\tilde{F}_i})_{an}$ for $i = 0, \dots, h$. Let s be the minimal integer such that $\dim \tilde{q}_s \leq \dim q - 2$. We have $\dim \tilde{q}_{s-1} \geq \dim q > 2(n+1)+2$. By Lemma 3.4, we have $\operatorname{ind} A_{\tilde{F}_s} = \operatorname{ind} \tilde{A} = 2^{n+1}$.

We set $E = \tilde{F}_s$. Since $\tilde{q}_E = q_E \perp -X\langle\langle Y,Z\rangle\rangle$, the forms q_E and $X\langle\langle Y,Z\rangle\rangle$ contain a common subform of dimension

$$\frac{1}{2}(\dim q + \dim(X\langle\langle Y,Z\rangle\rangle) - \dim(\tilde{q}_E)_{an}) = \frac{1}{2}(\dim q + 4 - \dim \tilde{q}_s)$$
$$\geq \frac{1}{2}(\dim q + 4 - (\dim q - 2)) = 3.$$

Hence, there exists a 3-dimensional E-form q_0 such that $q_0 \subset q_E$ and $q_0 \subset X\langle\langle Y,Z\rangle\rangle_E$. Clearly, $C_0(q_0) = (Y,Z)$. Hence, $\operatorname{ind}(A_E \otimes_E C_0(q_0)) = \operatorname{ind} \tilde{A}_E = 2^{n+1}$.

To complete the proof, it suffices to show that E/F is unirational. To prove this, let us write q in the form $q = x\langle 1, -y, -z\rangle \perp q_0$ with $x, y, z \in F^*$. Let us consider the field

$$K = \tilde{F}(\sqrt{X/x}, \sqrt{Y/y}, \sqrt{Z/z}) = F(X, Y, Z)(\sqrt{X/x}, \sqrt{Y/y}, \sqrt{Z/z}).$$

Clearly, K/F is purely transcendental. In the Witt ring $W(K)$, we have $\tilde{q}_K = q_K - X\langle\langle Y, Z\rangle\rangle_K = x\langle 1, -y, -z\rangle_K + q_0 - X\langle 1, -Y, -Z, YZ\rangle = q_0 - \langle XYZ\rangle$. Hence $\dim(\tilde{q}_K)_{an} \leq \dim q_0 + 1 = \dim q - 3 + 1 = \dim q - 2$. Since s is the minimal integer such that $\dim \tilde{q}_s \leq \dim q - 2$, it follows that the extension $(K \cdot \tilde{F}_s)/K$ is purely transcendental (see, e.g., [**16**, Cor. 3.9 and Prop. 5.13]), where $K \cdot \tilde{F}_s$ is the free composite of K and $\tilde{F}_s$ over $\tilde{F}$. Since K/F is purely transcendental, it follows that $(K \cdot \tilde{F}_s)/F$ is also purely transcendental. Hence $\tilde{F}_s/F$ is unirational. Since $E = \tilde{F}_s$, we are done. □

Lemma 3.6. *Let ρ be a Pfister neighbor of $\langle\langle a, b\rangle\rangle$ and n be a positive integer. Let ψ be a form such that $\dim_n \psi_{F(\rho)} < 2^{n-1}$. Then there exists an F-form μ such that $\dim \mu = \dim_n \psi_{F(\rho)}$ and $\psi_{F(\rho)} \equiv \mu_{F(\rho)} \pmod{I^n(F)}$.*

Proof. Let ξ be an $F(\rho)$-form such that $\dim \xi = \dim_n \psi_{F(\rho)}$ and $\psi_{F(\rho)} \equiv \xi \pmod{I^n(F(\rho))}$. By [**18**, Lemme 3.1], we have $\xi \in W_{nr}(F(\rho)/F)$. By the excellence property of $F(\rho)/F$ (see [**2**, Lemma 3.1]), there exists an F-form μ such that $\xi = \mu_{F(\rho)}$. □

Corollary 3.7. *Let ρ be a Pfister neighbor of $\langle\langle a, b\rangle\rangle$ and n be a positive integer such that $n \leq 5$. Let ψ be a form such that $\dim_n \psi_{F(\rho)} < 2^{n-1}$. Then there exist F-forms μ and γ such that $\dim \mu = \dim_n \psi_{F(\rho)}$ and $\psi \equiv \mu + \langle\langle a, b\rangle\rangle\gamma \pmod{I^n(F)}$.*

Proof. Let μ be a form as in Lemma 3.6. We have $(\psi - \mu)_{F(\rho)} \in I^n(F(\rho))$. By Corollary 2.4, we have $\psi - \mu \in \langle\langle a, b\rangle\rangle W(F) + I^n(F)$. Hence, there exists γ such that $\psi - \mu \in \langle\langle a, b\rangle\rangle\gamma + I^n(F)$. □

Proposition 3.8. *Let q be an F-form of dimension > 16 and ψ be a form over F such that $\dim_5 \psi_{F(q)} \leq 7$. Then $\dim_5 \psi = \dim_5 \psi_{F(q)}$. In particular, $\dim_5 \psi \leq 7$.*

Proof. By Lemma 3.2, we have $\dim_5 \psi \geq \dim_5 \psi_{F(q)}$. Hence, it suffices to verify that $\dim_5 \psi \leq \dim_5 \psi_{F(q)}$. As usually, we denote as $C_0(\psi)$ the even part of the Clifford algebra and as $c(\psi)$ the Clifford invariant of ψ. We start with the following case:

Case 1. either $\dim_5 \psi_{F(q)} \leq 6$ or $\dim_5 \psi_{F(q)} = 7$ and $\operatorname{ind} C_0(\psi) \neq 8$.

By the definition of $\dim_5 \psi_{F(q)}$, there exists an $F(q)$-form ϕ such that $\dim \phi = \dim_5 \psi_{F(q)} \leq 7$ and $\phi \equiv \psi_{F(q)} \pmod{I^5(F(q))}$. In particular, we have $c(\phi) = c(\psi_{F(q)})$. Since $\dim \phi \leq 7$ and $\dim q > 16$, the index reduction formula shows that $\operatorname{ind} C_0(\phi) = \operatorname{ind} C_0(\psi)$. By the assumption of Case 1, we see that

- either $\dim \phi \leq 6$,
- or $\dim \phi = 7$ and $\operatorname{ind} C_0(\phi) \neq 8$.

Since $\phi \equiv \psi_{F(q)} \pmod{I^5(F(q))}$, it follows that $\phi \in im(W(F) \to W(F(q))) + I^5(F(q))$. The principal theorem of [**18**] shows that ϕ is defined over F. In other words, there exists an F-form μ such that $\phi = \mu_{F(q)}$. Therefore, $\psi_{F(q)} \equiv \phi \equiv \mu_{F(q)} \pmod{I^5(F(q))}$. By Corollary 2.4, we see that $\psi \equiv \mu \pmod{I^5(F)}$. Hence, $\dim_5(\psi) \leq \dim \mu = \dim \phi = \dim_5(\psi_{F(q)})$. This completes the proof in Case 1.

Case 2. $\dim_5 \psi_{F(q)} = 7$ and $\operatorname{ind} C_0(\psi) = 8$.

Lemma 3.2 shows that we can change the ground field by an arbitrary unirational extension. After this, Lemma 3.5 (applied to $A = C_0(\psi)$, $n = 3$ and q) shows, that we can assume that *there exists a 3-dimensional subform* $q_0 \subset q$ *such that* $\operatorname{ind}(C_0(\psi) \otimes C_0(q_0)) = 16$.

Let $a, b \in F^*$ be such that q_0 is a Pfister neighbor of $\langle\langle a, b\rangle\rangle$.

By Corollary 3.3, we have $\dim_5 \psi_{F(q_0)} \leq 7$. By Corollary 3.7, there exists a form μ of dimension ≤ 7 and a form λ such that $\psi \equiv \mu + \langle\langle a, b\rangle\rangle\, \lambda \pmod{I^5(F)}$.

First, consider the case where $\dim \lambda$ is odd. Then $c(\psi) = c(\mu)+(a, b)$. Therefore $\operatorname{ind} C_0(\mu) = \operatorname{ind}(C_0(\psi) \otimes (a, b)) = \operatorname{ind}(C_0(\psi) \otimes C_0(q_0)) = 16$. On the other hand, $\dim \mu \leq 7$ and hence $\operatorname{ind} C_0(\mu) \leq 8$. We get a contradiction.

Now, we can assume that $\dim \lambda$ is even. Then $\lambda \equiv \langle\langle c\rangle\rangle \pmod{I^2(F)}$, where $c = d_\pm\lambda$. Hence, $\langle\langle a, b\rangle\rangle\, \lambda \equiv \langle\langle a, b, c\rangle\rangle \pmod{I^4(F)}$. Hence, $\psi - \mu \equiv \langle\langle a, b\rangle\rangle\, \lambda \equiv \langle\langle a, b, c\rangle\rangle \pmod{I^4(F)}$. Let $\pi = \langle\langle a, b, c\rangle\rangle$. We have $\psi \equiv \mu + \pi \pmod{I^4(F)}$.

Since $\pi_{F(\pi)}$ is hyperbolic, it follows that $\psi_{F(q,\pi)} \equiv \mu_{F(q,\pi)} \pmod{I^4(F(q, \pi))}$. Since $\dim_5 \psi_{F(q)} = 7$, there exists a 7-dimensional $F(q)$-form ξ such that $\psi_{F(q)} \equiv \xi \pmod{I^5(F(q))}$. This implies that $\mu_{F(q,\pi)} \equiv \psi_{F(q,\pi)} \equiv \xi_{F(q,\pi)} \pmod{I^4(F(q, \pi))}$. Since $\dim \mu + \dim \xi \leq 7 + 7 = 14 < 2^4$, APH shows that $\mu_{F(q,\pi)} = \xi_{F(q,\pi)}$. Hence, $\psi_{F(q,\pi)} \equiv \xi_{F(q,\pi)} \equiv \mu_{F(q,\pi)} \pmod{I^5(F(q, \pi))}$. Since $\dim q > 16$, Corollary 2.4 shows that $\psi_{F(\pi)} \equiv \mu_{F(\pi)} \pmod{I^5(F(\pi))}$. Hence $(\psi - \mu)_{F(\pi)} \in I^5(F(\pi))$.

By Corollary 2.4, there exists an F-form γ such that $\psi - \mu \equiv \pi\gamma \pmod{I^5(F)}$. Since $\psi - \mu \equiv \pi \pmod{I^4(F)}$, it follows that either π is hyperbolic or $\dim \gamma$ is odd. In any case, we can assume that $\dim \gamma$ is odd. Then $\gamma \equiv \langle k\rangle \pmod{I^2(F)}$, where $k = d_\pm\gamma$. Hence, $\psi - \mu \equiv \pi\gamma \equiv k\pi \pmod{I^5(F)}$. Therefore, $\xi \equiv \psi_{F(q)} \equiv (\mu + k\pi)_{F(q)} \pmod{I^5(F(q))}$. Since $\dim \xi + \dim \mu + \dim \pi = 7 + 7 + 8 < 2^5$, APH shows that $\xi = (\zeta_{F(q)})_{an}$, where $\zeta = (\mu \perp k\pi)_{an}$. Since $\dim \zeta \leq \dim(\mu \perp k\pi) \leq 7 + 8 < 2^4 < \dim q$, Hoffmann's theorem shows that $\zeta_{F(q)}$ is anisotropic. Hence, $\xi = \zeta_{F(q)}$. In particular, $\dim \zeta = 7$. We have $\psi_{F(q)} \equiv \xi \equiv \zeta_{F(q)} \pmod{I^5(F(q))}$. Since $\dim q > 16$, Corollary 2.4 shows that $\psi \equiv \zeta \pmod{I^5(F)}$. Hence, $\dim_5 \psi \leq \dim \zeta = 7$. On the other hand, $\dim_5 \psi \geq \dim_5 \psi_{F(q)} = 7$. The proof is complete. □

The essential part of the following theorem was proved by Ahmed Laghribi in [**18**].

THEOREM 3.9. (cf. [**18**, Théorème principal]). *Let q be a form of dimension > 16. Let ϕ be a form of dimension ≤ 7 over the field $F(q)$. Then the following conditions are equivalent.*

(1) *ϕ is defined over F,*
(2) *$\phi \in im(W(F) \to W(F(q)))$,*
(3) *$\phi \in im(W(F) \to W(F(q))) + I^5(F(q))$,*
(4) *$\phi \in W_{nr}(F(q)/F)$.*

PROOF. This theorem is proved in [**18**] except for the case where $\dim \phi = 7$ and $\operatorname{ind} C_0(\phi) = 8$. Implications (1)⇒(2)⇒(3) $\iff$ (4) are also proved in [**18**]. It suffices to prove implication (3)⇒(1).

Condition (3) shows that there exists a form ψ over F such that $\psi_{F(q)} \equiv \phi \pmod{I^5(F(q))}$. Therefore $\dim_5 \psi_{F(q)} \leq \dim \phi \leq 7$. By Proposition 3.8, we have $\dim_5 \psi \leq 7$. Hence there exists an anisotropic F-form μ of dimension ≤ 7 such that $\psi \equiv \mu \pmod{I^5(F)}$. Thus $\phi \equiv \psi_{F(q)} \equiv \mu_{F(q)} \pmod{I^5(F(q))}$. Since $\dim \phi + \dim \mu = 7 + 7 < 2^5$, APH shows that $\phi_{an} = (\mu_{F(q)})_{an}$. Since $\dim \mu < 8 < \dim q$, Hoffmann's theorem shows that $\mu_{F(q)}$ is anisotropic. Hence $\phi_{an} = \mu_{F(q)}$. Therefore ϕ_{an} is defined over F. Hence, ϕ is defined over F. □

Proof of theorem 1.7. (i)⇒(ii). Let $\phi = (q_{F(q)})_{an}$. By [**17**, Th. 7.13], it suffices to prove that ϕ is defined over F. Since $n \geq 5$, we have $\dim q \geq 2^n - 7 > 16$. Clearly, $\phi \in im(W(F) \to W(F(q)))$. Since q has maximal splitting, it follows that $\dim \phi = 2^n - \dim q \leq 7$. By Theorem 3.9, we see that ϕ is defined over F.

(ii)⇒(i). Obvious. □

4. Elementary properties of AMS-forms

In this section we start studying forms with absolutely maximal splitting (AMS-forms) defined in the introduction (Definition 1.10).

Lemma 4.1. *Let ϕ be an anisotropic form and n be an integer such that $2^{n-1} + 2^{n-3} < \dim \phi \leq 2^n$. Suppose that ϕ has maximal splitting. Then ϕ has absolutely maximal splitting.*

Proof. Let $m = \dim \phi - 2^{n-1}$. Clearly, $\dim \phi = 2^{n-1} + m$ and $2^{n-3} < m \leq 2^{n-1}$. Since ϕ has maximal splitting, we have $i_1(\phi) = m$. Let $F = F_0, F_1, \ldots, F_h$ be the generic splitting tower of ϕ. Let $\phi_i = (\phi_{F_i})_{an}$ for $i = 0, \ldots, h$. Let us fix $r > 1$. To prove that ϕ has absolutely maximal splitting, we need to verify that $i_r(\phi) < m$. Clearly, $i_r(\phi) = i_1(\phi_r)$. Thus, we need to verify that $i_1(\phi_r) < m$. In the case where $\dim \phi_r \leq 2^{n-2}$, we have $i_1(\phi_r) \leq \frac{1}{2} \dim \phi_r \leq \frac{1}{2} 2^{n-2} = 2^{n-3} < m$.

Thus, we can suppose that $\dim \phi_r > 2^{n-2}$. Since $r \geq 1$, we have $\dim \phi_r \leq \dim \phi_1 = \dim \phi - 2i_1(\phi) = 2^{n-1} + m - 2m = 2^{n-1} - m$. Hence, $2^{n-2} < \dim \phi_r \leq 2^{n-2} + (2^{n-2} - m)$. By Theorem 1.2, we have $i_1(\phi_r) \leq 2^{n-2} - m$. Since $m > 2^{n-3}$, we have $2^{n-2} - m < m$. Hence $i_1(\phi_r) \leq 2^{n-2} - m < m$. □

From the results proven in the next sections (see Theorem 7.1) it follows that in the dimension range we are interested in ($2^{n-1} + 2^{n-3} < \dim \phi \leq 2^n$), the form has maximal splitting if and only if it has absolutely maximal splitting.

Remark 4.2. We cannot change the strict inequality $2^{n-1} + 2^{n-3} < \dim \phi$ by $2^{n-1} + 2^{n-3} \leq \dim \phi$ in the formulation of the lemma. Indeed, for any $n \geq 3$ there exists an example of $(2^{n-1}+2^{n-3})$-dimensional form ϕ with maximal splitting which is not an AMS-form. The simplest example is the following:

$$\phi = \langle\langle x_1, x_2, \ldots, x_{n-3} \rangle\rangle \otimes \langle 1, 1, 1, 1, 1 \rangle \quad \text{over the field} \quad \mathbb{R}(x_1, \ldots, x_{n-3}).$$

In this case $i_1(\phi) = i_2(\phi) = 2^{n-3}$.

5. Motivic decomposition of AMS-Quadrics

In this section we will produce some "binary" motive related to an AMS-quadric.

Let X, Y and Z be smooth projective varieties over k of dimensions l, m and n, respectively. Then we have a natural (associative) pairing:

$$\circ : \mathrm{CH}^{n+b}(Y \times Z) \otimes \mathrm{CH}^{m+a}(X \times Y) \to \mathrm{CH}^{n+a+b}(X \times Z),$$

where $v \circ u := (\pi_{X,Z})_*(\pi^*_{X,Y}(u) \cap \pi^*_{Y,Z}(v))$, and $\pi_{X,Y} : X \times Y \times Z \to X \times Y$, $\pi_{Y,Z} : X \times Y \times Z \to Y \times Z$, $\pi_{X,Z} : X \times Y \times Z \to X \times Z$ are the natural projections. In particular, taking $X = \mathrm{Spec}(k)$, we get a pairing:

$$\mathrm{CH}^{n+b}(Y \times Z) \otimes \mathrm{CH}_r(Y) \to \mathrm{CH}_{r-b}(Z).$$

In this case, we will denote $v \circ u$ as $v(u)$.

THEOREM 5.1. (cf. [**33**, Proof of Statement 6.1]) *Let Q be an AMS-quadric. Let $P \subset Q$ be any subquadric of codimension $= i_1(q) - 1$. Then P possesses a Rost projector (in other words, $M(P)$ contains such direct summand N that $N|_{\overline{k}} \simeq \mathbb{Z} \oplus \mathbb{Z}(\dim(P))[2\dim(P)]$).*

PROOF. We say that "we are in the situation $(*)$", if we have the following data:

Q - some quadric; $P \subset Q$ - some subquadric of codimension d;
$\Phi \in \mathrm{CH}^m(Q \times Q)$, where $m := \dim(P)$.

In this case, let $\Psi \in \mathrm{CH}^{m+d}(P \times Q)$ denote the class of the graph of the natural embedding $P \subset Q$, and let $\Psi^\vee \in \mathrm{CH}^{m+d}(Q \times P)$ denote the dual cycle. We define $\varepsilon := \Psi^\vee \circ \Phi \circ \Psi \in \mathrm{CH}^m(P \times P)$.

The action on $\mathrm{CH}_*(P_{\overline{k}})$ identifies: $\mathrm{CH}^m(P_{\overline{k}} \times P_{\overline{k}}) = \prod_r \mathrm{End}(\mathrm{CH}_r(P_{\overline{k}}))$ (see [**29**, Lemma 7]), and we will denote as $\varepsilon_{(r)} \in \mathrm{End}(\mathrm{CH}_r(P_{\overline{k}}))$ the corresponding coordinate of $\varepsilon_{\overline{k}}$.

- If $0 \le s < m/2$, then $\mathrm{CH}_s(P_{\overline{k}}) = \mathbf{Z}$ with the generator l_s - the class of projective subspace of dimension s on $P_{\overline{k}}$;
- if $m/2 < s \le m$, then $\mathrm{CH}_s(P_{\overline{k}}) = \mathbf{Z}$ with the generator h^{m-s} - the class of plane section of codimension $m-s$ on $P_{\overline{k}}$;
- if $s = m/2$, then $\mathrm{CH}_s(P_{\overline{k}}) = \mathbf{Z} \oplus \mathbf{Z}$ with the generators $l^1_{m/2}$ and $l^2_{m/2}$ - the classes of $m/2$-dimensional projective subspaces from two different families.

This permits to identify $\mathrm{End}(\mathrm{CH}_s(P_{\overline{k}}))$ with $\mathbf{Z}$ if $0 \le s \le m$, $s \ne m/2$, and with $\mathrm{Mat}_{2\times 2}(\mathbf{Z})$, if $s = m/2$. We should mention, that since for an arbitrary field extension E/k, the natural map $\mathrm{CH}_s(P|_{\overline{k}}) \to \mathrm{CH}_s(P|_{\overline{E}})$ is an isomorphism (preserving the generators above), we have an equality: $(\varepsilon_E)_{(s)} = \varepsilon_{(s)}$ (in $\mathbf{Z}$, resp. $\mathrm{Mat}_{2\times 2}(\mathbf{Z})$).

We will need the following easy corollary of Springer's theorem. Under the degree of the cycle $A \in \mathrm{CH}_s(Q)$ we will understand the degree of the 0-cycle $A \cap h^s$.

LEMMA 5.2. *Let $0 \le s \le \dim(Q)/2$. Then the following conditions are equivalent:*

(1) *$q = (s+1) \cdot \mathbb{H} \perp q'$, for some form q';*
(2) *Q contains (projective subspace) $\mathbb{P}^s$ as a subvariety;*
(3) *$\mathrm{CH}_s(Q)$ contains cycle of odd degree.*

PROOF. (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are evident. (3) $\Rightarrow$ (1) Use induction on s. For $s = 0$ the statement is equivalent to the Theorem of Springer. Now if $s > 0$, then $\mathrm{CH}_0(Q)$ also contains a cycle of odd degree (obtained via intersection with h^s). So, $q = \mathbb{H} \perp q''$. And we have the natural degree preserving isomorphism: $\mathrm{CH}_s(Q) = \mathrm{CH}_{s-1}(Q'')$. By induction, $q'' = s \cdot \mathbb{H} \perp q'$. □

LEMMA 5.3. *In the situation of $(*)$, suppose, for some $0 \le s < m/2$, that $\varepsilon_{(s)} \in \mathbf{Z}$ is odd. Then for an arbitrary field extension E/k, if Q_E contains a projective space of dimension s, then it contains a projective space of dimension $s + d$.*

PROOF. Let E/k be such an extension that $l_s \in image(\mathrm{CH}_s(Q_E) \to \mathrm{CH}_s(Q_{\overline{E}}))$, and suppose that $\varepsilon_{(s)}$ is odd. We have: $\Psi \circ \Psi^\vee \circ \Phi(l_s) = \lambda \cdot l_s \subset \mathrm{CH}_s(Q_{\overline{E}})$, where

$\lambda \in \mathbf{Z}$ is odd (since $\Psi : \mathrm{CH}_s(P_{\overline{E}}) \to \mathrm{CH}_s(Q_{\overline{E}})$ is an isomorphism). On the other hand, the composition $\Psi \circ \Psi^\vee : \mathrm{CH}_{s+d}(Q_{\overline{E}}) \to \mathrm{CH}_s(Q_{\overline{E}})$ is given by the intersection with the plane section of codimension d, so it preserves the *degree* of the cycle. This implies that $\Phi(l_s) \in image(\mathrm{CH}_{s+d}(Q_E) \to \mathrm{CH}_{s+d}(Q_{\overline{E}}))$ has odd degree. By Lemma 5.2, Q_E contains a projective space of dimension $s+d$. □

LEMMA 5.4. *In the situation of* $(*)$*, suppose, for some* $m/2 < s \leq m$*, that* $\varepsilon_{(s)} \in \mathbf{Z}$ *is odd. Then for an arbitrary field extension* E/k*, if* Q_E *contains a projective space of dimension* $(m-s)$*, then it contains a projective space of dimension* $(m-s+d)$*.*

PROOF. Consider the cycle $\varepsilon^\vee \in \mathrm{CH}^m(P \times P)$ dual to ε. Since $(A \circ B)^\vee = B^\vee \circ A^\vee$, we have: $\varepsilon^\vee = \Psi^\vee \circ \Phi^\vee \circ \Psi$. On the other hand, $(\varepsilon^\vee)_{(s)} = \varepsilon_{(m-s)}$. Now, the statement follows from Lemma 5.3. □

LEMMA 5.5. *In the situation of* $(*)$*, if* $d > 0$*, then* $\varepsilon_{\overline{k}}(l^1_{m/2}) = \varepsilon_{\overline{k}}(l^2_{m/2}) = c \cdot h^{m/2}$*, where* $c \in \mathbf{Z}$*.*

PROOF. Clearly, $\Psi(l^i_{m/2}) = l_{m/2} \in \mathrm{CH}_{m/2}(Q_{\overline{k}})$. On the other hand, $\Phi \circ \Psi(l^i_{m/2}) \in \mathrm{CH}_{m/2+d}(Q_{\overline{k}})$, the later group is generated by $h^{m/2}$ (since $(m/2)+d > (m+d)/2$), and $\Psi^\vee(h^{m/2}) = h^{m/2}$. □

Let now Q be an AMS-quadric, and $P \subset Q$ be a subquadric of codimension $i_1(q)-1$. By the definition of AMS-quadrics, either $\dim(Q) = 0$, or $i_1(q) > 1$ and P is a proper subform of Q. Clearly, it is enough to consider the second possibility.

By the definition of $i_1(q)$, we have: $q_{k(Q)} = i_1(q) \cdot \mathbb{H} \perp q_1$. So, the quadric $Q_{k(Q)}$ contains an $(i_1(q)-1)$-dimensional projective subspace $l_{(i_1(q)-1)}$. Denote: $d := i_1(q)-1$, and $m := \dim(P)$. Let $\Phi \in \mathrm{CH}^m(Q \times Q)$ be the class of the closure of $l_d \subset \mathrm{Spec}(k(Q)) \times Q \subset Q \times Q$. Let us denote this particular case of $(*)$ as $(**)$.

LEMMA 5.6. *In the situation of* $(**)$*,* $\varepsilon_{\overline{k}} = P_{\overline{k}} \times l_0 + \sum_{0<i<m} b_i \cdot (h^{m-i} \times h^i) + a \cdot l_0 \times P_{\overline{k}}$*, where* $b_1, \ldots, b_{m-1}, a \in \mathbf{Z}$*.*

PROOF. If for some $0 \leq i < m/2$, the coordinate $\varepsilon_{(i)}$ is odd, then by Lemma 5.3, in the *generalized splitting tower* $k = F_0 \subset F_1 \subset \cdots \subset F_h$ for the quadrics Q (see [**16**]), there exists $0 \leq t < h$ such that $i_W(q_{F_t}) \leq i < i + i_1(q) - 1 < i_W(q_{F_{t+1}})$. Since q is an AMS-form, this can happen only if $i = 0$. In the same way, using Lemma 5.4, we get that for all $m/2 < i < m$, the coordinates $\varepsilon_{(i)}$ are even.

This implies that on the group $\mathrm{CH}_i(P_{\overline{k}})$, where $0 < i < m$, $i \neq m/2$, the map $\varepsilon_{\overline{k}}$ acts as some (integral) multiple of $h^{m-i} \times h^i$ (notice also that $h^{m-i} \times h^i$ acts trivially on all $\mathrm{CH}_j(P_{\overline{k}})$, $j \neq i$). The same holds for $i = m/2$ by Lemma 5.5.

Clearly, $P_{\overline{k}} \times l_0$ (resp. $l_0 \times P_{\overline{k}}$) acts on $\mathrm{CH}_0(P_{\overline{k}})$ (resp. $\mathrm{CH}_m(P_{\overline{k}})$) as a generator of $\mathrm{End}(\mathrm{CH}_0(P_{\overline{k}})) = \mathbf{Z}$ (resp. $\mathrm{End}(\mathrm{CH}_m(P_{\overline{k}})) = \mathbf{Z}$), and acts trivially on $\mathrm{CH}_j(P_{\overline{k}})$, $j \neq 0$ (resp. $j \neq m$). So, we need only to observe that $\varepsilon_{(0)} = 1$ (since $\Psi^\vee \circ \Phi \circ \Psi(l_0) = \Psi^\vee \circ \Phi(l_0) = \Psi^\vee(l_d) = l_0$)(this is the only place where we use the specifics of Φ). □

Now we can use ε to construct the desired projector in $\mathrm{End}(M(P))$, where $M(P)$ is a *motive* of the quadric P, considered as an object of the classical Chow-motivic category of Grothendieck $Chow^{eff}(k)$ (see [**3**],[**20**],[**31**],[**29**]). We remind that $\mathrm{End}(M(P))$ is naturally identified with $\mathrm{CH}^m(P \times P)$ with the composition given by the pairing $\circ$.

Take $\omega := \varepsilon - \sum_{0<i<m} b_i \cdot (h^i \times h^{m-i}) - [a/2] \cdot (h^m \times P) \in \mathrm{End}(M(P))$. Then $\omega_{\overline{k}}$ is a projector equal to either $(P_{\overline{k}} \times l_0 + l_0 \times P_{\overline{k}})$, or to $P_{\overline{k}} \times l_0$ (depending on the parity of a).

We have the following easy consequence of the Rost Nilpotence Theorem ([**29**, Corollary 10]):

Lemma 5.7. ([**33**, Lemma 3.12]) *If for some* $\omega \in \mathrm{End}(M(P))$, $\omega_{\overline{k}}$ *is an idempotent, then for some* r, ω^{2^r} *is an idempotent.*

The mutually orthogonal idempotents $P_{\overline{k}} \times l_0$ and $l_0 \times P_{\overline{k}}$ give the direct summands $\mathbb{Z}$ and $\mathbb{Z}(m)[2m]$ in $M(P_{\overline{k}})$. By lemma 5.7, we get a direct summand L in $M(P)$ such that either $L_{\overline{k}} = \mathbb{Z} \oplus \mathbb{Z}(m)[2m]$, or $L_{\overline{k}} = \mathbb{Z}$. The latter possibility is excluded by the following lemma.

Lemma 5.8. *Let* L *be a direct summand of* $M(P)$ *such that* $L_{\overline{k}} \simeq \mathbb{Z}$. *Then* P *is isotropic.*

Proof. Let $w \in \mathrm{CH}^{\dim(P)}(P \times P)$ be the projector, corresponding to L. We have: $\mathrm{End}(M(P_{\overline{k}})) = \prod_r \mathrm{End}(\mathrm{CH}_r(P_{\overline{k}}))$. So, if $\dim(P) > 0$, then the restriction $w_{\overline{k}}$ of our projector to $\overline{k}$ has no choice but to be $P_{\overline{k}} \times l_0 \in \mathrm{CH}^{\dim(P)}(P_{\overline{k}} \times P_{\overline{k}})$. Then, evidently, $degree(w \cap \Delta_P) = 1$, and on $P \times P$, and therefore also on P, we get a point of odd degree. By Springer's Theorem, P is isotropic. If $\dim(P) = 0$, then $\mathrm{End}(M(P))$ has a nontrivial projector if and only if $\det_{\pm}(p) = 1$ ($\Leftrightarrow$ p is isotropic). $\square$

From Lemma 5.7 it follows that ω^{2^r} is an idempotent, and by Lemma 5.8, $\omega^{2^r}|_{\overline{k}} = P_{\overline{k}} \times l_0 + l_0 \times P_{\overline{k}}$. Theorem 5.1 is proven. $\square$

6. Binary direct summands in the motives of quadrics

The following result was proven (but not formulated) by the second author in his thesis (see the proof of Statement 6.1 in [**33**]). We will reproduce its proof here for the reader's convenience.

Theorem 6.1. ([**33**]) ((*M*), see the end of Section 1) *Let* k *be a field of characteristic* 0, *and* P *be smooth anisotropic projective quadric of dimension* n *whose Chow-motive* $M(P)$ *contains a direct summand* N *such that* $N|_{\overline{k}} = \mathbb{Z} \oplus \mathbb{Z}(n)[2n]$. *Then* $n = 2^s - 1$ *for some* s.

Proof of Theorem 6.1. The construction we use here is very close to that used by V.Voevodsky in [**36**].

The category of *Chow-motives* $Chow^{eff}(k)$ which we used in the previous section is a full additive subcategory (closed under taking direct summands) in the triangulated category $DM_{-}^{eff}(k)$ - see [**34**]. The category $DM_{-}^{eff}(k)$ contains the "motives" of all smooth simplicial schemes over k. If P is a smooth projective variety over k, we denote as $\check{C}(P)^{\bullet}$ the *standard simplicial scheme* corresponding to the pair $P \to \mathrm{Spec}(k)$ (see Definition A.8). We will denote its motive by $\mathcal{X}_P$.

From the natural projection: $\check{C}(P)^{\bullet} \stackrel{pr}{\to} \mathrm{Spec}(k)$, we get a map: $\mathcal{X}_P \stackrel{M(pr)}{\longrightarrow} \mathbb{Z}$. By Theorem A.9, $M(pr)_{\overline{k}}$ is an isomorphism. From this point, we will denote $M(pr)$ simply as pr (since we will not use simplicial schemes themselves anymore).

By Theorem A.11, we get that in $DM_{-}^{eff}(k)$,

$$N := Cone[-1](\mathcal{X}_P \xrightarrow{\mu'} \mathcal{X}_P(n)[2n+1]),$$

where μ' is some (actually, the only) nontrivial [2] element from

$$\mathrm{Hom}(\mathcal{X}_P, \mathcal{X}_P(n)[2n+1]).$$

By Theorem A.15, $pr : \mathcal{X}_P \to \mathbb{Z}$ induces the natural isomorphism for all a, b:

$$pr_* : \mathrm{Hom}(\mathcal{X}_P, \mathcal{X}_P(a)[b]) \to \mathrm{Hom}(\mathcal{X}_P, \mathbb{Z}(a)[b]).$$

Denote: $\mu := pr_*(\mu') \in \mathrm{Hom}(\mathcal{X}_P, \mathbb{Z}(n)[2n+1])$.

SUBLEMMA 6.2. *The map*

$$(\mu')^* : \mathrm{Hom}(\mathcal{X}_P, \mathbb{Z}(c)[d]) \to \mathrm{Hom}(\mathcal{X}_P, \mathbb{Z}(c+n)[d+2n+1])$$

coincides with the multiplication by $\mu \in \mathrm{Hom}(\mathcal{X}_P, \mathbb{Z}(n)[2n+1])$.

PROOF. The maps $\Delta_{\mathcal{X}_P} : \mathcal{X}_P \to \mathcal{X}_P \otimes \mathcal{X}_P$, and $\pi_i : \mathcal{X}_P \otimes \mathcal{X}_P \to \mathcal{X}_P$ are mutually inverse isomorphisms (by Theorem A.13). Clearly, $\mu \cdot u = \Delta_{\mathcal{X}_P}(\mu \otimes u)$.

The map $\mu \otimes u : \mathcal{X}_P \otimes \mathcal{X}_P \to \mathbb{Z}(n)[2n+1] \otimes \mathbb{Z}(c)[d]$ coincides with the composition:

$$\begin{array}{ccccc} \mathcal{X}_P & \xrightarrow{\mu'} & \mathcal{X}_P(n)[2n+1] & \xrightarrow{pr(n)[2n+1]} & \mathbb{Z}(n)[2n+1] \\ \otimes & & \otimes & & \otimes \\ \mathcal{X}_P & \xrightarrow{id} & \mathcal{X}_P & \xrightarrow{u} & \mathbb{Z}(c)[d] \end{array}$$

which can be identified with the composition:

$$\mathcal{X}_P \xrightarrow{\mu'} \mathcal{X}_P(n)[2n+1] \xrightarrow{u} \mathbb{Z}(n+c)[2n+1+d],$$

which is equal to $(\mu')^*(u)$. □

SUBLEMMA 6.3. *Multiplication by μ induces a homomorphism*

$$\mathrm{Hom}(\mathcal{X}_P, \mathbb{Z}(c)[d]) \to \mathrm{Hom}(\mathcal{X}_P, \mathbb{Z}(c+n)[d+2n+1])$$

which is an isomorphism if $d - c > 0$, and which is surjective if $d = c$. The same holds for cohomology with $\mathbb{Z}/2$-coefficients.

PROOF. Since N is a direct summand in $M(P)$, $\mathrm{Hom}(N, \mathbb{Z}(a)[b]) = 0$, for $b - a > n = \dim(P)$, by Theorem A.2(1).

Consider Hom's from the exact triangle $N \to \mathcal{X}_P \xrightarrow{\mu'} \mathcal{X}_P(n)[2n+1] \to N[1]$ to $\mathbb{Z}(n+c)[2n+d+1]$. We have: $\mathrm{Hom}(N, \mathbb{Z}(n+c)[2n+d+1]) = 0$, if $d - c \geq 0$, and $\mathrm{Hom}(N, \mathbb{Z}(n+c)[2n+d]) = 0$, if $d - c > 0$. This, combined with Sublemma 6.2, implies the statement for $\mathbb{Z}$-coefficients. The case of $\mathbb{Z}/2$-coefficients follows from the five-lemma. □

We can also consider $\tilde{\mathcal{X}}_P := \mathrm{Cone}[-1](\mathcal{X}_P \xrightarrow{pr} \mathbb{Z})$.

SUBLEMMA 6.4. *Let a and b be integers such that $b > a$. Then*

- $\mathrm{Hom}(\mathcal{X}_P, \mathbb{Z}(a)[b])$ *is a 2-torsion group,*
- $\mathrm{Hom}(\mathcal{X}_P, \mathbb{Z}(a)[b])$ *embeds into* $\mathrm{Hom}(\mathcal{X}_P, \mathbb{Z}/2(a)[b])$,
- *the natural map $\tilde{\mathcal{X}}_P \xrightarrow{\delta} \mathcal{X}_P$ induces an isomorphism*

$$\mathrm{Hom}(\mathcal{X}_P, \mathbb{Z}/2(a)[b]) \xrightarrow{=} \mathrm{Hom}(\tilde{\mathcal{X}}_P, \mathbb{Z}/2(a)[b]).$$

[2] Since, otherwise, $\mathcal{X}_P$ would be a direct summand of N and hence also of $M(P)$, and by Lemma 5.8, P would be isotropic

Proof. For a finite field extension E/k we have the action of *transfers* on motivic cohomology:

$$\mathrm{Tr} : \mathrm{Hom}(X|_E, \mathbb{Z}(a)[b]) \to \mathrm{Hom}(X, \mathbb{Z}(a)[b]),$$

which is induced by the natural map $\mathbb{Z} \to M(\mathrm{Spec}(E))$ (given by the generic cycle on $\mathrm{Spec}(k) \times \mathrm{Spec}(E) = \mathrm{Spec}(E)$). The main property of the transfer is that $\mathrm{Tr} \circ j$ acts as multiplication by the degree $[E : k]$, where

$$j : \mathrm{Hom}(X, \mathbb{Z}(a)[b]) \to \mathrm{Hom}(X|_E, \mathbb{Z}(a)[b])$$

is the natural restriction.

A quadric P has a point E of degree 2, and over E, $\mathcal{X}_P$ becomes $\mathbb{Z}$ (by Theorem A.9), so we have that $\mathrm{Hom}(\mathcal{X}_P|_E, \mathbb{Z}(a)[b]) = 0$ for $b - a > 0$ (by Theorem A.2(1)).

Considering the composition $\mathrm{Tr} \circ j = \cdot[E : k]$ we get that $\mathrm{Hom}(\mathcal{X}_P, \mathbb{Z}(a)[b])$ is a 2-torsion group for $b > a$. In particular, the natural map $\mathrm{Hom}(\mathcal{X}_P, \mathbb{Z}(a)[b]) \to \mathrm{Hom}(\mathcal{X}_P, \mathbb{Z}/2(a)[b])$ is injective for $b > a$.

Since $\mathrm{Hom}(\mathbb{Z}, \mathbb{Z}/2(a)[b]) = 0$ for any $b > a$ (see Theorem A.2(1)), we also have that for $b > a$, $\delta^* : \mathrm{Hom}(\mathcal{X}_P, \mathbb{Z}/2(a)[b]) \to \mathrm{Hom}(\tilde{\mathcal{X}}_P, \mathbb{Z}/2(a)[b])$ is an isomorphism. □

We have the action of motivic cohomological operations Q_i on $\mathrm{Hom}(\mathcal{X}_P, \mathbb{Z}(*)[*'])$ and $\mathrm{Hom}(\tilde{\mathcal{X}}_P, \mathbb{Z}(*)[*'])$ (see Theorems A.5 and A.6). The differential Q_i acts without cohomology on $\mathrm{Hom}(\tilde{\mathcal{X}}_P, \mathbb{Z}/2(*)[*'])$ for any $i \leq [\log_2(n+1)]$ (see Theorem A.16).

Denote $\eta := \mu(mod 2)$, i.e. the image of μ in the cohomology with $\mathbb{Z}/2$ coefficients. From Sublemma 6.4 it follows that $\eta \neq 0$.

Denote $r = [\log_2(n)]$.

Sublemma 6.5. *$Q_i(\eta) = 0$, for all $i \leq r$.*

Proof. In fact, $Q_i(\eta) \in \mathrm{Hom}(\mathcal{X}_P, \mathbb{Z}/2(n + 2^i - 1)[2n + 2^{i+1}])$, and the latter group is an extension of 2-cotorsion in $\mathrm{Hom}(\mathcal{X}_P, \mathbb{Z}(n + 2^i - 1)[2n + 2^{i+1}])$, and 2-torsion in $\mathrm{Hom}(\mathcal{X}_P, \mathbb{Z}(n + 2^i - 1)[2n + 2^{i+1} + 1])$.

But, by Sublemma 6.3, the multiplication by μ induces surjections $\mathrm{Hom}(\mathcal{X}_P, \mathbb{Z}(2^i - 1)[2^{i+1} - 1]) \to \mathrm{Hom}(\mathcal{X}_P, \mathbb{Z}(n + 2^i - 1)[2n + 2^{i+1}])$ and $\mathrm{Hom}(\mathcal{X}_P, \mathbb{Z}(2^i - 1)[2^{i+1}]) \to \mathrm{Hom}(\mathcal{X}_P, \mathbb{Z}(n + 2^i - 1)[2n + 2^{i+1} + 1])$. Furthermore, the groups: $\mathrm{Hom}(\mathcal{X}_P, \mathbb{Z}(2^i - 1)[2^{i+1} - 1])$, $\mathrm{Hom}(\mathcal{X}_P, \mathbb{Z}(2^i - 1)[2^{i+1}])$ are zero.

In fact, from the exact triangle $N \to \mathcal{X}_P \to \mathcal{X}_P(n)[2n + 1] \to N[1]$, we get an exact sequence: $\mathrm{Hom}(N, \mathbb{Z}(2^i - 1)[2^{i+1} - 1]) \leftarrow \mathrm{Hom}(\mathcal{X}_P, \mathbb{Z}(2^i - 1)[2^{i+1} - 1]) \leftarrow \mathrm{Hom}(\mathcal{X}_P(n)[2n + 1], \mathbb{Z}(2^i - 1)[2^{i+1} - 1])$. The first group is zero since N is a direct summand in the motive of a smooth projective variety, and (consequently) $\mathrm{Hom}(N, \mathbb{Z}(a)[b]) = 0$ for $b > 2a$ (see Theorem A.2(2)). The third group is zero, since $n > 2^i - 1$ (see Theorem A.1). Hence the second is zero as well. The case of $\mathrm{Hom}(\mathcal{X}_P, \mathbb{Z}(2^i - 1)[2^{i+1}])$ follows in an analogous manner..

Thus, $Q_i(\eta) = 0$. □

Sublemma 6.6. *Let $0 \leq j \leq r$. Then Q_j is injective on* $\mathrm{Hom}(\tilde{\mathcal{X}}_P, \mathbb{Z}/2(c)[d])$, *if $d - c = n + 1 + 2^j$.*

Proof. Let $\tilde{v} \in \mathrm{Hom}(\tilde{\mathcal{X}}_P, \mathbb{Z}/2(c)[d])$, where $d - c = n + 1 + 2^j$. If $Q_j(\tilde{v}) = 0$, then $\tilde{v} = Q_j(\tilde{w})$, for some $\tilde{w} \in \mathrm{Hom}(\tilde{\mathcal{X}}_P, \mathbb{Z}/2(c - 2^j + 1)[d - 2^{j+1} + 1])$ (since Q_j acts without cohomology on $\mathrm{Hom}(\tilde{\mathcal{X}}_P, \mathbb{Z}/2(*)[*'])$, by Theorem A.16). Since

$(d-2^{j+1}+1)-(c-2^j+1)=n+1>0$, we have that $\delta^*:\mathrm{Hom}(\mathcal{X}_P,\mathbb{Z}/2(c-2^j+1)[d-2^{j+1}+1])\to\mathrm{Hom}(\tilde{\mathcal{X}}_P,\mathbb{Z}/2(c-2^j+1)[d-2^{j+1}+1])$ is an isomorphism, and there exists $w\in\mathrm{Hom}(\mathcal{X}_P,\mathbb{Z}/2(c-2^j+1)[d-2^{j+1}+1])$ such that $\tilde{w}=\delta^*(w)$.

By Sublemma 6.3, $w=\eta\cdot u$, for some $u\in\mathrm{Hom}(\mathcal{X}_P,\mathbb{Z}/2(c-2^j+1-n)[d-2^{j+1}-2n])$. By Theorem A.6(2), $Q_j(\eta\cdot u)=Q_j(\eta)\cdot u+\eta\cdot Q_j(u)+\sum\{-1\}^{x_i}\phi_i(\eta)\cdot\psi_i(u)$, where $x_i>0$, and ϕ_i,ψ_i are cohomological operations of some bidegree $(*)[*']$, where $*'>2*\geq 0$.

Note that $c-2^j+1-n=d-2^{j+1}-2n=:s$. But by Theorem A.18, we have that $\mathrm{Hom}(\mathbb{Z},\mathbb{Z}/2(a)[b])\stackrel{pr^*}{\to}\mathrm{Hom}(\mathcal{X}_P,\mathbb{Z}/2(a)[b])$ is an isomorphism for $a\geq b$. Hence, $u=pr^*(u_0)$, where $u_0\in\mathrm{Hom}(\mathbb{Z},\mathbb{Z}/2(s)[s])=\mathrm{K}^M_s(k)/2$. We have: $Q_j(u_0)=0$ and $\psi_i(u_0)=0$ (since $\mathrm{Hom}(\mathbb{Z},\mathbb{Z}/2(a)[b])=0$ for $b>a$).

But pr^* commutes with Q_j and ψ_i. So, $Q_j(u)=0$ and $\psi_i(u)=0$. That means: $Q_j(w)=Q_j(\eta\cdot u)=Q_j(\eta)\cdot u=0$, by Sublemma 6.5.

We get: $\tilde{v}=Q_j(\tilde{w})=Q_j\circ\delta^*(w)=\delta^*\circ Q_j(w)=0$. I.e., Q_j is injective on $\mathrm{Hom}(\tilde{\mathcal{X}}_P,\mathbb{Z}/2(c)[d])$. □

Denote $\tilde{\eta}:=\delta^*(\eta)\in\mathrm{Hom}(\tilde{\mathcal{X}}_P,\mathbb{Z}/2(n)[2n+1])$. Since $\eta\neq 0$, we have $\tilde{\eta}\neq 0$ (by Sublemma 6.4).

Sublemma 6.7. *Let $0\leq m<r$, and $\tilde{\eta}=Q_m\circ\cdots\circ Q_1\circ Q_0(\tilde{\eta}_m)$ for some $\tilde{\eta}_m\in\mathrm{Hom}(\tilde{\mathcal{X}}_P,\mathbb{Z}/2(n-2^{m+1}+m+2)[2n-2^{m+2}+m+4])$. Then there exists $\tilde{\eta}_{m+1}$ such that $\tilde{\eta}_m=Q_{m+1}(\tilde{\eta}_{m+1})$.*

Proof. Since Q_{m+1} acts without cohomology on $\mathrm{Hom}(\tilde{\mathcal{X}}_P,\mathbb{Z}/2(*)[*'])$, it is enough to show that $Q_{m+1}(\tilde{\eta}_m)=0$.

Denote $\tilde{v}:=Q_{m+1}(\tilde{\eta}_m)$. We have $\tilde{v}\in\mathrm{Hom}(\tilde{\mathcal{X}}_P,\mathbb{Z}/2(n+m+1)[2n+m+3])$. Since Q_i commutes with Q_j (by Theorem A.6(1)), we have: $Q_m\circ Q_{m-1}\circ\cdots\circ Q_0(\tilde{v})=Q_m\circ Q_{m-1}\circ\cdots\circ Q_0\circ Q_{m+1}(\tilde{\eta}_m)=Q_{m+1}\circ Q_m\circ Q_{m-1}\circ\cdots\circ Q_0(\tilde{\eta}_m)=Q_{m+1}(\tilde{\eta})=0$, by Sublemma 6.5.

But, for any $0\leq t\leq m$, $Q_{t-1}\circ\cdots\circ Q_0(\tilde{v})\in\mathrm{Hom}(\tilde{\mathcal{X}}_P,\mathbb{Z}(c)[d])$, where $d-c=n+1+2^t$, and Q_t is injective on $\mathrm{Hom}(\tilde{\mathcal{X}}_P,\mathbb{Z}(c)[d])$, by Sublemma 6.6. So, from the equality $Q_m\circ Q_{m-1}\circ\cdots\circ Q_0(\tilde{v})=0$, we get $\tilde{v}=0$. □

From Sublemma 6.7 it follows that $\tilde{\eta}=Q_r\circ\cdots\circ Q_1\circ Q_0(\tilde{\eta}_r)$. Denote $\tilde{\gamma}:=\tilde{\eta}_r$. We have $\tilde{\gamma}\in\mathrm{Hom}(\tilde{\mathcal{X}}_P,\mathbb{Z}/2(n-2^{r+1}+2+r)[2n-2^{r+2}+4+r])$.

But $(2n-2^{r+2}+4+r)-(n-2^{r+1}+2+r)=n-2^{r+1}+2$, and $r=[\log_2(n)]$, hence $2^r\leq n<2^{r+1}$. Since we know that $\mathrm{Hom}(\tilde{\mathcal{X}}_P,\mathbb{Z}/2(a)[b])=0$ for $a\geq b$ (by Theorem A.18), and $\tilde{\eta}\neq 0$, the only possible choice for n is $n=2^{r+1}-1$.

Theorem 6.1 is proven. □

Lemma 6.8. *Let $0\leq j\leq r$. Then Q_j is injective on $\mathrm{Hom}(\tilde{\mathcal{X}}_P,\mathbb{Z}/2(c)[d])$ provided $d-c=2^j$.*

Proof. Let $\tilde{v}\in\mathrm{Hom}(\tilde{\mathcal{X}}_P,\mathbb{Z}/2(c)[d])$, where $d-c=2^j$. If $Q_j(\tilde{v})=0$, then $\tilde{v}=Q_j(\tilde{w})$ for some $\tilde{w}\in\mathrm{Hom}(\tilde{\mathcal{X}}_P,\mathbb{Z}/2(c-2^j+1)[d-2^{j+1}+1])$ (since Q_j acts without cohomology on $\mathrm{Hom}(\tilde{\mathcal{X}}_P,\mathbb{Z}/2(*)[*'])$, by Theorem A.16). But $(c-2^j+1)=(d-2^{j+1}+1)$, and $\mathrm{Hom}(\tilde{\mathcal{X}}_P,\mathbb{Z}/2(c-2^j+1)[d-2^{j+1}+1])=0$, by Theorem A.18. □

Theorem 6.9. (compare with [**8**, Theorem 3.1]) ((*M*), see the end of Section 1) *Let k be a field of characteristic 0, P be a smooth n-dimensional anisotropic projective quadric over k, and N be a direct summand in $M(P)$ such that $N|_{\overline{k}}=$*

$\mathbb{Z} \oplus \mathbb{Z}(n)[2n]$. *Then* $n = 2^{s-1} - 1$, *and there exists* $\alpha \in \mathrm{K}_s^M(k)/2$ *such that for any field extension* E/k, *the following conditions are equivalent:*
1) $\alpha|_E = 0$; *2)* $P|_E$ *is isotropic.*

In particular, $\alpha \in \mathrm{Ker}(\mathrm{K}_s^M(k)/2 \to \mathrm{K}_s^M(k(P))/2) \neq 0$.

Proof. It follows from Theorem 6.1 that $n = 2^{r+1} - 1$ for some r, and there exists $\tilde{\gamma} \in \mathrm{Hom}(\tilde{\mathcal{X}}_P, \mathbb{Z}/2(r+1)[r+2])$ such that $\tilde{\eta} = Q_r \circ \cdots \circ Q_1 \circ Q_0(\tilde{\gamma})$. By Sublemma 6.4, the map $\delta^* : \mathrm{Hom}(\mathcal{X}_P, \mathbb{Z}/2(r+1)[r+2]) \to \mathrm{Hom}(\tilde{\mathcal{X}}_P, \mathbb{Z}/2(r+1)[r+2])$ is an isomorphism, and $\tilde{\gamma} = \delta^*(\gamma)$ for some $\gamma \in \mathrm{Hom}(\mathcal{X}_P, \mathbb{Z}/2(r+1)[r+2])$. Let τ be the only nontrivial element of $\mathrm{Hom}(\mathbb{Z}/2, \mathbb{Z}/2(1)) = \mathbf{Z}/2$. Denote as α the element corresponding to $\tau \circ \gamma$ via identification (by Theorem A.18) $\mathrm{Hom}(\mathcal{X}_P, \mathbb{Z}/2(r+2)[r+2]) = \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}/2(r+2)[r+2]) = \mathrm{K}_{r+2}^M(k)/2$. Then, by Theorem A.20, for any field extension E/k, $\alpha|_E = 0$ if and only if $\gamma|_E = 0$. But $\gamma|_E = 0 \Leftrightarrow \tilde{\gamma}|_E = 0$. By Lemma 6.8, $\tilde{\gamma}|_E = 0 \Leftrightarrow \tilde{\eta}|_E = 0$. By Sublemma 6.4, $\tilde{\eta}|_E = 0 \Leftrightarrow \eta|_E = 0 \Leftrightarrow \mu|_E = 0$. Finally, $\mu|_E = 0$ if and only if $\mathcal{X}_P|_E$ is a direct summand in N and, consequently, in $M(P)$, which by Lemma 5.8, is equivalent to $P|_E$ being isotropic. □

Remark 6.10.
1) Theorem 6.9 basically says that under the mentioned conditions, the quadric P is a *norm-variety* for $\alpha \in \mathrm{K}_s^M(k)/2$.
2) Taking into account the Milnor conjecture ([**36**]) and the definition of the *Rost projector*, we see that Theorem 6.9 implies Theorem 1.12.
3) It should be mentioned that in small-dimensional cases it is possible to prove the result (in arbitrary characteristic $\neq 2$) without the use of Voevodsky's technique. For example, the case $n = 7$ was considered in [**15**].

7. Properties of forms with absolutely maximal splitting

In this section we work with fields satisfying the condition $\mathrm{char}\, F = 0$. We begin with the following modification of Theorem 1.11.

Theorem 7.1. ((*M*), see the end of Section 1) *Let* ϕ *be an anisotropic quadratic form over a field* F *of characteristic* 0. *Suppose that* ϕ *is an AMS-form. Then*

(1) ϕ *has maximal splitting,*
(2) *the group* $H^s(F(\phi)/F)$ *is nontrivial, where* s *is the integer such that* $2^{s-1} < \dim \phi \leq 2^s$.

Proof. (1) Let ψ be subform of ϕ of codimension $i_1(q) - 1$. Let X be the projective quadric corresponding to ψ. By Theorem 5.1, X possesses a Rost projector. Theorem 6.1 shows that $\dim(X) = 2^{s-1} - 1$ for suitable s. Hence $\dim \psi = 2^{s-1} + 1$. By the definition of ψ, we have $\dim \phi - i_1(\phi) = \dim \psi - 1 = 2^{s-1}$. Therefore $\dim \phi = 2^{s-1} + m$, where $m = i_1(\phi)$. To prove that ϕ has maximal splitting, it suffices to verify that $m \leq 2^{s-1}$. This is obvious because $2^{s-1} + m = \dim \phi \geq 2i_1(\phi) = 2m$.

(2) Obvious in view of Theorem 6.9 and the isomorphism $k_s(F) \simeq H^s(F)$. □

Theorem 7.1 and Conjecture 1.1 make natural the following

Conjecture 7.2. *If an anisotropic quadratic form has absolutely maximal splitting, then it is a Pfister neighbor.*

In the proof of Theorem 1.8 we will need some deep results related to the Milnor conjecture.

Theorem 7.3. (see [**36**],[**26**]). ((*M*), see the end of Section 1) *Let F be a field of characteristic* 0. *Then for any $n \geq 0$*

1. *there exists an isomorphism* $e^n : I^n(F)/I^{n+1}(F) \xrightarrow{\simeq} H^n(F)$ *such that*
$$e^n(\langle\langle a_1, \dots, a_n \rangle\rangle) = (a_1, \dots, a_n).$$
2. *If ϕ is a Pfister neighbor of $\pi \in GP_n(F)$. Then $H^n(F(\phi)/F)$ is generated by $e^n(\pi)$.*
3. *If $\dim \tau > 2^n$, then $H^n(F(\tau)/F) = 0$.*
4. *The ideal $I^n(F)$ coincides with Knebusch's ideal $J_n(F)$. In other words, for any $\tau \in I^n(F) \backslash I^{n+1}(F)$, we have $\deg \tau = n$.*

We need also the following easy consequence of a result by Hoffmann.

Lemma 7.4. *Let ϕ be an anisotropic form such that $\dim \phi \leq 2^n$. Let τ be an anisotropic quadratic form and $F_0 = F, F_1, \dots, F_h$ be the generic splitting tower of τ. Let j be such that $\dim(\tau_{F_{j-1}})_{an} > 2^n$. Suppose that ϕ_{F_j} has maximal splitting. Then ϕ has maximal splitting.*

Proof. Obvious in view of [**4**, Lemma 5]. □

Proposition 7.5. ((*M*), see the end of Section 1) *Let F be a field of characteristic* 0. *Let ϕ be a quadratic form over F and n be such that $2^{n-1} < \dim \phi \leq 2^n$. Suppose that $H^n(F(\phi)/F) \neq 0$. Then $H^n(F(\phi)/F) \simeq \mathbf{Z}/2\mathbf{Z}$ and ϕ has maximal splitting.*

Proof. Let u be an arbitrary nonzero element of the group $H^n(F(\phi)/F)$. Since the homomorphism $e^n : I^n(F)/I^{n+1}(F) \to H^n(F)$ is an isomorphism, there exists an anisotropic $\tau \in I^n(F)$ such that $\tau \notin I^{n+1}(F)$ and $e^n(\tau) = u \in H^n(F(\phi)/F)$. Let $F_0 = F, F_1, \dots, F_h$ be the generic splitting tower of τ. Let $\tau_i = (\tau_{F_i})_{an}$. Since $\tau \in I^n(F) \backslash I^{n+1}(F)$, Item (4) of Theorem 7.3 shows that $\deg \tau = n$. Therefore, τ_{h-1} is a nonhyperbolic form in $GP_n(F_{h-1})$. Since $e^n(\tau) \in H^n(F(\phi)/F)$, we have $e^n((\tau_{h-1})_{F_{h-1}(\phi)}) = 0$. Hence, τ_{h-1} is hyperbolic over the function field of $\phi_{F_{h-1}}$. Since τ_{h-1} is an anisotropic form in $GP_n(F_{h-1})$, the Cassels–Pfister subform theorem shows that $\phi_{F_{h-1}}$ is a Pfister neighbor of $\tau_{F_{h-1}}$. Hence $\phi_{F_{h-1}}$ has maximal splitting. Lemma 7.4 shows that ϕ has maximal splitting.

Since $\phi_{F_{h-1}}$ is a Pfister neighbor of $\tau_{F_{h-1}}$, Item (2) of Theorem 7.3 shows that $|H^n(F_{h-1}(\phi)/F_{h-1})| \leq 2$. By Item (3) of Theorem 7.3, we have $H^n(F_{h-1}/F) = 0$. Hence $|H^n(F_{h-1}(\phi)/F)| \leq 2$. Since $H^n(F(\phi)/F) \subset H^n(F_{h-1}(\phi)/F)$, we get $|H^n(F(\phi)/F)| \leq 2$. Now, since $H^n(F(\phi)/F) \neq 0$, we have $H^n(F(\phi)/F) \simeq \mathbf{Z}/2\mathbf{Z}$. □

Corollary 7.6. ((*M*), see the end of Section 1) *Let $n \geq 5$ and ϕ be an anisotropic form such that $2^n - 7 \leq \dim \phi \leq 2^n$ Then the following conditions are equivalent:*

(a) *ϕ has maximal splitting,*
(b) *ϕ is a Pfister neighbor,*
(c) *$H^n(F(\phi)/F) \simeq \mathbf{Z}/2\mathbf{Z}$.*
(d) *$H^n(F(\phi)/F) \neq 0$.*

Proof. (a)⇒(b) follows from Theorem 1.7. (b)⇒(c) follows from Theorem 7.3; (c)⇒(d) is obvious; (d)⇒(a) is proved in Proposition 7.5. □

Proof of Theorem 1.8. Let ϕ and n be as in Theorem 1.8. If ϕ has maximal splitting, then Lemma 4.1 shows that ϕ has absolutely maximal splitting. Then Theorem 7.1 shows that $H^n(F(\phi)/F) \neq 0$. Conversely, if we suppose that $H^n(F(\phi)/F) \neq 0$, then Proposition 7.5 shows that ϕ has maximal splitting. □

Appendix A

In this section we will list some results of V.Voevodsky, which we use in the proof of Theorems 6.1 and 6.9.

We will assume everywhere that $char(k) = 0$.

First of all, we need some facts about triviality of motivic cohomology of smooth simplicial schemes. If not specified otherwise, under $\mathrm{Hom}(-,-)$ we will mean $\mathrm{Hom}_{DM^{eff}_{-}(k)}(-,-)$. We remind that $\mathrm{Hom}_{DM^{eff}_{-}(k)}(M(X), \mathbb{Z}(a)[b])$ is naturally identified with $H^{b,a}_{B}(X, \mathbb{Z})$ (see [**36**]).

Theorem A.1. ([**36**, Corollary 2.2(1)]) *Let $\mathcal{X}$ be smooth simplicial scheme over k. Then* $\mathrm{Hom}(M(\mathcal{X})(a)[b], \mathbb{Z}(c)[d]) = 0$*, for any $a > c$.*

In the case of smooth variety we have further restrictions on motivic cohomology:

Theorem A.2. ([**36**, Corollary 2.3]) *Let N be a direct summand in $M(X)$, where X is a smooth scheme over k. Then* $\mathrm{Hom}(N, \mathbb{Z}(a)[b]) = 0$ *in the following cases:*

1 *If $b - a > \dim(X)$;*
2 *If $b > 2a$.*

The same is true about cohomology with $\mathbb{Z}/2$-coefficients.

In [**36**] the *Stable homotopy category of schemes over* $\mathrm{Spec}(k)$, $\mathcal{SH}(k)$ was defined (see also [**25**]). $\mathcal{SH}(k)$ is a *triangulated* category, and there is a functor $S : SmSimpl/k \to \mathcal{SH}(k)$, and a triangulated functor $G : \mathcal{SH}(k) \to DM^{eff}_{-}(k)$ such that the composition $G \circ S : SmSimpl/k \to DM^{eff}_{-}(k)$ coincides with the usual *motivic functor*: $X \mapsto M(X)$ (here $SmSimpl/k$ is the category of smooth simplicial schemes over $\mathrm{Spec}(k)$). In [**36**], Section 3.3, the *Eilenberg-MacLane spectrum* $\mathbf{H}_{\mathbb{Z}/2}$ (as an object of $\mathcal{SH}(k)$) is defined, together with its shifts $\mathbf{H}_{\mathbb{Z}/2}(a)[b]$, for $a, b \in \mathbf{Z}$.

Theorem A.3. ([**36**, Theorem 3.12]) *If X is a smooth simplicial scheme, then there exist canonical isomorphisms*

$$\mathrm{Hom}_{\mathcal{SH}(k)}(S(X), \mathbf{H}_{\mathbb{Z}/2}(a)[b]) = \mathrm{Hom}_{DM^{eff}_{-}(k)}(M(X), \mathbb{Z}/2(a)[b]).$$

Definition A.4. ([**36**, p.31]) The *motivic Steenrod algebra* is the algebra of endomorphisms of $\mathbf{H}_{\mathbb{Z}/2}$ in $\mathcal{SH}(k)$, i.e.:

$$\mathcal{A}^{b,a}(k, \mathbb{Z}/2) = \mathrm{Hom}_{\mathcal{SH}(k)}(\mathbf{H}_{\mathbb{Z}/2}, \mathbf{H}_{\mathbb{Z}/2}(a)[b]).$$

The composition gives a pairing:

$$\mathrm{Hom}_{\mathcal{SH}(k)}(U, \mathbf{H}_{\mathbb{Z}/2}(c)[d]) \otimes \mathcal{A}^{b,a}(k, \mathbb{Z}/2) \to \mathrm{Hom}_{\mathcal{SH}(k)}(U, \mathbf{H}_{\mathbb{Z}/2}(c+a)[d+b]),$$

which is natural on U.

Let now $f : X \to Y$ be a morphism in $SmSimpl/k$. In $\mathcal{SH}(k)$ we have an *exact triangle*: $\mathrm{cone}(S(f))[-1] \stackrel{\delta'}{\to} S(X) \stackrel{S(f)}{\to} S(Y) \to \mathrm{cone}(f)$.

Theorem A.3 implies:

Theorem A.5. *We have an action of the motivic Steenrod algebra $\mathcal{A}^{*,*}(k,\mathbb{Z}/2)$ on $\oplus_{a,b}\operatorname{Hom}_{DM_-^{eff}(k)}(M(X),\mathbb{Z}/2(a)[b])$, $\oplus_{a,b}\operatorname{Hom}_{DM_-^{eff}(k)}(M(Y),\mathbb{Z}/2(a)[b])$, and $\oplus_{a,b}\operatorname{Hom}(\operatorname{cone}(M(f))[-1],\mathbb{Z}/2(a)[b])$, which is compatible with $M(f)^*$ and δ^*.*

We have some special elements $Q_i\in\mathcal{A}^{2^{i+1}-1,2^i-1}(k,\mathbb{Z}/2)$ (see [**36**, p.32]).

Theorem A.6. ([**36**, Theorems 3.17 and 3.14])

1) *$Q_i^2=0$, and $Q_iQ_j+Q_jQ_i=0$.*
2) *Let $u,v\in\operatorname{Hom}_{DM_-^{eff}(k)}(M(X),\mathbb{Z}(*)[*'])$, for a smooth simplicial scheme X. Then $Q_i(u\cdot v)=Q_i(u)\cdot v+u\cdot Q_i(v)+\sum\{-1\}^{n_j}\phi_j(u)\cdot\psi_j(v)$, where $n_j>0$, and $\phi_j,\psi_j\in\mathcal{A}(k,\mathbb{Z}/2)$ are some (homogeneous) elements of bidegree (b,a), where $b>2a\geq 0$.*
3) *$Q_i=[\beta,q_i]$, where β is Bockstein, and $q_i\in\mathcal{A}(k,\mathbb{Z}/2)$.*

Following [**36**], we define:

Definition A.7. ([**36**, p.32]) Margolis motivic cohomology $\tilde{H}M_i^{b,a}(U)$ are cohomology groups of the complex: $\operatorname{Hom}_{\mathcal{SH}(k)}(U,\mathbf{H}_{\mathbb{Z}/2}(a-2^i+1)[b-2^{i+1}+1])\stackrel{Q_i}{\to}\operatorname{Hom}_{\mathcal{SH}(k)}(U,\mathbf{H}_{\mathbb{Z}/2}(a)[b])\stackrel{Q_i}{\to}\operatorname{Hom}_{\mathcal{SH}(k)}(U,\mathbf{H}_{\mathbb{Z}/2}(a+2^i-1)[b+2^{i+1}-1])$, for any $U\in Ob(\mathcal{SH}(k))$.

If U is $\operatorname{Cone}[-1](S(f))$, for some morphism $f:X\to Y$ of simplicial schemes, then by Theorems A.3 and A.5, $\tilde{H}M_i^{b,a}(U)$ coincides with the cohomology of the complex

$$\mathrm{H}_{\mathcal{M}}^{b-2^{i+1}+1,a-2^i+1}(M(U),\mathbb{Z}/2)\stackrel{Q_i}{\to}\mathrm{H}_{\mathcal{M}}^{b,a}(M(U),\mathbb{Z}/2)\stackrel{Q_i}{\to}\mathrm{H}_{\mathcal{M}}^{b+2^{i+1}-1,a+2^i-1}(M(U),\mathbb{Z}/2),$$

where $\mathrm{H}_{\mathcal{M}}^{d,c}(*,\mathbb{Z}/2):=\operatorname{Hom}_{DM_-^{eff}(k)}(*,\mathbb{Z}/2(c)[d])$, and $M(U)=\operatorname{Cone}[-1](M(f))$. Since $\tilde{H}M_i^{b,a}(\operatorname{Cone}[-1](S(f)))$ depends only on $M(f)$, we can denote it simply as $\tilde{H}M_i^{b,a}(\operatorname{Cone}[-1](M(f)))$.

Let P be some smooth projective variety over $\operatorname{Spec}(k)$.

Definition A.8. The *standard simplicial scheme* $\check{C}(P)^\bullet$, corresponding to the pair $P\to\operatorname{Spec}(k)$ is the simplicial scheme such that $\check{C}(P)^n=P\times\cdots\times P$ ($n+1$-times), with faces and degeneration maps given by partial projections and diagonals.

In $SmSimpl/k$ we have a natural projection: $pr:\check{C}(P)^\bullet\to\operatorname{Spec}(k)$. Let us denote $\mathcal{X}_P:=M(\check{C}(P)^\bullet)$. We get the natural map $M(pr):\mathcal{X}_P\to\mathbb{Z}$.

Theorem A.9. ([**36**, Lemma 3.8]) *If P has a k-rational point, then $M(pr):\mathcal{X}_P\to\mathbb{Z}$ is an isomorphism.*

Remark A.10. 1) In the notations of [**36**, Lemma 3.8], one should take $X=P$, $Y=\operatorname{Spec}(k)$, and observe that the simplicial weak equivalence gives an isomorphism on the level of *motives*. 2) Actually, $M(pr)$ is an isomorphism if and only if P has a 0-cycle of degree 1 (see [**33**, Theorem 2.3.4]).

Theorem A.9 shows that $\mathcal{X}_P|_{\overline{k}}=\mathbb{Z}$, which means that $\mathcal{X}_P$ is a *form* of the Tate-motive.

Theorem A.11. ([**36**, Theorem 4.4]) *Let P be an anisotropic projective quadrics of dimension n. Let N be a direct summand in $M(P)$ such that $N|_{\overline{k}}=\mathbb{Z}\oplus\mathbb{Z}(n)[2n]$. Then in $DM_-^{eff}(k)$ there exists a distinguished triangle of the form:*

$$\mathcal{X}_P(n)[2n]\to N\to\mathcal{X}_P\stackrel{\mu'}{\to}\mathcal{X}_P(n)[2n+1].$$

REMARK A.12. Actually, Theorem 4.4 of [**36**] is formulated only for the case of the *Rost motive* (as a direct summand in the motive of the *minimal Pfister neighbour*). But the proof does not use any specifics of the Pfister form case, and works with any "binary" direct summand of dimension $= dim(P)$. At the same time, Theorem A.11 is a very particular case of [**33**, Lemma 3.23].

THEOREM A.13. ([**36**, Lemma 3.8]) *The natural diagonal map* $\Delta_{\mathcal{X}_P} : \mathcal{X}_P \to \mathcal{X}_P \otimes \mathcal{X}_P$ *is an isomorphism.*

REMARK A.14. One should observe that the same proof as in [**36**, Lemma 3.8] gives the *simplicial weak equivalence* $\check{C}(P)^\bullet \times \check{C}(P)^\bullet \stackrel{pr_1}{\to} \check{C}(P)^\bullet$ with the inverse - the diagonal map.

THEOREM A.15. ([**36**, Lemma 4.7])
$M(pr)_* : \mathrm{Hom}(\mathcal{X}_P, \mathcal{X}_P(a)[b]) \to \mathrm{Hom}(\mathcal{X}_P, \mathbb{Z}(a)[b])$ *is an isomorphism, for any* a, b.

Let us denote: $\tilde{\mathcal{X}}_P := \mathrm{Cone}[-1](M(pr))$. Since $\tilde{\mathcal{X}}_P$ comes from $\mathcal{S}H(k)$, it makes sense to speak of the *Margolis cohomology* $\tilde{H}M_i^{b,a}(\tilde{\mathcal{X}}_P)$ of $\tilde{\mathcal{X}}_P$.

Suppose now that P be a smooth projective quadric of dimension $\geq 2^i - 1$.

The following result of V.Voevodsky is the main tool in studying motivic cohomology of quadrics:

THEOREM A.16. ([**36**, Theorem 3.25 and Lemma 4.11]) *Let* P *be a smooth projective quadric of dimension* $\geq 2^i - 1$, *then* $\tilde{H}M_i^{b,a}(\tilde{\mathcal{X}}_P) = 0$, *for any* a, b.

REMARK A.17. In [**36**, Lemma 4.11], the result is formulated only for the case of a $(2^i - 1)$-dimensional *Pfister quadric* (corresponding to the form $\langle\langle a_1, \dots, a_i\rangle\rangle \perp - \langle a_{i+1}\rangle)$. But the proof does not use any specifics of the *Pfister case* (the only thing which is used is: for any $j \leq i$, P has a plane section of dimension $2^j - 1$, which is again a quadric). Thus, for any quadric P of dimension $\geq 2^i - 1$, the ideal I_P contains a $(v_i, 2)$-element (notations from [**36**, Lemma 4.11]).

The following statement is a consequence of the Beilinson-Lichtenbaum Conjecture for $\mathbb{Z}/2$-coefficients.

THEOREM A.18. ([**36**, Proposition 2.7, Corollary 2.13(2) and Theorem 4.1]) *The map* $M(pr)^* : \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}/2(a)[b]) \to \mathrm{Hom}(\mathcal{X}_P, \mathbb{Z}/2(a)[b])$ *is an isomorphism for any* $b \leq a$.

REMARK A.19. We should add that in [**36**, Theorem 4.1] it is proven that the condition $H90(n, 2)$ is satisfied for all n and all fields of characteristic 0 - see p.11 of [**36**]. Also, $\mathrm{Hom}(M(X), \mathbb{Z}/2(a)[b])$ can be identified with $H_B^{b,a}(X, \mathbb{Z}/2)$.

Motivic cohomology of $\mathcal{X}_P$ can be used to compute the kernel on Milnor's K-theory ($mod\, 2$):

THEOREM A.20. ([**35**, Lemma 6.4], [**36**, Theorem 4.1] ; or [**12**, Theorem A.1]) *Let* $\tau \in \mathrm{Hom}_{DM_-^{eff}(k)}(\mathbb{Z}/2, \mathbb{Z}/2(1)) = \mathbf{Z}/2$ *be the only nontrivial element. Let* P *be a smooth projective quadric over* k. *Then the composition*

$$\mathrm{Hom}(\mathcal{X}_P, \mathbb{Z}/2(m-1)[m]) \stackrel{\tau\circ}{\to} \mathrm{Hom}(\mathcal{X}_P, \mathbb{Z}/2(m)[m]) = \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}/2(m)[m]) = \mathrm{K}_m^M(k)/2$$

identifies the first group with the $\ker(\mathrm{K}_m^M(k)/2 \to \mathrm{K}_m^M(k(P))/2)$.

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