Excellent connections in the motives of quadrics

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Abstract

In this article we prove the Conjecture claiming that the connections in the motives of excellent quadrics are minimal ones for anisotropic quadrics of given dimension. This imposes severe restrictions on the motive of arbitrary anisotropic quadric. As a corollary we estimate from below the rank of indecomposable direct summand in the motive of a quadric in terms of its dimension. This generalises the well-known Binary Motive Theorem. Moreover, we have the description of Tate-motives involved. This, in turn, gives another proof of Karpenko's Theorem on the value of the first higher Witt index. But also other new relations among higher Witt indices follow.

1 Introduction

Let Q be smooth projective quadric of dimension n over the field k of characteristic not 2, and $M(Q)$ be its *motive* in the category $Chow(k)$ of Chow motives over k (see [14], or Chapter XII of [2]). Over the algebraic closure $\overline{k}$, our quadric becomes completely split, and so, cellular. This implies that $M(Q|_{\overline{k}})$ becomes isomorphic to a direct sum of Tate motives:

$$M(Q|_{\overline{k}}) \cong \oplus_{\lambda\in\Lambda(Q)}\mathbb{Z}(\lambda)[2\lambda],$$

where $\Lambda(Q) = \Lambda(n)$ is $\{i|\, 0 \leqslant i \leqslant [n/2]\}\bigsqcup\{n-i|\, 0 \leqslant i \leqslant [n/2]\}$. But over the ground field k our motive could be much less decomposable. The *Motivic Decomposition Type* invariant $MDT(Q)$ measures what kind of decomposition we have in $M(Q)$. Any direct summand N of $M(Q)$ also splits over $\overline{k}$, and $N|_{\overline{k}} \cong \sum_{\lambda\in\Lambda(N)} \mathbb{Z}(\lambda)[2\lambda]$, where $\Lambda(N) \subset \Lambda(Q)$ (see [14] for details). We say that $\lambda, \mu \in \Lambda(Q)$ are *connected*, if for any direct summand N of $M(Q)$, either both λ and μ are in $\Lambda(N)$, or both are out. This is an equivalence relation, and it splits $\Lambda(Q) = \Lambda(n)$ into disjoint union of *connected components*. This decomposition is called the *Motivic Decomposition Type*. It interacts in a nontrivial way with the *Splitting pattern*, and using this interaction one proves many results about both invariants. The (absolute) *Splitting pattern* $\mathbf{j}(q)$ of the form q is defined as an increasing sequence $\{\mathfrak{j}_0, \mathfrak{j}_1, \ldots, \mathfrak{j}_h\}$ of all possible Witt indices of $q|_E$ over all possible field extensions E/k. We will also use the (relative) *Splitting patter* $\mathbf{i}(q)$ defined as $\{\mathfrak{i}_0, \ldots, \mathfrak{i}_h\} := \{\mathfrak{j}_0, \mathfrak{j}_1 - \mathfrak{j}_0, \mathfrak{j}_2 - \mathfrak{j}_1, \ldots, \mathfrak{j}_h - \mathfrak{j}_{h-1}\}$.

Let us denote the elements $\{\lambda |\, 0 \leqslant \lambda \leqslant [n/2]\}$ of $\Lambda(n)$ as λ_{lo}, and the elements $\{n-\lambda |\, 0 \leqslant \lambda \leqslant [n/2]\}$ as λ^{up}. See the Appendix for the detailed explanation. The principal result relating the splitting pattern and the motivic decomposition type claims that all elements of $\Lambda(Q)$ come in pairs whose structure depends on the splitting pattern.

Proposition 1.1 ([14, Proposition 4.10], cf [2, Theorem 73.26]) *Let λ and μ be such that $\mathfrak{j}_{r-1} \leqslant \lambda, \mu < \mathfrak{j}_r$, where $1 \leqslant r \leqslant h$, and $\lambda + \mu = \mathfrak{j}_{r-1} + \mathfrak{j}_r - 1$. Then λ_{lo} is connected to μ^{up}.*

Consequently, any direct summand in the motive of anisotropic quadric consists of even number of Tate-motives when restricted to $\overline{k}$, in particular, of at least two Tate-motives. If it consists of just two Tate-motives we will call it *binary*. It can happen that $M(Q)$ splits into *binary motives*. As was proven by M.Rost ([12, Proposition 4]), this is the case for *excellent* quadrics, and, hypothetically, it should be the only such case. The *excellent* quadratic forms introduced by M.Knebusch ([8]) are sort of substitutes for the Pfister forms in dimensions which are not powers of two. Namely, if you want to construct such a form of dimension, say, m, you need first to present m in the form $2^{r_1} - 2^{r_2} + \ldots + (-1)^{s-1}2^{r_s}$, where $r_1 > r_2 > \ldots > r_{s-1} > r_s + 1 \geqslant 1$ (it is easy to see that such presentation is unique), and then choose pure symbols $\alpha_i \in \mathrm{K}^M_{r_i}(k)/2$ such that $\alpha_1 \vdots \alpha_2 \vdots \ldots \vdots \alpha_s$. Then the respective excellent form is an m-dimensional form $(\langle\langle\alpha_1\rangle\rangle - \langle\langle\alpha_2\rangle\rangle + \ldots + (-1)^{s-1}\langle\langle\alpha_s\rangle\rangle)_{an}$. In particular, if $m = 2^r$ one gets an r-fold Pfister form. It follows from the mentioned result of M.Rost that the only connections in the motives of excellent quadrics are binary ones coming from Proposition 1.1. At the same time, the experimental data suggested that in the motive of anisotropic quadric Q of dimension n we should have not only connections coming from the splitting pattern $\mathbf{i}(Q)$ of Q but also ones coming from the excellent splitting pattern:

Conjecture 1.2 ([14, Conjecture 4.22]) *Let Q and P be anisotropic quadrics of dimension n with P-excellent. Then we can identify $\Lambda(Q) = \Lambda(n) = \Lambda(P)$, and for $\lambda, \mu \in \Lambda(n)$,*

$$\lambda, \mu \ \textit{connected in} \ \Lambda(P) \quad \Rightarrow \quad \lambda, \mu \ \textit{connected in} \ \Lambda(Q).$$

Partial case of this Conjecture, where λ and μ belong to the *outer excellent shell* (that is, $\lambda, \mu < \mathfrak{j}_1(P)$), was proven earlier and presented by the author at the conference in Eilat, Feb. 2004. The proof used *Symmetric operations*, and the Grassmannian $G(1, Q)$ of projective lines on Q, and is a minor modification of the proof of [15, Theorem 4.4] (assuming $char(k) = 0$). Another proof using Steenrod operations and $Q^{\times 2}$ appears in [2, Corollary 80.13] (here $char(k) \neq 2$).

The principal aim of the current paper is to prove the whole conjecture for all field of characteristic different from 2.

Theorem 1.3 *Conjecture 1.2 is true.*

This Theorem shows that the connections in the motive of an excellent quadric are *minimal* among anisotropic quadrics of a given dimension. Moreover, for a given anisotropic quadric Q we get not just one set of such connections, but $h(Q)$ sets, where $h(Q)$ is a *height* of Q, since we can apply the Theorem not just to q but to $q_i := (q|_{k_i})_{an}$ for all fields $k_i, 0 \leqslant i < h$, from the *generic splitting tower of Knebusch* (see [7]). And the more splitting pattern of Q differs from the excellent splitting pattern, the more nontrivial conditions we get, and the more indecomposable $M(Q)$ will be.

As an application of this philosophy, we get the result bounding from below the rank of indecomposable direct summand in the motive of a quadric in terms of its dimension - see Theorem 2.1. This is a generalisation of the binary motive Theorem ([5, Theorem 6.1], see also other proofs in [15, Theorem 4.4] and [2, Corollary 80.11]) which claims that the dimension of a binary direct summand in the motive of a quadric is equal to $2^r - 1$, for some r, and which has many applications in the quadratic form theory. Moreover, we can describe which particular Tate-motives must be present in

$N|_{\overline{k}}$ depending on the dimension on N. An immediate corollary of this is another proof of the Theorem of Karpenko (formerly known as the Conjecture of Hoffmann) describing possible values of the first higher Witt index of q in terms of $\dim(q)$. This approach to the Hoffmann's Conjecture based on the Conjecture 1.2 is, actually, the original one introduced by the author in 2001, and it is pleasant to see it working, after all. But, aside from the value of the first Witt index, the Theorem 2.1 gives many other relations on higher Witt indices.

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2 Applications of the Main Theorem

For the direct summand N of $M(Q)$ let us denote as $rank(N)$ the cardinality of $\Lambda(N)$ (that is, the number of Tate-motives in $N|_{\overline{k}}$), as $a(N)$ and $b(N)$ the minimal and maximal element in $\Lambda(N)$, respectively, and as $\dim(N)$ the difference $b(N) - a(N)$. Unless otherwise stated, n will always be the dimension of a quadric.

Theorem 2.1 *Let N be indecomposable direct summand in the motive of anisotropic quadric with $\dim(N) + 1 = 2^{r_1} - 2^{r_2} + \ldots + (-1)^{s-1}2^{r_s}$, where $r_1 > r_2 > \ldots > r_{s-1} > r_s + 1 \geqslant 1$. Then:*

(1) *$rank(N) \geqslant 2s$;*

(2) *For $1 \leqslant k \leqslant s$, let $d_k = \sum_{i=1}^{k-1}(-1)^{i-1}2^{r_i-1} + \varepsilon(k) \cdot \sum_{j=k}^{s}(-1)^{j-1}2^{r_j}$, where $\varepsilon(k) = 1$, if k is even, and $\varepsilon(k) = 0$, if k is odd. Then*

$$(a(N) + d_k)_{lo} \in \Lambda(N), \quad \text{and} \quad (n - b(N) + d_k)^{up} \in \Lambda(N).$$

Remark: In particular, we get: if $rank(N) = 2$, that is, N is *binary*, then $\dim(N) = 2^r - 1$, for some r - the Binary Motive Theorem.

Proof: First, we reduce to the special case:

Lemma 2.2 *It is sufficient to prove Theorem* 2.1 *in the case* $\mathfrak{i}_1(q) = 1$*, and* $a(N) = 0$*,* $b(N) = \dim(Q)$*.*

Proof: It follows from [14, Corollary 4.14] that there exists $1 \leqslant t \leqslant h(q)$ such that $\mathfrak{j}_{t-1}(q) \leqslant a(N), (n - b(N)) < \mathfrak{j}_t(q)$, and

$$\dim(N) = n - \mathfrak{j}_{t-1}(q) - \mathfrak{j}_t(q) + 1.$$

Then we can pass to the field k_{t-1} from the generic splitting tower of Knebusch, and $N|_{k_{t-1}}$ shifted by $(-\mathfrak{j}_{t-1})[-2\mathfrak{j}_{t-1}]$ will be a direct summand in the motive of Q_{t-1}, where $q_{t-1} = (q|_{k_{t-1}})_{an}$. It follows from Corollary 4.2, that under this transformation *lower motives* λ_{lo} are transformed into lower motives $(\lambda - \mathfrak{j}_{t-1})_{lo}$, while *upper motives* λ^{up} are transformed into the upper ones $(\lambda - \mathfrak{j}_{t-1})^{up}$.

It can happen that $N|_{k_{t-1}}(-\mathfrak{j}_{t-1})[-2\mathfrak{j}_{t-1}]$ is decomposable, but it follows from [14, Corollary 4.14] that it should contain indecomposable submotive N' of the same dimension. Since we estimate the *rank* of N from below, it is sufficient to prove the statement for N' and $q' = q_{t-1}$. Thus, everything is reduced to the case $t = 1$. Considering the subform q'' of q' of codimension $\mathfrak{i}_1(q') - 1$, we get from [14, Theorem 4.15] that $M(Q'')$ contains a direct summand isomorphic to $N'(-a(N'))[-2a(N')]$, while $\mathfrak{i}_1(q'') = 1$ by [14, Corollary 4.9(3)]. Again, Corollary 4.2 shows that separation into upper and lower motives is preserved under these manipulations. Hence, we reduced everything to the case: $\mathfrak{i}_1(q) = 1$, and $a(N) = 0$, $b(N) = \dim(Q)$.

□

We will use the following observation describing the relation between the MDT of a form and of some anisotropic kernel of it.

Observation 2.3 *Let* ρ *be some quadratic form over* k*,* E/k *be some field extension,* $m = i_W(\rho|_E)$*, and* $\rho' = (\rho|_E)_{an}$*. Then* $\Lambda(\rho')$ *is naturally embedded into* $\Lambda(\rho)$ *by the rule:* $\lambda_{lo} \mapsto (\lambda + m)_{lo}$*,* $\lambda^{up} \mapsto (\lambda + m)^{up}$*, and connections in* $\Lambda(\rho')$ *imply ones in* $\Lambda(\rho)$*.*

Proof: It is sufficient to recall that by [13, Proposition 2], (see also [14, Proposition 2.1]),

$$M(X_\rho|_E) = \left(\oplus_{i=0}^{m-1}\mathbb{Z}(i)[2i] \oplus \mathbb{Z}(n-i)[2n-2i]\right) \oplus M(X_{\rho'})(m)[2m],$$

where X_ρ is the respective projective quadric, and $n = \dim(X_\rho)$. □

Now everything follows from excellent connections for Q and Q_1, where $q_1 = (q|_{k(Q)})_{an}$. Let P be anisotropic excellent quadric of dimension $= \dim(Q)$, and $\tilde{P}$ be anisotropic excellent quadric of dimension $= \dim(Q_1)$. By Observation 2.3, we can identify $\Lambda(Q_1)$ with the subset of $\Lambda(Q)$ by the rule: $\lambda_{lo} \mapsto (\lambda+1)_{lo}$ and $\lambda^{up} \mapsto (\lambda+1)^{up}$, and the connection between u and v in $\Lambda(Q_1)$ implies the connection between their images in $\Lambda(Q)$. We will apply inductively the following statement about excellent splitting patterns.

Lemma 2.4 *Let φ, ψ be anisotropic excellent forms (over some unrelated fields) of dimension $D+1$ and $D-1$, respectively, where $D = 2^{l_1} - 2^{l_2} + \ldots + (-1)^{m-1}2^{l_m}$, and $l_1 > l_2 > \ldots > l_{m-1} > l_m + 1 \geqslant 1$. Then:*
either: 1) *$D = 2^r$, for some r (that is, $m = 1$);*
or: 2) *For $\varphi' := \varphi_1$, and ψ' - one of: ψ, or ψ_1, we have:* $\dim(\varphi') = D'-1$, $\dim(\psi') = D'+1$, *where $D' = 2^{l_2} - \ldots + (-1)^{m-2}2^{l_m}$.*

Proof: If $\mathfrak{i}_1(\varphi) = 1$, then $\dim(\varphi) = 2^r + 1$, and $m = 1$.

If $\mathfrak{i}_1(\varphi) = 2$, then $D' = D - 2$, and $\dim(\varphi_1) = \dim(\varphi) - 4 = D' - 1$, so we can take $\psi' = \psi$.

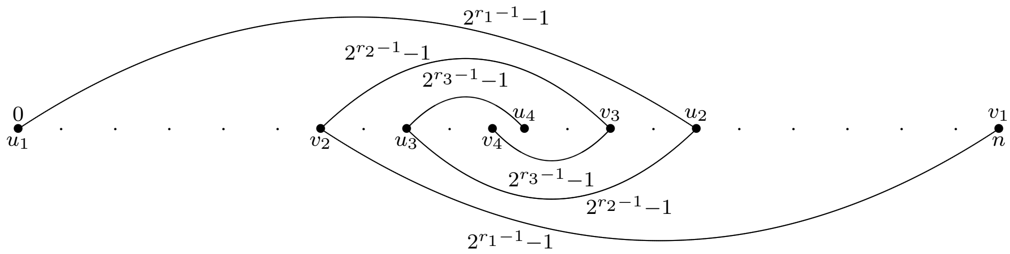
Finally, if $\mathfrak{i}_1(\varphi) > 2$, then $\mathfrak{i}_1(\psi) = \mathfrak{i}_1(\varphi) - 2$, and we can take $\psi' = \psi_1$. □

The above Lemma permits to pass from the pair (φ, ψ) of excellent anisotropic forms of dimension $(D+1, D-1)$ to the pair $(\varphi, \psi)^{(1)} = (\psi', \varphi')$ of excellent anisotropic forms of dimension $(D'+1, D'-1)$, where φ' and ψ' are some anisotropic kernels of the original forms (over some extensions from the Knebusch tower). In particular, by Observation 2.3, we have natural embeddings $\Lambda(\varphi') \subset \Lambda(\varphi)$, $\Lambda(\psi') \subset \Lambda(\psi)$, and connections in $\Lambda(\varphi')$, $\Lambda(\psi')$ imply connections in $\Lambda(\varphi)$, $\Lambda(\psi)$.

Since $\mathfrak{i}_1(Q) = 1$, and thus, $\dim(p) = \dim(N) + 2$, $\dim(\tilde{p}) = \dim(N)$, we can apply these considerations to our forms $p, \tilde{p}$, to get:

$$(p, \tilde{p}) \to (p, \tilde{p})^{(1)} \to (p, \tilde{p})^{(2)} \to \ldots \to (p, \tilde{p})^{(s-1)}.$$

This process will stop after $(s-1)$ steps, since $D^{(s-1)} = 2^{r_s}$. All our sets $\Lambda(P^{(j)}), \Lambda(\tilde{P}^{(j)})$ are naturally embedded into $\Lambda(Q)$ by Observation 2.3, and, by the Main Theorem (1.3), connections in the former imply connections in the latter. Let us say that λ is *the first* in the t^{th} shell of some quadric R, if $\lambda = \mathfrak{j}_{t-1}(R)$. Similarly, we say that λ is *the last* in the t^{th} shell of R, if $\lambda = \mathfrak{j}_t(R) - 1$. It follows from Proposition 1.1 that if λ and μ are the *first* and the *last* element in the 1^{st} shell of $P^{(j)}$ or $\tilde{P}^{(j)}$, then λ_{lo} is connected to μ^{up}, λ^{up} is connected to μ_{lo}, and $\mu^{up} - \lambda_{lo} = 2^{r_{j+1}-1} - 1$, for $0 \leqslant j \leqslant s-2$. But the last element in the 1^{st} shell of $P^{(j)}$ (respectively, $\tilde{P}^{(j)}$) will be the first one in the 1^{st} shell of $\tilde{P}^{(j+1)}$ (respectively, $P^{(j+1)}$). So, we get connections in $\Lambda(Q)$: $u_1 \leftrightarrow u_2 \leftrightarrow u_3 \leftrightarrow \ldots \leftrightarrow u_s$, where $u_1 = 0$, $u_{2k+1} \in \Lambda(Q)_{lo}$, $u_{2k} \in \Lambda(Q)^{up}$, and $u_{i+1} - u_i = (-1)^{i-1}(2^{r_i-1} - 1)$. And symmetric ones: $v_1 \leftrightarrow v_2 \leftrightarrow v_3 \leftrightarrow \ldots \leftrightarrow v_s$, where $v_1 = n$, $v_{2k+1} \in \Lambda(Q)^{up}$, $v_{2k} \in \Lambda(Q)_{lo}$, and $v_{i+1} - v_i = (-1)^i(2^{r_i-1} - 1)$. Since $0 = u_1$ and $n = v_1$ belong to $\Lambda(N)$, we have that $u_i, v_i \in \Lambda(N)$, for all $1 \leqslant i \leqslant s$. This proves (2) and (1).



□

As a corollary of Theorem 2.1 we get another proof of the Conjecture of Hoffmann.

Theorem 2.5 (N.Karpenko, [6]) *Let q be anisotropic quadratic form of dimension m. Then $(i_1(q) - 1)$ is a remainder modulo 2^r of $(\dim(q) - 1)$, for some $r < \log_2(\dim(q))$.*

Proof: Let N be indecomposable direct summand of $M(Q)$ such that $0_{lo} \in \Lambda(N)$ (by [14, Corollary 4.4] such N exists). Let $\dim(N)+1 = \sum_{i=1}^{s}(-1)^{i-1}2^{r_i}$, where $r_1 > r_2 > \ldots > r_{s-1} > r_s+1 \geqslant 1$. From [14, Corollary 4.14], $\dim(N) = \dim(Q) - \mathfrak{i}_1(Q) + 1$. Thus, $\dim(q) - 1 = \dim(Q) + 1 = \sum_{i=1}^{s}(-1)^{i-1}2^{r_i} + (\mathfrak{i}_1(Q) - 1)$.

By Theorem 2.1, we have: $(d_s)_{lo} \in \Lambda(N)$, where

$$d_s = \sum_{i=1}^{s-1} (-1)^{i-1} 2^{r_i - 1} + \varepsilon(s) \cdot (-1)^{s-1} 2^{r_s}.$$

But by [14, Theorem 4.13], $N(\mathfrak{i}_1(Q)-1)[2\mathfrak{i}_1(Q)-2]$ is also isomorphic to the direct summand of $M(Q)$. In particular, $\Lambda(N(\mathfrak{i}_1(Q)-1)[2\mathfrak{i}_1(Q)-2])_{lo} \subset \Lambda(Q)_{lo}$. But $\Lambda(N(\mathfrak{i}_1(Q)-1)[2\mathfrak{i}_1(Q)-2])_{lo} = \Lambda(N)_{lo} + (\mathfrak{i}_1(Q)-1)$, as separation into lower and upper elements is stable under Tate-shifts (Corollary 4.2). Thus, for any λ such that $\lambda_{lo} \in \Lambda(N)$, we should have: $\lambda + (\mathfrak{i}_1(Q)-1) \leqslant \dim(Q)/2$. Applying this to d_s, we get:

$$\sum_{i=1}^{s-1} (-1)^{i-1} 2^{r_i - 1} + \varepsilon(s) \cdot (-1)^{s-1} 2^{r_s} + (\mathfrak{i}_1(Q)-1) \leqslant$$
$$\sum_{i=1}^{s-1} (-1)^{i-1} 2^{r_i - 1} + 1/2((-1)^{s-1} 2^{r_s} + (\mathfrak{i}_1(Q)-1) - 1).$$

That is, $2^{r_s} > (\mathfrak{i}_1(Q)-1)$. Thus, $(\mathfrak{i}_1(Q)-1)$ is the remainder modulo 2^{r_s} of $\dim(q)-1$. □

One can apply similar considerations to elements d_k with $k < s$ and get other new relations for *higher Witt indices*.

Proposition 2.6 *Let* $\dim(q) - \mathfrak{i}_1(q) = 2^{r_1} - 2^{r_2} + \ldots + (-1)^{s-1} 2^{r_s}$, *where* $r_1 > r_2 > \ldots > r_{s-1} > r_s + 1 \geqslant 1$, *and elements* $d_k, 1 \leqslant k \leqslant s$ *be from Theorem* 2.1. *Then, for each* $1 \leqslant k \leqslant s$, *the elements* d_k *and* $d_k + (\mathfrak{i}_1(q)-1)$ *belong to the same (usual) shell of* Q.

These relations will be analysed in a separate text.

Another application is the characterisation of even-dimensional indecomposable direct summands in the motives of quadrics. An indecomposable direct summand appeares to be even-dimensional if and only if it is, sort of, "fat" (like the motive of even-dimensional quadric).

Theorem 2.7 *Let* N *be indecomposable direct summand in the motive of anisotropic quadric. Then the following conditions are equivalent:*

(1) $\dim(N)$ *is even;*

(2) *there exist i such that $(\mathbb{Z}(i)[2i] \oplus \mathbb{Z}(i)[2i])$ is a direct summand of $N|_{\overline{k}}$.*

Proof: $(2 \to 1)$ If $N|_{\overline{k}}$ contains $(\mathbb{Z}(i)[2i] \oplus \mathbb{Z}(i)[2i])$ then, clearly, $i = \dim(Q)/2$, and by [14, Theorem 4.19], the *dual direct summand*

$$N^{\vee} := \underline{\mathrm{Hom}}(N, \mathbb{Z}(\dim(Q))[2\dim(Q)])$$

is isomorphic to $N(k)[2k]$, where $k = \dim(Q) - a(N) - b(N)$. Since N and $N^{\vee}$ both contain the middle part of $M(Q)$, they must be isomorphic by [14, Lemma 4.2]. But $a(N^{\vee}) = \dim(Q) - b(N)$. Thus, $a(N) = \dim(Q) - b(N)$, and hence, $\dim(N) = b(N) - a(N) = 2b(N) - \dim(Q) = 2(b(N) - i)$ is even.

$(1 \to 2)$ If $\dim(N)$ is even, then in the presentation $\dim(N) + 1 = \sum_{i=1}^{s}(-1)^{i-1}2^{r_i}$ with $r_1 > r_2 > \ldots > r_{s-1} > r_s + 1 \geqslant 1$, r_s should be zero. Then for $d_s = \sum_{i=1}^{s-1}(-1)^{i-1}2^{r_i - 1} + \varepsilon(s) \cdot (-1)^{s-1} = \dim(N)/2$ we have that $(a(N) + d_s)_{lo}, (n - b(N) + d_s)^{up} \in \Lambda(N)$ are different elements of the same degree (one is upper, another is lower). □

The latter result implies that in the motives of even-dimensional quadrics all the Tate-motives living in the shells with higher Witt indices 1 are connected among themselves.

Corollary 2.8 *Let Q be even dimensional quadric, and s, t be such that $\mathfrak{i}_{t+1} = \mathfrak{i}_{s+1} = 1$. Then $(\mathfrak{j}_t)^{up}, (\mathfrak{j}_t)_{lo}, (\mathfrak{j}_s)^{up}$, and $(\mathfrak{j}_s)_{lo}$ are connected in $\Lambda(Q)$.*

Proof: It is sufficient to apply Theorem 2.7 to the quadrics Q_i, $i = s, t$, where $q_i = (q|_{k_i})_{an}$, and k_i is a field from the splitting tower of Knebusch, to see that the respective elements of $\Lambda(Q)$ are connected to the "middle part" $(n/2)_{lo}$, $(n/2)^{up}$, and thus, are connected among themselves. □

Remark: In particular, the motive of the even-dimensional quadric with the generic (relative) splitting pattern $(1, 1, \ldots, 1)$ is indecomposable. But this can be proven using the Binary Motive Theorem alone.

One should note, that nothing of this sort is true for the odd-dimensional quadrics with the generic splitting pattern. Indeed, if k is any field, $K = k(a, b_1, \ldots, b_m)$, and $q = \langle\langle a \rangle\rangle \cdot \langle b_1, \ldots, b_m \rangle \perp \langle 1 \rangle$, then the splitting pattern of Q is generic, but $M(Q)$ is decomposable (see [14, Theorem 6.1]).

3 Proof of the Main Theorem

We will first give a shorter but less transparent proof which works in arbitrary characteristic $\neq 2$, and then provide more transparent version for characteristic 0. The reason for such a scheme is that, this way, the reader will see what is really going on. Actually, the first proof is the derivative of the second, and the author would not be able to find it without having the second one. In the second case we will use Algebraic Cobordism theory, which is much richer than the Chow group theory, and this restricts our choice of characteristic. So, the reader who likes short proofs and does not like Cobordisms can read just the first proof, while the one who, may be, wants to prove some other statements is encouraged to read the second proof as well.

3.1 The General case

Let q be nondegenerate quadratic form of dimension $(n+2)$, and Q be respective (smooth) projective quadric of dimension n. From [14] one can see that motivic decomposition of quadrics with $\mathbb{Z}$-coefficients carries the same information as the one with $\mathbb{Z}/2$-coefficients, and $MDT(Q)$ can be reconstructed out of

$$\mathrm{image}(\mathrm{Ch}^n(Q \times Q) \to \mathrm{Ch}^n(Q \times Q|_{\overline{k}})),$$

where $\mathrm{Ch} = \mathrm{CH}/2$. More explicitly this idea is outlined in [3]. So, to prove the existence of *excellent connections* in $M(Q)$ we need to impose certain restrictions on the image above. The target group here has $\mathbb{Z}/2$-basis consisting of $h^i \times l_i$ and $l_i \times h^i$, $0 \leqslant i \leqslant [n/2]$ (plus $h^{n/2} \times h^{n/2}$ and $l_{n/2} \times l_{n/2}$, if n is even), where h^i is the class of the plane section of codimension i, and l_i is the class of projective subspace of dimension i (in the case $i = n/2$, we fix one of two families of middle-dimensional subspaces). Moreover, in the case n is even, the element $h^{n/2} \times h^{n/2}$ is always in the image, while if some element v defined over the ground field k has nontrivial coefficient at $l_{n/2} \times l_{n/2}$, the quadric Q must be hyperbolic (since the class $l^{1,2}_{n/2} = (\pi_1)_*((1 \times h^{n/2}) \cdot v)$ is defined over k). From now on we will assume that Q is not hyperbolic. In this case, to describe $MDT(Q)$ we need to determine, which elements $\overline{v}$ of the form:

$$\sum_{i=1}^{[n/2]} (\alpha_i \cdot (h^i \times l_i) + \beta_i \cdot (l_i \times h^i)),$$

where $\alpha_i, \beta_i \in \mathbb{Z}/2$, are defined over the field k.

The projector $h^i \times l_i$ gives the direct summand $\mathbb{Z}/2\mathbb{Z}(i)[2i] \in \Lambda(Q)_{lo}$, while $l_i \times h^i$ gives $\mathbb{Z}/2\mathbb{Z}(n-i)[2n-2i] \in \Lambda(Q)^{up}$. Thus, the above element $\overline{v}$ is a projector corresponding to the direct summand N of $M(Q)$ with $\Lambda(N)_{lo} = \{i | \alpha_i = 1\}_{lo}$, and $\Lambda(N)^{up} = \{i | \beta_i = 1\}^{up}$. In this light, the *connection* between certain elements of $\Lambda(Q)$ amounts to the equality between the respective α's and β's. Let us see what this means in the case of *excellent connections.*

Let $(n+2) = \sum_{i=1}^{s} (-1)^{i-1} 2^{r_i}$, where

$$r_1 > r_2 > \ldots > r_{s-1} > r_s + 1 \geqslant 1.$$

In the case n even, we put $s' = s$, in the case n - odd, we put $s' = s - 1$.

Definition 3.1 *We call the pair (a, b) excellent of degree t, if:*

(i) $n - (a+b) = 2^{r_t - 1} - 1$, $1 \leqslant t \leqslant s'$;

(ii) *a, b belong to the excellent shell number t:*

$$\sum_{i=1}^{t-1} (-1)^{i-1} 2^{r_i - 1} + \varepsilon(t) \cdot \sum_{i=t}^{s} (-1)^{i-1} 2^{r_i} \leqslant a, b$$
$$\leqslant \sum_{i=1}^{t} (-1)^{i-1} 2^{r_i - 1} + (1 - \varepsilon(t)) \cdot \sum_{i=t+1}^{s} (-1)^{i-1} 2^{r_i} - 1,$$

where $\varepsilon(t) = 1$, if t is even, and is 0, if t is odd.

Let P be anisotropic excellent quadric of dimension n. The following observation is straightforward from the computation of the excellent splitting pattern ([4])

Observation 3.2 *Let $0 \leqslant a, b \leqslant [n/2]$. Then the following conditions are equivalent:*

(1) *a_{lo} is connected to b^{up} in $\Lambda(P)$;*

(2) *(a,b) is an excellent pair.*

Thus, our Main Theorem amounts to:

Theorem 3.3 *Let Q be anisotropic quadric of dimension n, and*

$$v \in \mathrm{Ch}^n(Q \times Q) \quad \textit{with} \quad \overline{v} = \sum_{i=1}^{[n/2]} (\alpha_i \cdot (h^i \times l_i) + \beta_i \cdot (l_i \times h^i)).$$

Then $\alpha_b = \beta_a$, for all excellent pairs (a,b).

We will prove a more general statement, which has additional applications.

Theorem 3.4 *Let k be a field of characteristic not 2, and Q_1, Q_2 be two anisotropic quadrics of dimension n over k. Let $v \in \mathrm{Ch}^n(Q_1 \times Q_2)$ be cycle with*

$$\overline{v} = \sum_{i=1}^{[n/2]} (\alpha_i \cdot (h^i \times l_i) + \beta_i \cdot (l_i \times h^i)).$$

Then $\alpha_b = \beta_a$, for any excellent pair (a,b).

Proof: Let $(n+2) = \sum_{i=1}^{s}(-1)^{i-1}2^{r_i}$, where $r_1 > r_2 > \ldots > r_{s-1} > r_s + 1 \geqslant 1$. Let (a,b) be an excellent pair of degree t - see Definition 3.1. Take $M = \sum_{i=1}^{t}(-1)^{i-1}2^{r_i} - 1$. The next Lemma will be mostly used in the second variant of the proof, but here we will need part (2).

Let us denote $\mathcal{L}(M)$ the set of $l < M$ such that $\binom{M-l}{l} = 1 \in \mathbb{Z}/2$.

Lemma 3.5 *Let $M = \sum_{i=1}^{t}(-1)^{i-1}2^{r_i} - 1$, where*

$$r_1 > r_2 > \ldots > r_{t-1} > r_t > 0. \qquad \textit{Then}$$

(1) *All $l \in \mathcal{L}(M)$ are divisible by 2^{r_t}.*

(2) *The largest among such l's is*

$$L = L(M) = \sum_{i=1}^{t-1}(-1)^{i-1}2^{r_i - 1} - \varepsilon(t) \cdot 2^{r_t},$$

where $\varepsilon(t) = 1$, if t is even, and 0, if t is odd.

Proof: (1) Let 2^r be the minimal power of 2 in the binary presentation of l. If $r < r_t$, then the binary decomposition of $M-l$ will not contain 2^r, and so, $\binom{M-l}{l} = 0 \in \mathbb{Z}/2$. Thus, if $\binom{M-l}{l} = 1$, then l is divisible by 2^{r_t}.

(2) For the L as above, we have $M-L = L+2^{r_t}-1$. Since L is divisible by 2^{r_t}, this implies that $\binom{M-L}{L} = 1$. The next number divisible by 2^{r_t} is already greater than $M/2$, so our L is the largest element of $\mathcal{L}(M)$. $\square$

Let $L = L(M)$ as in Lemma 3.5. For each, $0 \leqslant l_1, l_2 \leqslant L$ consider the element

$$S_{n-(a'+b')}(\pi_1)_*((h^{a'} \times h^{b'}) \cdot v),$$

where $a' = a - L + l_1, b' = b - L + l_2$, and S_j are lower Steenrod operations (see [1]). Since the above element belongs to Ch_0, S_j commutes with push-forwards, and Q_1, Q_2 are anisotropic, we have that the degree of the above element modulo 4 is equal to the degree of the element

$$S_{n-(a'+b')}(\pi_2)_*((h^{a'} \times h^{b'}) \cdot v).$$

Thus, we get:

$$\sum_{0\leqslant l_1\leqslant L}\sum_{0\leqslant l_2\leqslant L} \binom{M-l_1}{l_1}\binom{M-l_2}{l_2} \deg(S_{n-(a'+b')}((\pi_1)_* - (\pi_2)_*)((h^{a'}\times h^{b'})\cdot v)) \equiv 0 \,(\,mod\, 4).$$

It remains to prove the following result:

Lemma 3.6

$$\sum_{0\leqslant l_1\leqslant L}\sum_{0\leqslant l_2\leqslant L} \binom{M-l_1}{l_1}\binom{M-l_2}{l_2} \deg(S_{n-(a'+b')}(\pi_1)_*((h^{a'}\times h^{b'})\cdot v)) \equiv 2\cdot\alpha_b \;(\,mod\, 4);$$

$$\sum_{0\leqslant l_1\leqslant L}\sum_{0\leqslant l_2\leqslant L} \binom{M-l_1}{l_1}\binom{M-l_2}{l_2} \deg(S_{n-(a'+b')}(\pi_2)_*((h^{a'}\times h^{b'})\cdot v)) \equiv 2\cdot\beta_a \;(\,mod\, 4).$$

Proof: From symmetry, it is sufficient to prove the first statement. We will need the following easy combinatorial fact:

Lemma 3.7 *For any $0 \leqslant r \leqslant M$, for any $N \geqslant 0$,*

$$\sum_{0\leqslant l\leqslant M} \binom{M-l}{l}\binom{N+l}{r-l} = \binom{M+N+1}{r} \in \mathbb{Z}/2.$$

Proof: It is sufficient to observe that

$$\sum_{0\leqslant l\leqslant M}\binom{M-l}{l}x^l(1+x)^l=\tfrac{(1+x)^{M+1}+(-1)^M x^{M+1}}{1+2x}.$$

□

Now, using the relation $S^\bullet = S_\bullet \cdot c_\bullet(T_X)$, and the multiplicative properties of $S^\bullet$ (see [1]), we have:

$$\sum_{0\leqslant l_1\leqslant L}\sum_{0\leqslant l_2\leqslant L}\binom{M-l_1}{l_1}\binom{M-l_2}{l_2}S_{n-(a'+b')}(\pi_1)_*((h^{a'}\times h^{b'})\cdot v)=$$
$$\sum_{0\leqslant l_1\leqslant L}\sum_{0\leqslant l_2\leqslant L}\binom{M-l_1}{l_1}\binom{M-l_2}{l_2}\sum_{j_1\geqslant 0}\binom{-(n+2)}{j_1}\sum_{k_1\geqslant 0}\binom{a-L+l_1}{k_1}\cdot$$
$$S^{n-(a+b)+2L-(l_1+l_2)-(j_1+k_1)}(\pi_1)_*((1\times h^{b-L+l_2})\cdot v)\cdot h^{a-L+l_1+j_1+k_1}=$$
$$\sum_{0\leqslant l_2\leqslant L}\binom{M-l_2}{l_2}\sum_{p_1\geqslant 0}\binom{M-(n+1)+a-L}{p_1}\cdot$$
$$S^{n-(a+b)+2L-l_2-p_1}(\pi_1)_*((1\times h^{b-L+l_2})\cdot v)\cdot h^{a-L+p_1},$$

(notice, that the sum over the set $0\leqslant l_1\leqslant L$ is the same as over $0\leqslant l_1\leqslant M$, since $\binom{M-l}{l}=0$, for $M\geqslant l>L$, and that $k_1+l_1\leqslant p_1<M$, since $a-L+p_1\leqslant n$). Since (a,b) is an excellent pair of degree t, $M-(n+1)+a-L\geqslant(\varepsilon(t)-1)\sum_{i=t+1}^{s}(-1)^{i-1}2^{r_i}\geqslant 0$. Thus, for the coefficient $\binom{M-(n+1)+a-L}{p_1}$ to be non-trivial, we must have: $p_1\leqslant M-(n+1)+a-L$. On the other hand, $(\pi_1)_*((1\times h^{b-L+l_2})\cdot v)\in \mathrm{Ch}^{b-L+l_2}$, and thus, S^j of this element will be zero, if $j>b-L+l_2$. But (provided, the above coefficient is non-zero), $n-(a+b)+2L-l_2-p_1=2^{r_t-1}-1+2L-l_2-p_1=M-2^{r_t-1}-l_2-p_1\geqslant M-2^{r_t-1}-l_2-M+(n+1)-a+L=b+L-l_2$. Hence, for our term to be non-zero, we should have: $l_2=L$, $p_1=M-(n+1)+a-L$, and in this case,

$$S^{n-(a+b)+2L-l_2-p_1}(\pi_1)_*((1\times h^{b-L+l_2})\cdot v)=$$
$$S^b(\pi_1)_*((1\times h^b)\cdot v)=((\pi_1)_*((1\times h^b)\cdot v))^2.$$

Multiplied by h^{a-L+p_1} this gives the element $h^n\cdot\alpha_b$ of degree $2\cdot\alpha_b$. The Lemma and the Main Theorem are proven. □

□

3.2 Algebraic Cobordism

The second variant of the proof will be using Algebraic Cobordism Ω^*. This is the universal oriented generalised cohomology theory on the category of smooth quasiprojective varieties over the field k of characteristic 0 constructed by M.Levine and F.Morel - see [10].

For any smooth quasiprojective X over k, the additive group $\Omega^*(X)$ is generated by the classes $[v : V \to X]$ of projective maps from smooth varieties V to X subject to certain relations, with the upper grading - the codimensional one. There is natural morphism of theories $pr : \Omega^* \to \mathrm{CH}^*$ given by: $pr([v]) := v_*^{CH}(1_V)$. The main properties of Ω^* are:

(1) $\Omega^*(\mathrm{Spec}(k)) = \mathbb{L} = \mathrm{MU}(pt)$ - the Lazard ring, and the isomorphism is given by the topological realisation functor;

(2) $\mathrm{CH}^*(X) = \Omega^*(X)/\mathbb{L}^{<0} \cdot \Omega^*(X)$.

On Ω^* there is the action of the Landweber-Novikov operations (see [10, Example 4.1.25]). Let $R(\sigma_1, \sigma_2, \ldots) \in \mathbb{L}[\sigma_1, \sigma_2, \ldots]$ be some polynomial, where we assume $\deg(\sigma_i) = i$. Then $S^R_{L-N} : \Omega^* \to \Omega^{*+\deg(R)}$ is given by:

$$S^R_{L-N}([v : V \to X]) := v_*(R(c_1, c_2, \ldots) \cdot 1_V),$$

where $c_j = c_j(\mathcal{N}_v)$, and $\mathcal{N}_v := -T_V + v^*T_X$ is the *virtual normal bundle*.

If $R = \sigma_i$, we will denote the respective operation simply as S^i_{L-N}. The following statement follows from the definition of Steenrod and Landweber-Novikov operations - see P.Brosnan [1], A.Merkurjev [11], and M.Levine [9]

Proposition 3.8 *There is commutative square:*

$$\begin{array}{ccc} \Omega^*(X) & \xrightarrow{S^i_{L-N}} & \Omega^{*+i}(X) \\ \downarrow & & \downarrow \\ \mathrm{Ch}^*(X) & \xrightarrow[S^i]{} & \mathrm{Ch}^{*+i}(X), \end{array}$$

where S^i is the upper Steenrod operation ($mod\, 2$) ([17, 1]).

In particular, using the results of P.Brosnan on S^i (see [1]), we get:

Corollary 3.9 (1) *$pr \circ S^i_{L-N}(\Omega^m) \subset 2 \cdot \mathrm{CH}^{i+m}$, if $i > m$;*

(2) $pr \circ (S^m_{L-N} - \Box)(\Omega^m) \in 2 \cdot \mathrm{CH}^{2m}$, *where* $\Box$ *is the square operation.*

This implies that (modulo 2 − torsion) we have well defined maps $\frac{pr \circ S^i_{L-N}}{2}$ and $\frac{pr \circ (S^m_{L-N} - \Box)}{2}$. In reality, these maps can be lifted to a well defined, so-called, *symmetric operations* $\Phi^{t^{i-m}} : \Omega^m \to \Omega^{m+i}$ - see [16]. Since over algebraically closed field all our varieties are cellular, and thus, the Chow groups of them are torsion-free, we will not need such subtleties, but we will keep the notation from [16], and denote our maps $\Omega^m \to \mathrm{CH}^{m+i}$ as $\phi^{t^{i-m}}$, and the (*mod* 2) version $\Omega^m \to \mathrm{Ch}^{m+i}$ as $\varphi^{t^{i-m}}$ (in our situation below there is no need to mod-out the 2-torsion even over the ground field, since all our maps end up in the CH_0 of quadrics, which are torsion-free, anyway).

Operations $\varphi^{q(t)}$ are "almost" additive. The following statement is clear.

Proposition 3.10 ([16, Proposition 2.8])

$$\varphi^{q(t)}(u' + u'') = \varphi^{q(t)}(u') + \varphi^{q(t)}(u'') + q(0) \cdot pr(u') \cdot pr(u'').$$

From the multiplicative properties of the Landweber-Novikov operations one gets (the "simplified version" of) the following proposition which we will use extensively.

Proposition 3.11 ([16, Proposition 3.15]) *Let* $u \in \mathbb{L}_r$, $r > 0$ *and* $w \in \Omega^m(X)$, *then, for* $i \geqslant (2r - m)$,

$$\varphi^{t^i}(u \cdot v) = \overline{\eta}_2(u) \cdot S^{i-(2r-m)}(pr(w)) \in \mathrm{Ch}^{i-2r}(X),$$

where $\overline{\eta}_2(u) = \frac{\deg(c_r(u))}{2}$ (*mod* 2) *is the Rost invariant, and* S^j *are Steenrod operations.*

It is worth mentioning the values of $\overline{\eta}_2$ on quadrics and projective spaces.

Lemma 3.12 (1) *For the* n*-dimensional quadric* Q_n*, we have:*

$$\overline{\eta}_2(Q_n) = \binom{-(n+2)}{n} \in \mathbb{Z}/2\mathbb{Z}.$$

(2) *For the* n*-dimensional projective space* $\mathbb{P}^n$*, we have:*

$$\overline{\eta}_2(\mathbb{P}^n) = \frac{1}{2}\binom{-(n+1)}{n} \in \mathbb{Z}/2\mathbb{Z}.$$

Proof: It is sufficient to recall that $c_\bullet(-T_{Q_n}) = (1+h)^{-(n+2)}(1+2h)$, while $c_\bullet(-T_{\mathbb{P}^n}) = (1+h)^{-(n+1)}$, and that $\deg(h^n \cdot [Q_n]) = 2$, while $\deg(h^n \cdot [\mathbb{P}^n]) = 1$. $\square$

In the light of Proposition 3.11, let me also recall the action of the Steenrod operations in the Chow groups of quadrics.

Lemma 3.13 *Let Q be smooth projective quadric of dimension n, and h^i and l_i be classes of plane section of codimension i, and of projective subspace of dimension i, respectively. Then*

(1) $S^r(h^i) = \binom{i}{r} h^{i+r}$;

(2) $S^r(l_i) = \binom{n+1-i}{r} l_{i-r}$.

Proof: This follows from the fact that $c_\bullet(\mathcal{N}_{h^i \subset Q_n}) = (1+h)^i$, while $c_\bullet(\mathcal{N}_{l_i \subset Q_n}) = (1+h)^{n+1-i}(1+2h)^{-1}$. $\square$

3.3 Relations from Symmetric Operations

Let us now give the other proof of the Main Theorem. To simplify the notations, we will prove Theorem 3.3, but the reader can see that exactly the same arguments work for Theorem 3.4. Here we will assume that characteristic of k is zero.

Let $v \in \mathrm{Ch}^n(Q \times Q)$. Using the surjection $\Omega^n(Q \times Q) \twoheadrightarrow \mathrm{Ch}^n(Q \times Q)$ we can lift v to some element $w \in \Omega^n(Q \times Q)$. Since over $\overline{k}$ quadric Q becomes cellular with $\Omega^*(Q|_{\overline{k}}) = \oplus_{i=0}^{[n/2]}(\mathbb{L} \cdot h^i \oplus \mathbb{L} \cdot l_i)$, we have:

$$\overline{w} = \sum_{i=0}^{[n/2]} (\tilde{\alpha}_i \cdot (h^i \times l_i) + \tilde{\beta}_i \cdot (l_i \times h^i)) + \tilde{\delta} \cdot (h^{n/2} \times h^{n/2}) +$$
$$\sum_{a=0}^{[n/2]} \sum_{b=0}^{[n/2]} x_{a,b} \cdot (l_a \times l_b) + \sum_{0 \leqslant a < b \leqslant [n/2]} (y_{a,b} \cdot (l_a \times h^b) + z_{a,b} \cdot (h^b \times l_a)),$$

where $\tilde{\alpha}_i, \tilde{\beta}_i, \tilde{\delta} \in \mathbb{Z} = \mathbb{L}_0$, $x_{a,b} \in \mathbb{L}_{n-(a+b)}$, $y_{a,b}, z_{a,b} \in \mathbb{L}_{b-a}$. Clearly, $\alpha_i = \tilde{\alpha}_i \,(\,mod\, 2)$, $\beta_i = \tilde{\beta}_i \,(\,mod\, 2)$, where α_i, β_i are coefficients from the decomposition of v - see 3.1. Let us denote $\gamma_{a,b} := \overline{\eta}_2(x_{a,b}) \,(\,mod\, 2)$. Since Q is not hyperbolic, the number $\gamma_{n/2,n/2} = \frac{x_{n/2,n/2}}{2}$ is integer. It appears that α_i, β_i and $\gamma_{a,b} \in \mathbb{Z}/2$ are the only bits of information about w we need.

We start by obtaining relations involving α's, β's and γ's. This will be done with the help of *symmetric operations* $\varphi^{t^r} : \Omega^m(Q) \to \mathrm{Ch}^{2m+r}(Q)$.

Lemma 3.14 *Consider the pair* $0 \leqslant c, d \leqslant [n/2]$ *such that* $c+2d \leqslant n$. *Then:*

$$w^1_{c,d} := h^c \cdot \varphi^{t^{n-(c+2d)}}(\pi_1)_*((1 \times h^d) \cdot w) \in \mathrm{Ch}_0(Q)$$

is an element of degree

$$\sum_{a=c}^{[n/2]} \binom{n+1-a}{a-c} \cdot \left(\binom{-(n+2-a-d)}{n-a-d} \cdot \beta_a + \gamma_{a,d}\right),$$

while

$$w^2_{d,c} := h^c \cdot \varphi^{t^{n-(c+2d)}}(\pi_2)_*((h^d \times 1) \cdot w) \in \mathrm{Ch}_0(Q)$$

is an element of degree

$$\sum_{a=c}^{[n/2]} \binom{n+1-a}{a-c} \cdot \left(\binom{-(n+2-a-d)}{n-a-d} \cdot \alpha_a + \gamma_{d,a}\right)$$

Remark: The conditions on c and d above are needed to ensure that the operations are defined.

Proof: From symmetry, it is sufficient to prove the first statement. Let us denote $h^c \cdot \varphi^{t^{n-(c+2d)}}(\pi_1)_*((1 \times h^d) \cdot ?)$ as $R(?)$. The degree of our element can be checked over $\overline{k}$. Using the fact that $\varphi^{t^l}|_{\Omega^m}$ is one half of the Chow trace of $(S^{l+m}_{L.N.} - \delta_{l-m,0}\square)$, and applying Lemmas 3.12 and 3.13, we get:

$$\begin{aligned} R(h^b \times l_b) =& \tfrac{1}{2} h^c \cdot (S^{n-(c+d)}_{L.-N.} - \delta_{n-(c+2d),0} \cdot \square)(\pi_1)_*((1 \times h^d) \cdot h^b \times l_b) = \\ & l_0 \cdot \left(\eta_2(\mathbb{P}^{b-d}) \cdot 2\binom{b}{n-c-b} - \delta_{b,d}\delta_{n-(c+2d),0}\right) = 0 \in \mathbb{Z}/2\mathbb{Z} \cdot l_0 \end{aligned}$$

(here we assume $\eta_2(\mathbb{P}^0) = 1/2$);

$$\begin{aligned} R(l_a \times h^a) &= l_0 \cdot \eta_2(Q_{n-a-d})\binom{n+1-a}{a-c} = l_0 \cdot \binom{-(n+2-a-d)}{n-a-d} \cdot \binom{n+1-a}{a-c}; \\ R(x_{a,b} \cdot l_a \times l_b) &= l_0 \cdot \eta_2(x_{a,b}) 2\eta_2(\mathbb{P}^{b-d})\binom{n+1-a}{a-c} = l_0 \cdot \delta_{b,d}\gamma_{a,b}\binom{n+1-a}{a-c}; \\ R(y_{a,b} \cdot (l_a \times h^b)) &= l_0 \cdot \eta_2(y_{a,b}) 2\eta_2(Q_{n-b-d})\binom{n+1-a}{a-c} = 0, \end{aligned}$$

since $\eta_2(Q_{n-b-d})$ is an integer; by similar reason, $R(h^{n/2}\times h^{n/2})=0$; finally,

$$R(z_{a,b}\cdot(h^a\times l_b))=l_0\cdot\eta_2(z_{a,b})2\eta_2(\mathbb{P}^{b-d})2\binom{a}{n-a-c}=0.$$

Taking into account that out of our generators only $h^d\times l_d$ projects non-trivially to Ch^* under $(\pi_1)_*((1\times h^d)\cdot ?)$, and using Proposition 3.10, we get:

$$R(w)\in \mathrm{Ch}_0(Q)\ \text{ has degree} \sum_{a=c}^{[n/2]}\binom{n+1-a}{a-c}\cdot\left(\binom{-(n+2-a-d)}{n-a-d}\cdot\beta_a+\gamma_{a,d}\right).$$

□

Corollary 3.15 *Let Q be anisotropic and $c+2d\leqslant n$. Then, for arbitrary $w\in\Omega^n(Q\times Q)$, we have equalities in $\mathbb{Z}/2\mathbb{Z}$:*

$$\sum_{a=c}^{[n/2]}\binom{n+1-a}{a-c}\cdot\left(\binom{-(n+2-a-d)}{n-a-d}\cdot\beta_a+\gamma_{a,d}\right)=0,\qquad \textit{and}$$

$$\sum_{a=c}^{[n/2]}\binom{n+1-a}{a-c}\cdot\left(\binom{-(n+2-a-d)}{n-a-d}\cdot\alpha_a+\gamma_{d,a}\right)=0.$$

Proof: It follows immediately from Lemma 3.14, since the classes $w^1_{c,d}$, $w^2_{c,d}$ are defined over the ground field, and so their degree must be even.

□

It appears that the Main Theorem follows from the above equations. But to see it we first need to organise them in a more convenient way.

Lemma 3.16 *Let $0\leqslant c,d\leqslant[n/2]$ and $M\in\mathbb{N}$ be such that:*

$$(n+1)-c\leqslant M\leqslant 2(n-(c+d))+1.\qquad \textit{Then}$$

$$\sum_{l=0}^{[n/2]-c}\binom{M-l}{l}\cdot\left(\gamma_{c+l,d}+\binom{-(n-(c+d)-l+2)}{n-(c+d)-l}\cdot\beta_{c+l}\right)=0.$$

Proof: Consider $\sum_{j=0}^{M+c-(n+1)} w^1_{c+j,d} \cdot \binom{M+c-(n+1)}{j}$. Notice, that $(c+j)+2d \leqslant M+2c-(n+1)+2d \leqslant 2n-2(c+d)+1+2(c+d)-n-1 = n$. Thus, $w^1_{c+j,d}$ is defined. We get the relation:

$$0 = \sum_{j=0}^{M+c-(n+1)} \sum_{a=c}^{[n/2]} \binom{n+1-a}{a-c-j} \binom{M+c-(n+1)}{j} \left(\gamma_{a,d} + \binom{-(n-a-d+2)}{n-a-d}\beta_a\right) =$$
$$\sum_{a=c}^{[n/2]} \binom{M-(a-c)}{(a-c)} \left(\gamma_{a,d} + \binom{-(n-a-d+2)}{n-a-d}\beta_a\right).$$

□

Now we have the freedom of picking M in the prescribed bounds. By choosing it appropriately, we will get relations with few nontrivial terms.

Lemma 3.17 *Let M, c, d be as in Lemma 3.16.*

(1) *Suppose for all $l \in \mathcal{L}(M)$ such that $l < L = L(M)$ (see Lemma 3.5), we have: $n-(c+d)-l \neq 2^r - 1$, for any r. Then*

$$\left(\gamma_{c+L,d} + \binom{-(n-(c+d)-L+2)}{n-(c+d)-L} \cdot \beta_{c+L}\right) = \sum_{L>l\in\mathcal{L}(M)} \gamma_{c+l,d}.$$

(2) *If, moreover, $n-(c+d)-L \neq 2^r - 1$, for any r, then*

$$\gamma_{c+L,d} = \sum_{L>l\in\mathcal{L}(M)} \gamma_{c+l,d}.$$

Proof: It is sufficient to notice that $\binom{-(j+2)}{j} = 1 \in \mathbb{Z}/2\mathbb{Z}$ if and only if $j = 2^r - 1$, for some r. □

Let us denote by $u^1(M, c+L, d)$ the relation from Lemma 3.17(1),(2). In the same way, we have "transposed" relations $u^2(M, a, b)$:

$$\left(\gamma_{a,b} + \binom{-(n-(a+b)+2)}{n-(a+b)} \cdot \alpha_b\right) = \sum_{L>l\in\mathcal{L}(M)} \gamma_{a,b-L+l}.$$

From now on we will fix $M = \sum_{i=1}^{t}(-1)^{i-1}2^{r_i} - 1$.

Let us say that $(a, b) \xrightarrow{M^1} (a', b)$, if $a' = a - L + l$, where $L = L(M)$, and $L > l \in \mathcal{L}(M)$. In the same way, we say that $(a, b) \xrightarrow{M^2} (a, b')$, if $b' = b - L + l$. Finally, we say that $(a, b) \xrightarrow{M^{i_1}, M^{i_2}} (a'', b'')$, if $(a, b) \xrightarrow{M^{i_1}} (a_1, b_1) \xrightarrow{M^{i_2}} (a'', b'')$. Clearly, M^1, M^2 gives the same result as M^2, M^1.

Lemma 3.18 *Let (a, b) be exc. pair of degree t, and $(a, b) \xrightarrow{M^{i_1},\ldots,M^{i_p}} (a'', b'')$. Then*

$$n - (a'' + b'') \neq 2^r - 1, \quad \textit{for any } r.$$

Proof: By Lemma 3.5(1), $L(M) - l$ is divisible by 2^{r_t}, for any $l \in \mathcal{L}(M)$. Hence $n - (a'' + b'') \equiv n - (a + b) \equiv 2^{r_t - 1} - 1\,(\mathit{mod}\, 2^{r_t})$. But if $p \geqslant 1$, then $n - (a'' + b'') > n - (a + b) = 2^{r_t - 1} - 1$. Thus, this number $\neq 2^r - 1$, for any r. □

Lemma 3.19 *Let (a, b) be excellent pair of degree t. Then*

$$\gamma_{a,b} + \beta_a = \sum_{L > l \in \mathcal{L}(M)} \gamma_{a+l-L,b}.$$

Proof: Take $c = a - L$, $d = b$. Let us show that the triple (M, c, d) satisfies the conditions of Lemmas 3.16 and 3.17. Evidently, $d \geqslant 0$, while $c \geqslant \sum_{i=1}^{t-1}(-1)^{i-1}2^{r_i - 1} + \varepsilon(t) \cdot \sum_{i=t}^{s}(-1)^{i-1}2^{r_i} - \sum_{i=1}^{t-1}(-1)^{i-1}2^{r_i - 1} + \varepsilon(t) \cdot 2^{r_t} = \varepsilon(t) \cdot \sum_{i=t+1}^{s}(-1)^{i-1}2^{r_i} \geqslant 0$. At the same time, $(n+1) - c \leqslant \sum_{i=1}^{s}(-1)^{i-1}2^{r_i} - 1 - \varepsilon(t) \cdot \sum_{i=t+1}^{s}(-1)^{i-1}2^{r_i} = M + (1 - \varepsilon(t)) \cdot \sum_{i=t+1}^{s}(-1)^{i-1}2^{r_i} \leqslant M$, and $n - (c + d) = 2^{r_t - 1} - 1 + \sum_{i=1}^{t-1}(-1)^{i-1}2^{r_i - 1} - \varepsilon(t) \cdot 2^{r_t} = \sum_{i=1}^{t}(-1)^{i-1}2^{r_i - 1} - 1$, so $2(n - (c + d)) + 1 = M$. The conditions of Lemma 3.17 follow from Lemma 3.18. So, we have the relation $u^1(M, a, b)$. □

Denote $a' := a - L + l,\ L > l \in \mathcal{L}(M)$.

Lemma 3.20 $\gamma_{a',b} = \sum_{L > l \in \mathcal{L}(M)} \gamma_{a',b-L+l}$.

Proof: Take $c = a'$, and $d = b - L$. Again, we need to show that the triple (M, c, d) satisfies the conditions of Lemmas 3.16 and 3.17. The latter follows from Lemma 3.18. As for the former, $a' \geqslant a - L \geqslant 0$, while $d \geqslant 0$ in the same way as $c \geqslant 0$ in the proof of Lemma 3.19 (from symmetry). And the same symmetry implies that $(n+1)-d \leqslant M$. Finally, $2(n-(c+d))+1 = 2(n-(a-L+l+b-L))+1 \geqslant 2(n-(a-L+b))+1 = M$ (as we know). Thus we have the relation $u^2(M, a', b)$. $\square$

As a corollary of Lemmas 3.19, 3.20 we get:

Lemma 3.21 *Let (a, b) be excellent pair of degree t. Then*

$$\gamma_{a,b} + \beta_a = \sum_{a',b'} \gamma_{a',b'},$$

where (a', b') runs over all pairs such that $(a, b) \stackrel{M^1,M^2}{\longrightarrow} (a', b')$.

And from the symmetry, we also have:

$$\gamma_{a,b} + \alpha_b = \sum_{a',b'} \gamma_{a',b'},$$

where (a', b') runs over all pairs such that $(a, b) \stackrel{M^2,M^1}{\longrightarrow} (a', b')$. Since both sets are the same, we get an equality:

$$\gamma_{a,b} + \beta_a = \sum_{a',b'} \gamma_{a',b'} = \gamma_{a,b} + \alpha_a,$$

and consequently,

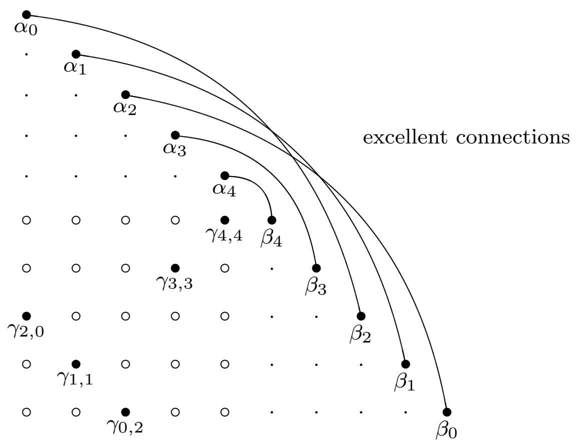
$$\alpha_b = \beta_a.$$

The Main Theorem is proven.

$\square$

Remark: In reality, one can prove that $\alpha_b = \gamma_{a,b} = \beta_a$, but this requires a little bit more of computations.

The following picture shows the excellent connections and pairs for the 9-dimensional anisotropic quadric (each $\cdot$ corresponds to a basis element in $\Omega^*(Q \times Q)$ (or $\mathrm{Ch}^*(Q \times Q)$) (of dimension $\leqslant \dim(Q)$ only), elements of dimension $= \dim(Q)$ and elements $l_a \times l_b$ corresponding to *excellent pairs* are denoted as $\bullet$, other elements $l_a \times l_b$ as $\circ$).



4 Appendix: Upper and lower motives

Let N be a direct summand in the motive of a quadric Q. Motives of quadrics with $\mathbb{Z}$-coefficients carry the same information as such motives with $\mathbb{Z}/2$-coefficients, so we will stick to the latter ones since in this language formulations are sort of simpler. So, from now on all the motives are the objects of $Chow(k, \mathbb{Z}/2)$. Recall that $\Lambda(Q)$ can be identified with the set of *standard projectors* of $M(Q|_{\overline{k}})$ as in [14, Section 4], which gives the canonical decomposition of $M(Q|_{\overline{k}})$ into a direct sum of Tate-motives. By [14, Theorem 5.6], there exists direct summand N' of $M(Q)$ isomorphic to N such that $N'|_{\overline{k}}$ is a direct sum of some part of these fixed Tate-motives. The respective subset of $\Lambda(Q)$ is denoted as $\Lambda(N)$. By [14, Lemma 4.1], $\Lambda(N)$ does not depend on the choice of N' and is well-defined (as long as Q is non-hyperbolic).

For a direct summand N of a non-hyperbolic quadric Q, let us define $\Lambda(N)^{up} := \Lambda(N) \cap \Lambda(Q)^{up}$, and $\Lambda(N)_{lo} := \Lambda(N) \cap \Lambda(Q)_{lo}$.

Let us call the motive N *anisotropic*, if N does not contain any Tate-motive $\mathbb{Z}/2(i)[2i]$ as a direct summand. In particular, if N is a direct summand in the motive of anisotropic quadric, then it is anisotropic ([14, Lemma 3.13]).

Proposition 4.1 *Let N be anisotropic direct summand in the motive of a quadric. Then:*

(1) *There is a complex*

$$0 \to \oplus_{\lambda \in \Lambda(N)^{up}} \mathbb{Z}/2(\lambda)[2\lambda] \stackrel{F^{up}}{\to} N \stackrel{F_{lo}}{\to} \oplus_{\mu \in \Lambda(N)_{lo}} \mathbb{Z}/2(\mu)[2\mu] \to 0,$$

which becomes a split exact sequence over $\overline{k}$.

(2) *For any complex of the form*

$$0 \to \oplus_{\lambda \in \Lambda^{up}} \mathbb{Z}/2(\lambda)[2\lambda] \stackrel{G^{up}}{\to} N \stackrel{G_{lo}}{\to} \oplus_{\mu \in \Lambda_{lo}} \mathbb{Z}/2(\mu)[2\mu] \to 0,$$

which is split exact over $\overline{k}$, *there exist unique identifications:* $\Lambda^{up} = \Lambda(N)^{up}$, $\Lambda_{lo} = \Lambda(N)_{lo}$, $G^{up}|_{\overline{k}} = F^{up}|_{\overline{k}}$, $G_{lo}|_{\overline{k}} = F_{lo}|_{\overline{k}}$.

Proof: (1) By the very definition of $\Lambda(Q)^{up}$ and $\Lambda(Q)_{lo}$, for $\lambda \in \Lambda(Q)^{up}$ and $\mu \in \Lambda(Q)_{lo}$, the canonical morphism $\mathbb{Z}/2(\lambda)[2\lambda] \to M(Q|_{\overline{k}})$ and $M(Q|_{\overline{k}}) \to \mathbb{Z}/2(\mu)[2\mu]$ are defined over the ground field k. So we get the needed maps F^{up} and F_{lo} for the *standard* motive N' isomorphic to N. Since these maps fit into the standard motivic decomposition of $N'|_{\overline{k}}$, the respective sequence is split exact over $\overline{k}$.

(2) If we have any other sequence of a similar sort, then $G_{lo} \circ F^{up} = 0$ and $F_{lo} \circ G^{up} = 0$, since otherwise we would have some composition $\mathbb{Z}/2(\lambda)[2\lambda] \stackrel{f^{\lambda}}{\to} N \stackrel{g_{\lambda}}{\to} \mathbb{Z}/2(\lambda)[2\lambda]$ (respectively, $\mathbb{Z}/2(\lambda)[2\lambda] \stackrel{g^{\lambda}}{\to} N \stackrel{f_{\lambda}}{\to} \mathbb{Z}/2(\lambda)[2\lambda]$) nonzero (since there are no nonzero maps between the Tate-motives of different weight), which would mean that $\mathbb{Z}/2(\lambda)[2\lambda]$ is a direct summand of N, because $\mathrm{End}_{Chow(k,\mathbb{Z}/2)}(\mathbb{Z}/2) = \mathbb{Z}/2$. This immediately identifies Λ^{up} with $\Lambda(N)^{up}$, and Λ_{lo} with $\Lambda(N)_{lo}$, since $\Lambda(N)^{up}$ and $\Lambda(N)_{lo}$ have no more than one Tate-motives of any given weight. And we also get canonical identifications: $G^{up}|_{\overline{k}} = F^{up}|_{\overline{k}}$, $G_{lo}|_{\overline{k}} = F_{lo}|_{\overline{k}}$. □

The following statement shows that separation of $\Lambda(N)$ into lower and upper part depends only on N and not on a particular presentation of N as a direct summand of a motive of some quadric.

Corollary 4.2 *Let* N *be a direct summand of* $M(Q)$, *and* N' *be a direct summand of* $M(Q')$, *where* N *is anisotropic motive, and* $N' \cong N(i)[2i]$. *Then* $\Lambda(N')$ *is naturally identified with* $\Lambda(N) + i$, *and this identification preserves the separation into upper and lower motives.*

Proof: This follows immediately from Proposition 4.1. □

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