Motives of quadrics with applications to the theory of quadratic forms

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0 Introduction

This text is the notes of my lectures at the mini-course "Méthodes géometriques en théorie des formes quadratiques" at the Université d'Artois, Lens, June 26-28, 2000. Although, some additional material is added. I tried to make the material more accessible for the reader. So, complicated technical proofs are presented in a separate Section.

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1 Grothendieck category of Chow motives

Let k be any field, and $SmProj(k)$ - the category of smooth projective varieties over k. We define the category of correspondences $\mathcal{C}(k)$ in the following way: the set $Ob(\mathcal{C}(k))$ is identified with the set $Ob(SmProj(k))$ (the object corresponding to X will be denoted as $[X]$), and if $X = \coprod_i X_i$ - is the decomposition into the disjoint union of connected components, then $Mor_{\mathcal{C}(k)}([X],[Y]) := \oplus_i \mathrm{CH}_{\dim(X_i)}(X_i \times Y)$. The composition of morphisms is defined as follows: if X, Y and Z - are smooth projective varieties over k, and $\varphi \in Mor_{\mathcal{C}(k)}([X],[Y])$, $\psi \in Mor_{\mathcal{C}(k)}([Y],[Z])$, then $\psi \circ \varphi \in Mor_{\mathcal{C}(k)}([X],[Z])$ is defined by the formula: $\psi \circ \varphi := \pi_{XZ*}(\pi^*_{XY}(\varphi) \cap \pi^*_{YZ}(\psi))$,

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where $\pi_{XY}, \pi_{YZ}, \pi_{XZ}$ are partial projections from $X \times Y \times Z$ onto $X \times Y, Y \times Z$ and $X \times Z$, respectively. $\mathcal{C}(k)$ is naturally a tenzor additive category, where $[X] \oplus [Y] := [X \coprod Y]$ and $[X] \otimes [Y] := [X \times Y]$. There is a natural functor $SmProj(k) \to \mathcal{C}(k)$, which sends X to $[X]$ and algebro-geometric morphism $f : X \to Y$ to the class of the graph $\Gamma_f \subset X \times Y$.

Now one can define the *category of effective Chow motives* $Chow^{eff}(k)$ as the pseudo-abelian envelope of the category $\mathcal{C}(k)$. In other words, the set $Ob(Chow^{eff}(k))$ consists of pairs $([X], p_X)$, where X is a smooth projective variety over k, and $p_X \in Mor_{\mathcal{C}(X)}(X, X)$ is a projector ($p_X \circ p_X = p_X$); $Mor_{Chow^{eff}(k)}(([X], p_X), ([Y], p_Y))$ is identified with the subgroup $p_Y \circ Mor_{\mathcal{C}(k)}([X], [Y]) \circ p_X \subset Mor_{\mathcal{C}(k)}([X], [Y])$, and the composition $\circ$ is induced from the category $\mathcal{C}(X)$. The category $Chow^{eff}(k)$ inherits the structure of the tenzor additive category from $\mathcal{C}(k)$. We have the natural functor of tenzor additive categories $\mathcal{C}(k) \to Chow^{eff}(k)$ sending $[X]$ to the pair $([X], \Delta_X)$. The composition $SmProj(k) \to \mathcal{C}(k) \to Chow^{eff}(k)$ will be called the *motivic functor* $X \mapsto M(X)$.

It appears that the object $M(\mathbb{P}^1) \in Chow^{eff}(k)$ is decomposable into a nontrivial direct sum: $M(\mathbb{P}^1) = ([\mathbb{P}^1], p_1) \oplus ([\mathbb{P}^1], p_2)$, where p_1 is defined by the cycle $\mathbb{P}^1 \times pt \subset \mathbb{P}^1 \times \mathbb{P}^1$ and p_2 by the cycle $pt \times \mathbb{P}^1 \subset \mathbb{P}^1 \times \mathbb{P}^1$. It is easy to see that $([\mathbb{P}^1], p_1)$ is isomorphic to the $M(\mathrm{Spec}(k))$; such object is called the *trivial Tate-motive* and will be denoted $\mathbb{Z}$. And the complimentary direct summand $([\mathbb{P}^1], p_2)$ is called the Tate-motive $\mathbb{Z}(1)[2]$. So, $M(\mathbb{P}^1) = \mathbb{Z} \oplus \mathbb{Z}(1)[2]$. For any nonnegative m, one can define: $\mathbb{Z}(m)[2m] := (\mathbb{Z}(1)[2])^{\otimes m}$. Tenzor product by the object $\mathbb{Z}(i)[2i]$ defines the additive functor $U \mapsto U(i)[2i] := U \otimes \mathbb{Z}(i)[2i]$. It is not difficult to show that $M(\mathbb{P}^m) = \mathbb{Z} \oplus \mathbb{Z}(1)[2] \oplus \ldots \oplus \mathbb{Z}(m)[2m]$.

The category of Chow motives $Chow(k)$ can now be defined as follows: $Ob(Chow(k))$ consists of pairs (A, l), where $A \in Ob(Chow^{eff}(k))$ and $l \in \mathbf{Z}$;

$$\mathrm{Hom}_{Chow(k)}((A,l),(B,m)) := \lim_{n \geqslant max(-l,-m)} \mathrm{Hom}_{Chow^{eff}(k)}(A(l+n)[2l+2n], B(m+n)[2m+2n]).$$

The natural functor $Chow^{eff}(k) \to Chow(k)$ sending A to the pair $(A, 0)$ is a full embedding, since the tenzor product with $\mathbb{Z}(1)[2]$ defines an isomorphism $\mathrm{Hom}_{Chow^{eff}(k)}(A, B) \cong \mathrm{Hom}_{Chow^{eff}(k)}(A(1)[2], B(1)[2])$. The composition $SmProj(k) \to Chow^{eff}(k) \to Chow(k)$ still will be called the *motivic functor* and denoted as M.

If X and Y are smooth projective varieties (connected, for simplicity), then $\mathrm{Hom}_{Chow(k)}(M(X), M(Y))$ is naturally identified with $\mathrm{CH}_{\dim(X)}(X \times Y)$, and $\mathrm{Hom}_{Chow(k)}(M(X), M(Y)(i)[2i])$ with $\mathrm{CH}_{\dim(X)-i}(X \times Y)$ (i here can be any integer). In particular, $\mathrm{Hom}_{Chow(k)}(\mathbb{Z}(i)[2i], M(X)) = \mathrm{CH}_i(X)$ and $\mathrm{Hom}_{Chow(k)}(M(X), \mathbb{Z}(i)[2i]) = \mathrm{CH}^i(X)$.

2 The motive and the Chow groups of a hyperbolic quadric

From this point we will assume that our base field k has characteristic different from 2.

Suppose quadratic form q is isotropic, that is: $q = \mathbb{H} \perp p$ for some quadratic form p. Then projective quadric Q has a k-rational point x, and the projective quadric of lines on Q passing through x will be isomorphic to P. This has the following consequence for the structure of the motive of Q.

Proposition 2.1 (M.Rost [22])
Let $q = \mathbb{H} \perp p$. Then $M(Q) \cong \mathbb{Z} \oplus M(P)(1)[2] \oplus \mathbb{Z}(n)[2n]$, where $n = \dim(Q)$.

Proof: Let z, z', u be k-rational points such that $z, z' \in Q$, $u \in \mathbb{P}(V_q)\backslash Q$ and z, z', u are colinear. Consider the cycle: $\Phi_z \in \mathrm{CH}^n(Q \times Q)$ defined as: $\{(x, y) | x, y, z \text{ are colinear}\}$. In the same way, the cycle Φ_u is defined.

We have: $\Phi_u = [\Delta_Q] + [\Gamma_{T_u}]$, where Γ_{T_u} is the graph of the reflection T_u from $O(q)$ with the center u. On the other hand, $\Phi_z = [\Delta_Q] + \Omega_2 + \Omega_3 + \Omega_4$, where $\Omega_2 = [Q \times z]$, $\Omega_3 = [z \times Q]$, and $\Omega_4 = \{(x, y) | x, y \in T_{Q,z} \cap Q; x, y, z \text{ are colinear}\}$. Let $\tau_u, \omega_2, \omega_3, \omega_4 \in \mathrm{End}_{Chow(k)}(M(Q))$ be the corresponding endomorphisms.

Since Φ_z, Φ_u belong to the algebraic family of cycles parametrized by $\mathbb{P}^1 = l(z, z', u)$, they are rationally equivalent. So, $\tau_u = \omega_2 + \omega_3 + \omega_4$. The maps ω_2 and ω_3 are projectors, giving direct summands $\mathbb{Z}$ and $\mathbb{Z}(n)[2n]$ of $M(Q)$, and all three ω_i are mutually orthogonal. So, $id = \tau_u^{\circ 2} = \omega_2 + \omega_3 + \omega_4^{\circ 2}$, and $\omega_4^{\circ 2}$ is a projector too. Thus, $M(Q) = \mathbb{Z} \oplus \mathbb{Z}(n)[2n] \oplus ([Q], \omega_4^{\circ 2})$.

The quadric P can be identified with the intersection $T_{Q,z} \cap T_{Q,z'} \cap Q \subset Q$ and also with the projective quadric of lines on Q passing through z (or through z'). We get cycle $\Psi \in \mathrm{CH}^{n-1}(Q \times P)$: $\{(x, l) | x \in l\}$. It defines maps $\psi : M(Q) \to M(P)(1)[2]$ and $\psi^\vee : M(P)(1)[2] \to M(Q)$.

Then, $\omega_4 = \psi^\vee \circ \psi$, and $\psi \circ \tau_u \circ \psi^\vee = id_{M(P)}$. But ψ and $\psi^\vee$ are orthogonal to ω_2 and ω_3. Thus, $id_{M(P)} = \psi \circ \tau_u \circ \psi^\vee = \psi \circ (\omega_2 + \omega_3 + \omega_4) \circ \psi^\vee = \psi \circ \omega_4 \circ \psi^\vee = \psi \circ \psi^\vee \circ \psi \circ \psi^\vee$. On the other hand, $\psi^\vee \circ \psi \circ \psi^\vee \circ \psi = \omega_4^{\circ 2}$. Thus, the maps ψ and $\psi^\vee \circ \psi \circ \psi^\vee$ define an isomorphism between $([Q], \omega_4^{\circ 2})$ and $M(P)(1)[2]$. □

Applying Proposition 2.1 inductively we get the following.

Proposition 2.2 (M.Rost [22])
Let Q be completely split quadric of dimension n. Then

$$M(Q) = \begin{cases} \sum_{i=0}^{n} \mathbb{Z}(i)[2i], & \textit{if } n \textit{ is odd};\\ (\sum_{i=0}^{n} \mathbb{Z}(i)[2i]) \oplus \mathbb{Z}(n/2)[n], & \textit{if } n \textit{ is even}. \end{cases}$$

In particular, we see that the motive of the smooth odd-dimensional completely split projective quadric is isomorphic to the motive of the projective space of the same dimension.

Because $\mathrm{CH}_i(\mathrm{Spec}(k)) = 0$ for $i \neq 0$, and $\mathrm{CH}_0(\mathrm{Spec}(k)) \cong \mathbf{Z}$, we get that

$$\mathrm{Hom}_{Chow(k)}(\mathbb{Z}(i)[2i], \mathbb{Z}(j)[2j]) \cong \begin{cases} 0 \text{ if } i \neq j;\\ \mathbf{Z} \text{ if i=j}. \end{cases} \qquad (*)$$

Thus, we can compute the Chow groups of a completely split quadric.

Observation 2.3 *Let Q be a completely split quadric of dimension n. Then*

$$\mathrm{CH}_r(Q) = \begin{cases} 0, & \textit{if } r < 0, \ \textit{or } r > n;\\ \mathbf{Z}, & \textit{if } 0 < r < n, \ \textit{and } r \neq n/2;\\ \mathbf{Z} \oplus \mathbf{Z}, & \textit{if } r = n/2. \end{cases}$$

In the situation of a completely split quadric the natural basis for $\mathrm{CH}_r(Q)$ is given by h^{n-r} - the class of plane section of codimension $n-r$ in the case $r > n/2$, by l_r - the class of projective subspace of dimension r, if $r < n/2$, and by $l^1_{n/2}$, $l^2_{n/2}$ - the classes of $n/2$-dimensional projective subspaces from the two different families for $r = n/2$.

Definition 2.4 *For arbitrary quadric Q we define linear function $\overline{\deg}_Q$: $\mathrm{CH}_*(Q|_{\overline{k}}) \to \mathbf{Z}/2$ by the rule that it takes the value 1 on each of the canonical generators described above.*

Remark: Clearly, the particular choice of the generators is important only in the case of $\mathrm{rank}(\mathrm{CH}_r(Q|_{\overline{k}})) = 2$, i.e. $r = n/2$.

If for some smooth projective variety X, the motive $M(X)$ is a direct sum of Tate-motives, then the pairing $\mathrm{End}_{Chow(k)}(M(X)) \otimes \mathrm{CH}_r(X) \to \mathrm{CH}_r(X)$ defines the natural identification: $\mathrm{End}_{Chow(k)}(M(X)) = \times_r \mathrm{End}_{\mathbf{Z}}(\mathrm{CH}_r(X))$ (this follows from (*)). In particular, since over the algebraically closed field quadric is always completely split, we get the item (2) of the following Proposition. In the same way, since $\mathrm{CH}_i(P) = 0$ for $i < 0$ and for $i > \dim(P)$, we get the item (1). The item (2) can be also obtained via the inductive application of (1).

Proposition 2.5 (M.Rost [23])

(1) *Let* $q = \mathbb{H} \perp p$, *then* $\mathrm{End}(M(Q)) = \mathbf{Z} \times \mathrm{End}(M(P)) \times \mathbf{Z}$, *where the first* $\mathbf{Z}$ *is identified with the* $\mathrm{End}_{\mathbf{Z}}(\mathrm{CH}_0(Q))$, *and the last* $\mathbf{Z}$ *- with the* $\mathrm{End}_{\mathbf{Z}}(\mathrm{CH}^0(Q))$.

(2) $\mathrm{End}(M(Q|_{\overline{k}})) = \times_r \mathrm{End}_{\mathbf{Z}}(\mathrm{CH}_r(Q|_{\overline{k}}))$.

We will also need the statement inverse to Proposition 2.1.

Proposition 2.6 *Suppose* q *is some quadratic form, such that* $M(Q)$ *contains* $\mathbb{Z}(l)[2l]$ *as a direct summand. Let* $m = \min(l, \dim(Q) - l)$. *Then* $q = (m+1) \cdot \mathbb{H} \perp q'$.

Proof: If $M(Q)$ contains $\mathbb{Z}(l)[2l]$ as a direct summand, then it also contains $\mathbb{Z}(\dim(Q) - l)[2\dim(Q) - 2l]$ (if $pr \in \mathrm{CH}^{\dim(Q)}(Q \times Q)$ is the corresponding projector, then we can consider the dual one: $pr^{\vee}$, obtained by switching the factors in $Q \times Q$). So, we can assume that $m = l \leqslant \dim(Q)/2$.

We have maps: $\varphi : M(Q) \to \mathbb{Z}(l)[2l]$ and $j : \mathbb{Z}(l)[2l] \to M(Q)$, s.t. $\varphi \circ j = id_{\mathbb{Z}(l)[2l]}$. Via identification: $\mathrm{Hom}(M(Q), \mathbb{Z}(l)[2l]) = \mathrm{CH}^l(Q)$ and $\mathrm{Hom}(\mathbb{Z}(l)[2l], M(Q)) = \mathrm{CH}_l(Q)$ our maps φ and j correspond to the cycles $A \in \mathrm{CH}^l(Q)$, and $B \in \mathrm{CH}_l(Q)$. Then $\varphi \circ j \in \mathrm{Hom}(\mathbb{Z}(l)[2l], \mathbb{Z}(l)[2l]) = \mathrm{CH}_0(\mathrm{Spec}(k)) = \mathbf{Z}$ is given by the degree of intersection $A \cap B \in \mathrm{CH}_0(Q)$. So, $degree(A \cap B) = 1$. This implies: if $l < \dim(Q)/2$, then $degree(B)$ is odd, and if $l = \dim(Q)/2$ then at least one of $degree(A), degree(B)$ is odd. Now, everything follows from:

Lemma 2.6.1 *Let $0 \leqslant l \leqslant \dim(Q)/2$, and Q has l-dimensional cycle of odd degree. Then $q = (l+1) \cdot \mathbb{H} \perp q'$.*

Proof: If $l = 0$ then, by Springer's Theorem (see [18, VII, Theorem 2.3]), we get a rational point on Q. So, q is isotropic.

Suppose the statement is proven for any quadratic form p, and for any $0 \leqslant a < l$. By taking the intersection of A with the plane section of codimension l, we get a zero-cycle of odd degree on Q. So, q is isotropic: $q = \mathbb{H} \perp q'$. Let x be any rational point on $Q \backslash A$ (the set of rational points on isotropic quadric is dense), then Q' can be identified with the projective quadric of lines on Q, passing through x. The union of all lines on Q passing through x is the cone over Q' with the vertex x, and it is just the intersection $R := Q \cap T_x$, where T_x is a tangent space to Q at x. We have natural projection $\pi : R\backslash x \to Q'$. Then $\pi_*(A \cap T_x)$ will be an $(l-1)$-cycle of odd degree on Q'. By induction, q' is l-times isotropic. So, q is $(l+1)$-times isotropic. Lemma is proven. □

Proposition 2.6 is proven. □

3 General theorems

Let now Q be an arbitrary smooth projective quadric. The following theorem, which will be called Rost Nilpotence Theorem in the sequel (RNT for short), gives a very important tool in the study of the motive of Q.

Theorem 3.1 (M.Rost,[23])
Let $\varphi \in \mathrm{End}(M(Q))$.

(1) *If $\varphi|_{\overline{k}} = 0$, then φ is nilpotent.*

(2) *If $\varphi|_{\overline{k}}$ is an isomorphism then φ is an isomorphism.*

As an immediate corollary we get:

Corollary 3.2 ([24, Lemma 3.12])
Let $\xi \in \mathrm{End}(M(Q))$ be some map, s.t. $\xi|_{\overline{k}}$ is a projector. Then, for some d, ξ^{2^d} is a projector.

Proof: Let $x := \xi^2 - \xi \in \mathrm{End}(M(Q)) = \mathrm{CH}^m(Q \times Q)$. Since $\xi|_{\overline{k}}$ is a projector, $x|_{\overline{k}} = 0$. In particular, $2^s \cdot x = 0$, for some s, since Q is hyperbolic over some Galois extension F/k of degree 2^s, and $\mathrm{Tr}_{F/k} \circ j_{F/k}(x) = [F : k] \cdot x$ (here $j_{F/k}$ and $\mathrm{Tr}_{F/k}$ is the restriction and corestriction maps on Chow groups). By Theorem 3.1, we have: $x^t = 0$ for some t.

That means, that for some large d: $\binom{2^d}{j} \cdot x^j = 0$ for any $j > 0$.

From the equality: $\xi^2 = (\xi + x)$ (and the fact that ξ and x commute), we get: $\xi^{2^{d+1}} = \sum_{0 \leqslant j \leqslant 2^d} \binom{2^d}{j} \xi^{2^d - j} \cdot x^j = \xi^{2^d}$. So, ξ^{2^d} is a projector. $\square$

Also we get:

Corollary 3.3 *If N is a direct summand of $M(Q)$, s.t. $N|_{\overline{k}} = 0$, then $N = 0$.*

We call a direct summand N of $M(Q)$ indecomposable if it can not be decomposed into the nontrivial direct sum $N = N_1 \oplus N_2$. Since $M(Q|_{\overline{k}})$ is a direct sum of $2[\dim(Q)/2] + 2$ indecomposable Tate-motives, we get in the light of Corollary 3.3:

Corollary 3.4 *Any direct summand of $M(Q)$ is a direct sum of finitely many indecomposable direct summands.*

For a direct summand N of $M(Q)$ we will denote as $j_N : N \to M(Q)$, and $\varphi_N : M(Q) \to N$ the corresponding natural morphisms, and as $p_N \in \mathrm{End}(M(Q))$ the corresponding projector $j_N \circ \varphi_N$. We can define:

$$\mathrm{CH}_r(N) := p_N \cdot \mathrm{CH}_r(Q) \subset \mathrm{CH}_r(Q),$$

where p_N acts on $\mathrm{CH}_r(Q)$ via pairing: $\mathrm{CH}^{\dim(Q)}(Q\times Q)\otimes\mathrm{CH}_r(Q) \to \mathrm{CH}_r(Q)$. In other words, $\mathrm{CH}_r(N) = \mathrm{Hom}(\mathbb{Z}(r)[2r], N)$.

$N|_{\overline{k}}$ being a direct summand of $M(Q|_{\overline{k}})$ is isomorphic to a direct sum of Tate-motives. In particular, $\mathrm{CH}_r(N|_{\overline{k}})$ is a free abelian group of rank $\leqslant 2$, and if $rank(\mathrm{CH}_r(N|_{\overline{k}})) = 2$, then $r = \dim(Q)/2$ (in particular, $\dim(Q)$ is even), and the natural embedding $\mathrm{CH}_r(N|_{\overline{k}}) \to \mathrm{CH}_r(Q|_{\overline{k}})$ is an isomorphism.

Also, the pairing $\mathrm{Hom}(\mathbb{Z}(r)[2r], N) \otimes \mathrm{Hom}(N, N) \to \mathrm{Hom}(\mathbb{Z}(r)[2r], N)$ defines an isomorphism $\mathrm{End}_{Chow(k)}(N|_{\overline{k}}) \to \times_r \mathrm{End}_{\mathbf{Z}}(\mathrm{CH}_r(N|_{\overline{k}}))$. For a given morphism $\psi \in \mathrm{End}(N)$ we denote as $\psi_{(r)} \in \mathrm{End}_{\mathbf{Z}}(\mathrm{CH}_r(N|_{\overline{k}}))$ the r-th component of $\psi|_{\overline{k}}$ in this decomposition.

Let us choose some basis for $\mathrm{CH}(N|_{\overline{k}})$. In the case $rank(\mathrm{CH}_r(N|_{\overline{k}})) = 1$, we choose arbitrary generator of this group (so, it is canonical up to sign), and in the case $rank(\mathrm{CH}_r(N|_{\overline{k}})) = 2$, we take $\varphi_N(l^1_{\dim(Q)/2})$ and $\varphi_N(l^2_{\dim(Q)/2})$ as basis elements. Now we can represent $\psi_{(r)}$ as a square matrix of size $\leqslant 2$.

Define the canonical linear function $\overline{\deg}_N : \mathrm{CH}(N|_{\overline{k}}) \to \mathbf{Z}/2$ by the rule that it takes the value 1 on each basis element.

Proposition 3.5 *Let N be indecomposable direct summand in $M(Q)$, and $\psi \in \mathrm{End}(N)$ be arbitrary morphism. Then*

$$\textit{either} \quad \overline{\deg}_N \circ \psi = \overline{\deg}_N, \quad \textit{or} \quad \overline{\deg}_N \circ \psi = 0.$$

In particular, to show that $M(Q)$ is decomposable it is sufficient to present morphism $\psi \in \mathrm{End}(M(Q))$ such that $\overline{\deg}_Q \neq \overline{\deg}_Q \circ \psi \neq 0$.

Examples: 1) Suppose the form q is isotropic: $q = q' \perp \mathbb{H}$, that is projective quadric Q posses the rational point z. Then the cycle $Q\times z \subset Q\times Q$ defines a morphism $\rho \in \mathrm{End}(M(Q))$ such that $\rho_{(0)} = 1$ and $\rho_{(r)} = 0$, for all $r \neq 0$. So, $\overline{\deg}_Q \circ \rho$ coincides with $\overline{\deg}_Q$ on the group $\mathrm{CH}_0(Q|_{\overline{k}})$ and is zero on the other Chow groups. Thus, $M(Q)$ is decomposable. Actually, ρ is a projector, defining the direct summand $\mathbb{Z}$ in the decomposition $M(Q) = \mathbb{Z} \oplus M(Q')(1)[2] \oplus \mathbb{Z}(m)[2m]$ (as usually, $m := \dim(Q)$).

2) Let $q = \langle\langle a, b \rangle\rangle$ be a two-fold Pfister form, and C be a conic defined by the form $\langle 1, -a, -b \rangle$. It is not difficult to show that $Q = C \times C$ as an algebraic variety. In particular we get (algebro-geometric!) map $Q \stackrel{pr_1}{\to} C \stackrel{\Delta}{\to} Q$. It induces the motivic map $\psi \in \operatorname{End}(M(Q))$. Clearly, $\psi_{(0)} = 1$ and $\psi_{(2)} = 0$. So, $M(Q)$ is decomposable. Actually, ψ is the projector, defining the direct summand $M(C)$ in the Rost decomposition $M(Q) = M(C) \oplus M(C)(1)[2]$.

Using Proposition 3.5 we can show that the existence of the reasonable maps between the indecomposable motives N_1 and N_2 implies their isomorphism. Namely, we have:

Theorem 3.6 (cf. [24, Lemma 3.25])
Let N_1 and N_2 be indecomposable direct summands in $M(Q_1)(d_1)[2d_1]$ and $M(Q_2)(d_2)[2d_2]$ respectively, for some d_1, d_2. Suppose there exist morphisms $\alpha : N_1 \to N_2$ and $\beta : N_2 \to N_1$, such that the map $\overline{\deg}_{N_1} \circ \beta \circ \alpha : \mathrm{CH}(N_1|_{\overline{k}}) \to \mathbf{Z}/2$ is nonzero. Then $N_1 \cong N_2$.

Corollary 3.7 *Let Q be a smooth projective quadric of dimension m, and N_1, N_2 be indecomposable direct summands of $M(Q)$. If for some $r \neq m/2$, $\mathbb{Z}(r)[2r]$ is a direct summand of $N_1|_{\overline{k}}$ and $N_2|_{\overline{k}}$, then $N_1 \cong N_2$.*

Proof: Under our assumptions, $rank(\mathrm{CH}_r(Q|_{\overline{k}})) = 1$ and the natural embeddings $\mathrm{CH}_r(N_1|_{\overline{k}}) \to \mathrm{CH}_r(Q|_{\overline{k}}) \leftarrow \mathrm{CH}_r(N_2|_{\overline{k}})$ are isomorphisms. Then the composition $\overline{\deg}_{N_1} \circ (\varphi_{N_1} \circ j_{N_2}) \circ (\varphi_{N_2} \circ j_{N_1}) : \mathrm{CH}_r(N_1|_{\overline{k}}) \to \mathbf{Z}/2$ is nonzero. According to Theorem 3.6, $N_1 \cong N_2$. $\square$

Theorem 3.8 ([24, Lemma 3.26])
Suppose Q_1, Q_2 be some smooth projective quadrics, and

$$\alpha \in \operatorname{Hom}(M(Q_1)(d_1)[2d_1], M(Q_2)(d_2)[2d_2]),$$
$$\beta \in \operatorname{Hom}(M(Q_2)(d_2)[2d_2], M(Q_1)(d_1)[2d_1])$$

be such morphisms, that the composition

$$\overline{\deg}_{Q_1} \circ \beta \circ \alpha : \mathrm{CH}_r(M(Q_1)(d_1)[2d_1]|_{\overline{k}}) \to \mathbf{Z}/2$$

is nonzero for some r. Then there exist indecomposable direct summands N_1 of $M(Q_1)(d_1)[2d_1]$, and N_2 of $M(Q_2)(d_2)[2d_2]$, such that $N_1 \simeq N_2$, and $\mathbb{Z}(r)[2r]$ is a direct summand in $N_i|_{\overline{k}}$.

Here are two important cases of such a situation.

Corollary 3.9 *Let Q_1, Q_2 be smooth projective quadrics such that $Q_1|_{k(Q_2)}$ and $Q_2|_{k(Q_1)}$ are isotropic (in other words, there exist rational maps $Q_1 \dashrightarrow Q_2$ and $Q_2 \dashrightarrow Q_1$). Then there are indecomposable direct summands N_1 of $M(Q_1)$ and N_2 of $M(Q_2)$ such that $N_1 \cong N_2$ and $N_1|_{\overline{k}}$ contains $\mathbb{Z}$ as a direct summand.*

Proof: The rational maps $Q_1 \dashrightarrow Q_2$ and $Q_2 \dashrightarrow Q_1$ define motivic maps $\alpha : M(Q_1) \to M(Q_2)$ and $\beta : M(Q_2) \to M(Q_1)$ such that $(\beta \circ \alpha)_{(0)} = 1$. Now we need only to apply Theorem 3.8. □

Corollary 3.10 *Let Q be smooth anisotropic projective quadric, and N be indecomposable direct summand of $M(Q)$ such that $N|_{\overline{k}}$ contains $\mathbb{Z}$ as a direct summand. Then for all $0 \leqslant i < i_1(Q)$, $N(i)[2i]$ is isomorphic to a direct summand of $M(Q)$.*

Proof: Let $0 \leqslant i < i_1(Q)$. Then the quadric $Q|_{k(Q)}$ has a projective subspace L of dimension i. Let $A \subset Q \times Q$ be the closure of $L \subset \mathrm{Spec}(k(Q)) \times Q \subset Q \times Q$. $\dim(A) = \dim(Q) + i$, so A defines a map $\alpha : M(Q)(i)[2i] \to M(Q)$. Let now $\rho^i : M(Q) \to M(Q)(i)[2i]$ be a map, defined by the plane section of codimension i, embedded diagonally into $Q \times Q$. It is easy to see that $(\rho^i \circ \alpha)_{(i)} = 1$. Hence, $\overline{\deg} \circ h^i \circ \alpha : \mathrm{CH}_i(M(Q)(i)[2i]) \to \mathbf{Z}/2$ is nonzero and, by Theorem 3.8, $M(Q)(i)[2i]$ contains indecomposable direct summand N_1, and $M(Q)$ contains indecomposable direct summand N_2 such that $N_1 \cong N_2$ and $\mathbb{Z}(i)[2i]$ is a direct summand of $N_1|_{\overline{k}}$. But, on the other hand, $N(i)[2i]$ is indecomposable direct summand of $M(Q)(i)[2i]$ and $\mathbb{Z}(i)[2i]$ is a direct summand of $N(i)[2i]|_{\overline{k}}$. By Corollary 3.7, $N_1 \cong N(i)[2i]$ (we can clearly assume that $\dim(Q) > 0$, so that the Chow group in question will not be the middle one). Thus, $M(Q)$ contains a direct summand isomorphic to $N(i)[2i]$. □

Theorem 3.11 *Let $N_1, \ldots, N_s$ be non-isomorphic indecomposable direct summands of $M(Q)$. Then $\oplus_{i=1}^{s} N_i$ is isomorphic to a direct summand of $M(Q)$.*

Example: Let $\alpha = \{a_1, \ldots, a_n\}$ be a pure symbol in $\mathrm{K}_n^M(k)/2$, and Q_α be a Pfister quadric, corresponding to the form $\langle\langle a_1, \ldots, a_n \rangle\rangle$. We can use results above to get Rost decomposition of $M(Q_\alpha)$.

Theorem 3.12 (M.Rost, [23])
Let Q_α be anisotropic. Then

$$M(Q_\alpha) \cong \oplus_{i=0}^{2^{n-1}-1} M_\alpha(i)[2i] = M_\alpha \otimes M(\mathbb{P}^{2^{n-1}-1}),$$

where M_α is indecomposable motive, and $M_\alpha|_{\overline{k}} = \mathbb{Z} \oplus \mathbb{Z}(2^{n-1}-1)[2^n-2]$.

Proof: Let M_α be indecomposable direct summand of $M(Q_\alpha)$ such that $\mathbb{Z}$ is a direct summand of $M_\alpha|_{\overline{k}}$. Then, by Corollary 3.10, $M_\alpha(i)[2i]$ is isomorphic to a direct summand of $M(Q)$, for any $0 \leqslant i < i_1(q_\alpha) = 2^{n-1}$. Clearly, for $i \neq j$, $M_\alpha(i)[2i]$ is not isomorphic to $M_\alpha(j)[2j]$ (since they are not isomorphic even over $\overline{k}$). By Theorem 3.11, $\oplus_{i=0}^{2^{n-1}-1} M_\alpha(i)[2i]$ is a direct summand of $M(Q)$. We will need the following easy Lemma.

Lemma 3.12.1 *Let Q be smooth projective quadric, and L be such direct summand of $M(Q)$ that $L|_{\overline{k}} = \mathbb{Z}$. Then Q is isotropic.*

Proof: Let $A \subset Q \times Q$ be the cycle representing the projector $p_L \in \mathrm{End}(M(Q)) = \mathrm{CH}^{\dim(Q)}(Q \times Q)$. Then $A|_{\overline{k}}$ must be rationally equivalent to $Q \times pt$. In particular, $[A \cap A^\vee|_{\overline{k}}]$ represents the class of a rational point on $Q \times Q|_{\overline{k}}$. So, the degree of the 0-cycle $[A \cap A^\vee]$ is 1 and, by Springer's theorem, Q is isotropic. $\square$

Lemma 3.12.1 implies that $M_\alpha|_{\overline{k}}$ consists of at least 2 Tate- motives. Then $\oplus_{i=0}^{2^{n-1}-1} M_\alpha(i)[2i]|_{\overline{k}}$ contains at least as many Tate-motives as $M(Q_\alpha)|_{\overline{k}}$ does. By Corollary 3.3, $M(Q) \cong \oplus_{i=0}^{2^{n-1}-1} M_\alpha(i)[2i]$. Clearly, $M_\alpha|_{\overline{k}} = \mathbb{Z} \oplus \mathbb{Z}(r)[2r]$, and $r = 2^{n-1}-1$. $\square$

The motive M_α is called the Rost-motive. For $n = 1$, $M_{\{a\}} = M(k(\sqrt{a}))$, and for $n = 2$, $M_{\{a,b\}} = M(C_{\{a,b\}})$, where $C_{\{a,b\}}$ is the conic, corresponding to the form $\langle 1, -a, -b \rangle$.

4 Indecomposable direct summands in the motives of quadrics

In this section we will present some results on the structure of indecomposable direct summands of the motives of quadrics.

Let Q be smooth projective quadric of dimension m. By Proposition 2.2, $M(Q|_{\overline{k}})$ is a direct sum of Tate-motives. Let us choose this decomposition in some fixed way. If $l \neq m/2$, then the direct summand $\mathbb{Z}(l)[2l]$ of $M(Q|_{\overline{k}})$ is defined uniquely. And for $l = m/2$, we choose the corresponding projectors as $(l^2_{m/2} - l^1_{m/2}) \times l^{\overline{2}}_{m/2} \subset (Q \times Q)|_{\overline{k}}$ and $l^1_{m/2} \times (l^2_{m/2} + l^1_{m/2})$, where $\overline{2} = \begin{cases} 2, \text{ if } m \equiv 0 \,(\,mod\, 4\,) \\ 1, \text{ if } m \equiv 2 \,(\,mod\, 4\,) \end{cases}$. We call the corresponding motives $L_{lo} \cong \mathbb{Z}(m/2)[m]$ and $L^{up} \cong \mathbb{Z}(m/2)[m]$ the *lower one* and the *upper one*, respectively. In particular, the restriction $\overline{\deg}_Q : \mathrm{CH}_{m/2}(L^{up}) \to \mathbf{Z}/2$ is zero, and the restriction $\overline{\deg}_Q : \mathrm{CH}_{m/2}(L_{lo}) \to \mathbf{Z}/2$ is surjective.

Let us denote the set of fixed Tate-motivic summands specified above as $\Lambda(Q)$. It follows from Definition 6.5, Theorem 6.6, that for arbitrary direct summand N of $M(Q)$, there exists direct summand N' isomorphic to N such that $N'|_{\overline{k}}$ being a summand of $M(Q)|_{\overline{k}}$ is a direct sum of some part of these fixed Tate-motives.

For the direct summand N of $M(Q)$ let us denote as $\Lambda(N)$ the subset of $\Lambda(Q)$ consisting of fixed Tate-motives from the decomposition of $N'|_{\overline{k}}$.

Lemma 4.1 *Let Q be smooth non-hyperbolic projective quadric. Then the subset $\Lambda(N) \subset \Lambda(Q)$ does not depend on the choice of N', and so, is correctly defined and depends only on the isomorphism class of N.*

Proof: Suppose that $N' \cong N \cong N''$, and the sets of fixed Tate-motives in the decomposition of $N'|_{\overline{k}}$ and $N''|_{\overline{k}}$ are different. Let $\mathbb{Z}(l)[2l]$ be some fixed Tate-motive from the decomposition of $N'|_{\overline{k}}$, which is not in the $N''|_{\overline{k}}$. Then $l = m/2$ (because in all other degrees there is only one Tate-motive available, and $N' \cong N''$). Also, $N'|_{\overline{k}}$ and $N''|_{\overline{k}}$ should contain only one Tate-motive of the type $\mathbb{Z}(m/2)[m]$ each. So, we assume that $N'|_{\overline{k}}$ contains L^{up}, and $N''|_{\overline{k}}$ contains L_{lo}. Now we can assume that N is indecomposable. If Q is not hyperbolic, then (by Lemma 3.12.1) both $N'|_{\overline{k}}$ and $N''|_{\overline{k}}$ should contain at least one more (this time, common) Tate-motive $\mathbb{Z}(r)[2r]$, where $r \neq m/2$. Then the map $\overline{\deg}_{N'} \circ (\varphi_{N'} \circ j_{N''}) \circ (\varphi_{N''} \circ j_{N'}) : \mathrm{CH}_r(N'|_{\overline{k}}) \to \mathbf{Z}/2$ is nonzero. By Proposition 3.5, $\overline{\deg}_{N'} \circ (\varphi_{N'} \circ j_{N''}) \circ (\varphi_{N''} \circ j_{N'}) : \mathrm{CH}_{m/2}(N'|_{\overline{k}}) \to \mathbf{Z}/2$ should be nonzero as well. But the map $(\varphi_{L_{lo}} \circ j_{L^{up}}) : L^{up} \to L_{lo}$ is zero - contradiction. □

Clearly, in the hyperbolic case, there is a problem only with the middle-dimensional part.

We can now state the more precise version of Corollary 3.7.

Lemma 4.2 *Let Q be smooth non-hyperbolic projective quadric, and N, M be non-isomorphic indecomposable direct summands of $M(Q)$. Then $\Lambda(N) \cap \Lambda(M) = \emptyset$.*

Proof: Suppose $\mathbb{Z}(i)[2i] \in \Lambda(N) \cap \Lambda(M)$. By Corollary 3.7, $i = m/2$. Pick from N, M the one with the minimal $rank(\mathrm{CH}_{m/2}(|_{\overline{k}}))$. Let it be M. Then the map $\overline{\deg}_M \circ (\varphi_M \circ j_N) \circ (\varphi_N \circ j_M) : \mathrm{CH}_i(M|_{\overline{k}}) \to \mathbf{Z}/2$ is nonzero, so, by Theorem 3.6, M must be isomorphic to N - a contradiction. □

Lemma 4.2, evidently, implies:

Theorem 4.3 *Let $\mathbb{Z}(i)[2i]$ and $\mathbb{Z}(j)[2j]$ be some elements of $\Lambda(Q)$. The following conditions are equivalent:*

(1) *For any direct summand N of $M(Q)$ the conditions: $\mathbb{Z}(i)[2i] \in \Lambda(N)$ and $\mathbb{Z}(j)[2j] \in \Lambda(N)$ are equivalent.*

(2) *There exists indecomposable direct summand N such that $\mathbb{Z}(i)[2i] \in \Lambda(N)$ and $\mathbb{Z}(j)[2j] \in \Lambda(N)$.*

If these conditions are satisfied we say that $\mathbb{Z}(i)[2i]$ and $\mathbb{Z}(j)[2j]$ are connected. Clearly, this is an equivalence relation.

Let $Z(Q)$ be the set of the isomorphism classes of indecomposable direct summands of $M(Q)$, and N_z be a representative of the class z. We have the following:

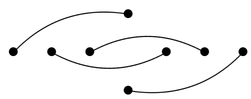
Corollary 4.4 *Let Q be non-hyperbolic quadric. Then:*

(1) $\Lambda(Q) = \bigsqcup_{z \in Z(Q)} \Lambda(N_z)$

(2) $M(Q) \cong \oplus_{z \in Z(Q)} N_z$.

And $\Lambda(N_z)$, for $z \in Z(Q)$ are exactly the connected components of $\Lambda(Q)$.

We can visualize this decomposition by denoting each Tate-motive from $\Lambda(Q)$ by a $\bullet$, and connecting the $\bullet$'s for which Tate-motives are *connected*.

Example: $M(Q_{\{a_1,a_2,a_3\}})$ will look as: 

(here we put L_{up} above L_{lo}, and the degrees of Tate-motives are increasing from left to right).

We already saw (Lemma 3.12.1) that the direct summand L of the anisotropic quadric can not be a *form* of a Tate-motive, that is, $L|_{\overline{k}}$ consists of at least 2 Tate-motives. It appears that $L|_{\overline{k}}$ is always the direct sum of even number of Tate-motives, and we can provide some restrictions on their degrees.

The following result is basic here.

Proposition 4.5 (cf. [25, proof of Statement])
Let Q be anisotropic quadric of dimension m with $i_1(q) = 1$. Let N be a direct summand of $M(Q)$, such that $N|_{\overline{k}}$ contains $\mathbb{Z}$. Then it contains $\mathbb{Z}(m)[2m]$.

In other words, if $i_1(q) = 1$, then $\mathbb{Z}$ is connected to $\mathbb{Z}(m)[2m]$.

Definition 4.6 *Let Q be a smooth projective quadric and N be some direct summand of $M(Q)$. Define:*

$$a(N) := \min(r|\ \mathrm{CH}_r(N|_{\overline{k}}) \neq 0);$$

$$b(N) := \max(r|\ \mathrm{CH}_r(N|_{\overline{k}}) \neq 0);$$

$$\mathrm{size}(N) := b(N) - a(N).$$

Clearly, $a(M(Q)) = 0$; $b(M(Q)) = \dim(Q)$ and $\mathrm{size}(M(Q)) = \dim(Q)$.

We can reformulate Proposition 4.5 as follows: If $i_1(Q) = 1$, then for arbitrary direct summand N of $M(Q)$, the condition $a(N) = 0$ is equivalent to the condition $b(N) = \dim(Q)$.

From Proposition 4.5 it is not difficult to deduce:

Corollary 4.7 *Let Q be smooth anisotropic projective quadric, and N be indecomposable direct summand of $M(Q)$ such that $a(N) = 0$. Then*

$$\mathrm{size}(N) = \dim(Q) - i_1(q) + 1.$$

Proposition 4.8 *Let Q be smooth anisotropic projective quadric of dimension m, and N be direct summand of $M(Q)$ such that the map $\overline{\deg}_Q : \mathrm{CH}_a(N|_{\overline{k}}) \to \mathbf{Z}/2$ with $0 \leqslant a \leqslant i_1(q)$ is nonzero (in other words, $\mathbb{Z}_{lo}(a)[2a]$ belongs to $\Lambda(N)$). Then $N|_{\overline{k}}$ contains $\mathbb{Z}(a)[2a] \oplus \mathbb{Z}(b)[2b]$ as a direct summand, where $b = m - i_1(q) + 1 + a$.*

Corollary 4.9 ([25, Corollary 3])

Let P, Q be smooth anisotropic quadrics over the field k. Then:

1) *If $q|_{k(P)}$ and $p|_{k(Q)}$ are isotropic, then*

$$\dim(q) - i_1(q) = \dim(p) - i_1(p).$$

2) *If $P \subset Q$ is a subquadric, s.t. $p|_{k(Q)}$ is isotopic, then*

$$codim(P \subset Q) < i_1(q).$$

3) *In the situation of 2), $i_1(p) = i_1(q) - codim(P \subset Q)$.*

Proof: 1) Since $q|_{k(P)}$ and $p|_{k(Q)}$ are isotropic, by Corollary 3.9, there exist isomorphic direct summands N of $M(Q)$, and M of $M(P)$ such that $a(N) = 0$. By Corollary 4.7, $\mathrm{size}(N) = \dim(Q) - i_1(q) + 1$, and $\mathrm{size}(M) = \dim(P) - i_1(p) + 1$. Since $N \simeq M$, we get the equality.

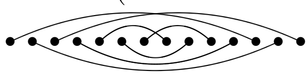
2) and 3) follow from 1), taking into account that $i_1(p) \geqslant 1$. □

Let $k = F_0 \subset F_1 \subset \ldots \subset F_{h(q)}$ be the generalized splitting tower of fields for the quadric Q (see [15, §5]). Applying Proposition 4.8 to the form $q_t := (q|_{F_t})_{an.}$, we get:

Proposition 4.10 ([25, Statement])

Let Q be smooth anisotropic quadric of dimension m, and N be a direct summand of $M(Q)$ such that the map $\overline{\deg}_Q : \mathrm{CH}_a(N|_{\overline{k}}) \to \mathbf{Z}/2$ is nonzero for some $i_W(q|_{F_t}) \leqslant a < i_W(q|_{F_{t+1}})$.
Then $N|_{\overline{k}}$ contains $\mathbb{Z}(a)[2a] \oplus \mathbb{Z}(b)[2b]$ as a direct summand, where $b = m - i_W(q|_{F_t}) - i_W(q|_{F_{t+1}}) + 1 + a$.

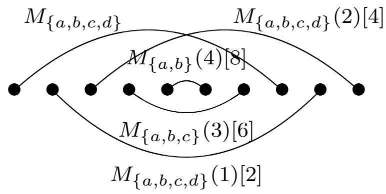
Proposition 4.10 shows that all Tate-motives in $M(Q|_{\overline{k}})$ come in pairs, and the structure of these pairs is determined by the splitting pattern of a quadric.

Example: For the motive of a quadric Q with the splitting pattern $(3,1,3)$ we have the following necessary connections (not to mess with the decomposition into connected components): 

Corollary 4.11 *Let Q be smooth anisotropic quadric, and N be direct summand of $M(Q)$. Then $N|_{\overline{k}}$ consists of even number of Tate-motives.*

Certainly, the binary connections, specified in Proposition 4.10, in general, are not all the existing connections among the elements of $\Lambda(Q)$. For example, if Q is generic quadric (given by the form $\langle x_1,\ldots,x_n\rangle$ over the field $k(x_1,\ldots,x_n)$), then $M(Q)$ is indecomposable, and so, all elements of $\Lambda(Q)$ are connected. Nevertheless, we have the situation, where all indecomposable direct summands are binary.

Example: Let Q be excellent quadric (see [16, Definition 7.7]). Then, by the result of M.Rost ([22, Proposition 4]), $M(Q)$ is a direct sum of binary Rost-motives. For example, if $q = (\langle\langle a,b,c,d\rangle\rangle \perp -\langle\langle a,b,c\rangle\rangle \perp \langle\langle a,b\rangle\rangle \perp -\langle 1\rangle)_{an.}$, then $M(Q)$ looks as:



Hypothetically, the excellent quadrics should be the only ones having such a property.

Conjecture 4.12 *Let Q be smooth anisotropic projective quadric. The following two conditions are equivalent:*

(1) *$M(Q)$ consists of binary motives.*

(2) *Q is excellent.*

At the same time, we have some results which guarantee that particular elements of $\Lambda(Q)$ are not connected. Namely, Corollary 3.10 together with Lemma 4.2 shows that the Tate-motives $\mathbb{Z}, \mathbb{Z}(1)[2], \ldots, \mathbb{Z}(i_1(q)-1)[2i_1(q)-2]$ all belong to different connected components of $\Lambda(Q)$. Here is the generalization of this result.

Theorem 4.13 ([25, Corollary 2])

Let $M(Q)$ be smooth projective quadric, and N be an indecomposable direct summand of $M(Q)$ such that $i_W(q|_{F_t}) \leqslant a(N) < i_W(q|_{F_{t+1}})$. Then for each $i_W(q|_{F_t}) \leqslant j < i_W(q|_{F_{t+1}})$, the motive $N(j - a(N))[2j - 2a(N)]$ is isomorphic to a direct summand of $M(Q)$.

Theorem 4.13 implies that if there exists direct summand N of $M(Q)$ such that $i_W(q|_{F_t}) \leqslant a(N) < i_W(q|_{F_{t+1}})$, then the Tate-motives $\mathbb{Z}(j)[2j]$, for different $i_W(q|_{F_t}) \leqslant j < i_W(q|_{F_{t+1}})$, are not connected. In particular, the binary connections specified above will be the only connections among the elements of $\Lambda(Q)$ if and only if, for arbitrary $1 \leqslant t \leqslant h(q)$, there exists direct summand N_t of $M(Q)$ such that $i_W(q|_{F_t}) \leqslant a(N_t) < i_W(q|_{F_{t+1}})$.

Combining Theorem 4.13 with Proposition 4.10 and Corollary 3.7, we get

Corollary 4.14 ([25, Statement])

Let Q be smooth projective anisotropic quadric, and N be indecomposable direct summand of $M(Q)$ such that $i_W(q|_{F_t}) \leqslant a(N) < i_W(q|_{F_{t+1}})$. Then

$$\text{size}(N) = \dim(Q) - i_W(q|_{F_t}) - i_W(q|_{F_{t+1}}) + 1.$$

In particular, $i_W(q|_{F_t}) \leqslant \dim(Q) - b(N) < i_W(q|_{F_{t+1}})$.

Corollary 4.14 shows that the size of the indecomposable direct summand is determined by the place where it starts and the splitting pattern of a quadric.

The following statement provides sufficient condition for the existence of a direct summand L with $a(L) = l$.

Theorem 4.15 ([25, Proposition 1])

Let Q and P be smooth projective quadrics, and $l \in \mathbb{N}$. Suppose, for arbitrary field extension E/k, the conditions: $i_W(p|_E) > 0$ and $i_W(q|_E) > l$ are equivalent. Then $M(Q)$ has an indecomposable direct summand L, and $M(P)$ has an indecomposable direct summand N, such that $a(L) = l$, $a(N) = 0$, and $L \cong N(l)[2l]$.

The natural question arises: if the converse is true as well?

Question 4.16 ([25, Question 1])

Are the following conditions equivalent?

(1) *Q contains a direct summand L with $a(L) = l$.*

(2) *There exists quadric P/k, such that, for arbitrary field extension E/k, the conditions: $i_W(p|_E) > 0$ and $i_W(q|_E) > l$ are equivalent.*

The following stronger version of Theorem 4.15 is often useful.

Theorem 4.17 *Let Q and P be smooth projective quadrics, and $n, m \in \mathbb{N}$. Suppose, for arbitrary field extension E/k, the conditions: $i_W(p|_E) > n$ and $i_W(q|_E) > m$ are equivalent. Suppose $M(P)$ has an indecomposable direct summand N such that $a(N) = n$. Then $M(Q)$ has an indecomposable direct summand $M \cong N(m-n)[2m-2n]$. In particular, $a(M) = m$.*

As a corollary we get the criterion of motivic equivalence for quadrics.

Theorem 4.18 ([24, Theorem 1.4.1], see also [13])
Let P and Q be smooth projective quadrics of the same dimension. Then the following conditions are equivalent:

(1) *$M(P) \cong M(Q)$;*

(2) *For arbitrary field extension E/k, $i_W(p|_E) = i_W(q|_E)$.*

Proof: (1) $\Rightarrow$ (2): By Proposition 2.1 and Proposition 2.6, $i_W(p|_E)$ is equal to the half the number of Tate-motives which split from $M(P|_E)$. Since $M(P|_E) \cong M(Q|_E)$, we get desired equality.

(2) $\Rightarrow$ (1): We can clearly assume that both our quadrics are not hyperbolic. Then, by Corollary 4.4, $M(Q) \cong \oplus_{z \in Z(Q)} N_z$, and $M(P) \cong \oplus_{y \in Z(P)} M_y$, where $Z(Q)$ and $Z(P)$ are sets of isomorphism classes of indecomposable direct summands of $M(Q)$ and $M(P)$, respectively. By Theorem 4.17, for each $z \in Z(Q)$ there exists $y(z) \in Z(P)$, such that $M_{y(z)} \cong N_z$, and vise-versa, for each $y \in Z(P)$ there exists $z(y) \in Z(Q)$, such that $N_{z(y)} \cong M_y$. This gives us bijection: $Z(Q) = Z(P)$, and an isomorphism $M(Q) \cong M(P)$. $\square$

Another restriction on the structure of the indecomposable direct summands comes from the fact that such motives are symmetric with respect to flipping over. That is, $N^\vee \cong N(j)[2j]$, for some j (here $N^\vee$ is the direct summand dual to N).

Theorem 4.19 ([25, Corollary 1])

Let Q be smooth projective anisotropic quadric of dimension m, and N be indecomposable direct summand of $M(Q)$. Then

$$N^{\vee} \cong N(r)[2r], \quad \textit{where } r = m - a(N) - b(N).$$

Proof: It is clear that proving the statement for N is equivalent proving it for $N^{\vee}$. So, we can assume that either $a(N) = b(N) = m/2$, or $b(N) \geqslant m/2$. In the former case, $N|_{\overline{k}} = \mathbb{Z}(m/2)[m] \oplus \mathbb{Z}(m/2)[m] = N^{\vee}|_{\overline{k}}$ (since it should consists of at least 2 Tate-motives), and so, $N \cong N^{\vee}$, by RNT. So, we can assume that $b(N) \geqslant m/2$. On the other hand, by Corollary 4.14, there exists $1 \leqslant t < h(Q)$ such that $i_W(q|_{F_t}) \leqslant a(N), (m - b(N)) < i_W(q|_{F_{t+1}})$. By Theorem 4.13, for $r = m - a(N) - b(N)$, $N(r)[2r]$ is isomorphic to a direct summand of $M(Q)$, and $a(N(r)[2r]) = m - b(N) = a(N^{\vee})$. By Corollary 3.7, $N^{\vee} \cong N(r)[2r]$. □

As we saw above, if N is indecomposable direct summand of the motive of anisotropic quadric, then $N|_{\overline{k}}$ consists of even number of Tate-motives. In the case, when $N|_{\overline{k}}$ is binary, we have severe restrictions on it's size.

Theorem 4.20 ([9, Theorem 6.1])

Let Q be smooth anisotropic projective quadric, and N be a direct summand of $M(Q)$, such that $N|_{\overline{k}} = \mathbb{Z}(a)[2a] \oplus \mathbb{Z}(b)[2b]$. Then $\mathrm{size}(N) = 2^r - 1$, *for some r.*

The proof of Theorem 4.20 uses the techniques developed by V.Voevodsky for the proof of Milnor conjecture (see [28]). In particular, one has to work in the bigger triangulated category of mixed motives $DM^{eff}(k)$ (see [27]) and use motivic cohomological operations of V.Voevodsky.

Remark: Originally, Theorem 4.20 was proven in the assumption that $char(k) = 0$, since at that time the technique of V.Voevodsky required such an assumption. Hopefully, due to the new results of V.Voevodsky ([29]), we can now just assume that $char(k) \neq 2$.

One can notice that the size of binary motives takes the same values as the size of Rost-motives. Moreover, we can state:

Conjecture 4.21 ([5, Conjecture 3.2],[26, Conjecture 2.8])

Let Q be smooth anisotropic quadric, and N be a binary direct summand of $M(Q)$. Then there exists $r \in \mathbb{N}$, and pure symbol $\alpha \in \mathrm{K}_r^M(k)/2$ such that $N \cong M_\alpha(j)[2j]$ for some j.

It is not difficult to show that Conjecture 4.21 implies Conjecture 4.12. Moreover, Theorem 4.20 shows that if $M(Q)$ consists of binary motives, then the splitting pattern of Q coincides with the splitting pattern of the excellent quadric of the same dimension. It gives some ground for the following important conjecture on the decomposition of the motive of a quadric. Let Q and P be some anisotropic quadrics of the same dimension. Then $\Lambda(Q)$ can be naturally identified with $\Lambda(P)$.

Conjecture 4.22 *Let Q be smooth anisotropic quadric, and P be excellent quadric of the same dimension. Let $\Lambda(Q) \stackrel{\varphi}{=} \Lambda(P)$ be the natural identification. Then: $\varphi(\lambda)$ connected to $\varphi(\mu) \Rightarrow \lambda$ connected to μ.*

Conjecture 4.22 says that aside from binary connections, corresponding to the splitting pattern of Q (Proposition 4.10), we should have binary connections, corresponding to the excellent splitting pattern. Moreover, we get not just one additional set of binary connections, but $h(Q)$ such sets, since we can apply Conjecture 4.22 to $q_t := (q|_{F_t})_{an.}$, for $1 \leqslant t \leqslant h(Q)$. In particular, the more splitting pattern of Q differs from the excellent splitting pattern, the less decomposable $M(Q)$ should be.

5 Some applications

In this section we list some applications of the technique described above.

Higher forms of the motives of quadrics

In Theorem 3.12 it was shown that the motive of a Pfister quadric $Q_{\{a_1,\ldots,a_n\}}$ decomposes into 2^{n-1} pieces isomorphic up to shift by the Tate-motive. This appears to be the particular case of the following general result.

Theorem 5.1 ([24, Theorem 4.1])
Let $\alpha = \{a_1, \ldots, a_n\} \in \mathrm{K}_n^M(k)/2$ be some pure symbol, p some (nondegenerate) quadratic form, and $r := \langle\langle \alpha \rangle\rangle \cdot p$. Then there exists some direct summand $F_\alpha(M(P))$ of $M(R)$ such that

$M(R) = F_\alpha(M(P)) \otimes M(\mathbb{P}^{2^n-1})$, *if* $\dim(p)$ *is even, and*

$M(R) = (F_\alpha(M(P)) \otimes M(\mathbb{P}^{2^n-1})) \oplus M(Q_\alpha)(a)[2a]$, *where* $a = \dim(R)/2 - 2^{n-1} + 1$, *if* $\dim(p)$ *is odd.*

Proof: We use the following well-known Lemma.

Lemma 5.2 *If form r is divisible by n-fold Pfister form $\langle\langle\alpha\rangle\rangle$, then $i_t(r)$, for $0 \leqslant t < h(r)$ as well as $(i_{h(r)}(r) + \dim(r)/2)$ is divisible by 2^n.*

Proof: Let $F_0 \subset \ldots \subset F_{h(r)}$ be the generalized splitting tower for r, and $0 \leqslant t \leqslant h(r)$. Then $r_{t-1} := (r|_{F_{t-1}})_{an.} = \langle\langle\alpha\rangle\rangle \cdot p_{t-1}$ and $r_t := (r|_{F_t})_{an.} = \langle\langle\alpha\rangle\rangle \cdot p_t$ for some forms p_{t-1}/F_{t-1} and p_t/F_t. If the difference $\dim(p_t) - \dim(p_{t-1})$ is odd, then one of the forms r_{t-1}, r_t is in I^{n+1} and another is not. Clearly, then $r_t \in I^{n+1}(F_t)$ and $r_{t-1} \notin I^{n+1}(F_{t-1})$. More precisely, $r_{t-1} \equiv \langle\langle\alpha\rangle\rangle\,(\,mod\, I^{n+1}(F_{t-1})\,)$. Then the form $\langle\langle\alpha\rangle\rangle|_{F_t}$ must be hyperbolic. But if $t < h(r)$, then F_t is obtained from k inductively by adjoining the generic points of quadrics of dimension $> 2^n - 2$. Hence $\langle\langle\alpha\rangle\rangle$ was hyperbolic already over the base field, r is hyperbolic, $0 = t = h(r)$ - contradiction. This shows that for $t < h(r)$, the difference $\dim(p_t) - \dim(p_{t-1})$ is even and $i_t(r)$ is divisible by 2^n. Since $i_{h(r)} = \dim(r)/2 - \sum_{t<h(r)} i_t(r)$, we get the statement. $\square$

Let $0 < t \leqslant h(r)$, and Y_t be the set of isomorphism classes of indecomposable direct summands N of $M(R)$, such that $i_W(r|_{F_{t-1}}) \leqslant a(N) < i_W(r|_{F_t})$. By Theorem 4.13 and Lemma 4.2, if Y_t is nonempty, then Y_t can be identified with the set of integers from the interval $[i_W(r|_{F_{t-1}}), i_W(r|_{F_t}) - 1]$, and, with this identification, $a(N_y) = y$. Also, $N_{y_2} = N_{y_1}(y_2 - y_1)[2(y_2 - y_1)]$, for any $y_1, y_2 \in Y_t$. If we put $y_0^t := i_W(r|_{F_{t-1}})$, then, by Lemma 5.2, for any $0 \leqslant t < h(r)$,

$$(\oplus_{y\in Y_t} N_y) \cong N_{y_0^t} \otimes (\oplus_{j=0}^{2^n-1}\mathbb{Z}(j)[2j]) \otimes (\oplus_{l=0}^{i_t(r)/2^n}\mathbb{Z}(l\cdot 2^n)[l\cdot 2^{n+1}]).$$

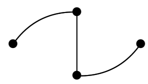
The same will be true for $t = h(r)$, if $i_{h(r)}(r)$ is divisible by 2^n, that is, $\dim(p)$ is even. Denote: $m := \dim(R)$. By Corollary 4.4,

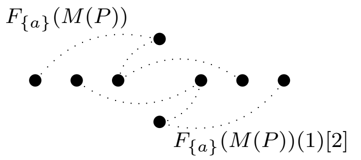
$$M(R) \cong \bigoplus_{z\in Z(Q)} N_z \cong \bigoplus_{0\leqslant j<i_W(r)} (\mathbb{Z}(j)[2j] \oplus \mathbb{Z}(m-j)[2m-2j]) \oplus (\bigoplus_{0<t\leqslant h(r)} \oplus_{y\in Y_t} N_y).$$

In the case $\dim(p)$ - even, we get that $M(R)$ is (uniquely, up to isomorphism) divisible by $(\oplus_{j=0}^{2^n-1}\mathbb{Z}(j)[2j])$ (we remind that $i_W(r)$ is also divisible by 2^n).

In the case $\dim(p)$ - odd, we need to show that $\oplus_{y\in Y_{h(r)}} N_y \cong M(Q_\alpha)(a)[2a]$, where $a = m/2 - 2^{n-1} + 1$. In other words, that $N_{y_0^{h(r)}} \cong M_\alpha(a)[2a]$. This follows from Theorem 4.15, since for arbitrary field extension E/k, $i_W(r|_E) > a$ $\Leftrightarrow$ the form $r|_E$ is hyperbolic $\Leftrightarrow$ $\langle\langle\alpha\rangle\rangle|_E$ is isotropic. □

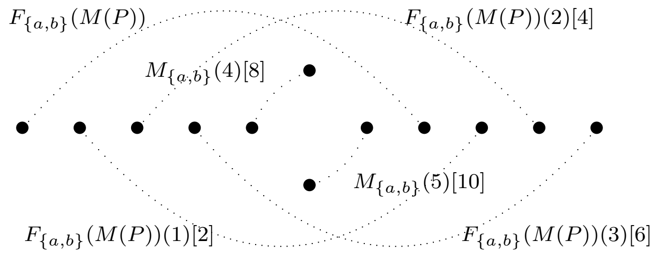
One can notice that $F_\alpha(M(P))|_{\overline{k}}$ consists of as many Tate-motives as $M(P)|_{\overline{k}}$, but they are 2^n-times further apart than the Tate-motives from $M(P)|_{\overline{k}}$. We would like to call $F_\alpha(M(P))$ the *higher form* of $M(P)$. So, we have some kind of action of the semigroup of pure symbols from $\mathrm{K}_*^M(k)/2$ on the motives of quadrics.

Example: 1) Let $p = \langle 1, -b, -c, -d\rangle$, where $\{-bcd\} \neq 0$ and $\{-bcd\}$ does not divide $\{b,c\} \in \mathrm{K}_*^M(k)/2$, and $\alpha = \{a\}$. Then $M(P)$ looks as: 

and $M(R)$ looks as: 

(we have dotted lines here since $F_{\{a\}}(M(P))$ is, in general, decomposable (when $\{a, -bcd\}$ divides $\{a,b,c\}$, or $\{a,-bcd\} = 0$)).

2) Let $p = \langle 1, -c, -d\rangle$, and $\alpha = \{a, b\}$. Then $M(P)$ looks as: • ⋯ • and

$M(R)$ looks as 

where $F_{\{a,b\}}(M(P)) \cong M_{\{a,b,c,d\}}$.

As we saw in the example 2) above, the Rost-motive is a particular case of a *higher form*. These are the higher forms of 0-dimensional quadrics. Namely, $M_{\{a_1,\dots,a_n\}} = F_{\{a_2,\dots,a_n\}}(M(k\sqrt{a_1}))$.

We expect that F_α act not only on the motives of quadrics, but also on all their direct summands. More precisely, we can state the following conjecture. Let $\varphi : \Lambda(P) \to \Lambda(F_\alpha(M(P)))$ be the natural identification such that the ordering of the degrees of the Tate-motives is preserved, and the degree of $\varphi(L^{up})$ is bigger than the degree of $\varphi(L_{lo})$.

Conjecture 5.3 *Under the natural identification* $\Lambda(P) \stackrel{\varphi}{=} \Lambda(F_\alpha(M(P)))$, $\varphi(\lambda)$ *is connected to* $\varphi(\mu) \Rightarrow \lambda$ *is connected to* μ. *In other words,* F_α *preserves the direct sum decomposition.*

Dimensions of anisotropic forms in I^n

Let $W(k)$ be the Witt-ring of quadratic forms over k, and $I \subset W(k)$ be the ideal of even-dimensional forms. I generates multiplicative filtration $W(k) \supset I \supset I^2 \supset \ldots \supset I^n \supset \ldots$ on $W(k)$, and due to the results of V.Voevodsky, the corresponding graded ring is isomorphic to Milnor's K-theory of k $(\,mod\, 2\,)$.

The important problem in quadratic form theory is to describe possible dimensions of anisotropic forms in I^n. Basic here is the following famous result.

Hauptsatz 1 (Arason-Pfister)
Let q *be anisotropic form in* I^n. *Then:*

(1) *Either* $q = 0$, *or* $\dim(q) \geqslant 2^n$.

(2) *If* $\dim(q) = 2^n$, *then* q *is proportional to a Pfister form.*

At the same time, A.Pfister proved that there are no 10-dimensional anisotropic forms in I^3 (see [21]), which showed that there are further restrictions on $\dim(q)$. And it was conjectured (see, for example, [11, Conjecture 9]) that the next possible dimension after 2^n is $2^n + 2^{n-1}$. In the case $n = 4$, this conjecture was proven by D.Hoffmann (see [4, Main Theorem]). Now we can prove it for all n.

Theorem 5.4 ([26, Main Theorem])
Let q *be anisotropic form in* I^n. *Then*

$$\dim(q) \text{ is either } \quad 0, \quad \text{or} \quad 2^n, \quad \text{or} \quad \geqslant 2^n + 2^{n-1}.$$

Remark: Originally, this theorem was proven under additional condition $char(k) = 0$. Then it was extended to the case of arbitrary characteristic ($\neq 2$) by P.Morandi, who proved, that if $d \in \mathbb{N}$ is the dimension of some anisotropic form from I^n in odd characteristic p, then it is the dimension of some anisotropic form from I^n in characteristic 0. Now, due to the new results of V.Voevodsky ([29]), we can drop this characteristic restriction in our original theorem.

Proof: Suppose it is not the case. Then there exists anisotropic $q \in I^n(k)$, such that $2^n < \dim(q) < 2^n+2^{n-1}$. Let us choose such counterexample of the smallest possible dimension (among all forms over all fields). Let $s \subset q$ be arbitrary 2^{n-1}-dimensional subform. By the result of D.Hoffmann ([2]), there exists field extension F/k, and anisotropic n-fold Pfister form $\langle\langle\alpha\rangle\rangle$ over F, such that $s|_F \subset \langle\langle\alpha\rangle\rangle$, and all forms anisotropic over k stay anisotropic after restricting to F. Let us denote: $q_1 := q|_F$, $q_2 := (q|_F \perp -\langle\langle\alpha\rangle\rangle)_{an.}$. Then $\dim(q_1) = \dim(q)$ and $\dim(q_2) \leqslant \dim(q)$.

Let E/F be some extension, such that $\dim((q_i|_E)_{an.}) < \dim(q)$. Then $(q_i|_E)_{an.}$ is not a counterexample, and by Hauptsatz, $(q_i|_E)_{an.}$ is either 0, or proportional to some n-fold Pfister form $\langle\langle\beta\rangle\rangle$. But $(q_{3-i}|_E)_{an.} = ((q_i|_E)_{an.} \perp \pm\langle\langle\alpha\rangle\rangle))_{an.}$. And, by the result of R.Elman and T.Y.Lam ([1]), $\dim((\langle\langle\alpha\rangle\rangle \perp \lambda\cdot\langle\langle\beta\rangle\rangle)_{an.})$ is either 2^{n+1}, or $2^{n+1} - 2^{i+1}$, where $0 \leqslant i \leqslant n$, for any n-fold Pfister forms $\langle\langle\alpha\rangle\rangle$ and $\langle\langle\beta\rangle\rangle$. So, such dimension is either $\geqslant 2^n + 2^{n-1}$, or $\leqslant 2^n$. Hence, $\dim((q_{3-i}|_E)_{an.}) < \dim(q)$ as well. Thus, the conditions $\dim((q_i|_E)_{an.}) < \dim(q)$ and $\dim((q_{3-i}|_E)_{an.}) < \dim(q)$ are equivalent.

In particular, $\dim(q_2) = \dim(q)$, and the forms $q_1|_{F(Q_2)}$ and $q_2|_{F(Q_1)}$ are isotropic. By Corollary 3.9, $M(Q_i)$ contains indecomposable direct summand N_i, such that $a(N_i) = 0$, and $N_1 \cong N_2$. Notice that since q_i is a counterexample of the smallest possible dimension, the height of q_i is 2. That is, the splitting pattern of q_i is $(j, 2^{n-1})$, where $0 < j < 2^{n-2}$. Then, by Corollary 4.7, $\text{size}(N_i) = 2^n + j - 1 \neq 2^r - 1$, for any r. By Theorem 4.20, N_i is not binary, and so, $\Lambda(N_i)$ must contain some Tate-motives $\mathbb{Z}(c)[2c]$ from the 2-nd shell (that is, with $j \leqslant \min(c, \dim(Q_i) - c)$). But then, for arbitrary field extension E/F, $N_i|_E$ splits into the direct sum of Tate-motives if and only if $q_i|_E$ is hyperbolic (by Proposition 2.1 and Proposition 2.6). Since $N_1 \cong N_2$, we get, that, for arbitrary field extension E/F, $q_1|_E$ is hyperbolic if and only if $q_2|_E$ is hyperbolic. This is impossible, since q_1 and q_2 differ by a non-hyperbolic Pfister form $\langle\langle\alpha\rangle\rangle$ (take, for example, $E = F_2$ - the last field from the generalized splitting tower for q_1). We get a contradiction, and

Theorem is proven. $\square$

The following Conjecture describes all possible dimensions of anisotropic forms in I^n.

Conjecture 5.5 ([26, Conjecture 4.11])

Let $q \in I^n(k)$ be anisotropic form. Then $\dim(q)$ *is either* $2^{n+1} - 2^{i+1}$, *where* $0 \leqslant i \leqslant n$, *or is even* $\geqslant 2^{n+1}$.

It is not difficult to show that all the values, prescribed by Conjecture 5.6, are indeed realized by appropriate forms.

Motivic decomposition and stable birational equivalence of 7-dimensional quadrics

As an illustration of the general methods described above, we will classify 7-dimensional quadrics in terms of motivic decomposition. This classification was an essential step in the proof of the criterion of O.Izhboldin for stable birational equivalence of 7-dimensional quadrics.

In [7], O.Izhboldin classified anisotropic 9-dimensional forms into 4 different types:

(1) q is a neighbor of some 4-fold Pfister form $\langle\langle\alpha(q)\rangle\rangle$.

(2) q is not a neighbor, and, for some $\lambda \in k^*$, λq differs by a 3-dimensional anisotropic form $r(q) = \langle 1\rangle \perp r'(q)$ from some 3-fold Pfister form $\langle\langle\beta(q)\rangle\rangle$.

(3) q is not a neighbor and q is a codimension 1 subform of the anisotropic form of the type $\langle\langle a\rangle\rangle \times \langle b_1, b_2, b_3, b_4, b_5\rangle$.

(4) all other forms.

Theorem 5.6 (O.Izhboldin, [7])

Let p and q be anisotropic 9-dimensional forms.

(a) *Suppose $p|_{k(Q)}$ and $q|_{k(P)}$ are isotropic (in other words, the quadrics P and Q are stably birationally equivalent). Then p and q have the same type.*

(*b*) *For 4 different types described above, P is stably birationally equivalent to Q if and only if:*

(1) $\alpha(p) = \alpha(q)$;

(2) $r(p) = r(q)$, *and* $\beta(p)|_{k(r(p))} = \beta(q)|_{k(r(p))}$.

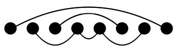
(3) *q is a codimension 1 subform of $\langle\langle a\rangle\rangle \times \langle b_1, b_2, b_3, b_4, b_5\rangle$, p is a codimension 1 subform of $\langle\langle c\rangle\rangle \times \langle d_1, d_2, d_3, d_4, d_5\rangle$, and these 10-dimensional forms contain proportional 9-dimensional subforms;*

(4) *q is proportional to p.*

The 4 classes above have the following motivic interpretation.

Proposition 5.7 (O.Izhboldin)
Let Q be smooth anisotropic quadric of dimension 7. Then the decomposition of $M(Q)$ into indecomposables looks as:

(*i*)  $\Leftrightarrow$ *Q is excellent.*

(*ii*)  $\Leftrightarrow$ *q is a neighbor of a 4-fold Pfister form $\langle\langle\alpha\rangle\rangle$, for some $a \in k^*$, $q|_{k\sqrt{a}}$ is completely split, and q is not excellent.*

(*iii*)  $\Leftrightarrow$ *q is a Pfister neighbor, and for any $a \in k^*$, $q|_{k\sqrt{a}}$ is not completely split.*

(*iv*)  $\Leftrightarrow$ *q is not a Pfister neighbor, and there exists a 3-dimensional anisotropic form $r_3(q)$ such that $(q \perp r_3(q))_{an.}$ is proportional to a 3-fold Pfister form $\langle\langle\beta\rangle\rangle$.*

(*v*)  $\Leftrightarrow$ *q is not a Pfister neighbor, and for some $a \in k^*$, $q|_{k\sqrt{a}}$ is completely split.*

(*vi*)  $\Leftrightarrow$ *q is not a Pfister neighbor, for any $a \in k^*$, $q|_{k\sqrt{a}}$ is not completely split, and q is not proportional to $(\langle\langle\beta\rangle\rangle \perp r)_{an.}$, for any 3-fold Pfister form $\langle\langle\beta\rangle\rangle$ and any 3-dimensional form r.*

We start with the forms of dimension 5 and 7.
Following B.Kahn, we introduce the notion of $\dim_n(q)$.

Definition 5.8 *For $n \in \mathbb{N}$ we define*

$$\dim_n(q) := \min(\dim(q') \,|\, q \perp q' \in I^n(k)).$$

If $\dim_n(q) < 2^{n-1}$, then the form q' with $\dim(q') = \dim_n(q)$, and $q \perp q' \in I^n(k)$ is defined uniquely, and it will be denoted as $r_n(q)$.

The element $\pi(q \perp r_n(q)) \in \mathrm{K}_n^M(k)/2$, in this case, will be denoted as $\omega_n(q)$ (here π is a natural projection $I^n(k) \to \mathrm{K}_n^M(k)/2$).

Example: For odd dimensional form q, $\dim_2(q) = 1$, $r_2(q) = \langle \det_{\pm}(q) \rangle$, and $\omega_2(q)$ corresponds to $C(q)$ via identification $\mathrm{K}_2^M(k)/2 = \mathrm{Br}_2(k)$.

The sequence of higher Witt indices $(i_1(q), \ldots, i_{h(q)}(q))$ will be denoted as $\mathbf{i}(q)$, and called the *splitting pattern* of q.

Proposition 5.9 *Let Q be smooth anisotropic quadric of dimension 3. Then the motive of Q looks as:*

(i)  $\Leftrightarrow$ *Q is excellent;*

(ii)  $\Leftrightarrow$ *Q is not excellent.*

Proof: Since $\mathbf{i}(q)$ is always $(1,1)$, by Proposition 4.10, we have the following necessary connections (not to mess with the indecomposable direct summands) in $\Lambda(Q)$: . If Q is excellent, then, by the result of M.Rost ([22, Proposition 4]), these are all the existing connections, and $M(Q)$ is a direct sum of binary Rost-motives. Conversely, let $M(Q)$ has only binary connections specified above. Consider the form $p := q \perp \langle \det_{\pm}(q) \rangle$. Then P is an Albert quadric, and for arbitrary field extension E/k, $i_W(q|_E) > 1$ if and only if $i_W(p|_E) > 1$. If there exists indecomposable direct summand M of $M(Q)$, such that $a(M) = 1$, then by Theorem 4.17 , M is isomorphic to some direct summand M' of $M(P)$. Then, by Theorem 4.13, $M'(1)[2]$ is also a direct summand of $M(P)$. Suppose, P is anisotropic. Then, if N is an indecomposable direct summand of $M(Q)$ such that $a(N) = 0$, then $\Lambda(N)$ does not contain $\mathbb{Z}(1)[2]$, or L_{lo}. By Proposition 4.10, N is binary of size 4 - in contradiction with Theorem 4.20. So, P is isotropic, and Q is excellent. □

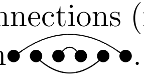
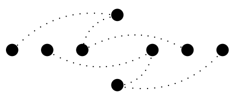
Proposition 5.10 *Let Q be smooth anisotropic quadric of dimension 5. Then the motive of Q looks as:*

(*i*)  $\Leftrightarrow$ *Q is excellent* $\Leftrightarrow$ $\mathbf{i}(q) = (3)$*;*

(*ii*)  $\Leftrightarrow$ $\dim_3(q) = 3 \Leftrightarrow q|_{k\sqrt{a}}$ *is completely split for some* $a \in k^*$*, and q is not excellent. In this case,* $\mathbf{i}(q) = (1, 1, 1)$*;*

(*iii*)  $\Leftrightarrow$ $\dim_3(q) > 3 \Leftrightarrow q|_{k\sqrt{a}}$ *is not completely split for any* $a \in k^*$*. In this case,* $\mathbf{i}(q) = (1, 1, 1)$*;*

Proof: The fact that $\mathbf{i}(p) = (3) \Leftrightarrow Q$ is excellent is well-known. By the result of M.Rost ([22, Proposition 4]), if Q is excellent, $M(Q)$ has the specified decomposition. Finally, if $M(Q)$ has a direct summand of the form , then, by Corollary 4.14, $\mathbf{i}(p) = (3)$.

Now we can assume that $\mathbf{i}(p) = (1, 1, 1)$. By Proposition 4.10, in $M(Q)$ we have connections (not to mess with the indecomposable direct summands) of the form . Let us show that $\mathbb{Z}$ is connected to $\mathbb{Z}(2)[4]$. Since there are no binary direct summands of size 5 (by Theorem 4.20), $\mathbb{Z}$ must be connected either to $\mathbb{Z}(1)[2]$, or to $\mathbb{Z}(2)[4]$. Suppose, $\mathbb{Z}(2)[4]$ is not connected to $\mathbb{Z}(1)[2]$. Then, for $q_1 := (q|_{k(Q)})_{an.}$, $M(Q_1)$ looks as , and, by Proposition 5.9, q_1 is excellent. In particular, $\dim_3(q_1) = 3$. Consider $p = q \perp \langle \det_{\pm}(q) \rangle$. Then $p \in I^2(k)$, and $\pi(p|_{k(Q)}) \in \mathrm{K}_2^M(k(Q))/2$ is a pure symbol (π here is the natural projection $I^n(F) \to \mathrm{K}_n^M(k)/2$). By the index-reduction formula of A.Merkurjev ([20]), $\pi(p) \in \mathrm{K}_2^M(k)/2$ is a pure symbol. Then, it is well-known (see, for example, [3]), that $p = \langle\langle a \rangle\rangle \cdot \langle b_1, b_2, b_3, b_4 \rangle$. By Theorem 5.1, $M(P)$ decomposes as:  In particular, if L is indecomposable direct summand of $M(P)$ such that $a(L) = 0$, then $L|_{\overline{k}}$ does not contain $\mathbb{Z}(1)[2]$. But $i_1(p) = 2$, and q is a codimension 1 subform in p. So, the forms $p|_{k(Q)}$ and $q|_{k(P)}$ are isotropic, and by Corollary 3.9, L is isomorphic to a direct summand of $M(Q)$. This shows that $\mathbb{Z}$ is not connected to $\mathbb{Z}(1)[2]$ (if $\mathbb{Z}(1)[2]$ is not connected to $\mathbb{Z}(2)[4]$). The conclusion is: in the case of a splitting pattern $(1, 1, 1)$, $\mathbb{Z}$ is always connected to $\mathbb{Z}(2)[4]$. So, in $M(Q)$ we have necessary connections (not to mess with the indecomposable direct summands) of the form: . If $M(Q)$ has decomposition as in (*ii*), then $\mathbb{Z}(1)[2]$ is not connected to $\mathbb{Z}(2)[4]$, and, as we saw above, then there exists $a \in k^*$, such that $q|_{k\sqrt{a}}$ is completely split. Conversely, if q is a codimension 1 subform of the anisotropic form $\langle\langle a \rangle\rangle \cdot \langle b_1, b_2, b_3, b_4 \rangle$, then, $\mathbb{Z}$ is not connected to $\mathbb{Z}(1)[2]$, and, if q is not excellent, $M(Q)$ decomposes into indecomposables as in (*ii*).

It is easy to see that for anisotropic 7-dimensional form q, $\dim_3(q) = 3$ if and only if q is non-excellent, and there exists $a \in k^*$ such that $q|_{k\sqrt{a}}$ is completely split. $\square$

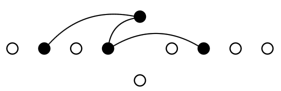
Lemma 5.11 *Let Q be anisotropic 7-dimensional quadric. Suppose, $\mathbb{Z}(1)[2]$ is not connected to $\mathbb{Z}(2)[4]$ in $\Lambda(Q)$. Then $\dim_3(q) \leqslant 3$.*

Proof: By the result of D.Hoffmann (see [2, Corollary 1]), $i_1(q) = 1$. Let $q_1 = (q|_{k(Q)})_{an.}$. Then $\dim(q_1) = 7$, and in $M(Q_1)$, $\mathbb{Z}$ is not connected to $\mathbb{Z}(1)[2]$. By Proposition 5.10, $\dim_3(q_1) \leqslant 3$. Then by the result of B.Kahn ([10, Theorem 2]), which, in our case, basically amounts to the index-reduction formula of A.Merkurjev, we get that $\dim_3(q) \leqslant 3$. $\square$

Lemma 5.12 *Let Q be anisotropic 7-dimensional quadric. Then the following conditions are equivalent:*

(a) $\mathbb{Z}(1)[2]$ *is not connected to* $\mathbb{Z}$ *and* $\mathbb{Z}(2)[4]$ *in* $\Lambda(Q)$.

(b) *there exists $a \in k^*$ such that $q|_{k\sqrt{a}}$ is completely split.*

Proof: $(a) \Rightarrow (b)$: Consider the form $p := q \perp \langle \det_{\pm}(q) \rangle$. Then $p \in I^2(k)$, and, by Lemma 5.11, $\pi(p) \in \mathrm{K}_2^M(k)/2$ is a pure symbol (possibly, zero). Suppose p is anisotropic. Then $\pi(p) \neq 0$, and the splitting pattern of p is $(1, 2, 2)$. Since $\mathbb{Z}$ is not connected to $\mathbb{Z}(1)[2]$ in $\Lambda(Q)$, there exists indecomposable direct summand L of $M(Q)$ such that $a(L) = 1$. But, for arbitrary field extension, E/k, $i_W(q|_E) > 1$ if and only if $i_W(p|_E) > 1$ (since $i_2(p) = 2 > 1$). Hence, by Theorem 4.17, L is isomorphic to a direct summand M of $M(P)$. Then $a(M) = 1$, $b(M) = 6$ (by Corollary 4.14), and so, by Theorem 4.20, M is not binary. Taking into account that $M(1)[2]$ is also a direct summand of $M(P)$ (by Theorem 4.13), we get that M must look as:

. Then the direct summand of $M(P)$ complimentary to $M \oplus M(1)[2]$ will be binary of size 8 - contradiction with Theorem 4.20. Hence p is isotropic. Since $\pi(p) \in \mathrm{K}_2^M(k)/2$ is a pure symbol, there exists $a \in k^*$, such that $p|_{k\sqrt{a}}$ is hyperbolic. Consequently, $q|_{k\sqrt{a}}$ is completely split.

$(b) \Rightarrow (a)$: If $q|_{k\sqrt{a}}$ is completely split. Then $p := q \perp \langle \det_{\pm}(q) \rangle$ is isotropic, and $p_{an.}$ is divisible by $\langle\langle a \rangle\rangle$. Then, by Lemma 5.2, $i_1(p_{an.}) > 1$, and so, for arbitrary field extension E/k, $i_W(p|_E) > 0$ if and only if $i_W(q|_E) > 1$. By Theorem 4.15, there are indecomposable direct summands M of $M(P_{an.})$ and L of $M(Q)$ such that $L \cong M(1)[2]$, and $a(L) = 1$ (respectively, $a(M) = 0$). Since $i_1(p_{an.}) > 1$, by Theorem 4.13 and Corollary 3.7, $M|_{\overline{k}}$ does not contain $\mathbb{Z}(1)[2]$. Thus, $L|_{\overline{k}}$ does not contain $\mathbb{Z}(2)[4]$. Evidently, $L|_{\overline{k}}$ does not contain $\mathbb{Z}$. So, $\mathbb{Z}(1)[2]$ is connected neither to $\mathbb{Z}$, nor to $\mathbb{Z}(2)[4]$. □

Lemma 5.13 *Let Q be anisotropic 7-dimensional quadric. Then the following conditions are equivalent:*

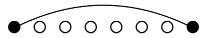
- (a) $\mathbb{Z}(2)[4]$ *is not connected to* $\mathbb{Z}$ *and* $\mathbb{Z}(1)[2]$ *in* $\Lambda(Q)$.
- (b) $\dim_3(q) \leqslant 3$, *and* $(q \perp r_3(q))_{an.}$ *is proportional to some anisotropic 3-fold Pfister form.*

Proof: (a) $\Rightarrow$ (b): By Lemma 5.11, $\dim_3(q) \leqslant 3$. Certainly, $\dim_3(q)$ is odd. If $\dim_3(q) = 1$, then q is excellent. Suppose $\dim_3(q) = 3$. Let $p := q \perp r_3(q) \in I^3(k)$. Since $\mathbb{Z}(2)[4]$ is not connected to $\mathbb{Z}$ and $\mathbb{Z}(1)[2]$ in $\Lambda(Q)$, we get direct summand L of $M(Q)$ with $a(L) = 2$. Since q is a codimension 3 subform of p, and for arbitrary field extension E/k, the conditions: $i_W(p|_E) > 2$ and $i_W(p|_E) > 5$ are equivalent, the conditions: $i_W(p|_E) > 2$ and $i_W(q|_E) > 2$ are equivalent as well. Then, by Theorem 4.17, L is isomorphic to a direct summand M of $M(P)$. By Theorem 4.13 and Theorem 3.11, $\oplus_{j=0}^{3} M(j)[2j]$ is isomorphic to a direct summand of $M(P)$. In particular, $\mathbb{Z}(l)[2l]$ with $2 \leqslant l \leqslant 8$ are not connected to $\mathbb{Z}$ in $\Lambda(P)$. Suppose p is anisotropic. Then the indecomposable direct summand N of $M(P)$ with $a(N) = 0$ must be binary of size 9 - in contradiction with Theorem 4.20. So, p is isotropic, and $p_{an.}$ is proportional to some 3-fold Pfister form.

(a) $\Rightarrow$ (b): Let $q = (\lambda \cdot \langle\langle \beta \rangle\rangle \perp -r_3(q))_{an.}$, where $\langle\langle \beta \rangle\rangle$ is some anisotropic 3-fold Pfister form, $\dim(r_3(q)) \leqslant 3$, and $\dim(q) = 9$. Then, for any field extension E/k, $i_W(q|_E) > 2 \Leftrightarrow i_W(p|_E) > 0$. By Theorem 4.15, the Rost-motive $M_\beta(2)[4]$ is a direct summand of $M(Q)$. In particular, $\mathbb{Z}(2)[4]$ is not connected to $\mathbb{Z}$ and to $\mathbb{Z}(1)[2]$. □

Lemma 5.14 (N.Karpenko, [14, Theorem 1.7])

Let Q be anisotropic 7-dimensional quadric. Then the following conditions are equivalent:

(a) *$M(Q)$ has binary direct summand of the form:* 

(b) *q is a neighbor of a 4-fold Pfister form.*

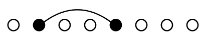
Proof: (b) $\Rightarrow$ (a): If q is a neighbor of a Pfister form $\langle\langle a_1, a_2, a_3, a_4\rangle\rangle$, then, by the result of M.Rost ([22, Proposition 4]), the binary Rost-motive $M_{\{a_1,a_2,a_3,a_4\}}$ is a direct summand of $M(Q)$.

(a) $\Rightarrow$ (b): Let N be the specified binary direct summand. Then, by the result of O.Izhboldin (see [5, Theorem 3.1], [9, Theorem 6.9]), there exists nonzero element $\alpha \in \mathrm{Ker}(\mathrm{K}_4^M(k)/2 \to \mathrm{K}_4^M(k(Q))/2)$ (again, due to the new results of V.Voevodsky ([29]), now the proof of [9, Theorem 6.9] works in arbitrary characteristic ($\neq 2$)). Due to the result of B.Kahn, M.Rost and R.J.Sujatha (see [12, Theorem 1]), α must be a pure symbol. Then $\langle\langle\alpha\rangle\rangle|_{k(Q)}$ is hyperbolic, and q is a neighbor of $\langle\langle\alpha\rangle\rangle$. $\square$

Lemma 5.15 *Let Q be anisotropic quadric of dimension 7. Then the following conditions are equivalent:*

(a) *Q is excellent;*

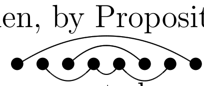
(b) $\mathbf{i}(q) = (1,3)$*;*

(c) *$M(Q)$ has binary direct summand of the form:*  .

Proof: It is well-known that (a) $\Leftrightarrow$ (b). Suppose Q is excellent (i.e, defined by a form $(\langle\langle a,b,c,d\rangle\rangle \perp -\langle\langle a,b,c\rangle\rangle \perp \langle 1\rangle)_{an.}$, where $\{a,b,c,d\} \neq 0$), then, by the result of M.Rost ([22, Proposition 4]), the binary motive $M_{\{a,b,c\}}(1)[2]$ is a direct summand of $M(Q)$. So, (a) $\Rightarrow$ (c). Finally, if $M(Q)$ has specified direct summand then, by Corollary 4.14, $i_2(q) = 3$, and $\mathbf{i}(q) = (1,3)$. Thus, (c) $\Rightarrow$ (b). $\square$

Now we can prove Proposition 5.7.

By the result of D.Hoffmann ([2, Corollary 1]), $i_1(q) = 1$. Hence, $\mathbf{i}(q)$ is either $(1,3)$, or $(1,1,1,1)$. By Lemma 5.15, $\mathbf{i}(q) = (1,3) \Leftrightarrow Q$ is excellent $\Leftrightarrow$ $M(Q)$ has a decomposition as in (i).

Now we can assume that $\mathbf{i}(q) = (1,1,1,1)$. Then, by Proposition 5.10, in $\Lambda(Q)$ we have necessary connections of the form: .

So, the question is: which of these pieces are connected, and which are not. We get 5 cases:

(1) all three $\mathbb{Z}$, $\mathbb{Z}(1)[2]$, $\mathbb{Z}(2)[4]$ are disconnected;

(2) $\mathbb{Z}(1)[2]$ is connected to $\mathbb{Z}(2)[4]$, but not to $\mathbb{Z}$;

(3) $\mathbb{Z}(1)[2]$ is connected to $\mathbb{Z}$, but not to $\mathbb{Z}(2)[4]$;

(4) $\mathbb{Z}$ is connected to $\mathbb{Z}(2)[4]$, but not to $\mathbb{Z}(1)[2]$;

(5) all three $\mathbb{Z}$, $\mathbb{Z}(1)[2]$, $\mathbb{Z}(2)[4]$ are connected.

Clearly, these cases correspond to the cases: $(ii), (iii), (iv), (v)$ and (vi) of Proposition 5.7, respectively. Applying Lemma 5.12, Lemma 5.13 and Lemma 5.14, we get the description of the corresponding quadrics in terms of quadratic form theory. Proposition is proven. □

Remark: We can notice that the Conjecture 4.22 is valid for quadics of dimension 3, 5 and 7.

We see that the 4 classes of forms of O.Izhboldin have the following motivic interpretation: (1) corresponds to the cases $(i), (ii)$, and (iii) of Proposition 5.7; (2) corresponds to (iv); (3) corresponds to (v); and (4) corresponds to (vi).

By Corollary 3.9, we know that q and p are stably birationally equivalent if and only if $M(Q)$ and $M(P)$ contain indecomposable direct summands N and L, such that $N \cong L$, and $a(N) = 0$. In particular, $\Lambda(N) = \Lambda(L)$. This shows that the type of a form is preserved under stable birational equivalence. To prove (b) one needs to analyze the corresponding direct summands more carefully.

6 Proofs

We start with some preliminary results.

Corollary 6.1 *Let N be a direct summand in $M(Q)$, and $\psi \in \mathrm{Hom}(N, N)$ be such that:*

1) $\psi|_{\overline{k}} = 0$. *Then $\psi^n = 0$ for some n.*

2) $\psi|_{\overline{k}}$ *is a projector. Then ψ^n is a projector for some n.*

3) $\psi|_{\overline{k}}$ *is an isomorphism. Then ψ is an isomorphism.*

Proof: Let $M(Q) = N \oplus M$. It is enough to consider $\varphi = \begin{pmatrix} \psi & 0 \\ 0 & \rho \end{pmatrix}$, where $\rho = 0$ in cases 1) and 2), and $= id_M$ in case 3). In case 1) apply Theorem 3.1(1), in case 2) - Corollary 3.2, in case 3) - Theorem 3.1(2). $\square$

Lemma 6.2 *Let L and N be direct summands in $M(Q)$ such that $p_L|_{\overline{k}} \circ p_N|_{\overline{k}} = p_N|_{\overline{k}} \circ p_L|_{\overline{k}} = p_L|_{\overline{k}}$. Then there exists direct summand $\tilde{L}$ in N, s.t. $\tilde{L}$ is isomorphic to L, and $p_L|_{\overline{k}} = p_{\tilde{L}}|_{\overline{k}}$.*

Proof: Let $j_L : L \to M(Q)$, $j_N : N \to M(Q)$, $\varphi_L : M(Q) \to L$, $\varphi_N : M(Q) \to N$ be s.t.: $\varphi_L \circ j_L = id_L$, $\varphi_N \circ j_N = id_N$, and $j_L \circ \varphi_L = p_L$, $j_N \circ \varphi_N = p_N$.

Take: $\alpha := \varphi_L \circ j_N : N \to L$, and $\beta := \varphi_N \circ j_L : L \to N$. If $\gamma := \alpha \circ \beta : L \to L$, then $\gamma|_{\overline{k}} = id_L$. By Corollary 6.1(2) and (1), $\gamma^s = id_L$, for some s. Consider $\psi := \varphi_N \circ p_L \circ j_N : N \to N$. Then ψ^s is a projector, $\psi^s = \beta \circ \tilde{\alpha}$, where $\tilde{\alpha} = \alpha \circ (\beta \circ \alpha)^{s-1}$, and $\tilde{\alpha} \circ \beta = id_L$. Then ψ^s defines direct summand $\tilde{L}$ in N, and for the corresponding projector in $M(Q)$, $p_{\tilde{L}} := j_N \circ \psi^s \circ \varphi_N$, we have: $p_{\tilde{L}}|_{\overline{k}} = p_L|_{\overline{k}}$. $\square$

Lemma 6.3 (cf. [24, Lemma 3.13])
Let N be a direct summand in $M(Q)$, $\dim(Q) = m$.

1) *Then there exists: $\kappa_{r,N} \in \mathrm{End}(N)$, s.t. $(\kappa_{r,N})_{(s)} = 0$, for any $s \neq r$, and $(\kappa_{r,N})_{(r)} = 2 \cdot id_{\mathrm{CH}_r(N|_{\overline{k}})}$.*

2) *If $rank(\mathrm{CH}_{m/2}(N|_{\overline{k}})) = 2$, then there exists: $\theta_{m/2,N} \in \mathrm{End}(N)$, s.t. $(\theta_{m/2,N})_{(m/2)} = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$; $(\theta_{m/2,N})_{(r)} = 0$, for any $r \neq m/2$.*

Proof: 1) Take:

$$\kappa_{r,N} := \begin{cases} \varphi_N \circ (h^r \times h^{m-r}) \circ j_N, \text{ if } r \neq m/2. \\ \varphi_N \circ (2 \cdot id_{M(Q)} - \sum_{0 \leqslant i < m/2} (h^i \times h^{m-i} + h^{m-i} \times h^i)) \circ j_N, \text{ if } r = m/2. \end{cases}$$

2) Take $\theta_{m/2,N} := \varphi_N \circ (h^{m/2} \times h^{m/2}) \circ j_N$. $\square$

Lemma 6.4 *Let N_i is a direct summand of $M(Q_i)$. Let for some odd number η, and some $\psi \in \mathrm{Hom}(N_1|_{\overline{k}}, N_2|_{\overline{k}})$, we have: $\eta \cdot \psi \in image(\mathrm{Hom}(N_1, N_2) \to \mathrm{Hom}(N_1|_{\overline{k}}, N_2|_{\overline{k}}))$. Then $\psi \in image(\mathrm{Hom}(N_1, N_2) \to \mathrm{Hom}(N_1|_{\overline{k}}, N_2|_{\overline{k}}))$.*

Proof: Let F/k be Galois extension of degree 2^n, such that $N_i|_F$ is a sum of Tate-motives (for example, one which splits both quadrics completely). We have: $\mathrm{Hom}(N_1|_F, N_2|_F) \to \mathrm{Hom}(N_1|_{\overline{k}}, N_2|_{\overline{k}})$ is an isomorphism. Let ψ_F be the corresponding element of $\mathrm{Hom}(N_1|_F, N_2|_F)$.

Since $\eta \cdot \psi_F \in image(\mathrm{Hom}(N_1, N_2) \to \mathrm{Hom}(N_1|_F, N_2|_F))$ we have: $\eta \cdot (\sigma(\psi_F) - \psi_F) = 0$, for any $\sigma \in Gal(F/k)$. Because $\mathrm{Hom}(N_1|_F, N_2|_F)$ has no torsion, we get: $\sigma(\psi_F) = \psi_F$. Then

$$2^n \cdot \psi_F = \sum_{\sigma \in Gal(F/k)} \sigma(\psi_F) \in image(\mathrm{Hom}(N_1, N_2) \to \mathrm{Hom}(N_1|_F, N_2|_F)).$$

Since $\eta \cdot \psi_F$, $2^n \cdot \psi_F \in image(\mathrm{Hom}(N_1, N_2) \to \mathrm{Hom}(N_1|_F, N_2|_F))$, we have: $\psi_F \in image(\mathrm{Hom}(N_1, N_2) \to \mathrm{Hom}(N_1|_F, N_2|_F))$, which implies:
$\psi \in image(\mathrm{Hom}(N_1, N_2) \to \mathrm{Hom}(N_1|_{\overline{k}}, N_2|_{\overline{k}}))$. □

Definition 6.5 *Let N and N' be indecomposable direct summands in $M(Q)$. We say that N' is a normal form of N, if N' is isomorphic to N, and either $m = \dim(Q)$ is odd, or m is even and $(p_{N'})_{(m/2)}$ is of one of the following types:*

1) 0; 2) $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$; 3) $\begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}$; 4) *id.*

Theorem 6.6 (cf. [24, proof of Lemma 3.21])
Each direct summand of $M(Q)$ has a normal form.

Proof: The case of odd-dimensional quadric is trivial. So, we can assume that $m := \dim(Q)$ is even.

$(p_N)_{(m/2)}$ is an idempotent, and if $rank((p_N)_{(m/2)})$ is 0, or 2, we get cases 1) and 4), respectively. Now, we can assume that $(p_N)_{(m/2)}$ is a projector in $\mathrm{Mat}_{2\times 2}(\mathbf{Z})$ of rank 1 (equivalently, $det((p_N)_{(m/2)}) = 0$ and $tr((p_N)_{(m/2)}) = 1$).

Sublemma 6.6.1 *Let N be indecomposable direct summand of $M(Q)$ such that $(p_N)_{(m/2)} \neq 0$. Let $\psi \in \mathrm{End}(M(Q))$ be such that $\psi|_{\overline{k}} \circ p_N|_{\overline{k}} = p_N|_{\overline{k}} \circ \psi|_{\overline{k}} = \psi|_{\overline{k}}$, and $tr(\psi_{(m/2)})$ is odd. Then*

$$2 \cdot \mathrm{End}(M(Q|_{\overline{k}})) \subset image(\mathrm{End}(M(Q)) \to \mathrm{End}(M(Q|_{\overline{k}}))).$$

Proof: If q is hyperbolic, then the map $\mathrm{End}(M(Q)) \to \mathrm{End}(M(Q|_{\overline{k}})$ is an isomorphism. So, we can assume that q is not hyperbolic.

Let $\tau \in \mathrm{Hom}(M(Q), M(Q)) = \mathrm{CH}^m(Q \times Q)$ be the morphism given by the graph of "reflection" τ_x (with any (rational) center $x \in \mathbb{P}^{m+1} \backslash Q$). Then $\tau^2 = id_{M(Q)}$. Also, $\tau_{(i)} = 1$, for any $i \neq m/2$, and $\tau_{(m/2)} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

Let $\psi_{(m/2)} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Then $degree(\psi(h^{m/2})) = a + b + c + d$. Since $h^{m/2}$ is defined over k, we have that if $a + b + c + d$ is odd, then on Q there is $m/2$-dimensional cycle of odd degree, which, by Lemma 2.6.1, implies that q is hyperbolic. So, we can assume that $a + b + c + d$ is even. Then in each pair (a, d), (b, c), one element is odd and another is even.

Changing ψ to $\psi - \sum_{i \neq m/2} [\psi_{(i)}/2] \cdot \kappa_{i,Q}$, we can assume that $\psi_{(i)}$ are either 0's or 1's, for all $i \neq m/2$. Notice, that this new ψ still satisfy the conditions of the sublemma. We have two cases:

A) a and b, or c and d are odd; B) a and c, or b and d are odd.

Let $\psi^\vee$ be the dual morphism. Put: $\tilde{\psi} := \begin{cases} \psi^\vee, & \text{if } m \equiv 2\,(\,mod\,4); \\ \tau \circ \psi^\vee \circ \tau, & \text{if } m \equiv 0\,(\,mod\,4). \end{cases}$

Then $\tilde{\psi}_{(m/2)} = \left(\begin{smallmatrix} d & b \\ c & a \end{smallmatrix}\right)$

A) Put $\varepsilon := \psi \circ \tilde{\psi} - \kappa_{m/2,Q} \circ (ad \cdot id + ab \cdot \tau))$.

B) Put $\varepsilon := \tilde{\psi} \circ \psi - \kappa_{m/2,Q} \circ (ad \cdot id + ac \cdot \tau))$.

It is easy to see, that: $\varepsilon_{(m/2)} = \left(\begin{smallmatrix} 0 & 0 \\ 2(cd-ab) & 0 \end{smallmatrix}\right)$, in the case A), and $= \left(\begin{smallmatrix} 0 & 2(bd-ac) \\ 0 & 0 \end{smallmatrix}\right)$, in the case B).

Clearly, $\varepsilon_{(i)} = \psi_{(i)} \cdot \psi^\vee_{(i)}$, for any $i \neq m/2$. At the same time, $\varepsilon^2_{(m/2)} = 0$.

Since $\varepsilon_{(i)} \in \{0, 1\}$, $\varepsilon^2|_{\overline{k}}$ is a projector. By Corollary 6.1(2), ε^{2r} is a projector. Since $\varepsilon^{2r}|_{\overline{k}} = \varepsilon^2|_{\overline{k}}$, and $\varepsilon^2|_{\overline{k}} \circ p_N|_{\overline{k}} = p_N|_{\overline{k}} \circ \varepsilon^2|_{\overline{k}} = \varepsilon^2|_{\overline{k}}$, by Lemma 6.2, we get a direct summand $\tilde{L}$ in N, s.t. $p_{\tilde{L}}|_{\overline{k}} = \varepsilon^2|_{\overline{k}}$. Since $\varepsilon^2_{(m/2)} = 0$, and $(p_N)_{(m/2)} \neq 0$, we have: $N \neq \tilde{L}$. Since N is indecomposable, we have: $\tilde{L} = 0$. In particular, $\varepsilon^2|_{\overline{k}} = p_{\tilde{L}}|_{\overline{k}} = 0$. This implies: $\varepsilon_{(i)} = 0$, for any $i \neq m/2$.

Since $(cd - ab)$ and $(bd - ac)$ are odd in respective cases, we have, using ε, $\tau \circ \varepsilon$, $\varepsilon \circ \tau$, $\tau \circ \varepsilon \circ \tau$, and Lemma 6.4, that for any $u \in 2 \cdot \mathrm{Mat}_{2\times 2}(\mathbf{Z})$, there exists $\varphi \in \mathrm{Hom}(M(Q), M(Q))$, s.t. $\varphi_{(i)} = 0$, for any $i \neq m/2$, and $\varphi_{(m/2)} = u$. Using also $\kappa_{i,Q} := h^i \times h^{m-i}$, for $i \neq m/2$, we get the statement. $\square$

Changing p_N by $\tau \circ p_N \circ \tau$, if nesessary (which does not change the isomorphism class of N), we can assume that in the case A): a and b are odd, and in the case B): b and d are odd. Since $(p_N)_{(m/2)}$ is an idempotent of $rank = 1$, we have: $(p_N)_{(m/2)} = \gamma\alpha\gamma^{-1}$, where $\alpha, \gamma \in \mathrm{Mat}_{2\times 2}(\mathbf{Z})$, and α is of type (2) in the case (A), and of type (3) in the case (B). Then it is easy to see that $u := (\gamma - id)$ is in $2 \cdot \mathrm{Mat}_{2\times 2}(\mathbf{Z})$.

That means that $p_N|_{\overline{k}} = \overline{f} \circ \pi \circ \overline{f}^{-1}$, where $\pi_{(m/2)} = \alpha$ is of type (2) in the case (A), and of type (3) in the case (B), and: $(\overline{f} - id) \in 2 \cdot \mathrm{End}(M(Q|_{\overline{k}}))$.

Take $\psi := p_N$, then $tr(\psi_{(m/2)}) = 1$, and we can apply Sublemma 6.6.1. From Sublemma 6.6.1 it follows that $\overline{f}$ is defined over k by some morphism φ. Then f is an isomorphism, by Theorem 3.1(2), since it is so over $\overline{k}$. $\rho := f^{-1} \circ p_N \circ f$ is a projector (since p_N is), so, $\rho = p_{N'}$ for some N'. Clearly, f defines an isomorphism between N and N', and $(p_{N'})_{(m/2)} = \alpha$ is of type (2), or (3). □

Proof of Theorem 3.11

Lemma 6.7.1 (cf. [24, Lemma 3.21])

Let N_1 and N_2 be nonisomorphic indecomposable direct summands of $M(Q)$. Let N_1' and N_2' be the corresponding normal forms.

Then: $p_{N_1'}|_{\overline{k}} \circ p_{N_2'}|_{\overline{k}} = p_{N_2'}|_{\overline{k}} \circ p_{N_1'}|_{\overline{k}} = 0$.

Proof: Let $\gamma := p_{N_1'} \circ p_{N_2'}$. Since N_i' is a *normal form*, we have: $\gamma_{(m/2)}$ is a projector in $\mathrm{Mat}_{2\times 2}(\mathbf{Z})$, and $\gamma_{(m/2)} \cdot (p_{N_i'})_{(m/2)} = (p_{N_i'})_{(m/2)} \cdot \gamma_{(m/2)} = \gamma_{(m/2)}$.

That means, $\gamma|_{\overline{k}}$ is a projector, and $\gamma|_{\overline{k}} \circ p_{N_i'}|_{\overline{k}} = p_{N_i'}|_{\overline{k}} \circ \gamma|_{\overline{k}} = \gamma|_{\overline{k}}$.

By Corollary 6.1, γ^s is a projector for some s, and if $\gamma^s = p_L$, for some L, then by Lemma 6.2, L is isomorphic to a direct summand in N_i'. Since N_i is indecomposable, we have that either L is isomorphic to N_i', or $L = 0$. Since N_1' is not isomorphic to N_2', we have: $L = 0$. This implies: $p_{N_1'}|_{\overline{k}} \circ p_{N_2'}|_{\overline{k}} = \gamma|_{\overline{k}} = \gamma^s|_{\overline{k}} = 0$

In the same way, considering $\delta := p_{N_2'} \circ p_{N_1'}$, we get: $p_{N_2'}|_{\overline{k}} \circ p_{N_1'}|_{\overline{k}} = 0$. □

Lemma 6.7.2 *Let L and N be direct summands of $M(Q)$, s.t. $p_L|_{\overline{k}} = p_N|_{\overline{k}}$. Then L is isomorphic to N.*

Proof: Consider $\alpha := \varphi_N \circ j_L$, and $\beta := \varphi_L \circ j_N$. Then $(\beta \circ \alpha)|_{\overline{k}} = id_{N|_{\overline{k}}}$ and $(\alpha \circ \beta)|_{\overline{k}} = id_{L|_{\overline{k}}}$. By Corollary 6.1(3), L is isomorphic to N. □

Lemma 6.7.3 *Let L_1, L_2 be direct summands of $M(Q)$, s.t. $p_{L_1}|_{\overline{k}} \circ p_{L_2}|_{\overline{k}} = p_{L_2}|_{\overline{k}} \circ p_{L_1}|_{\overline{k}} = 0$. Then there exists direct summand M of $M(Q)$, s.t. M is isomorphic to $L_1 \oplus L_2$, and $p_M|_{\overline{k}} = p_{L_1}|_{\overline{k}} + p_{L_2}|_{\overline{k}}$.*

Proof: Consider $\pi := p_{L_1} + p_{L_2}$. Then $\pi|_{\overline{k}}$ is a projector, and by Corollary 6.1, π^r is a projector for some r, i.e. there exists direct summand M of $M(Q)$, s.t. $\pi^r = p_M$.

By Lemma 6.2, there exists direct summand $\tilde{L_1}$ in M, s.t. $\tilde{L_1}$ is isomorphic to L_1, and $p_{\tilde{L_1}}|_{\overline{k}} = p_{L_1}|_{\overline{k}}$. Then, for the complimentary projector $p_{\tilde{L_2}} := p_M - p_{L_1}$, we have: $p_{\tilde{L_2}}|_{\overline{k}} = p_{L_2}|_{\overline{k}}$. By Lemma 6.7.2, $\tilde{L_2}$ is isomorphic to L_2, and so, $M \simeq L_1 \oplus L_2$. □

Now we can prove Theorem 3.11.

Let $N'_1, \ldots, N'_s$ be *normal forms* of $N_1, \ldots, N_s$. The statement follows from Lemma 6.7.1 and the inductive application of Lemma 6.7.3. □

Proof of Proposition 3.5

Sublemma 6.8.1 *Let N be a direct summand of $M(Q)$, and $\psi \in \mathrm{End}(N)$. Then either there exists an idempotent $\varepsilon \in \mathrm{End}(N)$ such that $(\varepsilon - \psi)|_{\overline{k}} \in \{2 \cdot \mathrm{End}(N|_{\overline{k}}) + \theta_{m/2,N} \cdot \mathbf{Z}\}$, or Q is (even-dimensional) hyperbolic.*

Proof: Let $\psi_{(r,\mathbf{Z}/2)} \in \mathrm{End}(\mathrm{CH}_r(N|_{\overline{k}})/2)$ be the map induced by ψ.

If $rank(\mathrm{CH}_r(N|_{\overline{k}})) = 1$, then $\psi_{(r,\mathbf{Z}/2)}$ is always a projector.

If $rank(\mathrm{CH}_r(N|_{\overline{k}})) = 2$, then $r = m/2$, and either $\overline{\deg}(\psi(h^{m/2})) = 1$, or one of: $\psi_{(m/2,\mathbf{Z}/2)}$, $(\psi + \theta_{m/2,N})_{(m/2,\mathbf{Z}/2)}$ is a projector. In the former case, Q is hyperbolic, by Lemma 2.6.1, since $\psi(h^{m/2}) \in \mathrm{CH}_{m/2}(Q)$ has odd degree.

Since any idempotent from $\mathrm{End}(N|_{\overline{k}}) \otimes \mathbf{Z}/2$ can be lifted to some idempotent of $\mathrm{End}(N|_{\overline{k}})$, we get: if Q is not hyperbolic, then there exists idempotent $\overline{\varepsilon} \in \mathrm{End}(N|_{\overline{k}})$ such that $(\psi|_{\overline{k}} - \overline{\varepsilon}) \in \{2 \cdot \mathrm{End}(N|_{\overline{k}}) + \theta_{m/2,N} \cdot \mathbf{Z}\}$.

Let there exists r (equal to $m/2$, certainly) with $rank(\mathrm{CH}_r(N|_{\overline{k}})) = 2$. Changing ψ by $\psi + \theta_{m/2,N}$, if nesessary, we can assume that $\psi|_{\overline{k}} \equiv \overline{\varepsilon} \pmod 2$. Then either 1) $tr(\psi_{(m/2)})$ is odd, or 2) $\psi_{(m/2,\mathbf{Z}/2)} = id$.

1) In this case, by Sublemma 6.6.1, there exists $\varepsilon \in \mathrm{End}(N)$, s.t. $\varepsilon|_{\overline{k}} = \overline{\varepsilon}$.

2) In this case, $tr(\psi_{(m/2)})$ is even, and $det(\psi_{(m/2)})$ is odd. Take:

$$\varepsilon' := -\psi \circ (\psi - tr(\psi_{(m/2)}) \cdot id_N) - (det(\psi_{(m/2)}) - 1) \cdot id_N \in \mathrm{End}(N),$$

then $(\varepsilon')_{(m/2)} = id$, and $(\psi - \varepsilon')_{(r)} \in 2 \cdot \mathrm{End}(\mathrm{CH}_r(N|_{\overline{k}}))$, for $r \neq m/2$.

If $rank(\mathrm{CH}_r(N|_{\overline{k}}) \leqslant 1$, for all r, take: $\varepsilon' := \psi$.

Put $\varepsilon := \varepsilon' - \sum_{r \neq m/2}([\varepsilon'_{(r)}/2] \cdot \kappa_{r,N})$, then $\varepsilon|_{\overline{k}} = \overline{\varepsilon}$. □

Let $\varepsilon \in \mathrm{End}(N)$ be an idempotent from Sublemma 6.8.1.

Since N is indecomposable, ε is either 0, or id_N, which implies: that either $\overline{\deg}_N \circ \varepsilon = \overline{\deg}_N$, or $\overline{\deg}_N \circ \varepsilon = 0$. Then the same is true for ψ. The hyperbolic case is evident. □

Proof of Theorem 3.6

Sublemma 6.9.1 *In the situation of Theorem* 3.6*, for* $\gamma := \beta \circ \alpha \in \mathrm{End}(N_1)$*, we have:* $\gamma|_{\overline{k}} \in \{id_{N_1|_{\overline{k}}} + 2 \cdot \mathrm{End}(N_1|_{\overline{k}}) + \theta_{m_1/2,N_1}|_{\overline{k}} \cdot \mathbf{Z}\}$.

Proof: Let $\varepsilon \in \mathrm{End}(N_1)$ be projector from Sublemma 6.8.1 such that $(\varepsilon - \gamma)|_{\overline{k}} \in \{2 \cdot \mathrm{End}(N_1|_{\overline{k}}) + \theta_{m_1/2,N_1} \cdot \mathbf{Z}\}$.

Clearly, $\overline{\deg}_{N_1} \circ \varepsilon = \overline{\deg}_{N_1} \circ \gamma \neq 0$. Hence, $\varepsilon \neq 0$. Since N_1 is indecomposable, we get: $\varepsilon = id_{N_1}$. □

As an evident consequence of Sublemma 6.9.1, we get:

Sublemma 6.9.2 *In the situation of Theorem* 3.6*, the map* $(\beta \circ \alpha)|_{\overline{k}} : \mathrm{CH}(N_1|_{\overline{k}})/2 \to \mathrm{CH}(N_1|_{\overline{k}})/2$ *is an isomorphism.* □

Sublemma 6.9.3 *In the situation of Theorem* 3.6*,* $\overline{\deg}_{N_1}(\beta \circ \alpha(y)) = \overline{\deg}_{N_1}(y)$*, for any* $y \in \mathrm{CH}(N_1|_{\overline{k}})$*, and* $\overline{\deg}_{N_2}(\alpha \circ \beta(z)) = \overline{\deg}_{N_2}(z)$*, for any* $z \in \mathrm{CH}(N_2|_{\overline{k}})$*. In particular, the conditions of Theorem* 3.6 *are symmetric with respect to* N_1 *and* N_2*.*

Proof: From Sublemma 6.9.1 it follows that $\overline{\deg}_{N_1}(\beta \circ \alpha(y)) = \overline{\deg}_{N_1}(y)$.

Since $(\beta \circ \alpha)|_{\overline{k}} : \mathrm{CH}(N_1|_{\overline{k}})/2 \to \mathrm{CH}(N_1|_{\overline{k}})/2$ is an isomorphism (by Sublemma 6.9.2), we have: $\alpha \circ \beta(\mathrm{CH}(N_2|_{\overline{k}})/2) = (\alpha \circ \beta)^2(\mathrm{CH}(N_2|_{\overline{k}})/2)$ is isomorphic to $\mathrm{CH}(N_1|_{\overline{k}})/2$. Let $\varepsilon \in \mathrm{End}(N_2)$ be the projector from Sublemma 6.8.1 such that $(\varepsilon - \alpha \circ \beta)|_{\overline{k}} \in \{2 \cdot \mathrm{End}(N_2|_{\overline{k}}) + \theta_{m_2/2,N_2} \cdot \mathbf{Z}\}$. Since N_2 is indecomposable ε is either 0, or id_{N_2}.

If $\varepsilon = 0$, then $(\alpha \circ \beta)^2|_{\overline{k}} \in 2 \cdot \mathrm{End}(N_2|_{\overline{k}})$, and $\mathrm{CH}(N_1|_{\overline{k}})/2 = 0$, which clearly contradicts to the assumptions of Theorem 3.6. So, $\varepsilon = id_{N_2}$. Then $\overline{\deg}_{N_2}(\alpha \circ \beta(z)) = \overline{\deg}_{N_2}(z)$ for any $z \in \mathrm{CH}(N_2|_{\overline{k}})$. □

Sublemma 6.9.4 *In the situation of Theorem* 3.6*, for any* r*,* $rank(\mathrm{CH}_r(N_1|_{\overline{k}})) = rank(\mathrm{CH}_r(N_2|_{\overline{k}}))$*.*

Proof: It follows from Sublemma 6.9.2, and the fact that the conditions of Theorem 3.6 are symmetric with respect to N_1 and N_2 (Sublemma 6.9.3).

□

Sublemma 6.9.5 *In the situation of Theorem* 3.6*,* $\overline{\deg}_{N_2} \circ \alpha = \overline{\deg}_{N_1} : \mathrm{CH}(N_1|_{\overline{k}}) \to \mathbf{Z}/2$*, and* $\overline{\deg}_{N_1} \circ \beta = \overline{\deg}_{N_2} : \mathrm{CH}(N_2|_{\overline{k}}) \to \mathbf{Z}/2$*.*

Proof: By Sublemma 6.9.3, it is enough to show that if $\overline{\deg}_{N_1}(y) = 0$, then $\overline{\deg}_{N_2}(\alpha(y)) = 0$, and if $\overline{\deg}_{N_2}(z) = 0$, then $\overline{\deg}_{N_1}(\beta(z)) = 0$.

$\mathrm{Ker}(\overline{\deg}_{N_1}) \subset \mathrm{CH}(N_1|_{\overline{k}})$ is generated by: $\{2 \cdot e_w\}_{w \in \Omega_1}$ and $l^1_{m_1/2} + l^2_{m_1/2} = h^{m_1/2}$. Consequently, if for some $y \in \mathrm{CH}(N_1|_{\overline{k}})$, we have: $\overline{\deg}_{N_1}(y) = 0$, and $\overline{\deg}_{N_2}(\alpha(y)) = 1$, then $rank(\mathrm{CH}_{m_1/2}(N_1|_{\overline{k}})) = 2$ and $\overline{\deg}_{N_2}(\alpha(h^{m_1/2})) = 1$. By Sublemma 6.9.4, $rank(\mathrm{CH}_{m_1/2}(N_2|_{\overline{k}})) = 2$, which implies: $m_1 = m_2$, and $\overline{\deg}_{N_2} : \mathrm{CH}_{m_2/2}(N_2|_{\overline{k}}) \to \mathbf{Z}/2$ coinsides with the usual degree ($mod\, 2$). Then on Q_2 there is $m_2/2$-dimensional cycle of odd degree. By Lemma 2.6.1, Q_2 is hyperbolic. Then any indecomposable direct summand in $M(Q_2)$ is a Tate-motive, which contradicts to the fact that $rank(\mathrm{CH}_{m_2/2}(N_2|_{\overline{k}})) = 2$. □

Sublemma 6.9.6 *In the situation of Theorem* 3.6*, if* $u \in \mathrm{Hom}(N_1, N_2)$ *then either* $\overline{\deg}_{N_2} \circ u = \overline{\deg}_{N_1}$*, or* $\overline{\deg}_{N_2} \circ u = 0$*. The same holds for any* $v \in \mathrm{Hom}(N_2, N_1)$*.*

Proof: If $\overline{\deg}_{N_2} \circ u \neq 0$, then, by Sublemma 6.9.5, the pair (u, β) satisfy the conditions of Theorem 3.6. Applying Sublemma 6.9.5 again, we get the statement.

□

Sublemma 6.9.7 *Let* $rank(\mathrm{CH}_r(N_1|_{\overline{k}})) = 2$*. Suppose* α*, and* β *are as in Theorem* 3.6*. Then there exists* α'*,* β'*, s.t.* $\beta'_{(r)} = \lambda \cdot \beta_{(r)}$*, where* λ *is odd,* $det(\alpha'_{(r)}) = det(\beta'_{(r)})$*,* $(\beta' \circ \alpha')_{(r)} = l \cdot id_{\mathrm{CH}_r(N_1|_{\overline{k}})}$*, where* l *is odd, and* $(\alpha' - \alpha)_{(r)} \in \{2 \cdot \mathrm{Mat}_{2\times 2}(\mathbf{Z}) + \left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right) \cdot \mathbf{Z}\}$*.*

Proof: Let $rank(\mathrm{CH}_r(N_1|_{\overline{k}})) = 2$. Then, by Sublemma 6.9.1, $\gamma_{(r)} \in \{id + 2 \cdot \mathrm{Mat}_{2\times 2}(\mathbf{Z}) + \left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right) \cdot \mathbf{Z}\}$. This implies that $det(\gamma_{(r)}) = \mu$ is odd, and $tr(\gamma_{(r)})$ is even.

In $\mathrm{End}(\mathrm{CH}_r(N_1|_{\overline{k}}))$ we have an equality: $-\gamma_{(r)} \circ (\gamma_{(r)} - tr(\gamma_{(r)})) = \mu \cdot id_{\mathrm{CH}_r(N_1|_{\overline{k}})}$. Take $\alpha'' := -\alpha \circ (\beta \circ \alpha - tr(\gamma_{(r)}) \cdot id_{N_1})$. It is easy to see that $(\beta \circ \alpha'')_{(r)} = \mu \cdot id_{\mathrm{CH}_r(N_1|_{\overline{k}})}$.

Let $\alpha' := \mu \cdot \alpha''$, and $\beta' := det(\alpha''_{(r)}) \cdot \beta$. Then $det(\alpha'_{(r)}) = det(\beta'_{(r)})$, and $(\beta' \circ \alpha')_{(r)} = (\mu^2 \cdot det(\alpha''_{(r)})) \cdot id_{\mathrm{CH}_r(N_1|_{\overline{k}})}$. Since $\overline{\deg}_{N_2} \circ \alpha = \overline{\deg}_{N_1}$, we have: $\alpha_{(r)} \circ \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right) \in \{2 \cdot \mathrm{Mat}_{2\times 2}(\mathbf{Z}) + \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right) \cdot \mathbf{Z}\}$, and $(\alpha - \alpha')_{(r)} \in \{id + 2 \cdot \mathrm{Mat}_{2\times 2}(\mathbf{Z}) + \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right) \cdot \mathbf{Z}\}$. □

Sublemma 6.9.8 *In the situation of Theorem* 3.6*:*

1) *For any* r*, such that* $rank(\mathrm{CH}_r(N_1|_{\overline{k}})) = 1$*, there exists* $\kappa_{r,1\to 2} \in \mathrm{Hom}(N_1, N_2)$ *such that* $(\kappa_{r,1\to 2})_{(r)} = 2$*, and* $(\kappa_{r,1\to 2})_{(s)} = 0$*, for any* $s \neq r$*.*

2) *If for some* r*,* $rank(\mathrm{CH}_r(N_i|_{\overline{k}})) = 2$*, then there exist* $\theta_{r,1\to 2}$ *and* $\kappa_{r,1\to 2} \in \mathrm{Hom}(N_1, N_2)$ *such that* $(\theta_{r,1\to 2})_{(r)} = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$*,* $(\kappa_{r,1\to 2})_{(r)} = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right)$*, and* $(\theta_{r,1\to 2})_{(s)} = 0 = (\kappa_{r,1\to 2})_{(s)}$*, for any* $s \neq r$*.*

Proof: 1) Let e_i be the generator of $\mathrm{CH}_r(N_i|_{\overline{k}})$. We can assume that N_i is in the *normal form* (see Definition 6.5). Also, we can assume that N_i is not a Tate-motive.

Using $h^{m_i - r}$ (and Sublemma 6.6.1 (with $\psi = p_{N_i}$), if $r = m_i/2$), we get: $2 \cdot \mathrm{CH}_r(M(Q_i)|_{\overline{k}}) \subset image(\mathrm{CH}_r(M(Q_i)) \to \mathrm{CH}_r(M(Q_i)|_{\overline{k}}))$. Since $\varphi_{N_i} \circ j_{N_i} = id_{N_i}$, we have: $2 \cdot \mathrm{CH}_r(N_i|_{\overline{k}}) \subset image(\mathrm{CH}_r(N_i) \to \mathrm{CH}_r(N_i|_{\overline{k}}))$. In particular, $2 \cdot e_i \in image(\mathrm{CH}_r(N_i) \to \mathrm{CH}_r(N_i|_{\overline{k}}))$. Let $2 \cdot e_i = \overline{g_i}$.

$\overline{\deg}_{N_2}(\alpha(e_1)) = \overline{\deg}_{N_1}(e_1) = 1$, and, by Sublemma 6.9.4, $\overline{\deg}_{N_1}(\beta(e_2)) = \overline{\deg}_{N_2}(e_2) = 1$. That means: $e_1 \in image(\mathrm{CH}_r(N_1) \to \mathrm{CH}_r(N_1|_{\overline{k}}))$ if and only if $e_2 \in image(\mathrm{CH}_r(N_2) \to \mathrm{CH}_r(N_2|_{\overline{k}}))$. We have two cases:
A) e_i is defined over k; B) e_i is not defined over k.

Define morphisms $u \in \mathrm{Hom}(N_1, \mathbb{Z}(r)[2r])$ and $v \in \mathrm{Hom}(\mathbb{Z}(r)[2r], N_2)$ in the following way: $u := h^{m_1 - r} \circ j_{N_1}$, and $v := \begin{cases} e_2 & \text{in the case A)} \\ g_2 & \text{in the case B)} \end{cases}$.

Since N_1 is in the normal form, $j_{N_1}(e_1)$ is either $\pm l_i$, or $\pm h^{m_1 - i}$. Moreover, these opportunities correspond to the cases B), and A), respectively (by Lemma 2.6.1, since N_1 is not a Tate-motive).

Define $\kappa_{r,1\to 2} := \pm v \circ u$. Clearly, $u(e_1)$ (as a map from $\mathbb{Z}(r)[2r]$ to $\mathbb{Z}(r)[2r]$) is equal to id times the degree of the intersection of $j_{N_1}(e_1)$ and $h^{m_1 - r}$. So, it is ± 2 in the case A), and ± 1 in the case B). Then $\kappa_{r,1\to 2}(e_1) = 2 \cdot e_2$, and so, $(\kappa_{r,1\to 2})_{(r)} = 2$. Since $\mathrm{Hom}(\mathbb{Z}(s)[2s], \mathbb{Z}(r)[2r]) = 0$, for $s \neq r$, we have: $u(\mathrm{CH}_s(N_1|_{\overline{k}})) = 0$, and $(\kappa_{r,1\to 2})_{(s)} = 0$, for any $s \neq r$.

2) Let now, for some r, $rank(\mathrm{CH}_r(N_i|_{\overline{k}})) = 2$. Then $m_1 = m_2$, and we can define: $\theta_{r,1\to 2} := \varphi_{N_2} \circ (h^{m_1/2} \times h^{m_1/2}) \circ j_{N_1}$. It is easy to see that $\theta_{r,1\to 2}$ has needed properties.

To define $\kappa_{r,1\to 2}$, observe that if α' and β' are as in Sublemma 6.9.7, then $\beta'_{(r)} = \lambda \cdot \beta_{(r)} = \lambda \cdot \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, and $\alpha'_{(r)} = \lambda \cdot \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$.

If the pair (α, β) satisfy the conditions of Theorem 3.6, then the pair $(\alpha, \beta + \theta_{r,2\to 1})$ satisfy them too (by Sublemma 6.9.6).

Then there exists odd μ, and $\alpha'' \in \mathrm{Hom}(N_1, N_2)$, $\beta'' \in \mathrm{Hom}(N_2, N_1)$, s.t. $\beta''_{(r)} = \mu \cdot (\beta + \theta_{r,2\to 1})_{(r)}$, $det(\alpha''_{(r)}) = det(\beta''_{(r)})$, and $(\beta'' \circ \alpha'')_{(r)} = l'' \cdot id_{\mathrm{CH}_r(N_1|_{\overline{k}})}$.

Take: $\varepsilon := \lambda \cdot \alpha'' - \mu \cdot \alpha' + \lambda\mu \cdot \theta_{r,1\to 2}$, then: $\varepsilon_{(r)} = \lambda\mu \cdot \left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right)$. The composition $\overline{\deg}_{N_2} \circ \varepsilon : \mathrm{CH}_r(N_1|_{\overline{k}}) \to \mathbf{Z}/2$ is zero. By Sublemma 6.9.6, for any s, $\overline{\deg}_{N_2} \circ \varepsilon : \mathrm{CH}_s(N_1|_{\overline{k}}) \to \mathbf{Z}/2$ is zero. Thus, for $s \neq r$, $\varepsilon_{(s)} \in \mathbf{Z}$ is even. Define: $\varepsilon' := \varepsilon - \sum_{s\neq r} (\varepsilon_{(s)}/2) \cdot \kappa_{s,1\to 2}$. Then $\varepsilon'_{(r)} = \varepsilon_{(r)}$, and $\varepsilon'_{(s)} = 0$, for $s \neq r$. By Lemma 6.4, there exists $\kappa_{r,1\to 2} \in \mathrm{Hom}(N_1, N_2)$, s.t. $\lambda\mu \cdot \kappa_{r,1\to 2}|_{\overline{k}} = \varepsilon'|_{\overline{k}}$. Clearly, $\kappa_{r,1\to 2}$ has desired properties. □

Sublemma 6.9.9 *Suppose $rank(\mathrm{CH}_r(N_i)) = 2$, and α, β satisfy the conditions of Theorem 3.6. Then there exists $\alpha_2 \in \mathrm{Hom}(N_1, N_2)$, such that $(\alpha - \alpha_2)_{(r)} \in \{2 \cdot \mathrm{Hom}(\mathrm{CH}_r(N_1|_{\overline{k}}), \mathrm{CH}_r(N_2|_{\overline{k}})) + \theta_{r,1\to 2} \cdot \mathbf{Z}\}$, and $(\alpha_2)_{(r)} = \eta \cdot A$, where η is odd and $A : \mathrm{CH}_r(N_1|_{\overline{k}}) \to \mathrm{CH}_r(N_2|_{\overline{k}})$ is an isomorphism.*

Proof: Since $rank(\mathrm{CH}_r(N_1|_{\overline{k}})) = rank(\mathrm{CH}_r(N_2|_{\overline{k}})) = 2$, we clearly have: $r = m_1/2 = m_2/2$.

For any morphism $w \in \mathrm{Hom}(N_i, N_j)$, we denote as $w^\vee$ *the dual* morphism $\varphi_{N_i} \circ (j_{N_j} \circ w \circ \varphi_{N_i})^\vee \circ j_{N_j} \in \mathrm{Hom}(N_j, N_i)$.

Let us denote as $\widetilde{w} \in \mathrm{Hom}(N_j, N_i)$ the following morphism: $\widetilde{w} := w^\vee$, if m is not divisible by 4, and $:= \tau_i \circ w^\vee \circ \tau_j$, if m is divisible by 4 (τ_i here is the morphism corresponding to the reflection on Q_i). If $w_{(r)} = \left(\begin{smallmatrix} x & y \\ z & t \end{smallmatrix}\right)$, then $\widetilde{w}_{(r)} = \left(\begin{smallmatrix} t & y \\ z & x \end{smallmatrix}\right)$.

Let $\beta_{(r)} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, and α' and β' be as in Sublemma 6.9.7. In particular, $\beta'_{(r)} = \lambda \cdot \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, and $\alpha'_{(r)} = \lambda \cdot \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$. Let $f := g.c.d.(c - b, d - a)$, and $f = u \cdot (d - a) + v \cdot (c - b)$. Then for $\psi := (u - v\tau_2) \circ \alpha' + (u + v\tau_2) \circ \widetilde{\beta'} - \lambda(ua + vb)\kappa_{r,1\to 2}$, we have: $\psi_{(r)} = \lambda \cdot \left(\begin{smallmatrix} 2f & 0 \\ 0 & 0 \end{smallmatrix}\right)$.

Let $\alpha_1 := \alpha' - \psi \circ ([(d-a)/2f] + [(c-b)/2f] \cdot \tau_1)$. Then $\alpha_1 = \lambda \cdot \left(\begin{smallmatrix} a_1 & b_1 \\ c_1 & d_1 \end{smallmatrix}\right)$, where $(a_1 - d_1), (b_1 - c_1) \in \{0, f\}$.

Thus, either 1) $a_1 = d_1$, or 2) $b_1 = c_1$, or 3) $a_1 - d_1 = b_1 - c_1$.

Considering $\alpha_1 \circ \tau_1$ (and $\alpha_2 \circ \tau_1$), we can reduce case 2) to case 1). So, it is enough to consider cases 1) and 3).

1) Take $\alpha_2 := \alpha_1 - \lambda((a_1 - 1) \cdot \theta_{r,1\to 2} - (b_1 - a_1 + 1)/2 \cdot \kappa_{r,1\to 2} \circ \tau_1)$. Then $(\alpha_2)_{(r)} = \lambda \cdot \left(\begin{smallmatrix} 1 & 0 \\ * & 1 \end{smallmatrix}\right)$ (notice that $(b_1 - a_1 + 1) \equiv (b + d + 1) \equiv 0\,(\,mod\,2)$, by Sublemma 6.9.5).

3) take $\alpha_2 := \alpha_1 - \lambda \cdot \kappa_{r,1\to 2} \circ ([(a_1 + d_1)/4] + [(b_1 + c_1)/4] \cdot \tau_1)$. Then $(\alpha_2)_{(r)} = \lambda \left(\begin{smallmatrix} a_2 & b_2 \\ c_2 & d_2 \end{smallmatrix}\right)$, where $a_2 - d_2 = b_2 - c_2$; $(a_2 + d_2), (b_2 + c_2) \in \{0, 2\}$, and $a_2 + b_2 + c_2 + d_2 = 2$ (by Sublemma 6.9.5, $(a + b) \equiv (a + c) \equiv (b + d) \equiv (c + d) \equiv 1\,(\,mod\,2)$).

Hence, in any case, $det((\alpha_2)_{(r)}/\lambda) = \pm 1$. □

Sublemma 6.9.10 *In the situation of Theorem* 3.6*, there exist* $\alpha_3 \in \mathrm{Hom}(N_1, N_2)$*, and some odd number* η*, that for all* s*,* $(\alpha_3)_{(s)} = \eta \cdot A_s$*, where* A_s *is invertible.*

Proof: If $rank(\mathrm{CH}_s(N_1|_{\overline{k}})) \leqslant 1$, for all s, take: $\alpha'' := \alpha$, and $\eta := 1$. If $rank(\mathrm{CH}_r(N_1|_{\overline{k}})) = 2$, for some r, take: $\alpha'' := \alpha_2$, and take η from the Sublemma 6.9.9 (here α_2, A are also from Sublemma 6.9.9).

In the light of Sublemma 6.9.9, $\overline{\deg}_{N_2} \circ \alpha'' : \mathrm{CH}_r(N_1|_{\overline{k}}) \to \mathbf{Z}/2$ is nonzero. By Sublemma 6.9.6, for all s with $rank(\mathrm{CH}_s(N_1|_{\overline{k}})) = 1$, $(\alpha'')_{(s)} = \lambda_s$ is odd. Define $\alpha_3 := \alpha'' - \sum(\lambda_s - \eta)/2 \cdot \kappa_{s,1\to 2}$, where sum is taken over all s, such that $rank(\mathrm{CH}_r(N_1|_{\overline{k}})) = 1$, and $\kappa_{s,1\to 2}$ are elements from Sublemma 6.9.8. Then, for any t, $(\alpha_3)_{(t)} = \eta \cdot A_t$, where A_t is invertible. □

Now we can prove Theorem 3.6. We start with the case $d_1 = d_2 = 0$.

From Sublemma 6.9.10 and Lemma 6.4 it follows that, in the situation of Theorem 3.6, there exists $\alpha_4 \in \mathrm{Hom}(N_1, N_2)$ such that $\alpha_4|_{\overline{k}}$ is an isomorphism. Since conditions of Theorem 3.6 are symmetric with respect to N_1 and N_2 (by Sublemma 6.9.3), we also have some $\beta_4 \in \mathrm{Hom}(N_2, N_1)$ such that $\beta_4|_{\overline{k}}$ is an isomorphism. Then $\beta_4 \circ \alpha_4$ and $\alpha_4 \circ \beta_4$ are isomorphisms, by Corollary 6.1(3).

Now the general case can be reduced to the case $d_1 = d_2 = 0$ since $M(Q_i)(d_i)[2d_i]$ is a direct summand in $M(Q'_i)$, where $q'_i := q_i \perp d_i \cdot \mathbb{H}$, by Proposition 2.1. □

Proof of Theorem 3.8

Sublemma 6.10.1 *Suppose N and L be indecomposable direct summands of $M(Q)$ in normal form, and $\gamma \in \mathrm{Hom}(N, L)$ be such map, that the composition $\overline{\deg}_Q \circ j_L \circ \gamma : \mathrm{CH}(N|_{\overline{k}}) \to \mathbf{Z}/2$ is nonzero. Then $N \simeq L$.*

Proof: Let $\overline{\deg}_Q \circ j_L \circ \gamma : \mathrm{CH}_r(N|_{\overline{k}}) \to \mathbf{Z}/2$ is nonzero for some r. In particular, $\mathrm{CH}_r(N|_{\overline{k}}) \neq 0 \neq \mathrm{CH}_r(L|_{\overline{k}})$. By Theorem 3.11, either $N \simeq L$, or $r = \dim(Q)/2$ (since only for such r, $rank(\mathrm{CH}_r(M(Q)|_{\overline{k}})) = 2$), and $rank(\mathrm{CH}_r(N|_{\overline{k}})) = rank(\mathrm{CH}_r(L|_{\overline{k}})) = 1$.

If N is not isomorphic to L, then, in the notations of Definition 6.5, N and L are of types (2), or (3). If L is of type (3), then $\overline{\deg}_Q \circ j_L : \mathrm{CH}_r(L|_{\overline{k}}) \to \mathbf{Z}/2$ is zero (since in such case $\mathrm{CH}_r(L|_{\overline{k}})$ is generated by the class of h^r). This is, clearly, not the case, so L is of type (2). Since L is not isomorphic to N, and N,L are indecomposable direct summands in *normal form*, we have by Lemma 6.7.1: N is of type (3). But then the generator h^r of $\mathrm{CH}_r(N|_{\overline{k}})$ is defined over k. This implies that on Q there exists r-dimensional cycle of odd degree (namely, $j_L \circ \gamma(h^r)$). By Lemma 2.6.1, Q is hyperbolic. Then N and L must be Tate-motives. Since $rank(\mathrm{CH}_r(N|_{\overline{k}})) = rank(\mathrm{CH}_r(L|_{\overline{k}})) = 1$, we have: $N \simeq \mathbb{Z}(r)[2r] \simeq L$. Contradiction. So, $N \simeq L$. □

Sublemma 6.10.2
Let N be indecomposable direct summand of $M(Q_2)(d_2)[2d_2]$, and the maps: $N \underset{\alpha}{\overset{\beta}{\rightleftarrows}} M(Q_1)(d_1)[2d_1]$ are such that the composition $\overline{\deg}_{Q_1} \circ \beta \circ \alpha$: $\mathrm{CH}(M(Q_1)(d_1)[2d_1]|_{\overline{k}}) \to \mathbf{Z}/2$ is nonzero. Then there exists direct summand L of $M(Q_1)(d_1)[2d_1]$ isomorphic to N.

Proof: Let $M(Q_1)(d_1)[2d_1] = \oplus_{a \in \Lambda_1} L^a$, where L^a are indecomposable direct summands in *normal form* (by Theorem 3.11, we can always find such decompositions). Let $\alpha_a := \alpha \circ j_{L^a} \in \mathrm{Hom}(L^a, N)$, and $\beta_c := \varphi_{L^c} \circ \beta$. We have: $\beta \circ \alpha = \sum_{a,c \in \Lambda_1} (p_{L^c} \circ \beta \circ \alpha \circ p_{L^a}) = \sum_{a,c \in \Lambda_1} (j_{L^c} \circ \beta_c \circ \alpha_a \circ \varphi_{L^a})$.

Since the composition $\overline{\deg}_{Q_1} \circ \beta \circ \alpha$: $\mathrm{CH}(M(Q_1)(d_1)[2d_1]|_{\overline{k}}) \to \mathbf{Z}/2$ is nonzero we have: for some a, c the composition $\overline{\deg}_{Q_1} \circ j_{L^c} \circ \beta_c \circ \alpha_a$: $\mathrm{CH}(L^a|_{\overline{k}}) \to \mathbf{Z}/2$ is nonzero.

By Sublemma 6.10.1, there exists an isomorphism $\psi : L^c \to L^a$. Take: $u := \alpha_a \circ \psi \in \mathrm{Hom}(L^c, N)$, and $v := \beta_c \in \mathrm{Hom}(N, L^c)$. Then the map

$\overline{\deg}_{Q_1} \circ j_{L^c} \circ v \circ u : \mathrm{CH}_r(L^c|_{\overline{k}}) \to \mathbf{Z}/2$ is nonzero. Since the composition $\mathrm{CH}_r(L^c|_{\overline{k}}) \stackrel{j_{L^c}}{\to} \mathrm{CH}_r(M(Q_1)(d_1)[2d_1]|_{\overline{k}}) \stackrel{\overline{\deg}_{Q_1}}{\to} \mathbf{Z}/2$ either coinsides with $\overline{\deg}_{L^c} : \mathrm{CH}_r(L^c|_{\overline{k}}) \to \mathbf{Z}/2$, or is zero, we get: $\overline{\deg}_{L^c} \circ v \circ u : \mathrm{CH}_r(L^c|_{\overline{k}}) \to \mathbf{Z}/2$ is nonzero. By Theorem 3.6, $L^c \simeq N$. □

Let $M(Q_2)(d_2)[2d_2] = \oplus_{b\in\Lambda_2} N_2^b$, where N^b are indecomposable direct summands. Let $\alpha^b := \varphi_{N^b} \circ \alpha$, and $\beta^b := \beta \circ j_{N^b}$. We have: $\beta \circ \alpha = \sum_{b\in\Lambda_2}(\beta \circ p_{N^b} \circ \alpha) = \sum_{b\in\Lambda_2}(\beta_b \circ \alpha_b)$.

So, if $\overline{\deg}_{Q_1} \circ \beta \circ \alpha : \mathrm{CH}_r(M(Q_1)(d_1)[2d_1]|_{\overline{k}}) \to \mathbf{Z}/2$ is nonzero, then for some $b \in \Lambda_2$, the map: $\overline{\deg}_{Q_1} \circ \beta_b \circ \alpha_b : \mathrm{CH}_r(M(Q_1)(d_1)[2d_1]|_{\overline{k}}) \to \mathbf{Z}/2$ is nonzero. By Sublemma 6.10.2, there exists some indecomposable direct summand L of $M(Q_1)(d_1)[2d_1]$ isomorphic to N^b. Clearly, $rank(\mathrm{CH}_r(N^b|_{\overline{k}})) \neq 0$, so, $\mathbb{Z}(r)[2r]$ is a direct summand in $N^b|_{\overline{k}}$. Theorem 3.8 is proven. □

Proof of Proposition 4.5

Suppose it is not the case. Changing N by $N^\vee$, we have: $\mathrm{CH}_m(N|_{\overline{k}}) \neq 0$ and $\mathrm{CH}_0(N|_{\overline{k}}) = 0$. The composition $\nu := (\varphi_N \otimes id) \circ \Delta_Q \circ j_N$ gives a section of the natural projection $\pi : N \otimes M(Q) \to N$.

Let $\beta_0 \in \mathrm{Hom}(\mathbb{Z}(m)[2m], M(Q)) = \mathrm{CH}_m(Q)$ be the morphism corresponding to the "generic cycle" on Q. Let $u := (\varphi_N \otimes id) \circ (\beta_0 \otimes id) \in \mathrm{Hom}(M(Q)(m)[2m], N \otimes M(Q))$, and $v := \Delta^\vee \circ (j_N \otimes id) \in \mathrm{Hom}(N \otimes M(Q), M(Q)(m)[2m])$, where $\Delta^\vee \in \mathrm{Hom}(M(Q \times Q), M(Q)(m)[2m])$ is dual to the "diagonal embedding" Δ via duality: $\mathrm{CH}^*(A \times B) \simeq \mathrm{CH}^*(B \times A)$.

Since $j_N \circ \varphi_N \circ \beta_0 = \beta_0$ (since $\mathrm{CH}_m(N|_{\overline{k}}) = \mathbf{Z}$, and hence $j_N \circ \varphi_N : \mathrm{CH}_m(Q|_{\overline{k}}) \to \mathrm{CH}_m(Q|_{\overline{k}})$ is the identity map), we have: $v \circ u = id \in \mathrm{End}(M(Q)(m)[2m])$. So, $N \otimes M(Q) = M(Q)(m)[2m] \oplus X$. Let $\varphi_X \in \mathrm{Hom}(N \otimes M(Q), X)$, and $j_X \in \mathrm{Hom}(X, N \otimes M(Q))$ be the corresponding projection and embedding. I.e.: $\varphi_X \circ j_X = id_X$, $j_X \circ \varphi_X + u \circ v = id_{N\otimes M(Q)}$, and $\varphi_X \circ u = 0$, $v \circ j_X = 0$.

In particular, $\pi \circ \nu = (\pi \circ j_X) \circ (\varphi_X \circ \nu) + (\pi \circ u) \circ (v \circ \nu)$. So, there exist maps $\alpha_1 : N \to M(Q)(m)[2m]$, $\beta_1 : M(Q)(m)[2m] \to N$, and $\alpha_2 : N \to X$, $\beta_2 : X \to N$, s.t. $\beta_1 \circ \alpha_1 + \beta_2 \circ \alpha_2 = id_N$.

We can assume that $m > 0$. Then, for arbitrary maps $\alpha_1' : M(Q) \to M(Q)(m)[2m]$, $\beta_1' : M(Q)(m)[2m] \to M(Q)$, the composition $\overline{\deg}_Q \circ \beta_1' \circ \alpha_1'$:

$\mathrm{CH}_m(Q|_{\overline{k}}) \to \mathbf{Z}/2$ is zero. Really, such degree is equal to the degree of some 0-cycle on Q, and Q is anisotropic. Since the maps $\overline{\deg_Q} \circ j_N$ and $\overline{\deg_N}$ coinside on $\mathrm{CH}_m(N|_{\overline{k}})$, we get: $\overline{\deg_N} \circ \beta_1 \circ \alpha_1 : \mathrm{CH}_m(N|_{\overline{k}}) \to \mathbf{Z}/2$ is zero.

Consider $E = k(Q)$. We have: $q|_E = \mathbb{H} \oplus q'$, where q' is anisotropic (since $i_1(q) = 1$). By [22, Proposition 1] (Proposition 2.1), $M(Q|_E) = \mathbb{Z} \oplus M(Q')(1)[2] \oplus \mathbb{Z}(m)[2m]$. And then $N|_E = N_{an} \oplus \mathbb{Z}(m)[2m]$, where N_{an} is a direct summand of $M(Q')(1)[2]$ (since $\mathrm{CH}_m(N|_{\overline{k}}) = \mathbf{Z}$, and $\mathrm{CH}_0(N|_{\overline{k}}) = 0$). Moreover, if $\tilde{j} : \mathbb{Z}(m)[2m] \to N|_E$ and $\tilde{\varphi} : N|_E \to \mathbb{Z}(m)[2m]$ be the corresponding maps, then $\tilde{j}$ coinsides with $(\varphi_N \circ \beta_0)|_E$. Then the map $\tilde{j} \otimes id_{M(Q)} : M(Q)(m)[2m]|_E \to N \otimes M(Q)|_E$ coinsides with $u|_E$. This implies that the complimentary direct summand $N_{an} \otimes M(Q)$ (in $(N \otimes M(Q))|_E$) is isomorphic to $X|_E$. Notice that $N_{an} \otimes M(Q)$ is a direct summand in $M(Q')(1)[2] \otimes M(Q)$. So, $\alpha_2|_E$ and $\beta_2|_E$ give us maps $\alpha_2' : N|_E \to M(Q' \times Q)(1)[2]$, and $\beta_2' : M(Q' \times Q)(1)[2] \to N|_E$, s.t. $\beta_2' \circ \alpha_2' = (\beta_2 \circ \alpha_2)|_E$.

If $\alpha_2' \circ \tilde{j} \in \mathrm{Hom}(\mathbb{Z}(m)[2m], M(Q' \times Q)(1)[2]) = \mathrm{CH}_{m-1}(Q' \times Q)$ is represented by the cycle A, and $\tilde{\varphi} \circ \beta_2' \in \mathrm{Hom}(M(Q' \times Q)(1)[2], \mathbb{Z}(m)[2m]) = \mathrm{CH}^{m-1}(Q' \times Q)$ is represented by the cycle B, then the composition $(\tilde{\varphi} \circ \beta_2') \circ (\alpha_2' \circ \tilde{j}) \in \mathrm{End}(\mathbb{Z}(m)[2m]) = \mathbf{Z}$ is given by the degree of the 0-cycle $A \cap B \in \mathrm{CH}_0(Q' \times Q)$. Since Q' is anisotropic, this number is even, by Springer's Theorem. Since $\tilde{j}$ and $\tilde{\varphi}$ are isomorphisms on CH_m, we have: the composition $\overline{\deg_N} \circ \beta_2' \circ \alpha_2' = \overline{\deg_N} \circ \beta_2 \circ \alpha_2 : \mathrm{CH}_m(N|_{\overline{k}}) \to \mathbf{Z}/2$ is zero. Since $\overline{\deg_N} \circ \beta_1 \circ \alpha_1 : \mathrm{CH}_m(N|_{\overline{k}}) \to \mathbf{Z}/2$ is zero as well, we get a contradiction with: $\beta_1 \circ \alpha_1 + \beta_2 \circ \alpha_2 = id_N$. □

Proof of Corollary 4.7

Sublemma 6.11.1 *Let Q be anisotropic quadric, and L be indecomposable direct summand of $M(Q)$ such that $a(L) = 0$. Then for any subquadric $P \subset Q$ with $\dim(P) = \dim(Q) - i_1(q) + 1$, we have:*

1) *$M(P)$ contains a direct summand isomorphic to L;*

2) *$p|_{k(Q)}$ and $q|_{k(P)}$ are isotropic.*

Proof: For any field extension E/k, we have: $p|_E$ is isotropic if and only if $q|_E$ is. In particular we get 2). So, we have rational (algebro-geometric) maps $f : Q \dashrightarrow P$, and $g : P \dashrightarrow Q$. Let $\alpha \in \mathrm{CH}^{\dim(P)}(Q \times P) =$

$\mathrm{Hom}(M(Q), M(P))$, and $\beta \in \mathrm{CH}^{\dim(Q)}(P \times Q) = \mathrm{Hom}(M(P), M(Q))$ be the closures of the graphs of f and g, respectively. Clearly, $\alpha(l_0) = l_0$, and $\beta(l_0) = l_0$. So, the composition $\overline{\deg}_Q \circ \beta \circ \alpha : \mathrm{CH}_0(Q|_{\overline{k}}) \to \mathbf{Z}/2$ is nonzero. By Theorem 3.8, $M(P)$ contains a direct summand isomorphic to L (notice that if M is indecomposable direct summand of $M(Q)$, s.t. $\mathrm{CH}_0(M) \neq 0$, then $M \simeq L$, by Lemma 6.7.1). □

Sublemma 6.11.2 *Let Q be anisotropic quadric, and L be indecomposable direct summand of $M(Q)$ with $a(L) = 0$. Then there exists subquadric $P \subset Q$ such that:*

1) $i_1(p) = 1$;

2) *$M(P)$ contains a direct summand isomorphic to L;*

3) *$p|_{k(Q)}$ and $q|_{k(P)}$ are isotropic.*

Proof: Use induction on the dimension of Q. The case of $\dim(Q) = 0$ is trivial. Suppose the statement is true for all quadrics of dimension $< \dim(Q)$. Consider P from Sublemma 6.11.1. Either $i_1(q) = 1$, in which case statement is trivial, or $\dim(P) < \dim(Q)$. Then there exists P', s.t. P' satisfy 1) and 2), and $p'|_{k(P)}$, $p|_{k(P')}$ are isotropic. Since we also have: $p|_{k(Q)}$, $q|_{k(P)}$ are isotropic, we get: P' satisfy 3) (since if $q_2|_{k(q_1)}$, $q_3|_{k(q_2)}$ are isotropic, then $q_3|_{k(q_1)}$ is). □

Now we can prove Corollary 4.7. Let us denote $c(N) := \dim(Q) - b(N) = a(N^{\vee})$.

Let P be quadric from Sublemma 6.11.2. Then L is also a direct summand in $M(P)$. By Proposition 4.5, $b(L) = \dim(P)$. In particular, by [22, Proposition 1], (Proposition 2.1), $L|_{k(P)}$ contains $\mathbb{Z}$ and $\mathbb{Z}(\dim(P))[2\dim(P)]$ as direct summands.

Since Q is anisotropic, P is also anisotropic. If $\dim(P) = 0$, then $rank(\mathrm{CH}_0(L|_{\overline{k}})) = 2$, and hence, $m = 0$ (since L is a direct summand in $M(Q)$). In this case everything is evident. If $b(L) = \dim(P) > 0$, then the map $\mathrm{CH}_{b(L)}(P) \to \mathrm{CH}_{b(L)}(P|_{\overline{k}})$ is surjective, and $\mathrm{CH}_{b(L)}(L|_{\overline{k}}) = \mathrm{CH}_{b(L)}(P|_{\overline{k}})$. So, the map $\mathrm{CH}_{b(L)}(L) \to \mathrm{CH}_{b(L)}(L|_{\overline{k}})$ is surjective. And the map $\overline{\deg}_L : \mathrm{CH}_{b(L)}(L) \to \mathbf{Z}/2$ is nonzero. If $b(L) < m/2$, then $\overline{\deg}_L =$

$\overline{\deg}_Q \circ j_L : \mathrm{CH}_{b(L)}(L) \to \mathbf{Z}/2$, and we get $b(L)$-dimensional cycle of odd degree on Q. By Lemma 2.6.1, Q is isotropic - contradiction. So, $b(L) \geqslant m/2$. Since $L|_{k(P)}$ contains $\mathbb{Z}(b(L))[2b(L)]$ as a direct summand, $L|_{k(Q)}$ also contains $\mathbb{Z}(b(L))[2b(L)]$ as a direct summand, by Sublemma 6.11.2(3). Then $b(L) > m - i_1(q)$, by Proposition 2.6 (since $b(L) \geqslant m/2$).

By Corollary 3.10, $M(Q)$ contains direct summand isomorphic to $L(i_1(q)-1)[2i_1(q)-2]$. This implies: $b(L) \leqslant \dim(Q) - i_1(q) + 1$. So, $b(L) = \dim(Q) - i_1(q) + 1$, and $c(L) = i_1(q) - 1$. □

Proof of Proposition 4.8

Let M be indecomposable direct summand of N such that $\overline{\deg}_Q : \mathrm{CH}_a(M|_{\overline{k}}) \to \mathbf{Z}/2$ is nonzero, and L be indecomposable direct summand of $M(Q)$ such that $a(L) = 0$. Let us show that $a(M) = a$ and $M \simeq L(a)[2a]$.

By Corollary 3.10, $L(a)[2a]$ is isomorphic to a direct summand of $M(Q)$, and both $M|_{\overline{k}}$ and $L(a)[2a]|_{\overline{k}}$ contain $\mathbb{Z}(a)[2a]$ as a direct summand.

If $a < m/2$, then by Corollary 3.7, $M \simeq L(a)[2a]$ and $a(M) = a$.

Suppose now $a = m/2$. Then $i_1(q) = m/2 + 1$ (so, Q is a Pfister quadric). Then all the motives $L|_{\overline{k}}, L(a)[2a]|_{\overline{k}}, M|_{\overline{k}}$ contain $\mathbb{Z}(a)[2a]$ as a direct summand (use Corollary 4.7). By Theorem 3.11, $L, L(a)[2a]$ and M can not be all pairwise nonisomorphic. Treating separately the evident case $a = m = 0$, we can assume that $a > 0$, and so, M is isomorphic either to L or to $L(a)[2a]$. Let us show that the first opportunity is impossible. Really, $b(L(a)[2a]) = m$, we have an equality: $\mathrm{CH}_m(L(a)[2a]|_{\overline{k}}) = \mathrm{CH}_m(Q|_{\overline{k}})$, and consequently the generator of $\mathrm{CH}_m(L(a)[2a]|_{\overline{k}})$ is defined over the base field k. Hence, the generator of $\mathrm{CH}_a(L|_{\overline{k}})$ is defined over the base field. Thus, the map $\overline{\deg}_Q : \mathrm{CH}_a(L|_{\overline{k}}) \to \mathbf{Z}/2$ should be trivial (otherwise, by Lemma 2.6.1, q would be hyperbolic). This implies that $\Lambda(L)$ does not contain L_{lo}. And so, $M \not\simeq L$ by Lemma 4.1. Hence $M \simeq L(a)[2a]$ and $a(M) = a$. But $\mathbb{Z}(a(M))[2a(M)] \oplus \mathbb{Z}(b(M))[2b(M)]$ is a direct summand of $M|_{\overline{k}}$, and hence, of $N|_{\overline{k}}$, and $b(M) = m - i_1(q) + 1 + a$ by Corollary 4.7. □

As a biproduct we get the following

Corollary 6.12 *Let Q be anisotropic quadric of dimension m, and M be indecomposable direct summand of $M(Q)$ such that $0 \leqslant a(M) < i_1(q)$. Then* $\mathrm{size}(M) = m - i_1(q) + 1$.

Proof of Theorem 4.13 and Corollary 4.14

Let Q be smooth projective quadric. We denote as $\underline{Q}^i$ the variety of flags $\pi_\bullet = (\pi_0 \subset \pi_1 \subset \ldots \subset \pi_i)$, where $\pi_j \subset Q$ is j-dimensional projective subspace. For example, $\underline{Q}^0 = Q$. Clearly, $\underline{Q}^i$ has a rational point if and only if the form q is $(i+1)$-times isotropic: $q = (i+1) \cdot \mathbb{H} \perp q'$.

We have the natural maps:

$$f_i : M(\underline{Q}^i)(i)[2i] \to M(Q), \text{ and } g_j : M(\underline{Q}^j) \otimes M(Q) \to M(\underline{Q}^{j+1})(j+1)[2j+2]$$

given by the cycles $\mathcal{F} \subset \underline{Q}^i \times Q$, and $\mathcal{G} \subset \underline{Q}^j \times Q \times \underline{Q}^{j+1}$, respectively, where $(\pi_\bullet, x) \in \mathcal{F} \Leftrightarrow x \in \pi_i$, and $(\pi_\bullet, x, \nu_\bullet) \in \mathcal{G} \Leftrightarrow \nu_\bullet^{\leqslant j} = \pi_\bullet$ and $x \in \nu_{j+1}$.

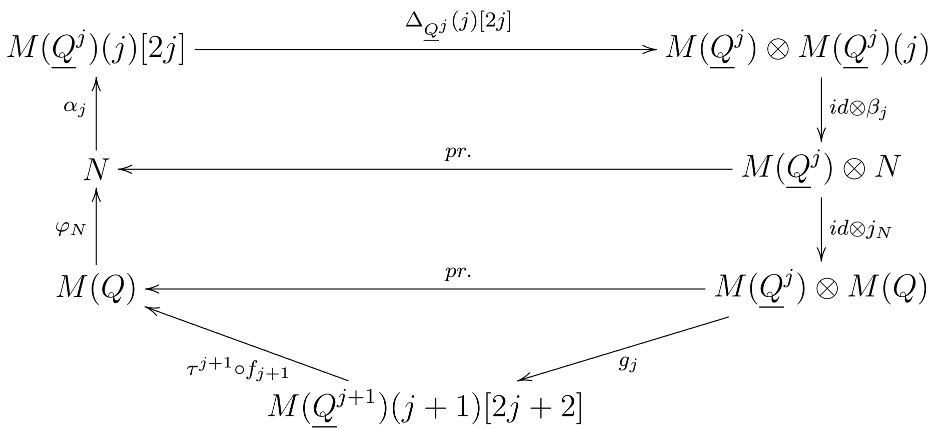
The following result is very useful in the applications. For example, it is used in the proof of the criterion of motivic equivalence for quadrics (see [24] and Theorem 4.18).

Theorem 6.13 *Let Q be smooth projective quadric, and N be a direct summand of $M(Q)$, such that $a(N) = i \leqslant \dim(Q)/2$. Then there exist maps $N \underset{\beta}{\overset{\alpha}{\rightleftarrows}} M(\underline{Q}^i)(i)[2i]$ such that the composition $\beta \circ \alpha : \mathrm{CH}_i(N|_{\overline{k}}) \to \mathrm{CH}_i(N|_{\overline{k}})$ is the identity.*

Proof: Let us prove by induction, that for every $0 \leqslant j \leqslant i$ there exist maps $N \underset{\beta_j}{\overset{\alpha_j}{\rightleftarrows}} M(\underline{Q}^j)(j)[2j]$ such that the composition $\beta_j \circ \alpha_j : \mathrm{CH}_i(N|_{\overline{k}}) \to \mathrm{CH}_i(N|_{\overline{k}})$ is the identity.

$(j = 0)$: We can take $\alpha_0 := j_N : N \to M(Q)$, and $\beta_0 := \varphi_N : M(Q) \to N$.

$(j \to j+1)$: Consider the following diagramm:



where $\tau : M(Q) \to M(Q)$ is the motive of the reflection from $O(q)$.

Denote $u := (id \otimes \beta_j) \circ \Delta_{\underline{Q}^j}(j)[2j] \circ \alpha_j$. The composition $pr \circ u$ is equal to $\beta_j \circ \alpha_j$, so, the map: $pr \circ u : \mathrm{CH}_i(N|_{\overline{k}}) \to \mathrm{CH}_i(N|_{\overline{k}})$ is the identity. But $a(N) = i$, so $\mathrm{CH}_s(N|_{\overline{k}}) = 0$, for $s < i$, and $pr : \mathrm{CH}_i(M(\underline{Q}^j|_{\overline{k}}) \otimes N|_{\overline{k}}) \to \mathrm{CH}_i(N|_{\overline{k}})$ is an isomorphism (the variety $\underline{Q}^j|_{\overline{k}}$ is rational). In particular, the group $\mathrm{CH}_i(M(\underline{Q}^j|_{\overline{k}}) \otimes N|_{\overline{k}})$ is generated by the elements of the form $l_0 \otimes \varphi_N(l_i)$, where l_0 is the class of a rational point on $\underline{Q}^j|_{\overline{k}}$, and $l_i \in \mathrm{CH}_i(Q|_{\overline{k}})$ is the class of a plane of dimension i. Then $u(\varphi_N(l_i)) = l_0 \otimes \varphi_N(l_i)$.

Clearly, the middle square of the diagramm is commutative. Finally, if $l_i = [A_i]$, then $f_{j+1} \circ g_j(l_0 \otimes l_i) = [B_i]$, where B_i is some plane of dimension i on Q such that $\mathrm{codim}(A_i \cap B_i \subset A_i) = j+1$. If $i < \dim(Q)/2$, then $[A_i] = [B_i] = \tau^{j+1}([B_i])$, and if $i = \dim(Q)/2$, then $[A_i] = \tau^{j+1}([B_i])$. Thus, $\tau^{j+1} \circ f_{j+1} \circ g_j(l_0 \otimes l_i) = l_i = pr(l_0 \otimes l_i)$. So, if we denote: $v := \varphi_N \circ \tau^{j+1} \circ f_{j+1} \circ g_j \circ (id \otimes j_N)$, then $v \circ u : \mathrm{CH}_i(N|_{\overline{k}}) \to \mathrm{CH}_i(N|_{\overline{k}})$ is the identity. It remains to put: $\alpha_{j+1} := g_j \circ (id \otimes j_N) \circ u$, and $\beta_{j+1} := \varphi_N \circ \tau^{j+1} \circ f_{j+1}$. Induction step is proven. □

Let $k = F_0 \subset F_1 \subset \cdots \subset F_h$ be the *generalized splitting tower* of M.Knebusch for Q (see [15]). We remind that the sequence $0 = i_W(q|_{F_0}) < i_W(q|_{F_1}) < \cdots < i_W(q|_{F_h})$ contains all possible values of $i_W(q|_E)$ for arbitrary field-extension E/k.

Sublemma 6.14.1 (cf. [24, Lemma 4.5])
Let $i_W(q|_{F_t}) \leqslant i < j < i_W(q|_{F_{t+1}})$, and N be indecomposable direct summand of $M(Q)$, s.t. $a(N) = i$. Then $N(j-i)[2j-2i]$ is isomorphic to a direct summand of $M(Q)$.

Proof: By Theorem 6.13, we have the map $\alpha_i : N \to M(\underline{Q}^i)(i)[2i]$, such that $\alpha_i(\varphi_N(l_i)) = l_0(i)[2i]$. Since $i_W(q|_{F_t}) \leqslant i < j < i_W(q|_{F_{t+1}})$, we have rational map $\underline{Q}^i \dashrightarrow \underline{Q}^j$, which gives us motivic map $\gamma : M(\underline{Q}^i) \to M(\underline{Q}^j)$, such that $\gamma(l_0) = l_0$ (l_0 here is the class of a rational point). Consider the composition $\varepsilon := f_j(i-j)[2i-2j] \circ \gamma(i)[2i] \circ \alpha_i : N \to M(Q)(i-j)[2i-2j]$, and the map $\eta := \varphi_N \circ \rho^{j-i} : M(Q)(i-j)[2i-2j] \to N$, where the map $\rho^{j-i} : M(Q)(i-j)[2i-2j] \to M(Q)$ is defined by the plane section of codimenion $(j-i)$ in Q, imbedded diagonally into $Q \times Q$. So, we have the pair of maps: $N \underset{\eta}{\overset{\varepsilon}{\rightleftarrows}} M(Q)(i-j)[2i-2j]$. Since $f_j(l_0(j)[2j]) = l_j$, and

$\rho^{j-i}(l_j(i-j)[2i-2j]) = l_i$, we get that the map $\overline{\deg}_N \circ \eta \circ \varepsilon : \mathrm{CH}_i(N|_{\overline{k}}) \to \mathbf{Z}/2$ is nonzero. By Sublemma 6.10.2, N is isomorphic to a direct summand of $M(Q)(i-j)[2i-2j]$. □

Sublemma 6.14.2 *Let Q be smooth anisotropic quadric of dimension m, and N be indecomposable direct summand of $M(Q)$. Then there exists $0 \leqslant t < h(q)$ such that, for fields $F_t \subset F_{t+1}$ from the generalized splitting tower of M.Knebusch for Q, we have: $i_W(q|_{F_{t+1}}) > a(N), c(N) \geqslant i_W(q|_{F_t})$, and $a(N) + c(N) \leqslant i_W(q|_{F_t}) + i_W(q|_{F_{t+1}}) - 1$.*

Proof: Let t and s be such that $i_W(q|_{F_t}) \leqslant a(N) < i_W(q|_{F_{t+1}})$ and $i_W(q|_{F_s}) \leqslant c(N) < i_W(q|_{F_{s+1}})$. Then applying Proposition 4.10 to N and $N^\vee$ we get: $c(N) \leqslant i_W(q|_{F_t}) + i_W(q|_{F_{t+1}}) - a(N) - 1$ and $a(N) \leqslant i_W(q|_{F_s}) + i_W(q|_{F_{s+1}}) - c(N) - 1$. This implies $t = s$ and $a(N) + c(N) \leqslant i_W(q|_{F_t}) + i_W(q|_{F_{t+1}}) - 1$. □

Now we can prove Theorem 4.13. Let Q, j, N be as in Theorem 4.13. Denote: $i = a(N)$. If $j > i$, then everything is contained in Sublemma 6.14.1.

Let $(i-j) > 0$. Let $F = F_r$ be a field in the *generalized splitting tower* of M.Knebusch, given by Sublemma 6.14.2. Then $i_W(q|_{F_{r+1}}) > a(N) \geqslant i_W(q|_{F_r})$. By conditions of Theorem 4.13, $i_W(q|_{F_{r+1}}) > a(N) - (i-j) \geqslant i_W(q|_{F_r})$. Also, $i_W(q|_{F_{r+1}}) > c(N) \geqslant i_W(q|_{F_r})$. On the other hand, from Sublemma 6.14.2 it follows that $i_W(q|_{F_{r+1}}) > c(N) + (i-j) \geqslant i_W(q|_{F_r})$. That means that the pair $(i', j') := (c(N), c(N) + (i-j))$ and indecomposable direct summand $N^\vee$ (with $a(N^\vee) = c(N)$) satisfy the conditions of Sublemma 6.14.1. By Sublemma 6.14.1, in $M(Q)$ there exists a direct summand L, isomorphic to $N^\vee(i-j)[2i-2j]$. Then $L^\vee$ will be isomorphic to $N(j-i)[2j-2i]$. Theorem 4.13 is proven. □

To prove Corollary 4.14, consider $l := i_W(q|_{F_{t+1}}) - 1 - a(N)$. By Sublemma 6.14.1, $N(l)[2l]$ is isomorphic to a direct summand M of $M(Q)$. Clearly, $a(M) = a(N) + l = i_W(q|_{F_{t+1}}) - 1$, and $c(M) = c(N) - l$. Since $a(M) \geqslant i_W(q|_{F_t})$, we have, by Sublemma 6.14.2, $c(M) \geqslant i_W(q|_{F_t})$. This implies: $a(N) + c(N) \geqslant i_W(q|_{F_t}) + i_W(q|_{F_{t+1}}) - 1$. Combined with Sublemma 6.14.2, this gives required result. □

Proof of Theorem 4.17 and Theorem 4.15

Let $k = F_0 \subset \ldots \subset F_{h(q)}$ be the generalized splitting tower for q, and $0 \leqslant t < h(q)$ be such number that $i_W(q|_{F_t}) \leqslant m < i_W(q|_{F_{t+1}})$. Let $k = E_0 \subset \ldots \subset E_{h(p)}$ be the generalized splitting tower for p, and $0 \leqslant s < h(p)$ be such number that $i_W(p|_{E_s}) \leqslant n < i_W(p|_{E_{s+1}})$. Denote as K the composite $F_t * E_s$ of the fields F_t and E_s. Using Theorem 4.13, we can assume that $m = i_W(q|_{F_t})$, and $n = i_W(p|_{E_s})$. Denote: $\tilde{q} := (q|_K)_{an.}$, and $\tilde{p} := (p|_K)_{an.}$. By the condition of the theorem, $\dim(\tilde{q}) = \dim((q|_{F_t})_{an.})$, and $\dim(\tilde{p}) = \dim((p|_{E_s})_{an.})$. Moreover, for arbitrary field extension G/K, the conditions: $i_W(\tilde{q}|_G) > 0$ and $i_W(\tilde{p}|_G) > 0$ are equivalent.

Let us denote: $m' := m + i_{t+1}(q) - 1$, and $n' := n + i_{s+1}(p) - 1$.

Because $a(N) = n$, by Theorem 6.13, we have the map $\alpha_n : M(P) \to M(\underline{P}^n)(n)[2n]$, which sends the class l_n to $l_0(n)[2n]$. From the conditions of the theorem we have rational maps $\underline{P}^n \dashrightarrow \underline{Q}^m$, and $\underline{P}^n \dashrightarrow \underline{Q}^{m'}$, which give us motivic maps $\lambda : M(\underline{P}^n) \to M(\underline{Q}^m)$, and $\lambda' : M(\underline{P}^n) \to M(\underline{Q}^{m'})$ sending the class of a rational point to the class of a rational point. Finally, we have the maps $f_m : M(\underline{Q}^m)(m)[2m] \to M(Q)$ and $f_{m'} : M(\underline{Q}^{m'})(m')[2m'] \to M(Q)$. Let $\varepsilon : M(P)(m-n)[2m-2n] \to M(Q)$ be the composition $f_m \circ \lambda(m)[2m] \circ \alpha_n(m-n)[2m-2n]$, and $\varepsilon' : M(P)(m'-n)[2m'-2n] \to M(Q)$ be the composition $f_{m'} \circ \lambda'(m')[2m'] \circ \alpha_n(m'-n)[2m'-2n]$.

Since $\tilde{p}|_{K(\tilde{q})}$ is isotropic, there exists morphism $\tilde{\gamma} : M(\tilde{Q}) \to M(\tilde{P})$, such that $\tilde{\gamma} : \mathrm{CH}_0(\tilde{Q}|_{\overline{K}}) \to \mathrm{CH}_0(\tilde{P}|_{\overline{K}})$ sends the class of a rational point to the class of a rational point. Since $M(\tilde{Q})(i_W(q|_K))[2i_W(q|_K)]$ is a direct summand of $M(Q|_K)$, and $M(\tilde{P})(i_W(p|_K))[2i_W(p|_K)]$ is a direct summand of $M(P|_K)$ (by Proposition 2.1), the morphism $\varepsilon|_K$ provides us with the morphism $\tilde{\varepsilon} : M(\tilde{P})(m)[2m] \to M(\tilde{Q})(m)[2m]$ (we remind, that $m = i_W(q|_{F_t}) = i_W(q|_K)$, and $n = i_W(p|_{E_s}) = i_W(p|_K)$), and the composition $\overline{\deg}_{\tilde{P}} \circ \tilde{\gamma} \circ \tilde{\varepsilon}(-m)[-2m] : \mathrm{CH}_0(\tilde{P}|_{\overline{K}}) \to \mathbf{Z}/2$ is nonzero. Let $\tilde{M}$ be indecomposable direct summand of $M(\tilde{Q})$ such that $a(\tilde{M}) = 0$, and $\tilde{N}$ be indecomposable direct summand of $M(\tilde{P})$ such that $a(\tilde{N}) = 0$. By Theorem 3.6, $\tilde{M} \simeq \tilde{N}$. In particular, $\mathrm{size}(\tilde{M}) = \mathrm{size}(\tilde{N})$. In the light of Corollary 4.7, we get an equality: $\dim(Q) - \dim(P) + n - m = m' - n'$. Denote this number as j.

Proposition 3.5, on it's part, gives us that $\overline{\deg}_{\tilde{N}} \circ \varphi_{\tilde{N}} \circ \tilde{\gamma} \circ \tilde{\varepsilon}(-m)[-2m] \circ j_{\tilde{N}} = \overline{\deg}_{\tilde{N}} : \mathrm{CH}(\tilde{N}|_{\overline{K}}) \to \mathbf{Z}/2$. In particular, $\overline{\deg}_{\tilde{N}} \circ \varphi_{\tilde{N}} \circ \tilde{\gamma} \circ \tilde{\varepsilon}(-m)[-2m] \circ j_{\tilde{N}}|_{CH_{b(\tilde{N})}} \neq 0$. By Corollary 4.7, $b(\tilde{N}) = \dim(\tilde{P}) - i_{s+1}(p) + 1$. So,

$\tilde{\varepsilon}(-m)[-2m](\tilde{h}^{i_{s+1}(p)-1}) = \mu \cdot \tilde{h}^{i_{t+1}(q)-1}$, where μ is odd, and $\tilde{h}$ is the class of hyperplane section in $\tilde{P}$ and $\tilde{Q}$, respectively. Then $\varepsilon(h^{n'}(m-n)[2m-2n]) = \mu \cdot h^{m'}$, where μ is odd and $h \in \mathrm{CH}^1$ is the class of hyperplane section in P and Q, respectively.

Let $\varepsilon^{\vee} : M(Q) \to M(P)(j)[2j]$ be the morphism dual to ε (the corresponding cycle is obtained by switching the factors in $P \times Q$). Denote as $(-,-)$ the natural composition pairings $\mathrm{CH}_r(M(Q)) \otimes \mathrm{CH}^r(M(Q)) \to \mathbf{Z}$ and $\mathrm{CH}_r(M(P)(j)[2j]) \otimes \mathrm{CH}^r(M(P)(j)[2j]) \to \mathbf{Z}$. We have tautological equality: $(\varepsilon^{\vee}(l_{m'}), h^{n'}(m'-n')[2m'-2n']) = (l_{m'}, \varepsilon(h^{n'}(m-n)[2m-2n]))$. Thus, $\varepsilon^{\vee}(l_{m'}) \equiv l_{n'}(m'-n')[2m'-2n'] \,(\,mod\,2)$.

Consider the diagram:

$$\begin{array}{ccccc} & & M(Q) & & \\ & \overset{\varepsilon'}{\nearrow} & & \overset{\varepsilon^{\vee}}{\searrow} & \\ M(P)(m'-n)[2m'-2n] & & \xleftarrow[\rho^{i_{s+1}(p)-1}]{} & & M(P)(m'-n')[2m'-2n'] \end{array}$$

where $\rho^{i_{s+1}(p)-1}$ is given by the plane section of codimension $i_{s+1}(p)-1$, embedded diagonally into $P \times P$.

Let $l_n \in \mathrm{CH}_n(P|_{\overline{k}})$ be the class of projective plane of dimension n on $P|_{\overline{k}}$. By the construction of ε', we have: $\varepsilon'(l_n(m'-n)[2m'-2n]) = l_{m'} \in \mathrm{CH}_{m'}(Q|_{\overline{k}})$. And we know that $\varepsilon^{\vee}(l_{m'}) \equiv l_{n'}(m'-n')[2m'-2n'] \,(\,mod\,2)$. So, the composition $\overline{\deg}_P \circ \rho^{i_{s+1}(p)-1} \circ \varepsilon^{\vee} \circ \varepsilon' : \mathrm{CH}_{m'}(M(P|_{\overline{k}})(m'-n)[2m'-2n]) \to \mathbf{Z}/2$ is nonzero. Then, by Theorem 3.8, $N(m'-n)[2m'-2n]$ is isomorphic to a direct summand of $M(Q)$. Since $a(N(m'-n)[2m'-2n]) = m'$, and $i_W(q|_{F_t}) \leqslant m, m' < i_W(q|_{F_{t+1}})$, by Theorem 4.13, $M := N(m-n)[2m-2n]$ is also isomorphic to a direct summand of $M(Q)$. Theorem is proven. $\square$

Theorem 4.15 is an evident corollary of Theorem 4.17. $\square$

7 Splitting patterns of small-dimensional forms

This section is devoted to the classification of the splitting patterns of small-dimensional forms. We will remind the definition of the *splitting pattern*. Let k be a field and q - a quadratic form defined over k. We construct the sequence of fields and quadratic forms in the following way. Set $k_0 := k$, $i_0(q) := i_W(q)$ - the Witt index of q, and $q_0 := q_{an.}$. Now if we have the field k_j and an anisotropic form q_j defined over k_j, we set: $k_{j+1} := k_j(Q_j)$; $i_{j+1}(q) := i_W(q_j|_{k_{j+1}})$; $q_{j+1} := (q_j|_{k_{j+1}})_{an.}$. Since $\dim(q_{j+1}) < \dim(q_j)$, this process will stop at some step h, namely, when $\dim(q_h) \leqslant 1$. This number h is called the *height* of q. As a result, we get a tower of fields $k = k_0 \subset k_1 \subset \ldots \subset k_h$, called the *generalized splitting tower of M.Knebusch* (see [15, §5]), and a sequence of natural numbers: $i_0(q), i_1(q), \ldots, i_h(q)$. The number $i_j(q)$ is called the *j-th higher Witt index* of q, and the set $\mathbf{i}(q) := (i_1(q), \ldots, i_h(q))$ will be called the *splitting pattern* of the quadric Q. Notice, that $i_j(q) \geqslant 1$ for each $j \geqslant 1$. We should stress, that our definition of the splitting pattern is not the one commonly used, since usually, the sequence $(i_1, i_1 + i_2, \ldots, i_1 + \ldots + i_h)$ is called by this name. But, it seems, that many properties of quadratic forms are much more transparent, when we see the higher Witt indices, rather than $i_W(q|_{k_t})$. I hope, the reader will agree with me after looking on the tables below. By this reason, in the current article we will stick to our nonstandard terminology.

It is an important question to describe all possible splitting patterns of quadrics. This problem was solved for all forms of dimension $\leqslant 10$ by D.Hoffmann - see [3]. With the help of the motivic methods as well as the methods developed by D.Hoffmann, O.Izhboldin, B.Kahn and A.Laghribi (see [2], [10], [6], [17]) we are able to describe all possible splitting patterns of forms of odd dimension $\leqslant 21$ as well as forms of dimension 12.

In many cases, we will be able to describe the class of forms having particular splitting pattern in terms of quadratic form theory.

The tools we will be using

In this section we list some known results on the structure of the splitting pattern as well as the structure of the motive of a quadric, which will be used in our computations.

We start with the last higher Witt index. By the result of M.Knebush, the quadrics of *height* 1 are exactly the Pfister quadrics and their hyperplane sections. Hence, we have the following restrictions on $i_{h(q)}(q)$:

Theorem 7.1 (M.Knebusch, [15, Theorem 5.8])

(1) *If* $\dim(q)$ *is even, then* $i_{h(q)}(q) = 2^d$ *for some* $d \geqslant 0$.

(2) *If* $\dim(q)$ *is odd, then* $i_{h(q)}(q) = 2^d - 1$ *for some* $d \geqslant 1$.

If $\dim(q)$ is even, the number $d+1$ is called the *degree* of q. The degree of any odd-dimensional form is zero, by definition.

The next important results of D.Hoffmann are related to the first higher Witt index.

Theorem 7.2 (D.Hoffmann, [2, Corollary 1])

Let q be anisotropic quadric of dimension $2^r + m$, where $0 < m \leqslant 2^r$. Then $i_1(q) \leqslant m$.

Theorem 7.3 (D.Hoffmann, [2])

Let $0 < m < 2^r$, and p is anisotropic quadratic form of dimension $2^r - m$ with the splitting pattern $\mathbf{i}(p)$. Then there is the field extension E/k, and anisotropic quadratic form q of dimension $2^r + m$ over E, such that the splitting pattern of q is $(m, \mathbf{i}(p))$.

Proof: By [2, Remark 1], there is the extension E of the field $k(y_1, \dots, y_r)$ such that $p|_E$ is isomorphic to a subform of $\langle\langle y_1, \dots, y_r\rangle\rangle|_E$, and E/k is unirational. Let q be an orthogonal complement to $p|_E$ in $\langle\langle y_1, \dots, y_r\rangle\rangle|_E$. Then $\mathbf{i}(q) = (m, i_1(p|_E), \dots, i_{h(p)}(p|_E))$. But higher Witt indices are clearly stable under rational, and hence, unirational extensions. So, $\mathbf{i}(q) = (m, \mathbf{i}(p))$. $\square$

We will need also results concerning the specialization of splitting patterns.

Definition 7.4 *Let $\mathbf{i} = (i_1, i_2, \dots, i_h)$ be a sequence of natural numbers.*

We say that the sequence $\mathbf{i}'$ is an elementary specialization of $\mathbf{i}$, if either $\mathbf{i}' = (i_2, \dots, i_h)$, or for some $1 \leqslant s < h$, $\mathbf{i}' = (i_1, \dots, i_{s-1}, i_s + i_{s+1}, i_{s+2}, \dots, i_h)$.

We say that the sequence $\mathbf{i}''$ is a specialization of $\mathbf{i}$, if it can be obtained from $\mathbf{i}$ by a (possibly empty) chain of elementary specializations.

Theorem 7.5 (M.Knebusch, [15, Corollary 5.6])

Let q be a quadratic form over the field k, and L/k, F/k - be such field extensions that there is a regular place $L \to F$. Then $\mathbf{i}(q|_F)$ is a specialization of $\mathbf{i}(q|_L)$. In particular, $\mathbf{i}(q|_F)$ is always a specialization of $\mathbf{i}(q)$.

We will also use repeatedly the following evident fact:

Theorem 7.6 *Let q be quadratic form such that $\mathbf{i}(q) = (i_1, i_2, \ldots, i_h)$, and $p = q \perp \langle a \rangle$, for some $a \in k^*$. Then $\mathbf{i}(p)$ is a specialization of*
$$\begin{cases} (1, i_1 - 1, 1, i_2 - 1, 1, \ldots, 1, i_h - 1), & \textit{if } \dim(q) \textit{ even} \\ (1, i_1 - 1, 1, i_2 - 1, 1, \ldots, 1, i_h - 1, 1), & \textit{if } \dim(q) \textit{ odd} \end{cases}, \textit{ where we omit zeroes.}$$

In our computations we will be using interplay between the splitting pattern of a quadrics and the structure of it's motive. So, we will need some facts concerning the later. The key tool here is Theorem 4.20, describing possible sizes of binary direct summands of $M(Q)$.

Let Q be some smooth quadric over k. Then $M(Q|_{\overline{k}})$ is a direct sum of Tate-motives. Namely,
$$M(Q|_{\overline{k}}) = \begin{cases} \oplus_{j=0}^{\dim(Q)} \mathbb{Z}(j)[2j], \text{ if } \dim(Q) \text{ is odd} \\ (\oplus_{j=0}^{\dim(Q)} \mathbb{Z}(j)[2j]) \oplus \mathbb{Z}(\dim(Q)/2)[\dim(Q)], \text{ if } \dim(Q) \text{ is even} \end{cases}.$$
Suppose Q is anisotropic and let $\mathbf{i}(q) = (i_1, i_2, \ldots, i_h)$. The spliting pattern separates our Tate-motives into different *shells*. We say that $\mathbb{Z}(m)[2m]$ belongs to *the shell number t*, if $\sum_{r=1}^{t-1} i_r \leqslant \min(l, \dim(Q) - l) < \sum_{r=1}^{t} i_r$. In the light of Proposition 2.1, Proposition 2.6, this condition is equivalent to the following: $\mathbb{Z}(l)[2l]$ is a direct summand of $M(Q|_{k_t})$, but is not a direct summand of $M(Q|_{k_{t-1}})$, where $k = k_0 \subset k_1 \subset \ldots \subset k_h$ is a generalized splitting tower of fields for Q. So, we have h different shells, where h is the *height* of Q, and the shell number t consists of $2i_t$ Tate-motives.

Now we can formulate the result which is very usefull in the splitting pattern computations.

Theorem 7.7 *Let Q be smooth anisotropic quadric of dimension m, and N be indecomposable direct summand of $M(Q)$, such that $a(N) = 0$. Then:*

(1) *If $t > 1$ and $i_t < i_1$, then $N|_{\overline{k}}$ does not contain Tate-motives from the shell number t.*

(2) *If i_2 is not divisible by i_1, then $N|_{\overline{k}}$ does not contain Tate-motives from the shell number 2.*

Proof: Let l be such number that $\mathbb{Z}(l)[2l]$ is a direct summand of $N|_{\overline{k}}$. By Proposition 4.10, we can assume that $l \geqslant m/2$. Let E/k be any field extension, and $j := i_1(q) - 1$. Then the following conditions are equivalent:

(a) $i_W(q|_E) > m - l$;

(b) $\mathbb{Z}(l)[2l]$ is a direct summand of $M(Q|_E)$;
(c) $\mathbb{Z}(l)[2l]$ is a direct summand of $N|_E$;
(d) $\mathbb{Z}(l+j)[2l+2j]$ is a direct summand of $N(j)[2j]|_E$;
(e) $\mathbb{Z}(l+j)[2l+2j]$ is a direct summand of $M(Q|_E)$;
(f) $i_W(q|_E) > m-l-j$.

The equivalence $(a) \Leftrightarrow (b)$ and $(e) \Leftrightarrow (f)$ follows from Proposition 2.1, Proposition 2.6. The equivalence $(b) \Leftrightarrow (c)$ and $(d) \Leftrightarrow (e)$ follows from the fact that $\mathbb{Z}(l)[2l]$ is a direct summand of $N|_{\overline{k}}$ (respectively, $\mathbb{Z}(l+j)[2l+2j]$ is a direct summand of $N(j)[2j]|_{\overline{k}}$), and N, $N(j)[2j]$ are direct summands of $M(Q)$ (by Theorem 4.13). The equivalence $(c) \Leftrightarrow (d)$ is evident.

The equivalence $(a) \Leftrightarrow (f)$ implies the first statement of the theorem (if $\mathbb{Z}(l)[2l]$ would belong to the shell number t, then i_t would be $> j = i_1(q)-1$).

To prove the second statement, consider the motive $L := \oplus_{j=0}^{i_1(q)-1} N(j)[2j]$. By Theorem 4.13, Theorem 3.11, L is isomorphic to a direct summand of $M(Q)$. By Theorem 4.19, L is self-dual, that is, $L^\vee \cong L$. Let M be complimentary direct summand. Then $M^\vee \cong M$ as well. Clearly, $N(j)[2j]|_{\overline{k}}$ contains as many Tate-motives from particular shell as $N|_{\overline{k}}$ does (since this number is equal to the number of Tate-motives which split from $N(j)[2j]$ (respectively N) over k_t but do not split over k_{t-1}). Since $i_2(q)$ is not divisible by $i_1(q)$, $L|_{\overline{k}}$ does not contain some of Tate-motives from the second shell. So, $M|_{\overline{k}}$ contains some Tate-motive from the second shell. Let $\mathbb{Z}(l)[2l]$ be such Tate-motive with the minimal possible l. Let M' be indecomposable direct summand of M, such that $\mathbb{Z}(l)[2l]$ is a direct summand of $M'|_{\overline{k}}$. We know that $M'|_{\overline{k}}$ contains no Tate-motives from the first shell. Hence, $a(M') = l$ (here is the only place, where we use the fact that the number of the shell is 2, but not bigger). Then, by Theorem 4.13, each Tate-motive $\mathbb{Z}(l'')[2l'']$ from the second shell will be a direct summand of $M''|_{\overline{k}}$, for some indecomposable direct summand M'', isomorphic to $M'(d)[2d]$, for some d. Since $M'(d)[2d]$ is not isomorphic to N, by Lemma 4.2, we get that $N|_{\overline{k}}$ does not contain Tate-motives from the second shell. □

Remark: In item (2) above, the fact that the number of the shell is 2 is essential. For example, if q is any codimension 1 subform of the form $\langle\langle e_1, e_2\rangle\rangle \cdot \langle a, b, -ab, -c, -d, cd\rangle$ over the field $k(a,b,c,d,e_1,e_2)$, then $\mathbf{i}(q) = (3,1,7)$, but $N|_{\overline{k}}$ contains Tate-motives from the third shell.

Previous theorem will be usually used in conjunction with the following one.

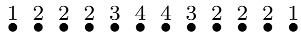
Theorem 7.8 *Let Q be smooth anisotropic quadric over k, and N be an indecomposable direct summand of $M(Q)$, such that $a(N) = 0$. Suppose that $N|_{\overline{k}}$ does not contain Tate-motives from the shells $2, 3, \ldots, h(Q)$. Then N is binary of size* $\dim(Q) - i_1(q) + 1$.

Proof: The fact that N is binary follows from Theorem 4.13, Lemma 4.2, and the statement about the size is valid for arbitrary indecomposable direct summand N with $a(N) = 0$, by Corollary 4.7. □

We will also use the following motivic result, which provides (in conjunction with Theorem 4.20) some sufficient conditions for all indecomposable direct summands of $M(Q)$ to "start" from the first shell.

Theorem 7.9 *Let q be anisotropic form over k, and $q_1 = (q|_{k(Q)})_{an.}$. Let N be an indecomposable direct summand of $M(Q)$ such that $a(N) = 0$, and L be an indecomposable direct summand of $M(Q_1)$ such that $a(L) = 0$. Suppose that $M(Q_1) = \oplus_{l=0}^{i_2(q)-1} L(l)[2l]$. Then either $M(Q) = \oplus_{j=0}^{i_1(q)-1} N(j)[2j]$, or N is binary of size* $\dim(Q) - i_1(q) + 1$.

Proof: By Theorem 4.13 and Theorem 3.11, $\oplus_{j=0}^{i_1(q)-1} N(j)[2j]$ is isomorphic to a direct summand of $M(Q)$. Let M be complimentary summand. If $M \neq 0$, then $M|_{\overline{k}}$ contains some Tate-motive from some shell number $\geqslant 2$. But the condition $M(Q_1) = \oplus_{l=0}^{i_2(q)-1} L(l)[2l]$ exactly says that any such Tate-motive is connected (even over $k(Q)$) to some Tate-motive from the second shell. So, if $M \neq 0$, then $M|_{\overline{k}}$ contains some Tate-motive $\mathbb{Z}(m)[2m]$ from the second shell. Since $M|_{\overline{k}}$, clearly, does not contain Tate-motives from the first shell, there exist indecomposable direct summand M' of $M(Q)$ such that $a(M') = m$ (we can assume $m < \dim(Q)/2$). By Theorem 4.13, Lemma 4.2, $N|_{\overline{k}}$ contains no Tate-motives from the second shell, and hence, no Tate-motives from the shells $3, \ldots, h$ (since they are all connected to the second shell). So, we have only two possibilities: either $M = 0$ and $M(Q) = \oplus_{j=0}^{i_1(q)-1} N(j)[2j]$, or $N|_{\overline{k}}$ does not contain Tate-motives from shells number $2, \ldots, h(Q)$, and so N is binary of size $\dim(Q) - i_1(q) + 1$, by Theorem 7.8. □

Sometimes we will draw the pictures of the motives of quadrics. In this case, each Tate-motive will be denoted as $\bullet$, and sometimes we will place number of the corresponding shell over it. For example, the motive of the quadric with the splitting pattern $(1,3,1,1)$ can be drawn as: 

The direct summand of $M(Q)$ then can be visualized as a collection of $\bullet$'s connected by dotted lines. For example, the direct summand L with $L|_{\overline{k}} = \mathbb{Z} \oplus \mathbb{Z}(2)[4] \oplus \mathbb{Z}(3)[6] \oplus \mathbb{Z}(5)[10]$ in $M(Q)$, where $q = \langle\langle a \rangle\rangle \cdot \langle b_1, b_2, b_3 \rangle \perp \langle c \rangle$, can be drawn as: 

The indecomposable direct summand of $M(Q)$ will be visualized as a collection of $\bullet$'s connected by solid lines. For example, the decomposition into indecomposables of the $M(Q)$, where $q = \langle\langle a \rangle\rangle \cdot \langle 1, -b_1, -b_2, -b_3 \rangle \perp \langle b_1 b_2 \rangle$, and $\{a, b_1, b_2, b_3\} \neq 0, \{a, -b_1 b_2 b_3\} \neq 0\,(\,mod\,2)$, will look as: 

Now we can list the splitting patterns of small-dimensional forms.

We start with the odd-dimensional forms.

Splitting patterns of odd-dimensional forms

I should mention again, that the splitting patterns of forms of dimension 3,5,7 and 9, as well as most cases of dimension 11 were classified by D.Hoffmann in [3]. Nevertheless, we included these cases below, to familirize the reader with the technique on simple examples.

$\underline{\dim(q) = 3}$**:** In this case, $\mathbf{i}(q) = (1)$ always.

$\underline{\dim(q) = 5}$**:** By Theorem 7.2, we have: $i_1(q) = 1$, and $\mathbf{i}(q) = (1,1)$ always.

$\underline{\dim(q) = 7}$**:** Let us show that $i_1(q) \neq 2$. First of all, this fact follows from the general result of O.Izhboldin, claimng that for the anisotropic form of dimension $2^r + 3$ the first higher Witt index does not equal to 2 - see [8, Corollary 5.13]. Alternatively, we can argue as follows. Let $i_1(q) = 2$, then $i_2(q) = 1$. Let N be an indecomposable direct summand in $M(Q)$ such that $a(N) = 0$. Then by Theorem 7.7, $N|_{\overline{k}}$ does not contain Tate-motives from the shell number 2, and so, N is binary of size 4 (by Theorem 7.8). This contradicts Theorem 4.20. So, $i_1(q) \neq 2$.

Since $7 = 2^2 + 3$, by Theorem 7.2, we have: either $i_1(q) = 1$, or $i_1(q) = 3$. In the first case, we have: $\mathbf{i}(q) = (1,1,1)$; in the second, $\mathbf{i}(q) = (3)$.

By the result of A.Pfister, $\mathbf{i}(q) = (3)$ if and only if q is a Pfister neighbour,.

Such forms clearly exist. Respectively, $\mathbf{i}(q) = (1, 1, 1)$ for all other anisotropic forms of dimension 7. The generic form $\langle a_1, \dots, a_7\rangle$ over the field $F = k(a_1, \dots, a_7)$ provides such an example (it is sufficient to notice that over $E := F(\sqrt{-a_1a_2})$, $i_W(q|_E) = 1$).

$\underline{\dim(q) = 9}$**:** Again, by Theorem 7.2, $i_1(q) = 1$, so $\mathbf{i}(q)$ is either $(1, 1, 1, 1)$, or $(1, 3)$. And both these cases exist in the light of Theorem 7.3. It remains to describe both classes of forms.

Let $\mathbf{i}(q) = (1, 3)$. Consider $p := q \perp \langle -\det_{\pm}(q)\rangle$. Then, on the one hand, $p \in I^2(k)$, and so, the splitting pattern of p is a specialization of $(1, 1, 1, 2)$. On the other hand, since p differs by 1-dimensional form from q, the splitting pattern of p is a specialization of $(1, 1, 2, 1)$ (by Theorem 7.6). Taking into account Theorem 7.1, we get that the splitting pattern of p is a specialization of $(1, 4)$. By the result of A.Pfister (see [21, Satz 14 and Zusatz]), anisotropic form from I^3 can not have dimension 10. So, $\dim(p_{an.}) = 8$, and $\mathbf{i}(p_{an.}) = (4)$. By the result of A.Pfister, $p_{an.}$ is isomorphic to $\lambda \cdot \langle\langle a, b, c\rangle\rangle$ for some $\lambda \in k^*$ and $\{a, b, c\} \neq 0 \in \mathrm{K}_3^M(k)/2$. Hence, $q = \lambda \cdot (\langle\langle a, b, c\rangle\rangle \perp \langle -d\rangle)$.

All other anisotropic 9-dimensional forms should have splitting pattern $(1, 1, 1, 1)$. The generic form $q = \langle a_1, \dots, a_9\rangle$ over the field $k(a_1, \dots, a_9)$ provides such an example (it is sufficient to notice that over the field $E = k(\sqrt{-a_1a_2}, \sqrt{-a_3a_4})$ we have: $i_W(q|_E) = 2$).

$\underline{\dim(q) = 11}$**:** Let us show that $i_1(q) \neq 2$. This fact is a particular case of the cited result of O.Izhboldin, since $11 = 2^3 + 3$. Alternatively, we can argue as follows: suppose $i_1(q) = 2$, then the splitting pattern of q is either $(2, 1, 1, 1)$, or $(2, 3)$. Let N be an indecomposable direct summand in $M(Q)$ such that $a(N) = 0$. By Theorem 7.7 and Theorem 7.8 N is binary of size 8 - a contradiction with Theorem 4.20.

By Theorem 7.2, $i_1(q)$ is either 1, or 3. If $i_1(q) = 1$, then $\mathbf{i}(q)$ is either $(1, 1, 1, 1, 1)$, or $(1, 1, 3)$, and we will see that both cases exist. If $i_1(q) = 3$, then $\mathbf{i}(q) = (3, 1, 1)$, and such quadrics also exist. Now we will describe respective classes of forms.

We start with $(3, 1, 1)$. By the result of B.Kahn (see [10, Remark after Theorem 4]), q must be a Pfister neighbour. And conversely, any 11-dimensional neighbour of an anisotropic Pfister form $\langle\langle a, b, c, d\rangle\rangle$ has such splitting pattern. Such forms clearly exist.

If $\mathbf{i}(q) = (1, 1, 3)$, then set $p := q \perp \langle\det_{\pm}(q)\rangle$. Then $p \in I^2(k)$, and the splitting pattern of p is a specialization of $(1, 1, 1, 1, 2)$. On the other hand, since p differs from q by a 1-dimensional form, $\mathbf{i}(p)$ is a specialization of $(1,1,1,2,1)$, in the light of Theorem 7.6. By Theorem 7.1, $\mathbf{i}(p)$ is a specialization of $(1,1,4)$ (actually, of $(2,4)$, by [21, Satz 14 and Zusatz]). That means, $p \in J^3(k)$, and since $J^3(k) = I^3(k)$, $p \in I^3(k)$. So, q is a codimension 1 subform of some anisotropic 12-dimensional form from $I^3(k)$. Conversely, if p is some anisotropic 12-dimensional form from $I^3(k)$, then $\mathbf{i}(p) = (2,4)$, and for arbitrary 11-dimensional subform q of p, the splitting pattern of q will be the specialization of $(1,1,3)$. And in our list of possible splitting patterns only the splitting pattern $(1,1,3)$ satisfies such conditions. To show that forms with the splitting pattern $(1,1,3)$ exist it is sufficient to construct a 12-dimensional anisotropic form from I^3. The form $\langle\langle e\rangle\rangle \cdot \langle a, b, -ab, -c, -d, cd\rangle$ over the field $k := F(a,b,c,d,e)$ provides such an example.

Finally, all other anisotropic forms of dimension 11 will have splitting pattern $(1,1,1,1,1)$. The generic form $\langle a_1, \ldots, a_{11}\rangle$ over the field $k(a_1, \ldots, a_{11})$ provides an example.

$\underline{\dim(q) = 13}$**:** We have: $i_1(q) \leqslant 5$. Let us show that $i_1(q) \neq 2, 3$, or 4. $i_1(q) \neq 4$, since otherwise $M(Q)$ would contain a binary direct summand of size 8 (in the light of Theorem 7.7 and Theorem 7.8), which contradicts Theorem 4.20.

If $i_1(q) = 2$, then $\mathbf{i}(q)$ is either $(2,1,1,1,1)$, or $(2,1,3)$. In the former case, we get a binary direct summand of $M(Q)$ of size 10, which contradicts Theorem 4.20. Suppose $\mathbf{i}(q) = (2,1,3)$. Actually, we can treat the cases $(2,1,3)$ and $(3,3)$ simultaneously. Let $p := q \perp \langle -\det(q)\rangle$. Then $\mathbf{i}(p)$ is a specialization of $(1,1,1,1,2,1)$, and using the fact that $p \in I^2(k)$, Theorem 7.1 and [21, Satz 14 and Zusatz], we get: $\mathbf{i}(p)$ is a specialization of $(1,2,4)$. If N is an indecomposable direct summand in $M(Q)$ such that $a(N) = 0$, then since for arbitrary field extension E/k the conditions: 1) $i_W(p) > 1$; and 2) $i_W(q) > 0$ are equivalent, $N(1)[2]$ must be isomorphic to some direct summand of $M(P)$ (by Theorem 4.15). If $\mathbf{i}(p) = (1,2,4)$, then $M(P)$ is indecomposable (by the inductive application of Theorem 7.9), which is impossible (since $\mathrm{rank}(\mathrm{CH}_l(Q|_{\overline{k}})) \leqslant 1$). So, $i_W(p) = 1$ and $\mathbf{i}(p) = (2,4)$. Then $M(P_{an.}) = L \oplus L(1)[2]$ (again, by the inductive application of Theorem 7.9), where L is indecomposable and $L|_{\overline{k}} = \mathbb{Z} \oplus \mathbb{Z}(2)[4] \oplus \mathbb{Z}(4)[8] \oplus \mathbb{Z}(5)[10] \oplus \mathbb{Z}(7)[14] \oplus \mathbb{Z}(9)[18]$. But then N must be isomorphic to L (since $M(P) = \mathbb{Z} \oplus M(P_{an.})(1)[2] \oplus \mathbb{Z}(\dim(P))[2\dim(P)]$). In the case $\mathbf{i}(q) = (2,1,3)$, we get: $\mathrm{size}(L) = 9 \neq 10 = \mathrm{size}(N)$ - a contradiction (we used

Corollary 4.7 here). In the case $\mathbf{i}(q) = (3,3)$, we get $N|_{\overline{k}}$ contains $\mathbb{Z}(2)[4]$, which is impossible, since $i_1(q) = 3$ and so $N(2)[4]$ is a direct summand of $M(Q)$. So, $\mathbf{i}(q)$ can not be $(2,1,3)$, or $(3,3)$ and $i_1(q) \neq 2$.

If $i_1(q) = 3$, then $\mathbf{i}(q)$ is either $(3,1,1,1)$, or $(3,3)$. In the former case, we get a binary direct summand in $M(Q)$ of size 9 - contradiction with Theorem 4.20. The case $(3,3)$ was treated above. So, $i_1(q) \neq 3$.

Thus, $i_1(q)$ is either 1, or 5. This gives splitting patterns $(1,1,1,1,1,1)$, $(1,1,1,3)$, $(1,3,1,1)$ and $(5,1)$. We will show that all of them are realized by the appropriate quadratic forms. Let us describe classes of forms corresponding to these 4 splitting patterns.

We start with the splitting pattern $(5,1)$. Then q is a Pfister neighbour, in the light of [16, Corollary 8.2] (see also [2, §4]). Conversely, any 13-dimensional neighbour of anisotropic 4-fold Pfister form has splitting pattern $(5,1)$. Such forms clearly exist.

Let $\mathbf{i}(q) = (1,3,1,1)$. We have $\dim((q|_{k(Q)})_{an.}) = 11$, and $(q|_{k(Q)})_{an.}$ is a Pfister neighbour. Then by the result of B.Kahn (see [10, Theorem 2]), there exists some 5-dimensional form $r_4(q)$ that $q \perp r_4(q) \in I^4(k)$. Clearly, $r_4(q)$ is anisotropic, since otherwise q would be a Pfister neighbour, and would have splitting pattern $(5,1)$. Conversely, let q and $r_4(q)$ be 13-dimensional and 5-dimensional anisotropic forms, s.t. $q \perp r_4(q) \in I^4(k)$. Then $\mathbf{i}((q \perp r_4(q))_{an.}) = (8)$. Since q differs from $(q \perp r_4(q))_{an.}$ by a 5-dimensional form, we get: $\mathbf{i}(q)$ is a specialization of $(1,3,1,1)$ (by Theorem 7.6). This means that $\mathbf{i}(q)$ is either $(1,3,1,1)$, or $(5,1)$. If $\mathbf{i}(q) = (5,1)$, then q is a Pfister neighbour, as we know. That is, there exists 3-dimensional form p, such that $q \perp p \in I^4(k)$. Then $r_4(q) \perp -p \in I^4(k)$. Since $\dim(r_4(q) \perp -p) = 8 < 16$, we get: $r_4(q) = p \perp \mathbb{H}$. But $r_4(q)$ is anisotropic - contradiction. Hence $\mathbf{i}(q) = (1,3,1,1)$. It remains to show that such anisotropic 13-dimensional forms do exist. Take $k := F(x_1, \ldots, x_4, a_1, \ldots, a_5)$, and $\tilde{q} := \langle\langle x_1, \ldots, x_4\rangle\rangle \perp \langle a_1, \ldots, a_5\rangle$. Let $k = k_0 \subset k_1 \subset \ldots \subset k_h$ be generalized splitting tower of M.Knebusch for $\tilde{q}$. Let $E = k(\sqrt{-a_1a_2})$. By the result of D.Hoffmann, there exists a field extension $\tilde{E}/E$ such that $\langle a_3, a_4, a_5\rangle|_{\tilde{E}}$ is a subform of anisotropic Pfister form $\langle\langle x_1, \ldots, x_4\rangle\rangle|_{\tilde{E}}$. That means that $\dim((\tilde{q}|_{\tilde{E}})_{an.}) = 13$. By the result of M.Knebusch ([15, Theorem 5.1]), there exists $0 < t < h$, such that $\dim((\tilde{q}|_{k_t})_{an.}) = 13$. Since k_t is obtained from k by adjoining the generic points of quadrics of dimension $\geqslant 13$, by the result of D.Hoffmann (see [2, Theorem 1]), $\langle a_1, \ldots, a_5\rangle|_{k_t}$ is anisotropic. So, we have proved the existence of the splitting pattern $(1,3,1,1)$.

Let $\mathbf{i}(q) = (1, 1, 1, 3)$. Consider $p := q \perp \langle -\det_{\pm}(q)\rangle$. Then $p \in I^2(k)$. So, $\mathbf{i}(p)$ is simultaneously a specialization of $(1, 1, 1, 1, 2, 1)$ and $(1, 1, 1, 1, 1, 2)$. Hence, by Theorem 7.1, $\mathbf{i}(p)$ is a specialization of $(1, 1, 1, 4)$, that is: $p \in I^3(k)$. Conversely, let q be such anisotropic 13-dimensional form that $q \perp \langle -\det_{\pm}(q)\rangle \in I^3(k)$. Then $\mathbf{i}(q)$ is a specialization of $(1, 1, 1, 3)$. As we know, the only possible specialization is $(1, 1, 1, 3)$ itself. To construct an example, consider the form $p := (\langle\langle a_1, a_2, a_3\rangle\rangle \perp -\langle\langle b_1, b_2, b_3\rangle\rangle)_{an.}$ over the field $k := F(a_1, a_2, a_3, b_1, b_2, b_3)$. Clearly, $p \in I^3(k)$. By the result of R.Elman and T.Y.Lam (see [1]), $\dim(p) = 14$. Then any subquadric of codimension 1 in p will have splitting pattern $(1, 1, 1, 3)$.

Finally, all other forms will have splitting pattern $(1, 1, 1, 1, 1, 1)$. The generic form provides an example.

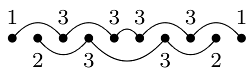
$\dim(q) = 15$**:** We know that $i_1(q) \leqslant 7$. If $i_1(q) = 7$, then q is a Pfister neighbour by [15, Theorem 5.8].

If $i_1(q) = 6$, or 5, or 4, then by the standard arguments, $M(Q)$ contains a binary direct summand of size 8, 9 and 10, respectively. This contradicts Theorem 4.20.

Suppose $i_1(q) = 3$. Then $\mathbf{i}(q)$ is either $(3, 1, 1, 1, 1)$, or $(3, 1, 3)$. In the former case, we get a binary direct summand in $M(Q)$ of size 11, which is impossible, by Theorem 4.20. We will show that the case $(3, 1, 3)$ is possible.

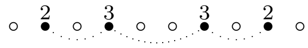
Suppose $i_1(q) = 2$. Then $\mathbf{i}(q)$ is either $(2, 1, 1, 1, 1, 1)$, or $(2, 1, 1, 3)$, or $(2, 3, 1, 1)$. In the first case, by Theorem 7.7 and Theorem 7.8, we get a binary direct summand in $M(Q)$ of size 12, which is impossible by Theorem 4.20. The same happens in the last case, since 2 does not divide 3.

To show that the case $(2, 1, 1, 3)$ is not possible, let us first study the motivic decomposition of the quadric with the splitting pattern $(1, 1, 3)$.

Lemma 7.10 *Let r be anisotropic quadratic form over some field F such that $\mathbf{i}(r) = (1, 1, 3)$. Then $M(R)$ decomposes as follows:* 

In particular, each Tate-motive from the shell number 3 is connected to some Tate-motives from the shells 1 or 2.

Proof: So, let r be such form over some field F. Consider $p := r \perp \langle \det_{\pm}(r)\rangle$. Then $\mathbf{i}(p)$ is simultaneously a specialization of $(1, 1, 1, 1, 2)$ and of $(1, 1, 1, 2, 1)$. By Theorem 7.1, $\mathbf{i}(p)$ is a specialization of $(1, 1, 4)$, and by [21, Satz 14 and Zusatz], a specialization of $(2, 4)$. Since r is anisotropic, we have:

$\mathbf{i}(p) = (2, 4)$. Let L be an indecomposable direct summand from $M(P)$ such that $a(L) = 0$. Then $M(P) = L \oplus L(1)[2]$ (by Theorem 7.9). Since $i_1(p) = 2$, and r is a codimension 1 subform of p, we have (by Corollary 3.10) that L is isomorphic to a direct summand N of $M(R)$. Let M be a complimentary direct summand. Then it should have the form:  If M would be decomposable, then in $M(R)$ there would be a direct summand M' of the form:  But then, by Theorem 4.13, $M'(-1)[-2]$ and $M'(1)[2]$ would be isomorphic to direct summands of $M(R)$ as well. We get: $\mathbb{Z}(2)[4]$ is contained in $M'(-1)[-2]|_{\overline{k}}$ and $L|_{\overline{k}}$. This contradicts the indecomposability of L (by Corollary 3.7). So, M is indecomposable and we get the desired picture for the decomposition of $M(R)$. □

Suppose now q be anisotropic form with the splitting pattern $(2, 1, 1, 3)$, and N be an indecomposable direct summand of $M(Q)$ such that $a(N) = 0$. Then, by Theorem 7.7, $N|_{\overline{k}}$ does not contain Tate-motives from the shells number 2 or 3. But, by Lemma 7.10, any Tate-motive from the shell number 4 is connected to some Tate-motive from the shells 2 or 3, so $N|_{\overline{k}}$ does not contain such motives either. Consequently, N is binary of size 12 - contradiction with Theorem 4.20.

So, we have proved, that $i_1(q) \neq 2$.

It remains to consider the case $i_1(q) = 1$. This gives splitting patterns: $(1, 1, 1, 1, 1, 1, 1)$; $(1, 1, 1, 1, 3)$; $(1, 1, 3, 1, 1)$; and $(1, 5, 1)$. All this patterns are realized by the appropriate quadrics.

Let us now describe classes of quadratic forms corresponding to particular splitting pattern.

The splitting pattern $\mathbf{i}(q) = (7)$ evidently corresponds to the case of a Pfister neighbour, that is: to a form of the type $\lambda \cdot (\langle\langle a, b, c, d\rangle\rangle \perp \langle -1\rangle)_{an.}$, where $\{a, b, c, d\} \neq 0 \in \mathrm{K}^M_4(k)/2$. Such forms clearly exist.

Let $\mathbf{i}(q) = (3, 1, 3)$. Consider $p := q \perp \langle \det(q)\rangle$. Then the splitting pattern of p is simultaneously a specialization of $(1, 2, 1, 1, 2, 1)$ and of $(1, 1, 1, 1, 1, 1, 2)$ (since p differs from q by a 1-dimensional form and $p \in I^2(k)$). By Theorem 7.1, $\mathbf{i}(p)$ is a specialization of $(1, 2, 1, 4)$. But, in the light of [21, Satz 14 and Zusatz], it should be a specialization of $(1, 1, 2, 4)$. So, it is a specialization of $(1, 3, 4)$. But 3 does not divide 4, so, by Theorem 7.7, $\mathbf{i}(p)$ is a specialization of $(4, 4)$ (otherwise, in the motive of the quadric with the splitting pattern $(3, 4)$ we would have a binary direct summand of size = 10, which contradicts Theorem 4.20). So, p is anisotropic (since q is anisotropic of dimension 15), and $\mathbf{i}(p)$ is either $(4,4)$, or (8). The last case is impossible since, in this case, $\mathbf{i}(q)$ would be (7). So, $\mathbf{i}(p) = (4,4)$. By the results of O.Izhboldin and B.Kahn ([6, Theorem 13.9] and [11, Theorem 2.12]), such form is isomorphic to $\langle\langle a,b\rangle\rangle \cdot \langle u,v,w,t\rangle$, and (up to a scalar) is a difference of a 4-fold and a 3-fold Pfister form, having (exactly) two common slots. Converesely, if p is a form of such type, then $\mathbf{i}(p) = (4,4)$, and so, $\mathbf{i}(q)$ is a specialization of $(3,1,3)$. Since $\mathbf{i}(q)$ is clearly not equal to (7), it is $(3,1,3)$.

Let $\mathbf{i}(q) = (1,5,1)$. We have $\dim((q|_{k(Q)})_{an.}) = 13$, and $(q|_{k(Q)})_{an.}$ is a Pfister neighbour. Then, by the result of B.Kahn (see [10, Theorem 2]), there exists such 3-dimensional form $r_4(q)$ that $q \perp r_4(q) \in I^4(k)$. That is, $q = (\lambda \cdot \langle\langle a,b,c,d\rangle\rangle \perp -r_4(q))_{an.}$, for some $\{a,b,c,d\} \neq 0 \in \mathrm{K}_4^M(k)/2$ and $\lambda \in k^*$. Conversely, if q is anisotropic 15-dimensional form of specified type, then $\mathbf{i}(q) = (1,5,1)$. The form $(\langle\langle a,b,c,d\rangle\rangle \perp \langle a,b,e\rangle)_{an.}$ over the field $k(a,b,c,d,e)$ gives an example.

Let $\mathbf{i}(q) = (1,1,3,1,1)$. We have $\dim((q|_{k(Q)})_{an.}) = 13$, and $(q|_{k(Q)})_{an.}$ differs by an anisotropic form of dimension 5 from some form from $I^4(k(Q))$. Then by the result of B.Kahn (see [10, Theorem 2]), there exists some 5-dimensional form $r_4(q)$ that $q \perp r_4(q) \in I^4(k)$. Clearly, $r_4(q)$ is anisotropic, since otherwise $\mathbf{i}(q)$ would be a specialization of $(1,5,1)$. Conversely, let q and $r_4(q)$ be 15-dimensional and 5-dimensional anisotropic forms, s.t. $q \perp r_4(q) \in I^4(k)$. Then $\mathbf{i}((q \perp r_4(q))_{an.}) = (8)$. Since q differs from $(q \perp r_4(q))_{an.}$ by a 5-dimensional form, in the light of Theorem 7.6, we get: $\mathbf{i}(q)$ is a specialization of $(1,1,3,1,1)$. This means that $\mathbf{i}(q)$ is either $(1,1,3,1,1)$, or $(1,5,1)$. If $\mathbf{i}(q)$ would be $(1,5,1)$, then there would exists 3-dimensional form p, such that $q \perp p \in I^4(k)$. Then $r_4(q) \perp -p \in I^4(k)$. Since $\dim(r_4(q) \perp -p) = 8 < 16$, we get: $r_4(q) = p \perp \mathbb{H}$. But $r_4(q)$ is anisotropic - contradiction. Hence $\mathbf{i}(q) = (1,1,3,1,1)$. It remains to show that such anisotropic 15-dimensional forms do exist. Take $k := F(x_1,\dots,x_4,a_1,\dots,a_5)$, and $\tilde{q} := \langle\langle x_1,\dots,x_4\rangle\rangle \perp \langle a_1,\dots,a_5\rangle$. Consider the field extension $E = k(\sqrt{-a_1a_2}, \sqrt{-a_3a_4}, \sqrt{-a_5})$. Then $\dim((\tilde{q}|_E)_{an.}) = 15$. By [15, Theorem 5.1], there exists $0 < s < h$, such that $\dim((\tilde{q}|_{k_s})_{an.}) = 15$. Since k_s is obtained from k by adjoining the generic points of quadrics of dimension $\geqslant 15$, by [2, Theorem 1], $\langle a_1,\dots,a_5\rangle|_{k_s}$ is anisotropic. Then the form $q := (\tilde{q}|_{k_s})_{an.}$ has the splitting pattern $(1,1,3,1,1)$.

Let $\mathbf{i}(q) = (1,1,1,1,3)$. Consider $p := q \perp \langle \det_{\pm}(q)\rangle$. Then the splitting pattern of p is simultaneously a specialization of $(1,1,1,1,1,2,1)$ and of $(1,1,1,1,1,1,2)$ (since p differs from q by a 1-dimensional form and $p \in I^2(k)$). By Theorem 7.1, $\mathbf{i}(p)$ is a specialization of $(1,1,1,1,4)$. In the light of [21, Satz 14 and Zusatz], it should be a specialization of $(1,1,2,4)$. Clearly, $\mathbf{i}(p)$ must be either $(1,1,2,4)$, or $(2,2,4)$, or $(1,2,4)$ (in the last case, p is isotropic). Conversely, if p has one of the above splitting patterns and q is of codimension 1 in p, then q is anisotropic and it's splitting pattern is a specialization of $(1,1,1,1,3)$. In our list of possible splitting patterns of forms of dimension 15 only the following three satisfy this property: $(1,1,1,1,3)$, $(3,1,3)$ and (7). But if $\mathbf{i}(q)$ would be $(3,1,3)$, or (7), then $\mathbf{i}(p)$ would be $(4,4)$, or (8). So, $\mathbf{i}(q) = (1,1,1,1,3)$. And the forms p, such that $i_h(p) = 4$ and $i_{h-1}(p) = 2$ can be described as: $p \in I^3(k)$, such that $\pi(p) \in \mathrm{K}_3^M(k)/2$ is not a *pure symbol* (here $\pi : I^3(k) \to \mathrm{K}_3^M(k)/2$ is a projection, induced by the isomorphism $\mathrm{K}_3^M(k)/2 \cong I^3(k)/I^4(k)$). This follows from the result of O.Izhboldin - see [6, Corollary 13.7]. The form $q = (\langle\langle a_1, a_2, a_3\rangle\rangle \perp -\langle\langle b_1, b_2, b_3\rangle\rangle)_{an} \perp \langle c\rangle$ over the field $k(a_1, a_2, a_3, b_1, b_2, b_3, c)$ provides an example.

Finally, the remaining forms will have the spliting pattern $(1,1,1,1,1,1,1)$. The generic form provides an example.

$\dim(q) = 17$**:** By Theorem 7.2, $i_1(q) = 1$. And so, the possible splitting patterns are: $(1,1,1,1,1,1,1,1)$, $(1,1,1,1,1,3)$, $(1,1,1,3,1,1)$, $(1,1,5,1)$, $(1,3,1,3)$, and $(1,7)$. Again, by Theorem 7.3, all these patterns are realized. Let us describe the corresponding classes of forms.

Let $\mathbf{i}(q) = (1,7)$. Consider $p := q \perp \langle \det_{\pm}(q)\rangle$. Then the splitting pattern of p is simultaneously a specialization of $(1,1,6,1)$ and $(1,1,1,1,1,1,1,2)$. So, it is a specialization of $(1,8)$. By the result of D.Hoffmann, p is isotropic and $p_{an.} = \lambda \cdot \langle\langle a,b,c,d\rangle\rangle$, for some $\{a,b,c,d\} \neq 0 \in \mathrm{K}_4^M(k)/2$, and $\lambda \in k^*$. Since q is anisotropic, we also have: $\{a,b,c,d,-\lambda\cdot\det(q)\} \neq 0 \in \mathrm{K}_5^M(k)/2$. Conversely, the form $\langle\langle a,b,c,d\rangle\rangle \perp \langle -e\rangle$, where $\{a,b,c,d,e\} \neq 0\,(\,mod\,2)$ has splitting pattern $(1,7)$. Such form clearly exists over the field $F(a,b,c,d,e)$.

Let $\mathbf{i}(q) = (1,3,1,3)$. Consider $p := q \perp \langle \det_{\pm}(q)\rangle$. Then the splitting pattern of p is simultaneously a specialization of $(1,1,2,1,1,2,1)$ and $(1,1,1,1,1,1,1,2)$. So, it is a specialization of $(1,1,2,1,4)$. But, in the light of [21, Satz 14 and Zusatz], it should be a specialization of $(1,1,1,2,4)$. So, it is a specialization of $(1,1,3,4)$. But 3 does not divide 4, so, by Theorem 7.7, Theorem 7.8 and Theorem 4.20 (applied to the form with the splitting pattern $(3,4)$), $\mathbf{i}(p)$ is a specialization of $(1,4,4)$. By the results of O.Izhboldin and D.Hoffmann ([6, Proposition 13.6], [4, Corollary 3.4]), there are no forms with the splitting patterns $(1,4,4)$, and $(1,8)$, so p is isotropic. Clearly, p can not have splitting pattern (8), so, $\mathbf{i}(p)=(4,4)$, and $p_{an.}=\langle\langle a,b\rangle\rangle\cdot\langle u,v,w,t\rangle$, where $\{a,b,uvwt\}\neq 0\,(\,mod\,2)$ and $\{a,b,-uv,-uw\}$ is not divisible by $\{a,b,uvwt\}$ $(\,mod\,2)$. Conversely, if p has specified type, and $q:=p\perp\langle c\rangle$ is anisotropic, then $\mathbf{i}(q)$ is a specialization of $(1,3,1,3)$. So, $\mathbf{i}(q)$ is either $(1,3,1,3)$, or $(1,7)$. In the last case, $\mathbf{i}(p)$ would be a specialization of $(1,1,6,1)$, which is not the case. So, $\mathbf{i}(q)=(1,3,1,3)$. Taking a,b,c,u,v,w,t generic, we get an example.

Let $\mathbf{i}(q)=(1,1,5,1)$. We have $\dim((q|_{k(Q)})_{an.})=15$, and $(q|_{k(Q)})_{an.}$ differs by a form of dimension 3 from some form from $I^4(k(Q))$. Then by the result of B.Kahn (see [10, Theorem 2]), there exists some 3-dimensional form $r_4(q)$ that $q\perp r_4(q)\in I^4(k)$. That is, $q=(\lambda\cdot\langle\langle a,b,c,d\rangle\rangle\perp -r_4(q))_{an.}$, for some $\{a,b,c,d\}\neq 0\in \mathrm{K}_4^M(k)/2$ and $\lambda\in k^*$ (by the result of D.Hoffmann, in I^4 there are no anisotropic forms of dimensions 18,20 and 22 - see [4, Main Theorem]). Conversely, if q is anisotropic 17-dimensional form of the specified type, then $\mathbf{i}(q)=(1,1,5,1)$. The form $(\langle\langle a,b,c,d\rangle\rangle\perp\langle a,e,f\rangle)_{an.}$ over the field $k(a,b,c,d,e,f)$ gives an example.

Let $\mathbf{i}(q)=(1,1,1,3,1,1)$. We have $\dim((q|_{k(Q)})_{an.})=15$, and $(q|_{k(Q)})_{an.}$ differs by an anisotropic form of dimension 5 from some form from $I^4(k(Q))$. Then by [10, Theorem 2], there exists some 5-dimensional form $r_4(q)$ that $q\perp r_4(q)\in I^4(k)$. Clearly, $r_4(q)$ is anisotropic, since otherwise q would have specialization of $(1,1,5,1)$ as a splitting pattern. Conversely, let q and $r_4(q)$ be 17-dimensional and 5-dimensional anisotropic forms, s.t. $q\perp r_4(q)\in I^4(k)$. Then $\mathbf{i}((q\perp r_4(q))_{an.})=(8)$ (since $\dim((q\perp r_4(q))_{an.})<24$ and $(q\perp r_4(q))_{an.}\in I^4(k)$). Since q differs from $(q\perp r_4(q))_{an.}$ by a 5-dimensional form, we get: $\mathbf{i}(q)$ is a specialization of $(1,1,1,3,1,1)$. This means that $\mathbf{i}(q)$ is either $(1,1,1,3,1,1)$, or $(1,1,5,1)$, or $(1,7)$. If $\mathbf{i}(q)=(1,5,1)$, then there exists 3-dimensional form p, such that $q\perp p\in I^4(k)$. Then $r_4(q)\perp -p\in I^4(k)$. Since $\dim(r_4(q)\perp -p)=8<16$, we get: $r_4(q)=p\perp\mathbb{H}$. But $r_4(q)$ is anisotropic - contradiction. The case $(1,7)$ can be treated in the same way. Hence $\mathbf{i}(q)=(1,1,1,3,1,1)$. It remains to show that such anisotropic 17-dimensional forms do exist. Take $k:=F(x_1,\dots,x_4,a_1,\dots,a_5)$, and $\tilde{q}:=\langle\langle x_1,\dots,x_4\rangle\rangle\perp\langle a_1,\dots,a_5\rangle$. Let $k=k_0\subset k_1\subset\ldots\subset k_h$ be the generalized splitting tower for $\tilde{q}$. We know (from the consideration of 15-dimensional forms), that there is t such that $(\tilde{q}|_{k_t})_{an.}$ has the splitting pattern $(1,1,3,1,1)$. On the other hand, if $E=k(\sqrt{-a_1a_2},\sqrt{-a_3a_4})$, then $\dim((\tilde{q}|_E)_{an.})=17$.

By [15, Theorem 5.1], the form $q := (\tilde{q}|_{k_{t-1}})_{an.}$ has the splitting pattern $(1,1,1,3,1,1)$.

Let $\mathbf{i}(q) = (1,1,1,1,1,3)$. Consider $p := q \perp \langle \det_{\pm}(q) \rangle$. Then the splitting pattern of p is simultaneously a specialization of $(1,1,1,1,1,1,2,1)$ and of $(1,1,1,1,1,1,1,2)$ (since p differs from q by a 1-dimensional form and $p \in I^2(k)$). By Theorem 7.1, $\mathbf{i}(p)$ is a specialization of $(1,1,1,1,1,4)$. In the light of [21, Satz 14 and Zusatz], it should be a specialization of $(1,1,1,2,4)$. Clearly, $i_h(p)$ must be 4 and $i_{h-1}(p)$ must be 2 (by Theorem 7.6). Conversely, if $i_h(p) = 4$, $i_{h-1}(p) = 2$, and q is anisotropic of codimension 1 in p, then $\mathbf{i}(q)$ is a specialization of $(1,1,1,1,1,3)$. In our list of possible splitting patterns of forms of dimension 17 only the following three satisfy this property: $(1,1,1,1,1,3)$, $(1,3,1,3)$ and $(1,7)$. But if $\mathbf{i}(q)$ would be $(1,3,1,3)$, or $(1,7)$, then $\mathbf{i}(p)$ would be $(4,4)$, or (8). So, $\mathbf{i}(q) = (1,1,1,1,1,3)$. And again, by the result of O.Izhboldin ([6, Corollary 13.7]), the forms p, such that $i_h(p) = 4$ and $i_{h-1}(p) = 2$ can be described as: $p \in I^3(k)$, such that $\pi(p) \in \mathrm{K}_3^M(k)/2$ is not a *pure symbol* (here $\pi : I^3(k) \to \mathrm{K}_3^M(k)/2$ is a projection, induced by the isomorphism $\mathrm{K}_3^M(k)/2 \cong I^3(k)/I^4(k)$). The form $q = \langle\langle a_1, a_2, a_3 \rangle\rangle \perp d \cdot \langle\langle b_1, b_2, b_3 \rangle\rangle \perp \langle c \rangle$ over the field $k(a_1,a_2,a_3,b_1,b_2,b_3,c,d)$ provides an example.

Finally, the remaining forms have the splitting pattern $(1,1,1,1,1,1,1,1)$. The generic form provides an example.

Let summarize our results (use the Definition 5.8).

$\dim(q)$	splitting pattern	description
3	(1)	-
5	(1,1)	-
7	(3) (1,1,1)	$\dim_3(q) = 1$ $\dim_3(q) > 1$
9	(1,3) (1,1,1,1)	$\dim_3(q) = 1$ $\dim_3(q) > 1$
11	(3,1,1) (1,1,3) (1,1,1,1,1)	$\dim_4(q) = 5$ $\dim_3(q) = 1$ $\dim_3(q) > 1$, $\dim_4(q) > 5$
13	(5,1) (1,3,1,1) (1,1,1,3) (1,1,1,1,1,1)	$\dim_4(q) = 3$ $\dim_4(q) = 5$ $\dim_3(q) = 1$ $\dim_3(q) > 1$, $\dim_4(q) > 5$

$\dim(q)$	splitting pattern	description
15	(7)	$\dim_4(q) = 1$
	(3,1,3)	$\dim_3(q) = 1$, $\omega_3(q)$ - is nonzero pure symbol
	(1,5,1)	$\dim_4(q) = 3$
	(1,1,3,1,1)	$\dim_4(q) = 5$
	(1,1,1,1,3)	$\dim_3(q) = 1$, $\omega_3(q)$ - is not a pure symbol
	(1,1,1,1,1,1,1)	$\dim_3(q) > 1$, $\dim_4(q) > 5$
17	(1,7)	$\dim_4(q) = 1$
	(1,3,1,3)	$\dim_3(q) = 1$, $\omega_3(q)$ - is nonzero pure symbol
	(1,1,5,1)	$\dim_4(q) = 3$
	(1,1,1,3,1,1)	$\dim_4(q) = 5$
	(1,1,1,1,1,3)	$\dim_3(q) = 1$, $\omega_3(q)$ - is not a pure symbol
	(1,1,1,1,1,1,1,1)	$\dim_3(q) > 1$, $\dim_4(q) > 5$

We can also describe possible splitting patterns of forms of dimension 19 and 21. Although, in these cases we will provide only hypothetical description of the respected classes of forms.

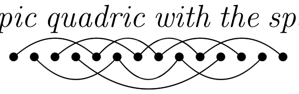
$\underline{\dim(q) = 19}$**:** By Theorem 7.2, $i_1(q) \leqslant 3$. Let us show that $i_1(q) \neq 2$. First of all, this is a particular case of the result of O.Izhboldin, since $19 = 2^4 + 3$. Alternatively, we can argue as follows. If $i_1(q) = 2$, then $\mathbf{i}(q)$ could be one of the following: $(2, 1, 1, 1, 1, 1, 1, 1)$, $(2, 1, 1, 1, 1, 3)$, $(2, 1, 1, 3, 1, 1)$, $(2, 1, 5, 1)$, $(2, 3, 1, 3)$, or $(2, 7)$. Let N be an indecomposable direct summand of $M(Q)$, such that $a(N) = 0$.

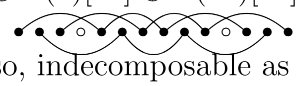
If $\mathbf{i}(q) = (2, 1, 1, 1, 1, 1, 1, 1)$, then we immediately get that N is binary of size $\dim(Q) - i_1(q) + 1 = 16$ - a contradiction with Theorem 4.20.

Let now $\mathbf{i}(q) = (2, 1, 1, 1, 1, 3)$. Then $N|_{\overline{k}}$ does not contain Tate-motives from the shells 2,3,4 and 5. At the same time, by Lemma 7.10, we know that each Tate-motive from the shell number 6 is connected to some Tate-motive from the shell number 4 or 5. So $N|_{\overline{k}}$ does not contain Tate-motives from the 6-th shell either, and, by Theorem 7.8, N is binary of size 16 - a contradiction with Theorem 4.20.

Let $\mathbf{i}(q) = (2, 7)$. Since $i_2(q)$ is not divisible by $i_1(q)$, by Theorem 7.7(2), N will be binary of size 16 - a contradiction with Theorem 4.20.

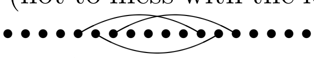
To exclude the case $\mathbf{i}(q) = (2, 3, 1, 3)$, let us first study the motivic decomposition of the quadric with the splitting pattern $(3, 1, 3)$.

Lemma 7.11 *Let R be anisotropic quadric with the splitting pattern* $(3, 1, 3)$*. Then it's motive decomposes as:* 

Proof: Let L be an indecomposable direct summand of $M(R)$ such that $a(L) = 0$. Then $L|_{\overline{k}}$ does not contain Tate-motives from the second shell, and since L is not binary (by Theorem 4.20), $L|_{\overline{k}}$ should contain Tate-motives from the third shell. Since $L(1)[2]$ and $L(2)[4]$ are also isomorphic to direct summands of $M(R)$ (by Theorem 4.13), $L|_{\overline{k}}$ must be $\mathbb{Z} \oplus \mathbb{Z}(4)[8] \oplus \mathbb{Z}(7)[14] \oplus \mathbb{Z}(11)[22]$, and in $M(R)$ we have indecomposables of the form:  The complimentary direct summand is binary, and so, indecomposable as well (by Lemma 3.12.1). $\square$

Let now q be a form with the splitting pattern $(2, 3, 1, 3)$. Since $i_2(q)$ is not divisible by $i_1(q)$, $N|_{\overline{k}}$ does not contain Tate-motives from the shell number 2. It also does not contain any Tate-motive from the shell number 3 (since $1 < 2$). So, if N is not binary, then $N|_{\overline{k}}$ contains Tate-motives from the shell number 4. But, by Lemma 7.11, each such Tate-motive is connected to some Tate-motive from the shell number 2. So, $N|_{\overline{k}}$ can not contain Tate-motives from the 4-th shell either. And N is binary of size 16 - a contradiction with Theorem 4.20.

Let $\mathbf{i}(q) = (2, 1, 1, 3, 1, 1)$. We know that $a(N) = 0$, $b(N) = 16$ (by Corollary 4.7), $N(1)[2]$ is a direct summand in $M(Q)$, and $N^\vee \cong N(1)[2]$ (by Theorem 4.19). In particular, if $\mathbb{Z}(l)[2l]$ is a direct summand of $N|_{\overline{k}}$, then $\mathbb{Z}(16-l)[32-2l]$ is a direct summand too. But in $M(Q)$ we have connections (not to mess with the idecomposable direct summands) of the following form:



If N is not binary, then $N|_{\overline{k}}$ must contain some Tate-motive from the shell number 4. But since any such $\mathbb{Z}(l)[2l]$ comes together with $\mathbb{Z}(16-l)[32-2l]$ and $\mathbb{Z}(l+7)[2l+14]$ (because of the connections above), we get that $N|_{\overline{k}}$ contains at least 4 Tate-motives from the shell number 4. But then $N(1)[2]|_{\overline{k}}$ contains another 4 from the same shell - a contradiction (the shell contains only 6 Tate-motives). So, N is binary of size 16 - a contradiction with Theorem 4.20.

Finally, let $\mathbf{i}(q) = (2, 1, 5, 1)$. By [10, Theorem 2], $\dim_4(q) = 3$. Consider $p := q \perp r_4(q) \in I^4(k)$. Since in I^4 there are no anisotropic forms of dimension 22,20 and 18, $p = \mathbb{H} \perp \mathbb{H} \perp \mathbb{H} \perp p_{an.}$, and p_{an} is proportional to the anisotropic 4-fold Pfister form. That means that $M(P_{an.})$ consists of binary Rost-motives, and because for any field extention E/k, $p_{an}|_E$ is isotropic if and only if $i_W(q|_E) > 3$, we get by Theorem 4.15, Theorem 4.13, that the shell number 3 of $M(Q)$ consists of the Rost-motives. In particular, $N|_{\overline{k}}$ does not contain any Tate-motive from the 3-rd shell, and so, N is binary of size 16 - contradiction with Theorem 4.20. So, we have proved that $i_1(q) \neq 2$.

The remaining possibilities are: $i_1(q) = 3$ and $i_1(q) = 1$. In the first case, we get splitting patterns: $(3,1,1,1,1,1,1)$, $(3,1,1,1,3)$, $(3,1,3,1,1)$ and $(3,5,1)$, and all these patterns are realized by the appropriate forms in the light of Theorem 7.3.

Let now $i_1(q) = 1$. If $\mathbf{i}(q) = (1,1,7)$, consider $p := q \perp \langle \det_{\pm}(q) \rangle$. Then $\mathbf{i}(p)$ is simultaneously a specialization of $(1,1,1,6,1)$ and $(1,1,1,1,1,1,1,1,2)$. So, it is a specialization of $(1,1,8)$. Since in I^4 there are no anisotropic forms of dimension 20 and 18, it should be a specialization of (8). That means that q is isotropic - a contradiction. So, this splitting pattern is not possible.

We will show that the remaining splitting patterns: $(1,1,1,1,1,1,1,1,1)$, $(1,1,1,1,1,1,3)$, $(1,1,1,1,3,1,1)$, $(1,1,1,5,1)$, and $(1,1,3,1,3)$ are realized by the appropriate forms.

The splitting pattern $(1,1,1,1,1,1,1,1,1)$ is realized by the generic form $\langle x_1, \ldots, x_{19} \rangle$ over the field $k(x_1, \ldots, x_{19})$.

To construct the form with the splitting pattern $(1,1,1,1,1,1,3)$, consider $\tilde{q} := \langle\langle a_1, a_2, a_3 \rangle\rangle \perp \lambda \cdot \langle\langle b_1, b_2, b_3 \rangle\rangle \perp \mu \cdot \langle\langle c_1, c_2, c_3 \rangle\rangle \perp \langle -1 \rangle$ over the field $F := k(a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3, \lambda, \mu)$. Let $F = F_0 \subset F_1 \subset \ldots \subset F_h$ be the generalized splitting tower for $\tilde{q}$. Then, for some t, $\dim((\tilde{q}|_{F_t})_{an.}) = 19$ (since it happens over the field $F(\sqrt{-\lambda\mu}, \sqrt{b_1 c_1})$). On the other hand, there exists s (clearly, equal to $t+1$), such that $\dim((\tilde{q}|_{F_s})_{an.}) = 17$, and $(\tilde{q}|_{F_s})_{an.}$ has splitting pattern $(1,1,1,1,1,3)$ (since it happens over the field $F(\sqrt{a_1})$). Consequently, $(\tilde{q}|_{F_t})_{an.}$ has splitting pattern $(1,1,1,1,1,1,3)$.

For the splitting pattern $(1,1,1,1,3,1,1)$, consider the form
$q := (\langle\langle a_1, a_2, a_3, a_4 \rangle\rangle \perp \langle -1, x_1, x_2, x_3, x_4 \rangle)_{an.}$ over the field
$F := k(a_1, a_2, a_3, a_4, x_1, x_2, x_3, x_4)$. Clearly, $\dim(q) = 19$. On the other hand, other the field $E = F(\sqrt{-a_1 x_1})$, $\dim((q|_E)_{an.}) = 17$. Hence $\dim((q|_{F(Q)})_{an.}) = 17$. And $\dim_4((q|_{F(Q)})_{an.}) = 5$. So, $(q|_{F(Q)})_{an.}$ has the splitting pattern $(1,1,1,3,1,1)$. Hence, q has the splitting pattern $(1,1,1,1,3,1,1)$.

For the splitting pattern $(1,1,1,5,1)$, consider the form
$q := \langle\langle a_1, a_2, a_3, a_4 \rangle\rangle \perp \langle x_1, x_2, x_3 \rangle$ over the field
$F := k(a_1, a_2, a_3, a_4, x_1, x_2, x_3)$. Clearly, q is anisotropic, and over the field $E = F(\sqrt{-x_1})$, $\dim((q|_E)_{an.}) = 17$. Hence $\dim((q|_{F(Q)})_{an.}) = 17$. And also, $\dim_4((q|_{F(Q)})_{an.}) = 3$. So, $(q|_{F(Q)})_{an.}$ has the splitting pattern $(1,1,5,1)$. Hence, q has the splitting pattern $(1,1,1,5,1)$.

Finally, for the splitting pattern $(1,1,3,1,3)$, consider the form $\tilde{q} := \langle\langle a_1, a_2, a_3, a_4\rangle\rangle \perp \lambda\cdot\langle\langle b_1, b_2, b_3\rangle\rangle \perp \langle\mu\rangle$ over the field $F := k(a_1, a_2, a_3, a_4, b_1, b_2, b_3, \lambda, \mu)$. Then for some t, $\dim((\tilde{q}|_{F_t})_{an.}) = 19$ (since it is so over the field $F(\sqrt{-\lambda}, \sqrt{a_1b_1}, \sqrt{a_2\mu})$). And $\dim((\tilde{q}|_{F_{t+1}})_{an.}) = 17$, since it is so over the field $F(\sqrt{-\lambda}, \sqrt{a_1b_1}, \sqrt{a_2b_2})$. But $\dim_3((\tilde{q}|_{F_{t+1}})_{an.}) = 1$ and $\omega_3((\tilde{q}|_{F_{t+1}})_{an.})$ is a nonzero pure symbol. So, $(q|_{F_{t+1}})_{an.}$ has the splitting pattern $(1,3,1,3)$, and $q := (\tilde{q}|_{F_t})_{an.}$ has the splitting pattern $(1,1,3,1,3)$.

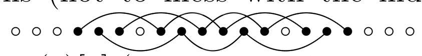
<u>$\dim(q) = 21$</u>**:** By Theorem 7.2, $i_1(q) \leqslant 5$. Let us show that $i_1(q)$ is not equal to 2, 3, or 4. If $i_1(q) = 4$, then $\mathbf{i}(q)$ would be $(4,1,1,1,1,1,1)$, $(4,1,1,1,3)$, $(4,1,3,1,1)$, or $(4,5,1)$. In the light of Theorem 7.7, in all these cases we get a binary direct summand of $M(Q)$ of size $\dim(Q) - 4 + 1 = 16$, which contradicts Theorem 4.20.

If $i_1(q) = 2$, then $\mathbf{i}(q)$ would be $(2,1,1,1,1,1,1,1,1)$, $(2,1,1,1,1,1,3)$, $(2,1,1,1,3,1,1)$, $(2,1,1,5,1)$, $(2,1,3,1,3)$, or $(2,1,7)$. Let N be an indecomposable direct summand of $M(Q)$ such that $a(N) = 0$.

If $\mathbf{i}(q) = (2,1,1,1,1,1,1,1,1)$, then N is binary of size 18 - a contradiction with Theorem 4.20.

The same will happen in the case $\mathbf{i}(q) = (2,1,1,1,1,1,3)$, since all Tate-motives from the shell number 7 are connected to some Tate-motives from the shells number 5 and 6 (by Lemma 7.10), and those shells are not connected to the shell number 1.

The nonexistence of the splitting pattern $(2,1,1,1,3,1,1)$ follows from the considerations we applied to the splitting pattern $(2,1,1,3,1,1)$ above (in $\dim = 19$)(with the only difference that N will be a binary direct summand of size 18 instead of 16, which still contradicts Theorem 4.20).

If $\mathbf{i}(q) = (2,1,3,1,3)$, Then, by Lemma 7.11, in $M(Q)$ we have connections (not to mess with the indecomposable direct summands) of the form:  . Since $a(N) = 0$, $b(N) = 18$, and $N^{\vee} \cong N(1)[2]$ (in other words, N is symmetric with respect to flipping over) (here $N^{\vee}$ is the direct summand of $M(Q)$ given by the dual projector), we see that $N|_{\overline{k}}$ does not contain any of the Tate-motives from the shells number 3 and 5. So, N is binary of size 18 - a contradiction with Theorem 4.20.

If $\mathbf{i}(q) = (2,1,7)$, consider $p := q \perp \langle -\det_{\pm}(q)\rangle$. Then $\mathbf{i}(p)$ is a specialization of $(1,1,1,1,6,1)$ and $p \in I^2(k)$. So, $\mathbf{i}(p)$ is a specialization of $(1,1,1,8)$. Since in I^4 there are no anisotropic forms of dimension 22 or 20 (by the result of D.Hoffmann), q must be isotropic - a contradiction.

Finally, if $\mathbf{i}(q) = (2, 1, 1, 5, 1)$, then by [10, Theorem 2], $\dim_4(q) = 3$, so there exists 3-dimensional form $r_4(q)$, such that $p := q \perp r_4(q) \in I^4(k)$. Since in I^4 there are no forms of dimension 18, 20 and 22, we get that p is anisotropic of dimension 24, and $\mathbf{i}(p) = (4, 8)$. But since q is a subform of codimension 3 in p, and $i_1(p) = 4 > 3$, we get that N will be isomorphic to a direct summand L of $M(P)$, such that $a(L) = 0$. But $\mathrm{size}(N) = \dim(Q) - i_1(q) + 1 = 18$, and $\mathrm{size}(L) = \dim(P) - i_1(p) + 1 = 19$ - a contradiction. So, we have proved that $i_1(q) \neq 2$.

Suppose $i_1(q) = 3$. We have the following possibilities for $\mathbf{i}(q)$: $(3, 1, 1, 1, 1, 1, 1, 1)$, $(3, 1, 1, 1, 1, 3)$, $(3, 1, 1, 3, 1, 1)$, $(3, 1, 5, 1)$, $(3, 3, 1, 3)$ and $(3, 7)$. Let N be an indecomposable direct summand of $M(Q)$ such that $a(N) = 0$.

If $\mathbf{i}(q) = (3, 1, 1, 1, 1, 1, 1, 1)$, then N is binary of size 17, which contradicts Theorem 4.20. The same happens in the case $\mathbf{i}(q) = (3, 1, 1, 1, 1, 3)$, since all Tate-motives from the shell number 6 are connected to some Tate-motives from the shells 4 and 5, and in the case $\mathbf{i}(q) = (3, 7)$, since 7 is not divisible by 3.

If $\mathbf{i}(q) = (3, 1, 5, 1)$, then by [10, Theorem 2], $\dim_4(q) = 3$, so there exists 3-dimensional form $r_4(q)$, such that $p := q \perp r_4(q) \in I^4(k)$. Since in I^4 there are no forms of dimension 18, 20 and 22, we get that p is anisotropic of dimension 24, and $\mathbf{i}(p) = (4, 8)$. But since q is a subform of codimension 3 in p, and $i_1(p) = 4 > 3$, we get that N will be isomorphic to a direct summand L of $M(P)$, such that $a(L) = 0$. But $\mathrm{size}(N) = \dim(Q) - i_1(q) + 1 = 17$, and $\mathrm{size}(L) = \dim(P) - i_1(p) + 1 = 19$ - a contradiction.

If $\mathbf{i}(q) = (3, 3, 1, 3)$, then consider $p := q \perp \langle \det_{\pm}(q) \rangle$. $\mathbf{i}(p)$ is a specialization of $(1, 2, 1, 2, 1, 1, 2, 1)$, and $p \in I^2(k)$. Hence $\mathbf{i}(p)$ is a specialization of $(1, 2, 1, 2, 1, 4)$, and finally, of $(1, 2, 4, 4)$. Clearly, then $\mathbf{i}(p)$ is either $(1, 2, 4, 4)$, or $(2, 4, 4)$ (in the last case p is isotropic). Let p'' be the form with the splitting pattern $(2, 4, 4)$. If L is an indecomposable direct summand of $M(P'')$ such that $a(L) = 0$, then, by the inductive application of Theorem 7.9, $M(P'') = L \oplus L(1)[2]$. In particular, $L|_{\overline{k}}$ contains $\mathbb{Z}(2)[4]$. Return to our original form q. Since $i_1(q) = 3$, we have that $N(1)[2]$ and $N(2)[4]$ are also direct summands of $M(Q)$. In particular, $N|_{\overline{k}}$ does not contain $\mathbb{Z}(2)[4]$. But for arbitrary field extension E/k, the conditions: $i_W(q|_E) > 0$ and $i_W(p|_E) > 1$ are equivalent (since $i_1(q) = 3$). So, by Theorem 4.15, $N(1)[2]$ is a direct summand of $M(P)$. Consider the field $F = k(P)$. Then $p|_F = \mathbb{H} \perp p''$, and $\mathbf{i}(p'') = (2, 4, 4)$. Since $M(P|_F) = \mathbb{Z} \oplus M(P'')(1)[2] \oplus \mathbb{Z}(20)[40]$, we get that

$N|_F$ is a direct summand of $M(P'')$ such that $a(N|_F) = 0$. In particular, the indecomposable direct summand L of $M(P'')$ described above should be a direct summand of $N|_F$. But $L|_{\overline{F}}$ does contain $\mathbb{Z}(2)[4]$ and $N|_{\overline{F}}$ does not - a contradiction. So, the splitting pattern $(3, 3, 1, 3)$ is not possible.

Finally, let $\mathbf{i}(q) = (3, 1, 1, 3, 1, 1)$. The only way for N not to be binary of size 17, is to contain Tate-motives from the shell number 4. That is $N|_{\overline{k}} = \mathbb{Z} \oplus \mathbb{Z}(5)[10] \oplus \mathbb{Z}(12)[24] \oplus \mathbb{Z}(17)[34]$. By the result of B.Kahn ([10, Theorem 2]), $\dim_4(q) = 5$. So, let $r_4(q)$ be such 5-dimensional form that $q \perp r_4(q) \in I^4(k)$. Let $p := (q \perp r_4(q))_{an.}$. We know that $\dim(p) \neq 18, 20$, or 22. Suppose $\dim(p) \geqslant 24$. Then $p|_{k(Q)}$ is isotropic, since $\dim((q|_{k(Q)})_{an}) = 15$. Suppose $q|_{k(P)}$ is isotropic. Then, by Corollary 3.9, $M(P)$ contains a direct summand L with $a(L) = 0$ isomorphic to N. But L has size $\dim(P) - i_1(p) + 1 = 19$ or 24, since $\mathbf{i}(p)$ is either $(4, 8)$, or $(1, 4, 8)$ (the last case does not exist, actually). In any case, it is not equal to $17 = \text{size}(N)$. So, $q|_{k(P)}$ is anisotropic, and $\mathbf{i}(q|_{k(P)})$ is a specialization of $(3, 1, 1, 3, 1, 1)$. So, $\mathbf{i}(q|_{k(P)})$ is either $(3, 1, 1, 3, 1, 1)$, or $(5, 3, 1, 1)$ (we already know that i_1 can't be 4 for 21-dimensional forms). But in the second case, for the indecomposable direct summand M of $M(Q|_{k(P)})$ with $a(M) = 0$ we would have $M|_{\overline{k(P)}}$ contains $\mathbb{Z}(\dim(Q) - i_1(q|_{k(P)}) + 1)[2(\dim(Q) - i_1(q|_{k(P)}) + 1)] = \mathbb{Z}(15)[30]$, but already $N|_{\overline{k(P)}}$ does not contain this Tate-motive, and M is clearly a direct summand in $N|_{k(P)}$. So, $\mathbf{i}(q|_{k(P)}) \neq (5, 3, 1, 1)$, and hence, $\mathbf{i}(q|_{k(P)}) = (3, 1, 1, 3, 1, 1)$. This means that by changing the field, we can assume that $\dim(p) < 24$, which means $\dim(p) = 16$, and p is a Pfister form up to a scalar multiple. Abusing notations, we will still call this new field k, and the new form q. But then we notice that for arbitrary field extension E/k, $p|_E$ is isotropic if and only if $i_W(q|_E) > 5$. By Theorem 4.15, in $M(Q)$ there is a direct summand N', such that $a(N') = 5$. But $N|_{\overline{k}}$ contains $\mathbb{Z}(5)[10]$ - a contradiction. So, the splitting pattern $(3, 1, 1, 3, 1, 1)$ is not possible, and we have proved that $i_1(q) \neq 3$.

The remaining values of $i_1(q)$ are 1 and 5. We will show that all the splitting patterns with such i_1 (which are provided by the already classified splitting patterns of forms of dimension 19 and 11) are realized by the appropriate forms.

If $i_1(q) = 5$, then it is a consequence of Theorem 7.3 that all the splitting patterns: $(5, 1, 1, 1, 1, 1)$, $(5, 1, 1, 3)$ and $(5, 3, 1, 1)$ are realized.

Let now $i_1(q) = 1$. The splitting pattern $(1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$ is realized by the generic form $\langle x_1,\ldots,x_{21}\rangle$ over the field $k(x_1,\ldots,x_{21})$.

To construct the form with the splitting pattern $(1,1,1,1,1,1,1,3)$, consider $\tilde q := \langle\langle a_1,a_2,a_3\rangle\rangle \perp \lambda\cdot\langle\langle b_1,b_2,b_3\rangle\rangle \perp \mu\cdot\langle\langle c_1,c_2,c_3\rangle\rangle \perp \langle\eta\rangle$ over the field $F := k(a_1,a_2,a_3,b_1,b_2,b_3,c_1,c_2,c_3,\lambda,\mu,\eta)$. Let $F=F_0\subset F_1\subset\ldots\subset F_h$ be the generalized splitting tower for $\tilde q$. Then, for some t, $\dim((\tilde q|_{F_t})_{an.}) = 21$ (since it happens over the field $F(\sqrt{-\lambda\mu},\sqrt{b_1c_1})$). On the other hand, $\tilde q$ over some field has a splitting pattern $(1,1,1,1,1,1,3)$ (as we saw, while considering forms of dimension 19). So, $(1,1,1,1,1,1,1,3)$ is a specialization of $\mathbf{i}((\tilde q|_{F_t})_{an.})$. But $\dim_3((\tilde q|_{F_t})_{an.}) = 1$. Consequently, $(\tilde q|_{F_t})_{an.}$ has splitting pattern $(1,1,1,1,1,1,1,3)$.

For the splitting pattern $(1,1,1,1,1,3,1,1)$, consider the form $q := \langle\langle a_1,a_2,a_3,a_4\rangle\rangle \perp \langle x_1,x_2,x_3,x_4,x_5\rangle$ over the field $F := k(a_1,a_2,a_3,a_4,x_1,x_2,x_3,x_4,x_5)$. Clearly, $\dim(q) = 21$. On the other hand, other the field $E = F(\sqrt{-x_5})$, $\dim((q|_E)_{an.}) = 19$, and $\mathbf{i}((q|_E)_{an.}) = (1,1,1,1,3,1,1)$. So, $(1,1,1,1,1,3,1,1)$ is a specialization of $\mathbf{i}(q)$. But for arbitrary field extension K/F, $i_W(q|_K) > 5 \Leftrightarrow i_W(q|_K) > 7$. Hence, q has the splitting pattern $(1,1,1,1,1,3,1,1)$.

For the splitting pattern $(1,1,1,1,5,1)$, consider the form $\tilde q := \langle\langle a_1,a_2,b_1,b_2\rangle\rangle \perp -\langle\langle a_1,a_2,c_1,c_2\rangle\rangle \perp \langle x_1,x_2,x_3\rangle$ over the field $F := k(a_1,a_2,b_1,b_2,c_1,c_2,x_1,x_2,x_3)$. For some t, $\dim((\tilde q|_{F_t})_{an.}) = 21$ (since it is so over the field $F(\sqrt{b_1x_1},\sqrt{b_2x_2},\sqrt{-c_1x_3})$). On the other hand, over the field $E = F(\sqrt{b_1c_1})$, $\dim((\tilde q|_E)_{an.}) = 19$, and $\mathbf{i}((\tilde q|_E)_{an.}) = (1,1,1,5,1)$. Put $q := (\tilde q|_{F_t})_{an.}$. Then $(1,1,1,1,5,1)$ is a specialization of $\mathbf{i}(q)$. Since for arbitrary field extension K/F_t, $i_W(q|_K) > 4 \Leftrightarrow i_W(q|_K) > 8$, we get: $\mathbf{i}(q) = (1,1,1,1,5,1)$.

For the splitting pattern $(1,1,1,3,1,3)$, consider the form $\tilde q := \langle\langle a_1,a_2,a_3,a_4\rangle\rangle \perp \lambda\cdot\langle\langle b_1,b_2,b_3\rangle\rangle \perp \langle\mu\rangle$ over the field $F := k(a_1,a_2,a_3,a_4,b_1,b_2,b_3,\lambda,\mu)$. Then there exists t with $\dim((\tilde q|_{F_t})_{an.}) = 21$ (since it is so over the field $F(\sqrt{-\lambda},\sqrt{a_1b_1})$). On the other hand, we know that for some s (evidently, equal to $t+1$), $\dim(\tilde q|_{F_s})_{an.}) = 19$, and $\mathbf{i}(\tilde q|_{F_s})_{an.}) = (1,1,3,1,3)$. Consequently, for $q := (\tilde q|_{F_t})_{an.}$ we get $\mathbf{i}(q) = (1,1,1,3,1,3)$.

For the splitting pattern $(1,3,1,1,1,1,1,1)$, consider the form $\tilde q := \langle\langle a_1,a_2,a_3,a_4,a_5\rangle\rangle \perp \langle x_1,\ldots,x_{13}\rangle$ over the field $F = k(a_1,a_2,a_3,a_4,a_5,x_1,\ldots,x_{13})$. Then, for some t, $\dim((\tilde q|_{F_t})_{an.}) = 21$, since it is so over the field E obtained by the adjoining to F the square roots of $a_1x_1, a_2x_2, a_3x_3, a_4x_4, a_5x_5, -a_1a_2x_6, -a_1a_3x_7, -a_1a_4x_8, -a_1a_5x_9, -a_2a_3x_{10}$, $-a_2a_4x_{11}$ and $-a_2a_5x_{12}$. If we add also the square root of $-a_3a_4x_{13}$, then the dimension of the anisotropic part of $\tilde{q}$ will be 19. And finally, over the field $K = F(\sqrt{a_1})$, $\dim((\tilde{q}|_K)_{an.}) = 13$, and $(\tilde{q}|_K)_{an.}$ is generic, so $\mathbf{i}((\tilde{q}|_K)_{an.}) = (1,1,1,1,1,1)$. Then for $q := (\tilde{q}|_{F_t})_{an.}$, $(1,3,1,1,1,1,1,1)$ is a specialization of $\mathbf{i}(q)$. Since for arbitarary field extension E/F_t, $i_W(q|_E) > 1 \Leftrightarrow i_W(q|_E) > 3$, we get: $\mathbf{i}(q) = (1,3,1,1,1,1,1,1)$.

For the splitting pattern $(1,3,1,1,1,3)$, consider the form $\tilde{q} := \langle\langle a_1,a_2,a_3,a_4,a_5\rangle\rangle \perp \langle\langle b,c_1,c_2\rangle\rangle \perp -\langle\langle b,d_1,d_2\rangle\rangle \perp \langle e\rangle$ over the field $F = k(\sqrt{-1})(a_1,a_2,a_3,a_4,a_5,b,c_1,c_2,d_1,d_2,e)$. For some t, $\dim((\tilde{q}|_{F_t})_{an.}) = 21$, since it is so over the field $E = F(\sqrt{a_1 b}, \sqrt{a_2 c_1}, \sqrt{a_3 c_2}, \sqrt{a_4 d_1}, \sqrt{a_5 d_2})$. And $\dim((\tilde{q}|_{E\sqrt{a_1a_2a_3e}})_{an.}) = 19$. On the other hand, $\dim((\tilde{q}|_{F\sqrt{a_1}})_{an.}) = 13$, and $\mathbf{i}((\tilde{q}|_{F\sqrt{a_1}})_{an.}) = (1,1,1,3)$. So, for $q := (\tilde{q}|_{F_t})_{an.}$, $(1,3,1,1,1,3)$ is a specialization of $\mathbf{i}(q)$. Since for arbitarary field extension E/F_t, $i_W(q|_E) > 1 \Leftrightarrow i_W(q|_E) > 3$ and $i_W(q|_E) > 7 \Leftrightarrow i_W(q|_E) > 9$, we have: $\mathbf{i}(q) = (1,3,1,1,1,3)$.

For the splitting pattern $(1,3,1,3,1,1)$, consider the form $\tilde{q} := \langle\langle a_1,a_2,a_3,a_4,a_5\rangle\rangle \perp -\langle\langle b_1,b_2\rangle\rangle \cdot \langle 1,-c_1,-c_2\rangle \perp \langle d\rangle$ over the field $F = k(a_1,a_2,a_3,a_4,a_5,b_1,b_2,c_1,c_2,d)$. For some t, $\dim((\tilde{q}|_{F_t})_{an.}) = 21$, since it is so over the field $E = F(\sqrt{a_1b_1}, \sqrt{a_2b_2}, \sqrt{a_3c_1}, \sqrt{a_4c_2})$. At the same time, $\dim((\tilde{q}|_{E(\sqrt{a_5d})})_{an.}) = 19$. On the other hand, $\dim((\tilde{q}|_{F(\sqrt{a_1})})_{an.}) = 13$, and $\mathbf{i}((\tilde{q}|_{F(\sqrt{a_1})})_{an.}) = (1,3,1,1)$. So, for $q := (\tilde{q}|_{F_t})_{an.}$, $(1,3,1,3,1,1)$ is a specialization of $\mathbf{i}(q)$. Since for arbitarary field extension E/F_t, $i_W(q|_E) > 1 \Leftrightarrow i_W(q|_E) > 3$, and $i_W(q|_E) > 5 \Leftrightarrow i_W(q|_E) > 7$, we have: $\mathbf{i}(q) = (1,3,1,3,1,1)$.

For the splitting pattern $(1,3,5,1)$ take $q = \langle\langle a_1,a_2\rangle\rangle \cdot \langle b_1,b_2,b_3,b_4,b_5\rangle \perp \langle -b_1b_2b_3b_4b_5\rangle$ over the field $F = k(a_1,a_2,b_1,b_2,b_3,b_4,b_5)$. Then, on the one hand, q has codimension 1 subform $p' = \langle\langle a_1,a_2\rangle\rangle \cdot \langle b_1,b_2,b_3,b_4,b_5\rangle$, so, $\mathbf{i}(p') = (4,4,2)$, and hence, $\mathbf{i}(q)$ is a specialization of $(1,3,1,3,1,1)$. On the other hand, q is itself a subform of codimension 3 in the form $p'' = \langle\langle a_1,a_2\rangle\rangle \cdot \langle b_1,b_2,b_3,b_4,b_5,-b_1b_2b_3b_4b_5\rangle$ from $I^4(F)$. So, $\mathbf{i}(p'') = (4,8)$, and hence, $\mathbf{i}(q)$ is a specialization of $(1,1,1,1,5,1)$. Consequently, $\mathbf{i}(q)$ is a specialization of $(1,3,5,1)$. Since q is anisotropic, $\mathbf{i}(q) = (1,3,5,1)$ (it is the only specialization possible - check the list).

The following table contains the list of possible splitting patterns we obtained. We should stress that the description of the respected classes of forms is only hypothetical.

$\dim(q)$	splitting pattern	**hypothetical** description
19	(3,5,1)	$\dim_4(q) = 3, \dim_5(q) = 13 \Leftrightarrow$ excellent
	(3,1,3,1,1)	$\dim_4(q) = 5$ and $\begin{cases} \text{either } \dim_5(q) = 13, \\ \text{or } q \perp \langle \det_{\pm}(q) \rangle \text{ is divisible} \\ \quad \text{by a 2-fold Pfister form} \end{cases}$
	(3,1,1,1,3)	$\dim_5(q) = 13$, $\dim_3(q) = 1$
	(3,1,1,1,1,1,1)	$\dim_5(q) = 13$, $\dim_4(q) > 5$, $\dim_3(q) > 1$
	(1,1,3,1,3)	$\dim_3(q) = 1$, $\omega_3(q)$ is a nonzero pure symbol
	(1,1,1,5,1)	$\dim_4(q) = 3, \dim_5(q) > 13$
	(1,1,1,1,3,1,1)	$\dim_4(q) = 5, \dim_5(q) > 13$ and $q \perp \langle \det_{\pm}(q) \rangle$ is not divisible by a two-fold Pfister form
	(1,1,1,1,1,1,3)	$\dim_3(q) = 1$, $\omega_3(q)$ is not a pure symbol, $\dim_5(q) > 13$
	(1,1,1,1,1,1,1,1,1)	$\dim_3(q) > 1, \dim_4(q) > 5, \dim_5(q) > 13$
21	(5,3,1,1)	$\dim_5(q) = 11, \dim_4(q) = 5$
	(5,1,1,3)	$\dim_5(q) = 11, \dim_3(q) = 1$
	(5,1,1,1,1,1)	$\dim_5(q) = 11, \dim_4(q) > 5, \dim_3(q) > 1$
	(1,3,5,1)	$\dim_4(q) = 3$, and $(q \perp r_4(q))\vert_{k(r_4(q))}$ is hyperbolic
	(1,3,1,3,1,1)	$\dim_4(q) = 5$ and $\begin{cases} \text{either } \dim_5(q) = 13, \\ \text{or } \dim_5(q) > 11, (q \perp \langle \det_{\pm}(q) \rangle)_{an.} \\ \quad \text{is divisible by a 2-fold Pfister form} \end{cases}$
	(1,3,1,1,1,3)	$\dim_5(q) = 13, \dim_3(q) = 1$
	(1,3,1,1,1,1,1,1)	$\dim_5(q) = 13, \dim_4(q) > 5, \dim_3(q) > 1$
	(1,1,1,3,1,3)	$\dim_3(q) = 1$ and $\omega_3(q)$ is a nonzero pure symbol
	(1,1,1,1,5,1)	$\dim_4(q) = 3$, and $(q \perp r_4(q))\vert_{k(r_4(q))}$ is not hyperbolic
	(1,1,1,1,1,3,1,1)	$\dim_4(q) = 5, \dim_5(q) > 13$, $(q \perp \langle \det_{\pm}(q) \rangle)_{an.}$ is not divisible by a 2-fold Pfister form
	(1,1,1,1,1,1,1,3)	$\dim_3(q) = 1$, $\omega_3(q)$ is not a pure symbol, $\dim_5(q) > 13$
	(1,1,1,1,1,1,1,1,1,1)	$\dim_3(q) > 1, \dim_4(q) > 5, \dim_5(q) > 13$

Splitting patterns of even-dimensional forms

I should mention, that the cases of forms of dimension 2,4,6,8 and 10 were classified by D.Hoffmann (see [3]). We still included these cases below.

$\underline{\dim(p) = 2}$**:** $\mathbf{i}(p) = (1)$.

$\underline{\dim(p) = 4}$**:** Either $\mathbf{i}(p) = (2)$, and p is a 2-fold Pfister form (up to scalar), or $\mathbf{i}(p) = (1, 1)$ - p is any other form, for example, the generic one.

dim$(p) = 6$**:** By Theorem 7.2, $i_1(p) \leqslant 2$. If $i_1(p) = 2$, then $\mathbf{i}(p) = (2, 1)$, and p is a Pfister neighbour. If $i_1(p) = 1$, then either $\mathbf{i}(p) = (1, 2)$, which corresponds to the case of Albert forms, or $\mathbf{i}(p) = (1, 1, 1)$, which happens, if $p \notin I^2(k)$, and p is not a Pfister neighbour. The generic form $\langle x_1, \ldots, x_6\rangle$ over the field $k(x_1, \ldots, x_6)$ provides an example.

dim$(p) = 8$**:** Clearly, $i_1(p) \leqslant 4$. If $i_1(p) = 4$, then p is proportional to a 3-fold Pfister form. By Theorem 7.7, Theorem 7.8 and Theorem 4.20, $i_1(p) \neq 3$. If $i_1(p) = 2$, then $\mathbf{i}(p) = (2, 2)$, again, by Theorem 7.7, Theorem 7.8 and Theorem 4.20. It is well known, that in this case, p is proportional to a difference of a 3-fold Pfister form and a 2-fold Pfister form, having exactly one common slot. Finally, let $i_1(p) = 1$. Then all the cases: $(1, 2, 1), (1, 1, 2)$, and $(1, 1, 1, 1)$ are realized by the appropriate forms. If $\mathbf{i}(p) = (1, 2, 1)$, then p is proportional to a difference of a 3-fold Pfister form and a 1-fold Pfister form, having no common slots. The case $(1, 1, 2)$ corresponds to the form from $I^2(k)$, such that $\omega_2(p) \in \mathrm{K}_2^M(k)/2$ is not a pure symbol (follows from the Merkurjev's index-reduction formula). And finally, all other forms have the splitting pattern $(1, 1, 1, 1)$. The generic form provides an example.

dim$(p) = 10$**:** By Theorem 7.2, $i_1(p) \leqslant 2$. For $i_1(p) = 2$, all the splitting patterns: $(2, 2, 1), (2, 1, 2)$, and $(2, 1, 1, 1)$ are realized by Theorem 7.3. Let us describe the respective classes of forms. Since $i_1(p) = 2$, by the result of O.Izhboldin ([6, proof of Conjecture 0.10]), either p is divisible by some binary form $\langle\langle a\rangle\rangle$, or p is a Pfister neighbour. If $\mathbf{i}(p) = (2, 2, 1)$, then p is clearly divisible by $\langle\langle \det_{\pm}(p)\rangle\rangle$ (by Theorem 7.1). And vise-versa, if p is divisible by $\langle\langle a\rangle\rangle$ then $i_s(p)$ are divisible by 2, for all $s < h(p)$. Since $i_1(p) \neq 4$, $\mathbf{i}(p)$ must be $(2, 2, 1)$. Consequently, the cases $(2, 1, 2)$ and $(2, 1, 1, 1)$ correspond to Pfister neighbours. In the first case, $p \in I^2(k)$. In the second, $p \notin I^2(k)$, and p is not divisible by $\langle\langle \det_{\pm}(p)\rangle\rangle$, or, which is equivalent, $\dim_3(p) > 2$. And vise-versa, if p is a Pfister neighbour, $p \in I^2(k)$, then $\mathbf{i}(p)$ is a specialization of $(2, 1, 2)$, and there are no nontrivial specializations at our disposal. Similarly, if p is a Pfister neighbour, $p \notin I^2(k)$, and $\dim_3(p) > 2$, then $\mathbf{i}(p)$ is not equal to $(2, 2, 1)$, or $(2, 1, 2)$ but is a specialization of $(2, 1, 1, 1)$. So, $\mathbf{i}(p) = (2, 1, 1, 1)$.

Let now $i_1(p) = 1$. The case $(1, 4)$ is not possible by the result of A.Pfister ([21, Satz 14 and Zusatz]). The other cases: $(1, 2, 2), (1, 1, 2, 1), (1, 1, 1, 2)$, and $(1, 1, 1, 1, 1)$ are all realized by the appropriate forms.

It is well known that the case $(1, 2, 2)$ corresponds to the difference of a 3-fold Pfister form and a 2-fold Pfister form having no common slots.

Let $\mathbf{i}(p) = (1, 1, 2, 1)$. Then there exists $c \in k^*$ such that $p \perp c \cdot$

$\langle\langle \det_\pm(p)\rangle\rangle \in I^3(k)$. Also, clearly, p is not divisible by any binary form. Conversely, let $\dim_3(p) = 2$, and p is not divisible by a binary form $\langle\langle \det_\pm(p)\rangle\rangle$. Then $\mathbf{i}(p)$ is a specialization of $(1,1,2,1)$, but not $(2,2,1)$, and $\det_\pm(p) \neq 1$, so $i_{h(p)}(p) = 1$. Hence, $\mathbf{i}(p) = (1,1,2,1)$. The form $p = \langle\langle a_1,a_2,a_3\rangle\rangle \perp \langle b_1,b_2\rangle$ over the field $k(a_1,a_2,a_3,b_1,b_2)$ provides an example.

If $\mathbf{i}(p) = (1,1,1,2)$, then $p \in I^2(k)$, and $\omega_2(p)$ is not a pure symbol (othewise, we get a splitting pattern $(1,2,2)$), and p is not a Pfister neighbour. Conversely, any form, satisfying these conditions has the splitting pattern $(1,1,1,2)$. Such forms clearly exist: take $p = \langle\langle a_1,a_2\rangle\rangle \perp \lambda \cdot \langle b_1,b_2,-b_1b_2,-c_1,-c_2,c_1c_2\rangle$ over the field $F = k(a_1,a_2,b_1,b_2,c_1,c_2,\lambda)$, then $p \in I^2$, and at the same time, over the fields: $E_1 = F(\sqrt{b_1c_1})$, $E_2 = F(\sqrt{a_1})$, and $E_3 = F(\sqrt{b_1},\sqrt{c_1})$, the dimension of the anisotropic part of p is $8, 6$, and 4, respectively. So, $\mathbf{i}(p) = (1,1,1,2)$. Finally, all the other forms have the splitting pattern $(1,1,1,1,1)$. The generic form provides an example.

$\underline{\dim(p) = 12}$**:** By Theorem 7.2, $i_1(p) \leqslant 4$. If $i_1(p) = 4$, then p is a Pfister neighbour by the result of B.Kahn ([10, Theorem 2]). So, $\mathbf{i}(p) = (4,2)$ if and only if $p = \lambda\cdot(\langle\langle a_1,a_2,a_3,a_4\rangle\rangle \perp -\langle\langle a_1,a_2\rangle\rangle)_{an.}$ for some $\{a_1,a_2,a_3,a_4\} \neq 0 \in \mathrm{K}_4^M(k)/2$. And $\mathbf{i}(p) = (4,1,1)$ if and only if p is a Pfister neighbour and $p \notin I^2(k)$.

By Theorem 7.7, Theorem 7.8 and Theorem 4.20, $i_1(p) \neq 3$.

Let $i_1(p) = 2$. We have the following possibilities for $\mathbf{i}(p)$: $(2,4),(2,2,2)$, $(2,1,2,1),(2,1,1,2)$, and $(2,1,1,1,1)$. The case $(2,1,1,1,1)$ is not possible by Theorem 7.7, Theorem 7.8 and Theorem 4.20. The same applies to the case $\mathbf{i}(p) = (2,1,1,2)$, since the motive of a quadric with the splitting pattern $(1,2)$ (Albert one) is indecomposable (by Theorem 7.9), and so, the Tate-motives from the shell number 4 of $M(P)$ are connected to ones from the shell number 3.

Consider the case $\mathbf{i}(p) = (2,1,2,1)$. Then $\mathbf{i}(p|_{k\sqrt{\det_\pm(p)}})$ must be a specialization of $(2,4)$. Then for some $c \in k^*$, $p \perp c\cdot\langle\langle \det_\pm(p)\rangle\rangle$ belongs to $I^3(k)$. Really, consider $p' = p \perp \langle\langle \det_\pm(p)\rangle\rangle$. Then $p' \in I^2(k)$, and $\omega_2(p')|_{k\sqrt{\det_\pm(p)}} = 0$. So, by the result of A.Merkurjev (see [19]), there exists $c \in k^*$, such that $\omega_2(p') = \{c, \det_\pm(p)\}$. Then $p' \perp -\langle\langle c, \det_\pm(p)\rangle\rangle \in I^3(k)$. Hence, $p'' := p \perp c\cdot\langle\langle \det_\pm(p)\rangle\rangle \in I^3(k)$. So, $\mathbf{i}(p'')$ is a specialization of $(1,2,4)$. It must be either $(1,2,4)$, or $(2,4)$ (by Theorem 7.7, Theorem 7.8 and Theorem 4.20, there are no forms with the splitting pattern $(3,4)$). By Corollary 4.9(2), $p|_{k(p'')}$ is anisotopic. At the same time, $\langle\langle \det_\pm(p)\rangle\rangle|_{k(p'')}$ is, clearly, not hyperbolic. So, $\mathbf{i}(p|_{k(p'')}) = (2,1,2,1)$ (there are no other specializations possible with $i_{h(p)} = 1$). This means, that by changing the field, we can assume that p'' is isotropic and for $r := (p'')_{an.}$, $\mathbf{i}(r) = (2,4)$ (while still having $\mathbf{i}(p) = (2,1,2,1)$). But now p and r have common subform of codimension 1, and since $i_1(p) > 1$, $i_1(r) > 1$, we get that $p|_{k(r)}$ and $r|_{k(p)}$ are isotropic. By Corollary 3.9, $M(P)$ and $M(R)$ contain isomorphic direct summands N and L with $a(N) = 0$. From Theorem 7.9 we know that $M(R) = L \oplus L(1)[2]$. Hence, $L|_{\overline{k}} = \mathbb{Z} \oplus \mathbb{Z}(2)[4] \oplus \mathbb{Z}(4)[6] \oplus \mathbb{Z}(5)[10] \oplus \mathbb{Z}(7)[14] \oplus \mathbb{Z}(9)[18]$. But the Tate-motive $\mathbb{Z}(2)[4]$ belongs to the second shell of $M(P)$, so it is not contained in $N|_{\overline{k}}$ by Theorem 7.7 - a contradiction. So, the case $(2,1,2,1)$ is not possible.

The remaining cases $(2,4)$ and $(2,2,2)$ are possible. By the result of A.Pfister, $\mathbf{i}(p) = (2,4)$ if and only if $p = (\langle\langle a, b_1, b_2\rangle\rangle \perp -\langle\langle a, c_1, c_2\rangle\rangle)_{an.}$, where $\{a, b_1, b_2\}$ and $\{a, c_1, c_2\}$ have exactly one common slot.

The forms with the splitting pattern $(2,2,2)$ are not classified at the moment. Although, hypothetically, p must have form $\langle\langle a\rangle\rangle \cdot \langle b_1, \ldots, b_6\rangle$, where $\{a, -b_1 \cdot \ldots \cdot b_6\} \neq 0$ and p is not a Pfister neighbour (the last two conditions are clearly nesessary, so, the question is about the divisibility by a binary form). Clearly, the specified forms have the splitting pattern $(2,2,2)$.

Let $\mathbf{i}(p) = (1,2,2,1)$. Then, by Theorem 7.1 and Theorem 7.5, $p|_{k\sqrt{\det_{\pm}(p)}}$ must be hyperbolic. But then $i_1(p)$ must be divisible by 2 - contradiction. So, such splitting pattern does not exist.

We will show that all the other possibilities: $(1,2,1,2)$, $(1,2,1,1,1)$, $(1,1,2,2)$, $(1,1,1,2,1)$, $(1,1,1,1,2)$, and $(1,1,1,1,1,1)$ are realized.

Let $\mathbf{i}(p) = (1,2,1,2)$. Then, as we saw above, $p_1 := (p|_{k(P)})_{an.}$ is a Pfister neighbour. So over the field $k(P)$ there exists 6-dimensional form $\tilde{r}$ with the trivial discriminant such that $p_1 \perp \tilde{r}$ is proportional to an anisotropic 4-fold Pfister form. Then $\tilde{r} \in W_{nr}(k(P)/k)$, by the standard arguments (see, for example, [10]). By the result of B.Kahn ([10, Theorem 2]), $\tilde{r}$ is defined over k, so there exists 6-dimensional form r over k, such that $r|_{k(P)} = \tilde{r}$. Then $p \perp r$ must be in $I^4(k)$. Really, if it would not be, then $p_1 \perp \tilde{r}$ would not be in $I^4(k(P))$ either (since $\dim(p) > 8$). It is also clear that $\det_{\pm}(r) = 1$. So, up to a scalar, p differs from some Pfister form by an anisotropic form of dimension 6 with trivial discriminant (we use here the fact that $\dim \neq 18$ for anisotropic forms from I^4 - see [4]). Conversely, if p is such a form, then, by Theorem 7.6, $\mathbf{i}(p)$ is a specialization of $(1,2,1,2)$. Since $\omega_2(p)$ is not a pure symbol, we have: $i_{h(p)-1}(p)=1$, and $\mathbf{i}(p)$ must be $(1,2,1,2)$.

Let us show that such forms really exist. Consider the form $\tilde{p}=\langle\langle a_1,a_2,a_3,a_4\rangle\rangle\perp\lambda\cdot\langle -b_1,-b_2,b_1b_2,c_1,c_2,-c_1c_2\rangle$ over the field $F=k(a_1,a_2,a_3,a_4,b_1,b_2,c_1,c_2,\lambda)$. Let $F=F_0\subset\ldots\subset F_{h(\tilde{p})}$ be the generalized splitting tower for $\tilde{p}$. Then, for some t, the form $p:=(\tilde{p}|_{F_t})_{an.}$ has dimension 12. Really, it follows from the fact that $\dim((\tilde{p}|_E)_{an.})=12$ for $E=F(\sqrt{b_1},\sqrt{\lambda},\sqrt{a_1c_1},\sqrt{a_2c_2})$. Then $\mathbf{i}(p)=(1,2,1,2)$, as we saw above.

Let $\mathbf{i}(p)=(1,2,1,1,1)$. This is, actually, the complicated variant of the previous case (the difference is that we cannot use [10, Theorem 2] here, but, hopefully, we now have the results of O.Izhboldin and A.Laghribi, which permit to handle the problem). Let us do it in a separate Lemma.

Lemma 7.12 *Let p be anisotropic form of dimension 12. Then the following conditions are equivalent:*

(1) $\mathbf{i}(p)=(1,2,1,1,1)$;

(2) $p=(r\perp d\cdot\langle\langle\gamma\rangle\rangle)_{an.}$, *where r is 6-dimensional form with the splitting pattern* $(1,1,1)$, $d\in k^*$, *and* $\gamma\in \mathrm{K}_4^M(k)/2$ *is a nonzero pure symbol.*

Proof: We know that $p_1:=(p|_{k(P)})_{an.}$ is a Pfister neighbour. So over the field $k(P)$ there exists 6-dimensional form r'' such that $p_1\perp r''$ is proportional to an anisotropic 4-fold Pfister form $\langle\langle\alpha''\rangle\rangle$, where $\alpha''\in\mathrm{K}_4^M(k(P))/2$. Then $r''\in W_{nr}(k(P)/k)$, and $\alpha''\in\mathrm{H}^4_{nr}(k(P)/k,\mathbf{Z}/2)$. By the result of O.Izhboldin ([6, Theorem 0.5, Theorem 0.6]), there exists $\alpha\in\mathrm{K}_4^M(k)/2=\mathrm{H}^4_{et}(k,\mathbf{Z}/2)$, such that $\alpha|_{k(P)}=\alpha''$. Under the projection $\pi: I^4(k)\to\mathrm{K}_4^M(k)/2$, α can be lifted to some form $q\in I^4(k)$. Let $k=k_0\subset\ldots\subset k_{h(q)}$ be generalized splitting tower for q. Then $q_{h(q)-1}:=(q|_{k_{h(q)-1}})_{an.}$ is proportional to some 4-fold Pfister form (by the result of B.Kahn, M.Rost, and R.J.Sujatha - see [12]), and, consequently, $\alpha|_{k_{h(q)-1}}$ is a nonzero pure symbol. Notice, that for any $1\leqslant s<h(q)$, $k_s=k_{s-1}(q_{s-1})$, where q_{s-1} is a form of dimension $\geqslant 24$. Denote $F:=k_{h(q)-1}$. Then $\mathbf{i}(p|_F)=\mathbf{i}(p)$ (by [2], since $\dim(q_{s-1})>16$). At the same time, $(p|_{F(P)})_{an.}$ is a neighbour of the Pfister form $\langle\langle\alpha|_{F(P)}\rangle\rangle$. We know that $i_W(p|_{F(\langle\langle\alpha|_F\rangle\rangle)(P)})=3$. Hence, either $i_W(p|_{F(\langle\langle\alpha|_F\rangle\rangle)})=3$, or $p|_{F(\langle\langle\alpha|_F\rangle\rangle)}$ is anisotropic and $i_1(p|_{F(\langle\langle\alpha|_F\rangle\rangle)})=3$ (we remind that $(p|_{F(P)})_{an.}$ is a neighbour of $\langle\langle\alpha|_{F(P)}\rangle\rangle$). The last case is impossible, since $i_1\neq 3$ for 12-dimensional forms. So, $i_W(p|_{F(\langle\langle\alpha|_F\rangle\rangle)})=3$. In particular, for any $\{a\}\in\mathrm{K}_1^M(F)/2$ dividing $\alpha|_F$, $i_W(p|_{F(\langle\langle a\rangle\rangle)})\geqslant 3$. Pick any such a. Then there exists

$c \in F^*$ such that $i_W(p|_F \perp c\cdot\langle\langle a\rangle\rangle) \geqslant 2$, and so, $i_W(p|_F \perp c\cdot\langle\langle\alpha|_F\rangle\rangle) \geqslant 2$, and for $\tilde{r} := (p|_F \perp c\cdot\langle\langle\alpha|_F\rangle\rangle)_{an.}$, $\dim(\tilde{r}) \leqslant 24$. We know that for some $\lambda \in F(P)^*$, $\dim((p|_{F(P)} \perp \lambda\cdot\langle\langle\alpha|_{F(P)}\rangle\rangle)_{an.}) = 6$. Then $\dim((p|_{F(P)} \perp \lambda\cdot\langle\langle\alpha|_{F(P)}\rangle\rangle)_{an.} \perp -\tilde{r}|_{F(P)}) \leqslant 30$. But $(p|_{F(P)} \perp \lambda\cdot\langle\langle\alpha|_{F(P)}\rangle\rangle)_{an.} \perp -\tilde{r}|_{F(P)} \in I^5(F(P))$. So, $(p|_{F(P)} \perp \lambda\cdot\langle\langle\alpha|_{F(P)}\rangle\rangle)_{an.} \perp -\tilde{r}|_{F(P)}$ is hyperbolic. We have two possibilities: either $\dim(\tilde{r}) > 6$, or $\dim(\tilde{r}) = 6$. Suppose $\dim(\tilde{r}) > 6$. We know that $\dim((\tilde{r}|_{F(P)})_{an}) = 6$, and $\mathbf{i}((\tilde{r}|_{F(P)})_{an}) = (1,1,1)$. Let F_t be a field from the generalized splitting tower of $\tilde{r}$, such that $height((\tilde{r}_{F_t})_{an.}) = 4$ (in other words, $t = h(\tilde{r}) - 4$). Denote $r' := (\tilde{r}_{F_t})_{an.}$. Then $\dim(r') \geqslant 10$ (since if $\dim(r')$ would be 8, then $\dim((r'|_{F_t(P)})_{an.})$ would be 8 as well ($12 > 8$)). Then $\mathbf{i}(r') = (m,1,1,1)$, where $m > 1$. Consequently, by Theorem 7.7, Theorem 7.8 and Theorem 4.20, $\dim(r') - m$ is a power of 2. In particular, either $\dim(r') = 10$, or $\dim(r') \geqslant 26$. Since $\dim(\tilde{r}) \leqslant 24$, we have $\dim(r') = 10$. But then r' must be a neighbour of some 4-fold Pfister form $\langle\langle\beta\rangle\rangle$, as we saw above. And $\langle\langle\beta\rangle\rangle|_{F_t(P)}$ is hyperbolic, since $r'|_{F_t(P)}$ is isotropic. In particular, $p|_{F_t}$ must be a Pfister neighbour, and so, $\mathbf{i}(p|_{F_t})$ should be a specialization of $(4,1,1)$. But $\dim((p|_{F(P)})_{an.}) = 10$, and there is a regular place $F(P) \to F_t$ (since $\dim((\tilde{r}|_{F(P)})_{an.}) = 6 < 10$). So, $\dim((p|_{F_t(P)})_{an.}) = 10$ - a contradiction (with Theorem 7.5). This implies: $\dim(\tilde{r}) = 6$. So, we have shown that $p|_F = (\tilde{r} \perp -c\cdot\langle\langle\alpha|_F\rangle\rangle)_{an.}$, where $\alpha|_F \in \mathrm{K}_4^M(F)/2$ is a nonzero pure symbol, and $\tilde{r}$ is a 6-dimensional form with the splitting pattern $(1,1,1)$. But then $\tilde{r} \in W_{nr}(F/k)$, and since F is obtained from k by adjoining the function fields of forms of dimension > 16, we get by the result of A.Laghribi ([17, Théorème principal]), that $\tilde{r}$ is defined over k by some form r. Clearly, $\mathbf{i}(r) = (1,1,1)$. Then $(p \perp -r)_F \in I^4(F)$. But since $\dim(q_{s-1}) > 8$, for all $1 \leqslant s < h(q)$, we must have $p \perp -r \in I^4(k)$, and $p = (r \perp -d\langle\langle\gamma\rangle\rangle)_{an.}$ for some $d \in k^*$ and nonzero pure symbol $\gamma \in \mathrm{K}_4^M(k)/2$ (we used here the fact that in $I^4(k)$ there are no anisotropic forms of dimension 18 - see [4]). Conversely, if p has such a form (with $\mathbf{i}(r) = (1,1,1)$), then $\mathbf{i}(p)$ must be a specialization of $(1,2,1,1,1)$. Since $\mathbf{i}((p|_{k(\langle\langle\gamma\rangle\rangle)})_{an.}) = (1,1,1)$, and $i_1(p) \neq 3$, we get: $\mathbf{i}(p) = (1,2,1,1,1)$. □

Let us show that such forms really exist. Consider the form $\tilde{p} = \langle\langle a_1,a_2,a_3,a_4\rangle\rangle \perp \langle b_1,b_2,b_3,b_4,b_5,b_6\rangle$ over the field $F = k(a_1,\ldots,a_4,b_1,\ldots,b_6)$. Let $F = F_0 \subset \ldots \subset F_{h(\tilde{p})}$ be the generalized splitting tower for $\tilde{p}$. Then, for some t, the form $p := (\tilde{p}|_{F_t})_{an.}$ has dimension 12. Really, it follows from the fact that $\dim((\tilde{p}|_E)_{an.}) = 12$, for

$E = F(\sqrt{b_5}, \sqrt{b_6}, \sqrt{a_1b_1}, \sqrt{a_2b_2}, \sqrt{a_3b_3}, \sqrt{a_4b_4})$. Then, by the evident part of Lemma 7.12, $\mathbf{i}(p) = (1,2,1,1,1)$.

The classification of the forms with the splitting pattern $(1,1,2,2)$ depends on the hypothetical classification of forms with the splitting pattern $(2,2,2)$ above, and so, is itself hypothetical. Let $\mathbf{i}(p) = (1,1,2,2)$. Then $p \in I^2(k)$ and, by the index reduction formula of A.Merkurjev (see [20]), $\omega_2(p) \in \mathrm{K}_2^M(k)/2$ is a nonzero pure symbol. Also, p is not divisible by any binary form $\langle\langle a\rangle\rangle$, since $i_1(p) = 1$. Hypothetically, the converse should be also true. That is, the anisotropic 12-dimensional form from $I^2(k)$, for which $\omega_2(p)$ is a pure symbol, and which is not divisible by a binary form, should have the splitting pattern $(1,1,2,2)$. It is evident, that for such form, $\mathbf{i}$ is either $(1,1,2,2)$, or $(2,2,2)$, but we do not know, if the nondivisibility by a binary form quarantees that $\mathbf{i}(p)$ is not $(2,2,2)$. Let us construct the example of the form p with $\mathbf{i}(p) = (1,1,2,2)$. Take $p := \langle\langle a_1,a_2,a_3\rangle\rangle \perp \lambda \cdot \langle\langle b_1,b_2\rangle\rangle$ over the field $F := k(a_1,a_2,a_3,b_1,b_2,\lambda)$. Then p is anisotropic, $p \in I^2(F)$, and $\omega_2(p) = \{b_1,b_2\} \neq 0 \in \mathrm{K}_2^M(F)/2$ is a pure symbol. So, $\mathbf{i}(p)$ is either $(1,1,2,2)$, or $(2,2,2)$. But if $E = F\sqrt{-\lambda}$, then $i_W(p|_E) = 1$ (by the result of R.Elman and T.Y.Lam ([1]), since $\{a_1,a_2,a_3\}|_E$ and $\{b_1,b_2\}|_E$ have no common slots). This shows that $\mathbf{i}(p) = (1,1,2,2)$.

Let $\mathbf{i}(p) = (1,1,1,2,1)$. Then for some $c \in k^*$, $p \perp c \cdot \langle\langle \det_\pm(p)\rangle\rangle \in I^3(k)$. Conversely, if $\dim_3(p) = 2$, then $\mathbf{i}(p)$ is a specialization of $(1,1,1,2,1)$, and $i_{h(p)}(p) = 1$. But $(1,1,1,2,1)$ itself is the only possible specialization satisfying this condition. As an example of such p we can take any codimension 2 subform of the form $(\langle\langle a_1,a_2,a_3\rangle\rangle \perp -\langle\langle b_1,b_2,b_3\rangle\rangle)_{an.}$ over the field $F = k(a_1,a_2,a_3,b_1,b_2,b_3)$.

Let $\mathbf{i}(p) = (1,1,1,1,2)$. Then $p \in I^2(k)$, $\omega_2(p) \in \mathrm{K}_2^M(k)/2$ is not a pure symbol (since $i_{h(p)-1}(p) = 1$), and $\dim_4(p) > 6$ (since, otherwise, $\mathbf{i}(p)$ would be a specialization of $(1,2,1,1,1)$). Conversely, all the forms satisfying these 3 conditions have splitting pattern $(1,1,1,1,2)$. Really, the first two condition give us: $i_{h(p)}(p) = 2$ and $i_{h(p)-1}(p) = 1$. So, $\mathbf{i}(p)$ is either $(1,1,1,1,2)$, or $(1,2,1,2)$. The last possibility is excluded since $\dim_4(p) > 6$. The form $\langle -a_1,-a_2,a_1a_2,d_1,d_2,-d_1d_2\rangle \perp \lambda\cdot\langle b_1,b_2,-b_1b_2,-c_1,-c_2,c_1c_2\rangle$ over the field $k(a_1,a_2,b_1,b_2,c_1,c_2,d_1,d_2,\lambda)$ provides an example (just observe that p is anisotopic, and for $F = k\sqrt{d_1}$, $\mathbf{i}((p|_F)_{an}) = (1,1,1,2)$ - see the corresponding case in dimension 10).

Finally, all the other forms have the splitting pattern $(1,1,1,1,1,1)$. They clearly can be described as: $p \notin I^2(k)$, $\dim_3(p) > 2$, $\dim_4(p) > 6$. The generic form provides an example.

Let us summarize our results.

$\dim(q)$	splitting pattern	description
2	(1)	-
4	(2)	$\dim_2(p) = 0$
	(1,1)	$\dim_2(p) > 0$
6	(2,1)	$\dim_3(p) = 2$
	(1,2)	$\dim_2(p) = 0$
	(1,1,1)	$\dim_2(p) > 0, \dim_3(p) > 2$
8	(4)	$\dim_3(p) = 0$
	(2,2)	$\dim_2(p) = 0$ and $\omega_2(p)$ is a nonzero pure symbol
	(1,2,1)	$\dim_3(p) = 2$
	(1,1,2)	$\dim_2(p) = 0$ and $\omega_2(p)$ is not a pure symbol
	(1,1,1,1)	$\dim_2(p) > 0, \dim_3(p) > 2$
10	(2,2,1)	p is divisible by a binary form $\langle\langle \det_\pm(p) \rangle\rangle$
	(2,1,2)	$\dim_4(p) = 6, \dim_2(p) = 0$
	(2,1,1,1)	$\dim_4(p) = 6$, $\dim_2(p) > 0, \dim_3(p) > 2$
	(1,2,2)	$\dim_2(p) = 0$, $\omega_2(p)$ is a nonzero pure symbol
	(1,1,2,1)	$\dim_3(p) = 2$, p is not divisible by $\langle\langle \det_\pm(p) \rangle\rangle$
	(1,1,1,2)	$\dim_2(p) = 0$, $\omega_2(p)$ is not a pure symbol, $\dim_4(p) > 6$
	(1,1,1,1,1)	$\dim_2(p) > 0, \dim_3(p) > 2, \dim_4(p) > 6$
12	(4,2)	$\dim_4(p) = 4, \dim_2(p) = 0$
	(4,1,1)	$\dim_4(p) = 4, \dim_2(p) > 0$
	(2,4)	$\dim_3(p) = 0$
	(2,2,2)	** $\dim_3(p) > 0, \dim_4(p) > 4$, p is divisible by a binary form
	(1,2,1,2)	$\dim_4(p) = 6, \dim_2(p) = 0$
	(1,2,1,1,1)	$\dim_4(p) = 6, \dim_2(p) > 0$
	(1,1,2,2)	** $\dim_2(p) = 0$, $\omega_2(p)$ is a pure symbol, and p is not divisible by a binary form
	(1,1,1,2,1)	$\dim_3(p) = 2$
	(1,1,1,1,2)	$\dim_2(p) = 0$, $\omega_2(p)$ is not a pure symbol, $\dim_4(p) > 6$
	(1,1,1,1,1,1)	$\dim_2(p) > 0, \dim_3(p) > 2, \dim_4(p) > 6$

** only hypothetically

Some conclusions

Let us list couple of observations concerning computations above.

Although, in arbitrary dimension, there is no even hypothetical description of the set of possible splitting patterns, there is a conjecture describing elementary pieces of such splitting patterns, that is, higher Witt indices.

Conjecture 7.13 (D.Hoffmann)[1]

Let q be anisotropic form. Then $i_1(q) - 1$ is the remainder of $\dim(q) - 1$ *under the division by some power of 2.*

Remarks: 1) Conjecture 7.13 claims, in particular, that higher Witt indices for odd-dimensional forms are always odd, and for even-dimensional forms are either even, or 1.

2) For each d, and each s, where $2^s < d$, there exists anisotropic form q of dimension d over some field F such that $i_1(q) - 1$ is exactly the remainder of $d - 1$ divided by 2^s. Really, let $d - 1 = 2^s \cdot n + r$, where $0 \leqslant r < 2^s$. Consider $F := k(a_1, \ldots, a_s, x_1, \ldots, x_{n+1})$, and let q be any $(2^s - r - 1)$-codimensional subform of $p := \langle\langle a_1, \ldots, a_s \rangle\rangle \cdot \langle x_1, \ldots, x_{n+1} \rangle$. By Lemma 5.2, $i_1(p)$ is divisible by 2^s, on the other hand, $i_W(p|_{F\sqrt{-x_1x_2}}) = 2^s$. So, $i_1(p) = 2^s$. By Corollary 4.9(3), $i_1(q) = r + 1$.

Our computations show:

Theorem 7.14 *Conjecture 7.13 is valid for all forms of dimension* $\leqslant 22$.

Proof: We just need to note that, by Corollary 4.9(3), if $\dim(p)$ is even, $i_1(p) > 1$, and q is any codimension 1 subform of p, then $i_1(q) = i_1(p) - 1$. Thus, if Conjecture 7.13 is valid for q, then it is valid for p. But in the case of odd dimensional forms we have complete classification of $\mathbf{i}(q)$ up to dimension 21. □

Also, looking on the tables above, it is not difficult to guess the description of forms with the "generic" splitting pattern $(1, 1, \ldots, 1)$.

Conjecture 7.15 *The following conditions are equivalent:*

(1) $\mathbf{i}(q) = (1, 1, \ldots, 1)$;

(2) $\dim_s(q) \geqslant 2^{s-1} - 1$, *for all* $2 \leqslant s \leqslant \log_2(\dim(q) - 2) + 1$

[1] After the article was originally submitted this conjecture was proven by N.Karpenko

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