Direct summands in the motives of quadrics
*Preliminary version.*
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# 1 Introduction

In this paper we will investigate the structure of undecomposable direct summands in the motives of quadrics. We will work mostly in the category of *Chow motives* $Chow(k)$ (see, for example, [4], 2.2), but since we rely heavily on the results from [4], we will also be using bigger *triangulated category of mixed motives* $D_{-}^{eff.}(k)$ of V.Voevodsky (see [5]) (though, in most cases it will be done just to make terminology compatible with that of [4]). By the latest reason our considerations are restricted to the case of characteristic 0.

Let $Q$ be $n$-dimensional quadric over the field $k$. If $Q$ is *hyperbolic*, then the *Chow motive* of $Q$ is very simple - it is isomorphic to the direct sum of *Tate motives* $\oplus_{i=0,\ldots,n}\mathbb{Z}(i)[2i]$ $(\oplus\mathbb{Z}(n/2)[n]$ if $n$ is even), so, all undecomposable direct summands in this case are given by the *Tate motives* above. In particular, this happens if $k$ is algebraically closed.

For arbitrary $Q$ and $k$ the situation is more delicate, but it follows from the Rost Nilpotence Theorem (see [3], Proposition 9) (see also [4], Lemma 3.10) that for any direct summand $N$ in the motive of $Q$, $N$ is defined up to isomorphism by it's restriction to $\overline{k}$ (see [4], Lemma 3.21), and this restriction is isomorphic to $\oplus_{i\in I(N)}\mathbb{Z}(i)[2i]$, so $N$ is defined up to isomorphism by the set $I(N)$ (plus the $Q$ itself). The natural problem arises - to describe possible sets $I(N)$ for various direct summands $N$.

It appears that if $N$ is undecomposable, then $I(N)$ has a symmetry, coming from the isomorphism $\underline{\mathrm{Hom}}(N,\mathbb{Z}(\dim(N))[2\dim(N)]) \backsimeq N$ (see Corollary 1 ), and, consequently, consists of even number of elements (if $Q$ is anisotropic). Moreover, there is interaction between the *splitting pattern* of $Q$ (or, the set of *higher Witt indices*) and the possible decomposition of it's motive - see the Statement . The Statement can be also used to answer the question: "When subform $p\subset q$ is isotropic over $k(Q)$?"; as one could expect, this happens iff $codim(p\subset q)<\mathbf{i}_1(q)$, where $\mathbf{i}_1$ is 1-st *higher Witt index* - see Corollary 3 . Another interesting question is - to describe minimal elements $i$ of $I(N)$ for all possible $N$ (ans so, describe the number of undecomposable direct summands in the motive of $Q$). Using Statement , we show in Proposition 1 that if there exists such quadric $P$, that $p$ is isotropic if and only if $q$ is $i+1$ times isotropic, then $i$ is minimal for some set $I(N)$. We believe, that this condition should be also nesessary - see Question 1 . Finally, we improve Lemma 4.5 and Proposition 3.4 from [4] - see Corollary 2 and Corollary 4 .

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# 2

We need to make some brief introduction into the terminology of [4].

In the Voevodsky's category $DM_{-}^{eff}(k)$ we have not only the motives of all smooth projective varieties (as in $Chow(k)$), but also of all smooth simplicial schemes. If $P/k$ is smooth variety (of finite type) over $k$, then we denote as $\mathcal{X}_P$ the smooth simplicial scheme with $\mathcal{X}_P^n = P \times_k P \times_k \cdots \times_k P$ - $n+1$-times, where the maps of faces and degenerations are given by partial projections and partial diagonals. Also we will denote in the same way the image of $\mathcal{X}_P$ in $DM_{-}^{eff.}$ (actually, we will denote motives of all smooth varieties and simplicial schemes in the same way as objects themselves, so, omitting $M(-)$). As soon as $P$ has a rational point (or even the 0-cycle of degree 1), the motive $\mathcal{X}_P$ is isomorphic to the trivial *Tate motive* $\mathbb{Z}$. In particular, over $\overline{k}$, the motive $\mathcal{X}_P$ will be isomorphic to $\mathbb{Z}$, so we can say that $\mathcal{X}_P$ is a *form* of $\mathbb{Z}$. More generally, for two smooth (connected) varieties $P$ and $R$ we have: the motives $\mathcal{X}_P$ and $\mathcal{X}_R$ are isomorphic if and only if $R$ has a 0-cycle of degree 1 over $k(P)$ and $P$ has a 0-cycle of degree 1 over $k(R)$ (see [4], 2.3 for details). The last statement justifies the following notation: we will write $\mathcal{X}_P \geqslant \mathcal{X}_R$ if $R$ has a 0-cycle of degree 1 over $k(P)$.

If we have *triangulated* category $\mathcal{D}$ and $X, Y, Z \in \mathcal{D}$, then we say that $Z$ is an *elementary extension* of $X$ and $Y$ iff there exists an *exact* triangle either of the form $X \to Z \to Y \to X[1]$, or of the form $Y \to Z \to X \to Y[1]$. If we have objects $X_1, \ldots, X_m, Z \in \mathcal{D}$, then (inductively) $Z$ is called an *extension* of $\{X_j\}_{1 \leqslant j \leqslant m}$, iff there exist $1 \leqslant i \leqslant m$ and an exact triangle either of the form $X_i \to Z \to Y \to X_i[1]$, or of the form $Y \to Z \to X_i \to Y[1]$, s.t. $Y$ is an *extension* of $\{X_j\}_{j \neq i, 1 \leqslant j \leqslant m}$.

If $Q$ is a quadric of dimension $n$, then we can consider the smooth projective varieties $Q^i$ of $i$-dimensional projective planes on $Q$, $i = 0, \ldots, [n/2]$. The Theorem 3.1 from [4] states that the motive of $Q$ is an *extension* (in the sense specified above, with $\mathcal{D} = DM_{-}^{eff.}(k)$) of the motives $\mathcal{X}_Q$, $\mathcal{X}_{Q^1}(1)[2]$,..., $\mathcal{X}_{Q^{[n-1/2]}}([n-1/2])[2[n-1/2]]$, $\mathcal{X}_{Q^{[n-1/2]}}(n-[n-1/2])[2n-2[n-1/2]]$, ..., $\mathcal{X}_{Q^1}(n-1)[2n-2]$, $\mathcal{X}_Q(n)[2n]$ (plus $k\sqrt{det(Q)} \times \mathcal{X}_{Q^{[n/2]}}$, if $n$ is even). Moreover, in the abovementioned Theorem 3.1 it is specified in which order the elementary pieces appear, so (the motive of) $Q$ is a *total object* in some *Postnikov system* with *graded parts* as above. This *Postnikov system* appears to be compatible with the ring of endomorphisms of the motive $Q$, i.e. any endomorphism of $Q$ extends uniquely to the endomorphism of the whole system (see [4], Theorem 3.7). This shows that for any direct summand $N$ in $Q$, the corresponding projector $p$ in $\mathrm{End}_{DM_{-}^{eff.}}(Q)$ gives us the decomposition of the *Postnikov system* into a direct sum of two, and provides us with the projectors $p_i \in \mathrm{End}_{DM_{-}^{eff.}}(\mathcal{X}_{Q^i}(i)[2i])$ and $p'_j \in \mathrm{End}_{DM_{-}^{eff.}}(\mathcal{X}_{Q^j}(n-j)[2n-2j])$. But for arbitrary smooth $P/k$, we have $\mathrm{End}_{DM_{-}^{eff.}}(\mathcal{X}_P) = \mathbb{Z}$ (see [4], Theorem 2.3.2, Theorem 2.3.3 (1)). This shows that $p_i$ and $p'_j$'s are either $0 \cdot Id$, or $1 \cdot Id$, and $N$ is an *extension* of some number of "elementary pieces" from the same set: $\mathcal{X}_Q$, $\mathcal{X}_{Q^1}(1)[2]$, ... , $\mathcal{X}_{Q^1}(n-1)[2n-2]$, $\mathcal{X}_Q(n)[2n]$ (namely, those ones, for which the corresponding projector ($p_i$ or $p'_j$) is identity). Over $\overline{k}$, these "elementary pieces" are becoming *Tate motives*, and their weights give you the set $I(N)$, i.e.: $I(N) = (\cup_{p_i=1} i) \cup (\cup_{p'_j=1} n-j) \cup (n/2,$ taken $0, 1$ or $2$ times, if $n$ is even$)$. This permits one to translate from *Chow-motivic* terminology to that of [4].

# 3

For a quadric $Q$ over $k$ we will define (following M.Knebusch, see [1], Definition 5.4) it's *higher Witt indices* $\mathbf{i}_0(q), \ldots, \mathbf{i}_s(q)$ inductively in the following way: $q_0 := q$, $k_0 := k$, $\mathbf{i}_t(q) := i_W(q_t)$, $k_{t+1} := k_t((q_t)_{anis.})$, and $q_{t+1} := ((q_t)_{anis.})|_{k_{t+1}}$.

Let $Q$ be a quadric of dimension $n$, which has *higher Witt indices* $\mathbf{i}_1, \ldots, \mathbf{i}_s$. Denote $\bar{\mathbf{i}} := (\mathbf{i}_1, \ldots, \mathbf{i}_s)$. For $0 \leqslant i < n/2$, let $1 \leqslant j(i, \bar{\mathbf{i}}) \leqslant s$ be such that $\mathbf{i}_1 + \cdots + \mathbf{i}_{j(i,\bar{\mathbf{i}})-1} \leqslant i < \mathbf{i}_1 + \cdots + \mathbf{i}_{j(i,\bar{\mathbf{i}})}$, and let $i^{\perp}_{\bar{\mathbf{i}}} := 2(\mathbf{i}_1 + \cdots + \mathbf{i}_{j(i,\bar{\mathbf{i}})-1}) + \mathbf{i}_{j(i,\bar{\mathbf{i}})} - i$. Clearly $j(i, \bar{\mathbf{i}}) = j(i^{\perp}_{\bar{\mathbf{i}}}, \bar{\mathbf{i}})$.

Let $N$ be a direct summand in the motive of $Q$, and $0 \leqslant i < n/2$. We say that "$N$ contains $(i)$" iff $N$ contains $\mathcal{X}_{Q^i}(i)[2i]$ as an elementary piece in the sense of [4], Lemma 3.23. Similarly, we say that "$N$ contains $(i)'$ " iff it contains $\mathcal{X}_{Q^i}(n-i)[2n-2i]$ as an elementary piece. Certainly, the above two conditions can be reformulated as: "$I(N)$ contains $i$" and "$I(N)$ contains $n-i$", respectively.

The main restriction on the structure of $N$, which we use in this paper, is provided by the following:

Statement . *Suppose $Q$ is anisotropic.*

(1) *Let $N$ be a direct summand in $Q$. Then $N$ contains $(i)$ iff it contains $(i^{\perp}_{\bar{\mathbf{i}}})'$*

(2) *Suppose $N$ is undecomposable. If $N$ contains $(i)$, but does not contain any $(m)$, $0 \leqslant m < i$, then $N$ contains $(i^{\perp}_{\bar{\mathbf{i}}})'$ and does not contain any $(l)'$, $0 \leqslant l < i^{\perp}_{\bar{\mathbf{i}}}$.*

*Proof of the Statement*

(1) First of all, we can change $k$ by $k(Q^{\mathbf{i}_1+\cdots+\mathbf{i}_{j(i,\bar{\mathbf{i}})-1}-1})$, and assume that $j(i,\bar{\mathbf{i}})=1$ (i.e., $\mathcal{X}_{Q^i}=\mathcal{X}_Q$).

By [4], Lemma 4.5, there are undecomposable direct summands $M(v)[2v]$, $0 \leqslant v \leqslant \mathbf{i}_1-1$ of $Q$, s.t. $M(v)[2v]$ *contains* $(v)$. It is enough to prove the statement for $N=M(i)[2i]$, or which is the same, for $M$ and $i=0$.

It is clear, that $M$ can't contain $(m)'$, with $m<\mathbf{i}_1-1$ (look on $M(\mathbf{i}_1-1)[2\mathbf{i}_1-2]$). So, if $M$ does not *contain* $(\mathbf{i}_1-1)'$, that means that it does not contain any $\mathcal{X}_{Q^m}(n-m)[2n-2m]$ with $\mathcal{X}_{Q^m}=\mathcal{X}_Q$. Suppose it is the case.

Let $S$ be a plane section of $Q$ of codimension $\mathbf{i}_1-1$. Since $\mathcal{X}_S=\mathcal{X}_Q=\mathcal{X}_{Q^{\mathbf{i}_1-1}}$, we have a map $\varphi: S(\mathbf{i}_1-1)[2\mathbf{i}_1-2]\rightarrow Q$, s.t, over $\overline{k}$, $\varphi|_{\overline{k}}$ maps $(\mathbb{Z})(\mathbf{i}_1-1)[2\mathbf{i}_1-2]$ to $\mathbb{Z}(\mathbf{i}_1-1)[2\mathbf{i}_1-2]$ isomorphically. Let $\psi: Q\rightarrow S(\mathbf{i}_1-1)[2\mathbf{i}_1-2]$ is the map given by the cycle "$S$", embedded "diagonally" to $Q\times S$ (the map, dual to the embedding). Clearly, $\psi|_{\overline{k}}$ maps $\mathbb{Z}(\mathbf{i}_1-1)[2\mathbf{i}_1-2]$ to $(\mathbb{Z})(\mathbf{i}_1-1)[2\mathbf{i}_1-2]$ isomorphically. By [4], Lemma 3.26, it follows that $S(\mathbf{i}_1-1)[2\mathbf{i}_1-2]$ contains a direct summand, isomorphic to $M(\mathbf{i}_1-1)[2\mathbf{i}_1-2]$, i.e., $S$ contains one isomorphic to $M$. And, by our assumption, $M$ does not contain any $\mathcal{X}_{S^m}(n'-m)[2n'-2m]$ with $\mathcal{X}_{S^m}=\mathcal{X}_S$ ($n'=n-\mathbf{i}_1+1$ is the dimension of $S$). Since, $\dim(S)<\dim(Q)$, repeating this procedure, if nesessary, we get a quadric $S'$, which contains a direct summand, isomorphic to $M$, and $\mathbf{i}_1(S')=1$ (in particular, $M$ does not *contain* $(0)'_{S'}$).

So, finally, we can assume, that $\mathbf{i}_1(Q)=1$, $M$ *contains* $(0)$, but not a $(0)'$.

Lemma 1 .

*Let $N$ be a direct summand of $Q$. Then the natural map $Q \times N \to N$ has a splitting $N \to Q \times N$, i.e.: $N$ is a direct summand of $Q \times N$.*

*Proof of the Lemma 1*

We have an *exact triangle* in $DM^{eff}(k)$: $R^1 \to Q \to \mathcal{X}_Q \to R^1[1]$.

From this we get an *exact triangle*: $R^1 \times N \to Q \times N \to \mathcal{X}_Q \times N \to R^1[1] \times N$. We have: $\mathcal{X}_Q \times N = N$ (since $\mathcal{X}_Q \times Q = Q$, and $N$ is a direct summand in $Q$).

$R^1[1] \times N$ is a direct summand in $R^1[1] \times Q$ and the later is the direct summand in $Q[1] \times Q$ (since the map $Q \times Q \to \mathcal{X}_Q \times Q = Q$ has a splitting - the *diagonal*).

So, $\mathrm{Hom}(N, R^1[1] \times N)$ is a subgroup in $\mathrm{Hom}(Q, Q[1] \times Q) = \mathrm{Hom}(Q \times Q \times Q, \mathbb{Z}(2n)[4n+1]) = 0$, since $Q \times Q \times Q$ is a smooth projective variety and $4n + 1 > 2(2n)$.

Lemma 1 is proven.

□

Let's take $N = M^{\vee}$ - dual to $M$ via duality $\underline{\mathrm{Hom}}(-, \mathbb{Z}(n)[2n])$.

Since $M$ does not *contain* $(0)'$, but *contains* $(0)$, we have that $N$ does not *contain* $(0)$, but *contains* $(0)'$. In particular, $N$ is a direct summand in $R^1$ (notations as above) (since for corresponding projector $p_N$ we have $(p_N)_0 = 0$). But $Q \times R^1 = \underline{Q^1}(1)[2] \oplus Q(n)[2n]$ (since we have an exact triangle $R^1 \to Q\langle 1 \rangle(1)[2] \to \mathcal{X}_Q(n)[2n+1] \to R^1[1]$, the composition $Q(n)[2n] = Q \times \mathcal{X}_Q(n)[2n] \to Q \times R^1 \to Q \times Q$ has a splitting - the map dual to the *diagonal* via duality $\underline{\mathrm{Hom}}(-, \mathbb{Z}(2n)[4n])$, and $Q \times Q\langle 1 \rangle$ is isomorphic to $\underline{Q^1}$ - see [4], Claim 3.2). So, by Lemma 1 , $N$ is a direct summand in $\underline{Q^1}(1)[2] \oplus Q(n)[2n]$. Let $\rho_1 : N \to \underline{Q^1}(1)[2]$, $\rho_2 : N \to Q(n)[2n]$, and $\pi_1 : \underline{Q^1}(1)[2] \to N$, $\pi_2 : Q(n)[2n] \to N$ be corresponding maps. So, $\pi_1 \circ \rho_1 + \pi_2 \circ \rho_2 = id_N$. As we know, $N$ contains $\mathcal{X}_Q(n)[2n]$. Since $Q$ is anisotropic, over $\overline{k}$, $\rho_2|_{\overline{k}}$ should send $\mathbb{Z}(n)[2n]$ to $\mathbb{Z}(n)[2n]$ via multiplication by an even number (otherwise, the composition $\mathbb{Z}(n)[2n] \to Q \stackrel{proj.}{\to} N \stackrel{\rho_2}{\to} Q(n)[2n]$ would give us a 0-cycle of odd degree on $Q$). So, over $\overline{k}$, $(\pi_1 \circ \rho_1)|_{\overline{k}}$ should act on $\mathbb{Z}(n)[2n]$ via multiplication by an odd number. Let $K = k(Q)$. Then over $K$, $q|_K = \mathbb{H} \perp p$, where $\mathbb{H}$ is hyperbolic plane and $p/K$ is anisotropic (since $\mathbf{i}_1(Q) = 1$). Hence, $\underline{Q^1}|_K = P \oplus P(1)[2] \oplus \underline{P^1}(2)[4] \oplus P(n-1)[2n-2] \oplus P(n)[2n]$. Also, $N_K = N' \oplus \mathbb{Z}(n)[2n]$, and we get the maps: $\alpha : \mathbb{Z}(n)[2n] \to P \oplus P(1)[2] \oplus \underline{P^1}(2)[4] \oplus P(n-1)[2n-2] \oplus P(n)[2n]$ and $\beta : P \oplus P(1)[2] \oplus \underline{P^1}(2)[4] \oplus P(n-1)[2n-2] \oplus P(n)[2n] \to \mathbb{Z}(n)[2n]$, s.t. $\beta \circ \alpha : \mathbb{Z}(n)[2n] \to \mathbb{Z}(n)[2n]$ is a multiplication by an odd number.

Lemma 2 .

*Let $P/K$ be anisotropic quadric, and $0 \leqslant m \leqslant \dim(P)/2$. Suppose for some $l$ we have maps: $\alpha : \mathbb{Z}(l)[2l] \to \underline{P^m}$, $\beta : \underline{P^m} \to \mathbb{Z}(l)[2l]$. Then the composition $\beta \circ \alpha : \mathbb{Z}(l)[2l] \to \mathbb{Z}(l)[2l]$ is a multiplication by an even number.*

*Proof of the Lemma 2*

We have the natural identification: $\mathrm{Hom}(\mathbb{Z}(l)[2l], \underline{P^m}) = \mathrm{CH}_l(\underline{P^m})$, and $\mathrm{Hom}(\underline{P^m}, \mathbb{Z}(l)[l]) = \mathrm{CH}^l(\underline{P^m})$, and if $\alpha$ is represented by a cycle $A$, and $\beta$ by cycle $B$, then the composition $\beta \circ \alpha : \mathbb{Z}(l)[2l] \to \mathbb{Z}(l)[2l]$ is a multiplication by the degree of the intersection $A \cap B \in CH_0(\underline{P^m})$. If this number would be odd, then we would have a point of odd degree on $\underline{P^m}$, and, because of the natural projection $\underline{P^m} \to P$, also on $P$. By Springer's theorem, we then would have a rational point on $P$ - contradiction. So, the degree of $A \cap B$ is even.

Lemma 2 is proven.

□

Using Lemma 2 , we get a contradiction. So, (1) is proven.

(2) follows from (1) applied to $N$ and $N^{\vee}$ (the dual to $N$ via duality $\underline{\mathrm{Hom}}(-, \mathbb{Z}(n)[2n],\ n = \dim(Q))$. Really, clearly, by (1), $N$ will contain $(i^{\perp}_{\bar{\mathbf{i}}})'$. Let $N$ *contains* $(l)'$, where $l < i^{\perp}_{\bar{\mathbf{i}}}$. Let $i_1 = l^{\perp}_{\bar{\mathbf{i}}}$. We have: $(i^{\perp}_{\bar{\mathbf{i}}})^{\perp}_{\bar{\mathbf{i}}} = i$, and if $j(i, \bar{\mathbf{i}}) = j(i^{\perp}_{\bar{\mathbf{i}}}, \bar{\mathbf{i}}) > j(l, \bar{\mathbf{i}})$, then $i > i_1$, and so, $N^{\vee}$, *containing* $(l)$, by (1), should *contain* also $(i_1)'$ (i.e., $N$ *contains* $(i_1)$), which is not the case by the condition. So, $j(i, \bar{\mathbf{i}}) = j(i^{\perp}_{\bar{\mathbf{i}}}, \bar{\mathbf{i}}) = j(l, \bar{\mathbf{i}})$. Now, we can change $k$ by $K = k(Q^{\mathbf{i}_1 + \cdots + \mathbf{i}_{j(i,\bar{\mathbf{i}})-1} - 1})$, and assume that $j(i, \bar{\mathbf{i}}) = 1$. Then we have by [4], Lemma 4.5, that for each $0 \leqslant u \leqslant \mathbf{i}_1 - 1$, $(u)$ is contained in an undecomposable direct summand isomorphic to $N(u - i)[2u - 2i]$. Taking $u = \mathbf{i}_1 - 1$, and applying (1), we get that $l$ can't be $< i$.

Statement is proven.

□

From the result above we immediately get that undecomposable direct summands should have some kind of symmetry (the simplest consequence of which is: if $Q$ is anisotropic, and $N$ undecomposable, then $N|_{\overline{k}}$ consists of even number of *Tate motives*).

Corollary 1 .

*Let $N$ be an undecomposable direct summand in the motive of anisotropic quadric $Q$. Let lowest term of $N$ is $(i)$, and highest - $(j)'$. Let $M = N(-i)[-2i]$. Define the dimension of $M$ as $d = n - i - j$, where $n = \dim(Q)$. Then $\underline{\mathrm{Hom}}(M, \mathbb{Z}(d)[2d])$ is isomorphic to $M$.*

*Proof of Corollary 1*

Clearly, to prove this statement for $N$ is the same as to prove it for $N^{\vee} = \underline{\mathrm{Hom}}(N, \mathbb{Z}(n)[2n])$. Evidently, $j = i^{\perp}_{\bar{\mathbf{i}}}$, by Statement (2). Changing $N$ by $N^{\vee}$, if nesessary, we can assume that $i \leqslant i^{\perp}_{\bar{\mathbf{i}}}$. Take $S$ to be a plane section of codimension $l = i^{\perp}_{\bar{\mathbf{i}}} - i$. We have: $\mathcal{X}_{Q^i} = \mathcal{X}_{Q^{i+l}} = \mathcal{X}_{Q^{i^{\perp}_{\bar{\mathbf{i}}}}}$. By [4], Lemma 4.5, $Q$ contains direct summand isomorphic to $N(l)[2l]$. Let $\varphi : S \to Q$ be map, given by the embedding $S \subset S \times Q$, and $\psi : Q \to S(l)[2l]$ be a map dual to it (via duality $\underline{\mathrm{Hom}}(-, \mathbb{Z}(n)[2n])$). Let $\rho_N : N \to Q$, $\rho_{N(l)[2l]} : N(l)[2l] \to Q$, and $\pi_N : Q \to N$, $\pi_{N(l)[2l]} : Q \to N(l)[2l]$ be maps defining $N$ and $N(l)[2l]$ as direct summands in $Q$. Consider $\varepsilon := \psi(-l)[-2l] \circ \rho_{N(l)[2l]}(-l)[-2l] \circ \pi_N : Q \to S$. It is easy to see, that over $\overline{k}$, $\varphi \circ \varepsilon : Q \to Q$ maps $\mathbb{Z}(i)[2i]$ to itself isomorphically. Hence, by [4], Lemma 3.26, $S$ contains a direct summand $N_1$ isomorphic to $N$. The lowest term of $N_1$ is $\mathcal{X}_{S^i}(i)[2i]$, and the highest term should be $\mathcal{X}_{S^i}(n_1 - i)[2n_1 - i]$, where $n_1 = n - l = \dim(S)$ (really, $i = i^{\perp}_{\bar{\mathbf{i}}} - l$). Since $N_1$ *contains* $(i)$ and $(i)'$, $N_1^{\vee}$ (dual to $N_1$ via duality $\underline{\mathrm{Hom}}(-, \mathbb{Z}(n_1)[2n_1])$) should also contain $(i)$ and $(i)'$. By [4], Lemma 3.21, that means that $N_1^{\vee}$ is isomorphic to $N_1$, i.e.: $\underline{\mathrm{Hom}}(M, \mathbb{Z}(d)[2d])$ is isomorphic to $M$.

□

Also, we can improve a bit the Lemma 4.5 from [4].

Corollary 2 .

*Let $N$ be an undecomposable direct summand in $Q$, with the lowest term $(i)$. Then for any $m$, s.t. $\mathcal{X}_{Q^m} = \mathcal{X}_{Q^i}$, there exist an undecomposable direct summand of $Q$, isomorphic to $N(m-i)[2m-2i]$.*

*Proof of Corollary 2* By Statement (2), the highest term of $N$ will be $(i^{\perp}_{\bar{\mathbf{i}}})'$. Since $\mathcal{X}_{Q^i} = \mathcal{X}_{Q^{i^{\perp}_{\bar{\mathbf{i}}}}}$, it is equivalent to prove the statement for $N$ or for $N^{\vee} = \underline{\mathrm{Hom}}(N, \mathbb{Z}(n)[2n])$. Changing $N$ by $N^{\vee}$, if nesessary, we can assume that $i \leqslant i^{\perp}_{\bar{\mathbf{i}}}$. Then $Q\langle i\rangle'$ contains $N$, and from [4], Lemma 4.5 follows the statement for required $m \geqslant i$. Moreover, if $r = \mathbf{i}_1 + \cdots + \mathbf{i}_{j(i,\bar{\mathbf{i}})} - 1$, then $N(r-i)[2r-2i]$ *contains* $(r)$ and $(\mathbf{i}_1 + \cdots + \mathbf{i}_{j(i,\bar{\mathbf{i}})-1})'$. Again, applying [4], Lemma 4.5 to $(N(r-i)[2r-2i])^{\vee}$ (and then, dualizing back), we get the statement for required $m < i$.

Corollary 2 is proven.

□

One more application of the Statement explains, in which cases the subform of $q$ will be isotropic over the generic point of $Q$, and computes the 1-st *higher Witt index* for such subforms in terms of that for $q$.

Corollary 3 .

(1) *Let $P$ and $Q$ be such anisotropic quadrics that $\mathcal{X}_P = \mathcal{X}_Q$ (in other words, $P$ has a rational point over $k(Q)$, and $Q$ has a rational point over $k(P)$. Then* $\dim(P) - \mathbf{i}_1(P) = \dim(Q) - \mathbf{i}_1(Q)$.

(2) *Let $Q$ be anisotropic quadric, and $P \subset Q$ - a subquadric of codimension $i$. Then the following conditions are equivalent:*

  (a) $\mathcal{X}_Q = \mathcal{X}_P$.

  (b) $0 \leqslant i < \mathbf{i}_1(Q)$.

  *Moreover, if these conditions are satisfied, then* $\mathbf{i}_1(P) = \mathbf{i}_1(Q) - i$.

*Proof of Corollary 3* (1) Since $\mathcal{X}_P = \mathcal{X}_Q$, we have direct summands $N$ of $Q$, and $M$ of $P$, s.t. $N$ *contains* $\mathcal{X}_Q$, $M$ contains $\mathcal{X}_P$, and $M \simeq N$. To construct such summands, consider rational maps: $f : Q \to P$, $g : P \to Q$. Closure of their graphs in $Q \times P$ and $P \times Q$, respectively, gives us motivic maps $\phi : Q \to P$ and $\psi : P \to Q$, s.t. $\phi$ and $\psi$, restricted to $\overline{k}$ map $\mathbb{Z}$ to $\mathbb{Z}$ isomorphically. By [4], Lemma 3.26, this implies the existence of specified motives $M$ and $N$. From Statement it follows, that the *dimension* (see Corollary 1 ) of $N$ is $\dim(Q) - \mathbf{i}_1(Q) + 1$, and the dimension of $M$ is $\dim(P) - \mathbf{i}_1(Q) + 1$. The isomorphism $M \simeq N$ completes the proof.

(2) Clearly, from the existence of rational point on $P$ follows the existence of such on $Q$. Hence, $\mathcal{X}_P \geqslant \mathcal{X}_Q$. Also, from the existence of $i$-dimensional projective subspace on $Q$ follows the existence of rational point on $P$. Hence, $\mathcal{X}_{Q^i} \geqslant \mathcal{X}_P \geqslant \mathcal{X}_Q$.

If $0 \leqslant i < \mathbf{i}_1(Q)$, then $\mathcal{X}_{Q^i} = \mathcal{X}_Q$, and from the above unequality we get: $\mathcal{X}_Q = \mathcal{X}_P$. So, $(b) \Rightarrow (a)$.

In the other direction: if $\mathcal{X}_Q = \mathcal{X}_P$, then by (1), $\dim(P) - \mathbf{i}_1(P) = \dim(Q) - \mathbf{i}_1(Q)$. Since $\mathbf{i}_1(P) \geqslant 1$, we get: $i \leqslant \mathbf{i}_1(Q)$.

The last statement is evident in the light of (1).

□

The following interesting question in the study of direct summands of $Q$ arises: for which $i$ we have a direct summand $N$, "starting" from $(i)$ (that is: $N|_{\overline{k}}$ contains $\mathbb{Z}(i)[2i]$, but does not contain any $\mathbb{Z}(j)[2j]$ with $j < i$)? We can give here some sufficient condition (see Proposition 1 below), which, we believe, should be also nesessary one (see Question 1 ). Our Statement is very useful here.

Lemma 3 .

*Let $Q/k$ be some quadric, and $K/k$ be some field, such that $K$ has a smooth point over $k(Q)$. Then $\bar{\mathbf{i}}(Q/K) = \bar{\mathbf{i}}(Q/k)$.*

*proof of Lemma 3*

We just need to check, that over $K(Q^{\mathbf{i}_1+\cdots+\mathbf{i}_t-1})$, $Q^{\mathbf{i}_1+\cdots+\mathbf{i}_t}$ has no *smooth point* for any $1 \leqslant t \leqslant s$. But $Q$ has a *smooth point* over $k(Q^{\mathbf{i}_1+\cdots+\mathbf{i}_t-1})$, and $K$ has a *smooth point* over $k(Q)$. By *transitivity*, from the existence of a $K(Q^{\mathbf{i}_1+\cdots+\mathbf{i}_t-1})$-point on $Q^{\mathbf{i}_1+\cdots+\mathbf{i}_t}$ would follow the existence of $k(Q^{\mathbf{i}_1+\cdots+\mathbf{i}_t-1})$-point there, which is not the case.

Lemma 3 is proven.

□

Lemma 4 .

*Let $Q$ be a quadric, and $\mathbf{i}_1, \ldots, \mathbf{i}_s$ be it's higher Witt indices. Let $1 \leqslant t \leqslant s$, and $S$ be a plane section of $Q$ of codimension $\mathbf{i}_t(Q)-1$. Let $i = \mathbf{i}_1+\cdots+\mathbf{i}_{t-1}$. Then $Q$ contains an undecomposable direct summand with "lowest" term $(i)$ iff $S$ does.*

*Proof of Lemma 4*

Let $\dim(Q) = n$.

$(\rightarrow)$ If $Q$ contains such a summand $N$, then by [4], Lemma 4.5, it contains also one isomorphic to $N(\mathbf{i}_t(Q)-1)[2\mathbf{i}_t(Q)-2]$, with lowest term $(i+\mathbf{i}_t-1)$ (since $\mathcal{X}_{Q^{i-1}} \neq \mathcal{X}_{Q^i} = \cdots = \mathcal{X}_{Q^{i+\mathbf{i}_t(Q)-1}}$). Let $\varphi : S \rightarrow Q$ be a map, corresponding to the inclusion $S \subset Q$, and $\varphi^{\vee} : Q(-\mathbf{i}_t(Q)+1)[-2\mathbf{i}_t(Q)+2] \rightarrow S$ be the dual map via duality $\underline{\mathrm{Hom}}(-, \mathbb{Z}(n-\mathbf{i}_t+1)[2n-2\mathbf{i}_t+2])$. Let $j_N : N \rightarrow Q$, $j_{N(\mathbf{i}_t-1)[2\mathbf{i}_t-2]} : N(\mathbf{i}_t-1)[2\mathbf{i}_t-2] \rightarrow Q$, and $\pi_N : Q \rightarrow N$, and $\pi_{N(\mathbf{i}_t-1)[2\mathbf{i}_t-2]} : Q \rightarrow N(\mathbf{i}_t-1)[2\mathbf{i}_t-2]$ be maps, realizing $N$ and $N(\mathbf{i}_t-1)[2\mathbf{i}_t-2]$ as direct summands in $Q$. Then we have the pair of maps: $\varphi : S \rightarrow Q$, and $\psi := \varphi^{\vee} \circ j_{N(\mathbf{i}_t-1)[2\mathbf{i}_t-1]}(1-\mathbf{i}_t)[2-2\mathbf{i}_t] \circ \pi_N : Q \rightarrow S$. It is easy to see, that for $\psi \circ \varphi : S \rightarrow S$ we have: $(\psi \circ \varphi)_i = 1$ (see [4], Theorem 3.7 for notations). That means (see [4], Lemma 3.26) that $Q$ and $S$ contain isomorphic direct summands, *containing* $(i)$. Such summands should be isomorphic to $N$, so the $(i)$ is the "lowest" term in them.

$(\leftarrow)$ If $S$ contains such a summand $M$, then $M(-i)[-2i]$ is a direct summand in $\underline{S^i}$ (see [4], Claim 3.2 and Lemma 4.6). Really, we need only to check, that $M$ "is contained" in $S\langle i\rangle'$, i.e.: $i \leqslant i^{\perp}_{\bar{\mathbf{i}}(S)}$ (by the Statement ). Let

$K = k(Q^{i-1})$. Suppose $M$ is not contained in $S\langle i\rangle'$, then $i > i^{\perp}_{\bar{\mathbf{i}}(S)}$, and by Lemma 3 , $i > i^{\perp}_{\bar{\mathbf{i}}(S|_K)}$.

But over $K$, $Q|_K$ is $(i-1)$-times isotropic. Then $Q$ contains undecomposable direct summand $N$, whose "lowest" term is $(i)$, then, by the Statement (2), the "highest" term of $N$ will be $(i+\mathbf{i}_t-1)'$. By $(\rightarrow)$, $S|_K$ also contains direct summand $M'$ isomorphic to $N$, and $M'$ contains $(i)$ and $(i)'$ as it's "lowest" and "highest" terms. By the statement, we get contradiction with the assumption that $i > i^{\perp}_{\bar{\mathbf{i}}(S|_K)}$. So, $M(-i)[-2i]$ is a direct summand in $\underline{S^i}$.

Since $\mathcal{X}_{\underline{S^i}} = \mathcal{X}_{S^i} = \mathcal{X}_{Q^i} = \mathcal{X}_{Q^{i+\mathbf{i}_t-1}}$, we have the map $\varepsilon' : \underline{S^i}(i+\mathbf{i}_t-1)[2i+2\mathbf{i}_t-2] \rightarrow Q$, which over $\overline{k}$ maps $(\mathbb{Z})(i+\mathbf{i}_t-1)[2i+2\mathbf{i}_t-2]$ to $\mathbb{Z}(i+\mathbf{i}_t-1)[2i+2\mathbf{i}_t-2]$ isomorphicallyically. Since $M(-i)[-2i]$ is a direct summand in $\underline{S^i}$, we get a map $\varepsilon'' : M(\mathbf{i}_t-1)[2\mathbf{i}_t-2] \rightarrow Q$ with the same property. Let $\varepsilon := \varepsilon'' \circ \pi_M(\mathbf{i}_t-1)[2\mathbf{i}_t-2] : S(\mathbf{i}_t-1)[2\mathbf{i}_t-2] \rightarrow Q$, where $\pi_M : S \rightarrow M$ is the natural projection. We can see, that $\varepsilon|_{\overline{k}}$ still maps $(\mathbb{Z})(i+\mathbf{i}_t-1)[2i+2\mathbf{i}_t-2]$ to $\mathbb{Z}(i+\mathbf{i}_t-1)[2i+2\mathbf{i}_t-2]$ isomorphically.

Let $\varphi : S \rightarrow Q$ be natural map (corresponding to the embedding $S \subset Q$), and $\varphi^{\vee} : Q(1-\mathbf{i}_t)[2-2\mathbf{i}_t] \rightarrow S$ be map dual to $\varphi$ via duality $\underline{\mathrm{Hom}}(-, \mathbb{Z}(n-\mathbf{i}_t+1)[2n-2\mathbf{i}_t+2])$. Then $\varphi^{\vee} \circ \varepsilon(1-\mathbf{i}_t)[2-2\mathbf{i}_t] : S \rightarrow S$ over $\overline{k}$ maps $\mathbb{Z}(i)[2i]$ to $\mathbb{Z}(i)[2i]$ isomorphically. By [4], Lemma 3.26 that means that $Q(1-\mathbf{i}_t)[2-2\mathbf{i}_t]$ and $S$ contain isomorphic undecomposable direct summands $N_1$ and $N_2$, containing $\mathcal{X}_{Q^{i+\mathbf{i}_t-1}}(i)[2i]$ and $\mathcal{X}_{S^i}$, respectively. So, they should be isomorphic to $M$ (see [4], Lemma 3.21). In particular, $\mathcal{X}_{Q^{i+\mathbf{i}_t-1}}(i)[2i]$ is the "lowest" term in $N_1$, and $\mathcal{X}_{Q^{i+\mathbf{i}_t-1}}(i+\mathbf{i}_t-1)[2i+2\mathbf{i}_t-2]$ is the "lowest term in $N_1(\mathbf{i}_t-1)[2\mathbf{i}_t-2]$. By the Statement (2), $\mathcal{X}_{Q^i}(n-i)[2n-2i]$ is the "highest" term in it. If $(N_1(\mathbf{i}_t-1)[2\mathbf{i}_t-2])^{\vee}$ is the summand dual to $N_1(\mathbf{i}_t-1)[2\mathbf{i}_t-2]$ via duality $\underline{\mathrm{Hom}}(-, \mathbb{Z}(n)[2n])$, then it is evidently undecomposable and it's "lowest" term is $(i)$.

Lemma 4 is proven.

□

Proposition 1 .

*Let $Q$ and $P$ be such quadrics, that for some $i$, $\mathcal{X}_{Q^i} = \mathcal{X}_P$. Then there exists a direct summand $N$ in $Q$, starting from $\mathcal{X}_{Q^i}(i)[2i]$ (i.e. $N$ contains no $\mathcal{X}_{Q^l}(l)[2l]$ with $l < i$, but contains $\mathcal{X}_{Q^i}(i)[2i]$ ).*

*Proof*

Changing $P$, if nesessary, by it's plane section, we can assume, that $\mathbf{i}_1(P) = 1$. By Lemma 4 , we can also assume, that $\mathbf{i}_{j(i,\bar{\mathbf{i}}(Q))} = 1$, i.e.

$\mathcal{X}_{Q^{i-1}} \neq \mathcal{X}_{Q^i} \neq \mathcal{X}_{Q^{i+1}}$. Really, first of all we can assume, by [4], Lemma 4.5 and our Statement , that $i = \mathbf{i}_1 + \cdots + \mathbf{i}_{t-1}$ for some $t$. If the corresponding *higher Witt index* $\mathbf{i}_t(Q)$ is $> 1$, change $Q$ by it's plane section $S$ of codimension $\mathbf{i}_t - 1$. Since $\dim(S) < \dim(Q)$, after few such steps we should get the quadric $S'$ with $\mathbf{i}_{j(i,\bar{\mathbf{i}}(S'))} = 1$. And the existence of the required direct summand for $Q$ follows from that for $S'$ (by Lemma 4 ).

Let $\dim(Q) = n$, $\dim(P) = m$.

Since $\mathcal{X}_{Q^i} = \mathcal{X}_P$, we have a motivic map $\varphi : P(i)[2i] \to Q$, s.t. $\varphi|_{\overline{k}}$ sends $(\mathbb{Z})(i)[2i]$ to $\mathbb{Z}(i)[2i]$ isomorphically. Let $\varphi^\vee : Q(m+2i-n)[2(m+2i-n)] \to P(i)[2i]$ will be map dual to $\varphi$ with respect to duality $\underline{\mathrm{Hom}}(-, \mathbb{Z}(m+2i)[2m+4i])$.

Let $K = k(Q^{i-1})$. Then over $K$, $q$ is (precisely) $i$-times isotropic (since $\mathcal{X}_{Q^{i-1}} \neq \mathcal{X}_{Q^i}$); let $R/K$ be anisotropic part of $Q|_K$.

Then $Q_K = \oplus_{l=0,\ldots,i-1}(\mathbb{Z}(l)[2l] \oplus \mathbb{Z}(n-l)[2n-2l]) \oplus R(i)[2i]$, and $\varphi|_K(-i)[-2i]$ gives a map $\psi : P|_K \to R$.

But $\mathcal{X}_{R/K} = \mathcal{X}_{Q^i/K} = \mathcal{X}_{P/K}$, that means that there exist a map: $\rho : R \to P$, s.t. for the composition $\alpha = \rho \circ \psi : P \to P$, we have $\alpha_0 = 1$ (in the sense of [4], Theorem 3.7, and the text after the Corollary 3.9)(i.e., $\alpha|_{\overline{k}}$ sends $\mathbb{Z}$ to $\mathbb{Z}$ isomorphically). By [4], Lemma 3.26, that means, that $R$ and $P|_K$ contain isomorphic direct summands $N$ and $M$, containing $(0)$ and $(0)$, respectively. But by the Statement , $M$ should contain $(0)'$ (since, by the Lemma 3 , $\mathbf{i}_1(P/K) = \mathbf{i}_1(P) = 1$). and it will be the "highest" elementary piece of $M$, and in the same way, the "highest" elementary piece of $N$ will be $(0)'$ (again, by the Lemma 3 , $\mathbf{i}_1(R) = \mathbf{i}_t(Q) = 1$). Since $M$ is isomorphic to $N$, and $\dim(R) = n - 2i$, $\dim(P) = m$, we get that: $m = n - 2i$, and $\psi|_{\overline{k}}$ sends $\mathbb{Z}(m)[2m]$ to itself via multiplication by an odd number (hence, $\varphi(-i)[-2i]$ does the same). Then $\varphi^\vee(n-m-3i)[2n-2m-6i]|_{\overline{k}}$ maps $\mathbb{Z}$ to $\mathbb{Z}$ via multiplication by an odd number. By [4], Lemma 3.20 we can find a map $\rho : Q(-i)[-2i] \to P$, which, over $\overline{k}$, will map $\mathbb{Z}$ to $\mathbb{Z}$ isomorhically. Hence, for $\varepsilon := \rho \circ \varphi(-i)[-2i] : P \to P$ we have $\varepsilon_0 = 1$, and by [4], Lemma 3.26, $P$ and $Q(-i)[-2i]$ have isomorphic direct summands, containing $\mathcal{X}_P$ and $\mathcal{X}_{Q^i}$, respectively. That means that $Q$ contains a direct summand starting from $\mathcal{X}_{Q^i}(i)[2i]$.

Proposition 1 is proven.

□

In connection with Proposition 1 it is natural to ask the following:

Question 1 .

*Are the following conditions equivalent?*

(1) *$Q$ contains a direct summand with the lowest term $(i)$.*

(2) *There exists quadric $P/k$, s.t. $\mathcal{X}_P = \mathcal{X}_{Q^i}$.*

In a meantime, we can characterize those $i$, for which there exists a direct summand $N$ starting from $(i)$, in the following way:

Proposition 2 .

*Let $Q$ be a quadric, and $0 \leqslant i < n/2$, where $n = \dim(Q)$. Then the following conditions are equivalent:*

1) *There exists undecomposable direct summand $N$ in $Q$, with the lowest term $(i)$.*

2) *The natural map $\alpha_i : Q \to \mathbb{Z}(i)[2i]$ (corresponding to a plane section of codimension $i$) is a composition $Q \xrightarrow{u} \mathcal{X}_{Q^i}(i)[2i] \to \mathbb{Z}(i)[2i]$, for some $u$ ($\mathcal{X}_{Q^i}(i)[2i] \to \mathbb{Z}(i)[2i]$ here is a natural projection).*

3) *The map $\alpha_i : Q \to \mathbb{Z}(i)[2i]$ is a composition $Q \xrightarrow{v} Q^i(i)[2i] \to \mathbb{Z}(i)[2i]$, for some $v$, where $Q^i(i)[2i] \to \mathbb{Z}(i)[2i]$ is again a natural projection.*

4) *There exists a subvariety $T$ of $Q$ of codimension $i$ and of degree not divisible by 4, s.t. $Q^i$ has a rational point over $k(T)$.*

*Proof of the Proposition 2*

$(1 \to 2)$ Follows from [4], Lemma 3.23.

$(2 \to 3)$ In $\mathrm{DM}^{eff}(k)$ we have the following *exact triangle*: $Q^i \to \mathcal{X}_{Q^i} \to Y \to Q^i[1]$, where $Y$ is an "extension" of $Q^i \times Q^i[1]$, $Q^i \times Q^i \times Q^i[2]$, etc ... . Since $\mathrm{Hom}_{\mathrm{DM}^{eff}}(Q, Q^i \times \cdots \times Q^i(i)[2i+p]) = 0$, for any positive $p$, we have that any map $u : Q \to \mathcal{X}_{Q^i}(i)[2i]$ can be lifted to the map $v : Q \to Q^i(i)[2i]$.

$(3 \to 4)$ The map $v : Q \to Q^i(i)[2i]$ is given by some cycle $V \subset Q \times Q^i$ of dimension $n-i$, and the composition $Q \xrightarrow{v} Q^i(i)[2i] \to \mathbb{Z}(i)[2i]$ is given by the cycle $W = (\pi_1)_*(V) \subset Q$, where $\pi_1 : Q \times Q^i \to Q$ is the projection on the first factor. Since the composition coincides with $\alpha_i$ (given by plane section of codimension $i$), we have that the degree of $W$ is 1. Hence $W$ should contain irreducible component $T$ of odd degree in odd multiplicity. That means that over $k(T)$, $Q^i$ has a point of odd degree, and by Springer theorem it has a rational point over that field.

$(4 \to 1)$ The rational map $T \to Q^i$ give us cycle $V \subset Q \times Q^i$ of dimension $n-i$, and so, a map $v : Q \to Q^i(i)[2i]$. Consider the standard map $\varphi : Q^i(i)[2i] \to Q$ (given by the cycle $\Phi \subset Q^i \times Q$, $\Phi = \{(l,x) : x \in l\}$). The composition $Q^i(i)[2i] \stackrel{\varphi}{\to} Q \stackrel{\alpha_i}{\to} \mathbb{Z}(i)[2i]$ coincides with the natural projection $Q^i(i)[2i] \to \mathbb{Z}(i)[2i]$. That means that the composition $Q \stackrel{v}{\to} Q^i(i)[2i] \stackrel{\varphi}{\to} Q \stackrel{\alpha_i}{\to} \mathbb{Z}(i)[2i]$ is given by the cycle $T \subset Q$. Consider the map $\rho := \varphi \circ v : Q \to Q$. Since $\mathrm{Hom}(\mathbb{Z}(j)[2j], Q^i(i)[2i]) = 0$ for any $j < i$, we have that over $\overline{k}$, $\rho|_{\overline{k}}$ maps $\mathbb{Z}(j)[2j]$ to 0 for such $j$. On the other hand $\rho|_{\overline{k}} : \mathbb{Z}(i)[2i] \to \mathbb{Z}(i)[2i]$ is a multiplication by the degree of $T$ divided by 2, which is odd. By [4], Lemma 3.12 and Lemma 3.25, there is a direct summand of $Q$, starting from $(i)$.

□

The Proposition 2 permits us to clarify the picture in the Proposition 3.4 from [4].

We remind, that $\beta_i : \mathbb{Z}(n-i)[2n-2i] \to Q$ is a natural map given by the plane section of codimension $i$ in $Q$. Then we can define the natural map $\oplus\beta'_i : \oplus_{0\leqslant i<n/2}\mathcal{X}_{Q^i}(n-i)[2n-2i] \to Q$, where $\beta'_i$ is a composition of natural projection $\mathcal{X}_{Q^i}(n-i)[2n-2i] \to \mathbb{Z}(n-i)[2n-2i]$ and $\beta_i$. Let $P' = \mathrm{Cone}(\oplus\beta'_i)$. By [4], Proposition 3.4, $P'$ is an extension of $\mathcal{X}_{Q^i}(i)[2i]$, $0 \leqslant i < n/2$, and also $k(\sqrt{det(Q)}) \times \mathcal{X}_{Q^{n/2}}(n/2)[n]$ (if $n$ is even).

Corollary 4 .

*Let $Q$ be anisotropic quadric. The following conditions are equivalent:*

(1) *$P'$ is isomorphic to a direct sum $\oplus_{0\leqslant i<n/2}\mathcal{X}_{Q^i}(i)[2i]$ $(\oplus k(\sqrt{det(Q)}) \times \mathcal{X}_{Q^{n/2}}(n/2)[n]$, if $n$ is even).*

(2) *$Q$ consists of "binary motives" (i.e., motives, consisting of just two elementary pieces).*

*Proof* $(1 \to 2)$ Since $P'$ is a direct sum we have that the map $\alpha_i : Q \to \mathbb{Z}(i)[2i]$ can be lifted to the map $\alpha'_i : Q \to \mathcal{X}_{Q^i}(i)[2i]$. By Proposition 2 , that means that there is an undecomposable direct summand of $Q$, starting from $i$. Since it is true for all $0 \leqslant i \leqslant n/2$, all undecomposable direct summands of $Q$ are binary (they contain only one "simple piece" from the lower half of $Q \Rightarrow$ only one from the upper half as well).

$(2 \to 1)$ The decomposition of $Q$ into the binary motives gives the decomposition of the map $\oplus\beta'_i : \oplus_{0\leqslant i<n/2}\mathcal{X}_{Q^i}(n-i)[2n-2i] \to Q$, which

gives a decomposition of $P'$ into the direct sum of elementary components of $P'$ (i.e. $\mathcal{X}_Q, \mathcal{X}_{Q^1}(1)[2]$, etc. ...).

□

*Remark* (1) and (2) in the Corollary 4 should be equivalent to (3): $Q$ is *Excellent* quadric. In the one direction it is a result of M.Rost (see [2], Proposition 4). In the other: we know only (from the proof of the Statement 6.1 from [4]) that $Q$ should have *excellent splitting pattern*, and our binary motives are "of the Rost-motive size".

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