Subtle Characteristic Classes

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Abstract

We construct new *subtle Stiefel–Whitney classes* of quadratic forms. These classes are much more informative than the ones introduced by Milnor. In particular, they see all the powers of the fundamental ideal of the Witt ring, contain the Arason invariant and it's higher analogues. Moreover, the new classes allow to treat the J-invariant of quadrics. This invariant, introduced in [12], has been so far completely isolated from characteristic classes. In addition, our classes allow to describe explicitly the structure of some motives associated with quadratic forms.

1 Introduction

This work grew out of attempts to develop a sufficiently generic motivic homotopic approach to the classification of algebro-geometric structures and to apply it to quadratic forms.

Here we restrict our considerations to the classification of torsors of algebraic groups over a field. Usually such torsors are classified by the étale cohomology, whereas the Zariski topology is quite inadequate, especially over a field. The situation changes if one consider the large Zariski site instead of the small one, since then torsors are split by appropriate schemes. Here some new fenomenon appears. Namely, the sheaf represented by a torsor does not surject to the base. Therefore, when passing to sheaves we get a torsor not over the base, but over it's part. It is called the support of the corresponding sheaf. Consequently, one obtains the natural arrow from this support to the respective classificator. These arrows are very interesting invariants of torsors which should play an important role in the motivic homotopic approach to the classification. Actually we get the whole family of classificators, supports, and arrows indexed by appropriate topologies. We mainly consider the Nisnevich topology since in this case we have a well-developed motivic homotopy theory, namely the theory of F. Morel and V. Voevodsky. Our results demonstrate that this way we get sufficiently rich invariants.

As soon as we have the above arrow to the classificator we can apply any cohomology functor to it getting a homomorphism from the cohomology of the classificator to the cohomology of the support. This invariant, introduced in §2 for arbitrary groups, is the main object we study in the current paper. The main novelty of our approach is that we consider the Nisnevich classificator where traditionally an étale one was studied, and that we (respectively) substitute the "point" by the "support of the torsor". This permits to produce invariants which are much more informative.

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In §3 we focus on the orthogonal case. The torsors here are in one-to-one correspondence with quadratic forms. As the cohomology functor we take the motivic $\mathbb{Z}/2$-cohomology and compute it for the Nisnevich classificator $BO(n)$. Theorem 3.1.1 says that the result is the polynomial ring over motivic cohomology of the base field with canonical generators $u_1, \dots, u_n$. These are our *subtle* Stiefel–Whitney classes. The description we get is much simpler than that for the motivic cohomology of the étale classificator (this cohomology is computed by N. Yagita in [21]). The classes u_i's by means of pullback, associated with the canonical arrow to the classificator $BO(n)$, give the subtle Stiefel–Whitney classes of any individual quadratic form q. They take values in the motivic $\mathbb{Z}/2$-cohomology of the Chech simplicial scheme related to the torsor X_q.

Originally, in the context of quadratic forms, the Stiefel–Whitney classes w_i were introduced algebraically by J. Milnor in [6] (the paper where the Milnor's K-theory was also introduced). They allow to identify $K_2^M(k)/2$ with the second component of the graded Witt ring. The classes w_i were interpreted as pullbacks of some étale cohomology classes of the corresponding classificator by H. Esnault, B. Kahn, and E. Viehweg in [2] (see also J. F. Jardine [3]). The drawback of the classes of J. Milnor though is that they are trivial on $I^3(k)$ (provided (-1) is a square in k), which makes their use for the classification of quadratic forms quite limited. In contrast, our classes are non-trivial on any power I^n of the fundamental ideal. Moreover, they distinguish if the form is in I^n, or not (see Theorem 3.2.27), and so distinguish the triviality of torsors (see Cor. 3.2.32). We establish explicit relations between our *subtle* classes and classical ones as well as with the Chern classes. It appears that our classes are "approximately" equal to square roots of the Chern classes, while are obtained from w_i's by "dividing" the latter by some powers of $\tau \in \mathrm{H}^{0,1}_{\mathcal{M}}(\operatorname{Spec} k, \mathbb{Z}/2)$. In particular, the topological realization identifies u_i and w_i with the topological Stiefel–Whitney class. Since multiplication by τ is usually a very non-injective map in the motivic cohomology, we see why Milnor's classes loose so much information.

We compute the action of the Steenrod operations on our classes which appears to be as simple as in the topological case provided (-1) is a square in k (see Prop. 3.1.9). We also describe the behavior of the subtle classes under addition of quadratic forms (see Prop. 3.1.10). After that it becomes possible to compute these classes effectively. In particular, we describe them completely in the case of a Pfister form (see Theorem 3.2.26). Here we are exploiting the fact that the motivic cohomology of the respective Chech simplicial scheme is known in this case. This computation enables us to show that the subtle Stiefel–Whitney classes do remember the Arason invariant and higher invariants identifying I^r/I^{r+1} with $K_r^M(k)/2$. This is done in Theorem 3.2.34.

Finally, we use our classes to describe the motive of the torsor X_q and the motive of the highest quadratic Grassmannian as an explicit extension of twisted motives of the simplicial Chech scheme, related to X_q (see Theorem 3.2.2 and Theorem 3.2.13). It appears that these motives have poly-binary structure (are tensor products of motives each of which over algebraic closure decomposes as a sum of just two Tate-motives). This allow to relate subtle Stiefel–Whitney classes with the J-invariant of q (see 3.2.22 and 3.2.23). Thus, our classes connect Stiefel–Whitney classes with the J-invariant. These areas were previously completely isolated from each other.

Of course, there are similar subtle versions of other characteristic classes and these would be a cornestone of the classification of respective structures. As for quadratic forms, the subtle Stiefel–Whitney classes should serve as a zero-order step in the homotopic classification of quadratic forms in the same way as the J-invariant is the zero-order, but the most important step of the EDI-invariant (see [14]). Thus, the task now is to build those next layers on top of what we have. Probably, it will involve some sort of higher subtle characteristic classes.

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2 Torsors and Their Classificators

Let k be a field, $S = \operatorname{Spec} k$, and G be a smooth linear algebraic group over S. We are going to consecutively describe the problem of classification of G-torsors on the pre-homotopic level, that is, on the level of spaces, then on the level of simplicial homotopic category $\mathcal{H}_s = \mathcal{H}_s(S)$, and finally on the level of homotopic category $\mathcal{H} = \mathcal{H}(S)$.

2.1 Algebraic torsors

The use of the notion of G-torsor in various situations will require a certain degree of precision from us. Let us achieve it through the following chain of definitions.

2.1.1 Definition. *An "algebraic G-torsor" is a non-empty S-scheme P together with an action $\alpha : G \times_S P \to P$ such that the map $(\alpha, \pi_P) : G \times P \to P \times P$ is an isomorphism. Here π_P is the projection $G \times P \to P$.*

2.1.2. Example. In our main example $char(k) \neq 2$, q is a non-degenerate quadratic form over k, and G is the orthogonal group of q. With each non-degenerate quadratic form p with $\operatorname{rk} p = \operatorname{rk} q$ one can associate the algebraic G-torsor X represented by the functor $\operatorname{Iso}(p \to q)$. It is known that each algebraic G-torsor can be obtained this way and that p can be recovered from X and G-action.

2.2 Torsor classificators on pre-homotopic level

Let T be a site represented by the category Sm_S with the Nisnevich topology and Shv /T be the category of sheaves of sets on T. Our category of spaces Spc is defined as the category of simplicial objects in Shv /T.

We identify the group G with the respective representable sheaf of groups, that is, with a group in Shv /T, and also with the constant group in Spc. In addition, if X is a G-scheme, then the corresponding sheaf in Shv /T and the space Spc inherit the action of G. At the same time, it is not true that, this way, an algebraic torsor always produces a torsor. To clarify this we recall the definition of torsor in a Grothendieck topos E.

2.2.1 Definition. *An object $P \in \mathrm{E}$ together with an action $\alpha : G \times P \to P$ is called a "formal G-torsor" if $(\alpha, \pi_P) : G \times P \to P \times P$ is an isomorphism, where π_P is the projection. A formal torsor (P, α) is called a "torsor" if the unique map $P \to 1$ to the final object is an epimorphism.*

Clearly, each algebraic G-torsor X induces formal G-torsors (also denoted as X) in Shv /T and Spc. Setting aside the question of which formal torsors can be obtained this way, we mention only that the replacement of an algebraic torsor by a formal one looses no information in the sense that it does not glue objects (is a conservative functor).

2.2.2. Example. We continue with the example 2.1.2. Assume that p is not isomorphic to q over k. That means that $X(k) = \emptyset$. Moreover, in the Nisnevich topology, any covering $\{U_i\}$ of S has a section, and so $\prod \Gamma(U_i, X) = \emptyset$. Thus, the arrow $X \to S$ is not an epimorphism and X is not a torsor.

2.2.3. Let $X \in \mathrm{E}$. Since morphisms in toposes posses a mono-epi decomposition, one has a well-defined notion of the image of any morphism and we can set

$$\operatorname{Supp} X = \operatorname{Im} p_X,$$

where $p_X : X \to 1$ is the unique map to the final object. The object $\operatorname{Supp} X$ is called the support of X.

Let P be a formal G-torsor. We can restrict G and P to $\operatorname{Supp} P$. In other words, we consider G and P in the topos $\mathrm{E} / \operatorname{Supp} P$. Here P becomes a torsor. Below we show that $\operatorname{Supp} P$ is a very interesting invariant of P.

2.2.4. In the category of sets the classificator of G-torsors on the pre-homotopic level is represented by the topos of G-sets. In a more complicated topos the classificator of torsors should be represented by something like an internal topos. Then the torsor P would induce an arrow from $\operatorname{Supp} P$ to this classificator. It would be exactly the arrow we need. But, at this stage, we would like to avoid such constructions. Hopefully, this is possible since in [7] the classificator was constructed on the homotopic level.

2.3 Torsor classificators on $\mathcal{H}_s$-level

The homotopic category in the algebro-geometric context was introduced by F. Morel and V. Voevodsky in [7]. Among other things, their results permit to describe torsors as homotopy classes of maps to the classificator (which they produced). These results form an indispensable part of our approach as well.

The homotopic category $\mathcal{H}_s(\mathrm{T})$ is obtained from the category of spaces Spc via localization with respect to the class of weak equivalences W_s ([7, Def 2.1.2, p. 48]). A morphism of spaces $f : U \to V$ belongs to W_s if and only if for each point e of the generalized space (that is, *topos*) $\mathrm{E} = \mathrm{Shv}/\mathrm{T}$ the morphism of fibers $f_e : U_e \to V_e$ is a weak equivalence of simplicial sets. Here the point e of the topos E, that is, a morphism from the usual point $\bullet$ to E, by definition, is represented by the pullback functor $e^* : \mathrm{Shv}/\mathrm{T} \to \mathrm{Shv}/\bullet$, where the latter category is the category of sets. In this paper we need no explicit description of points of the generalized space $\mathrm{Shv}(\mathrm{Sm}_S)_{Nis}$. We only mention that with each choice of a scheme $X \in \mathrm{Sm}_S$ together with a point $x \in X$ one can associate a point e of the topos E. Then $e^*(F)$ is the fiber of F in the hensilization of x. As far as we understand this construction gives essentially all the points of E.

The class W_s is a part of the simplicial model structure (C_s, W_s, F_s) (see [7, p. 48]). The cofibrations, that is, the elements of C_s, are just embeddings. Thus, any object of Spc is cofibrant. The fibrations are defined via the lifting property w. r. to the acyclic cofibrations.

2.3.1. With each smooth S-scheme X one can associate a space EX (see [15, p. 9] and [7, Exa 4.1.11]). By definition $(EX)_n = X^{n+1}$ (products over S), with faces and degeneration maps given by partial projections and partial diagonals. In other words, we connect by a segment (even by two) each pair of point, glue up all triangles, etc.

2.3.2. An $\mathcal{H}_s$-classificator of G-torsors BG is constructed by F. Morel and V. Voevodsky in [7]. The space BG is introduced together with an isomorphism of functors $B_\bullet \mapsto P(B_\bullet, G)$ and $B_\bullet \mapsto \mathcal{H}_s(B_\bullet, BG)$, where $P(B_\bullet, G)$ is the set of isomorphism classes of torsors over $B_\bullet \in \mathrm{Spc}$. For $U \in \mathrm{T}$, by definition, $BG(U)$ is the nerve of the category having one object and the group $G(U)$ of arrows. Thus, the space BG is represented by the simplicial scheme with $BG_{n+1} = G^n$ and the standard faces and degenerations maps. To complete the description of the classificator it remains to specify the morphism of functors $P(B_\bullet, G) \to \mathcal{H}_s(B_\bullet, BG)$. For this one needs the space EG (see 2.3.1) as well as the morphism of spaces $EG \to BG$,

$$(g_0, \dots, g_n) \mapsto (g_n^{-1} g_{n-1}, \dots, g_2^{-1} g_1, g_1^{-1} g_0).$$

This map identifies the spaces $G \backslash EG$ and BG. In [7, proof of Lemma 4.1.12], to each torsor $P_\bullet$ over $B_\bullet$, a hat $B_\bullet \overset{p}{\leftarrow} \tilde{P}_\bullet \overset{f}{\to} BG$ is assigned where $\tilde{P}_\bullet = (EG \times_G P_\bullet)$. The correspondence $P_\bullet \mapsto f \circ p^{-1}$ provides the needed $\mathcal{H}_s$-arrow $B_\bullet \to BG$.

2.3.3 Theorem. (F. Morel, V. Voevodsky, [7, Proposition 4.1.15]) *The above correspondence defines a bijection*

$$P(B_\bullet, G) \cong \mathcal{H}_s(B_\bullet, BG).$$

2.3.4 Proposition. *There exists a unique* Spc*-arrow* $p : EX \to \mathrm{Supp}\, X$*. This arrow is a weak equivalence. Thus,* EX *is a homotopic model for the support of* X*.*

Proof. The existence is the consequence of the locality of the product $(\mathrm{Supp}(X \times Y) = (\mathrm{Supp}\, X) \times (\mathrm{Supp}\, Y))$, while the uniqueness follows from the injectivity of the embedding $\mathrm{Supp}\, X \subset 1$. To check that $p \in W_s$ it is sufficient to consider the fibers, where everything is reduced to the fact that, for a non-empty set X, the simplicial set EX is weakly equivalent to the point. □

2.3.5. Notice, that $E = \mathrm{cosk}_n$ for $n = 0$. The functor cosk_n is the right adjoint to the truncation i_* from the category of simplicial objects to the category of n-bounded simplicial objects and should be denoted $i^!$. For $n = 0$ the truncation has the form $Z_\bullet \mapsto Z_0$, and the adjointness means that $\mathrm{Hom}(Z_\bullet, EX) = \mathrm{Hom}(Z_0, X)$.

2.3.6. The space EX is locally fibrant by [7, Lemma 2.1.15], that is, fibrant in each fiber. Indeed, if t is a point, then $(EX)_t = E(X_t)$ is Kan's set, since there everything is glued up tautologically. More accurate argument uses the adjointness from 2.3.5.

2.3.7. S is the final object of $\mathcal{H}_s$. Indeed, in Spc each object is cofibrant, while the final one is fibrant. Hence, $\mathcal{H}_s$-arrows to S are obtained as a quotient of Spc-arrows to S, that is, as a quotient of the one-element set.

2.3.8. Let $Y_\bullet$ be locally fibrant. Then each $\mathcal{H}_s$-arrow $f : T_\bullet \to Y_\bullet$ can be represented by some $\mathcal{H}_s$-composition $f = q \circ p^{-1}$ corresponding to a hat $T_\bullet \overset{p}{\leftarrow} \tilde{T}_\bullet \overset{q}{\to} Y_\bullet$, where p is a local fibration and a weak equivalence. In such a situation, f depends only on the classes of p and q w.r.to the s-homotopic equivalence. Moreover, for two $\mathcal{H}_s$-arrows $T_\bullet \to Y_\bullet$ one can find hats with the common $\tilde{T}_\bullet$ and p [7, Prop. 2.1.13].

2.3.9. EX is a subobject of S in $\mathcal{H}_s$. In other words, the unique (see 2.3.7) $\mathcal{H}_s$-arrow $EX \to S$ is a monomorphism. One needs to check that, for arbitrary space $T_\bullet$, the set $\mathcal{H}_s(T_\bullet, EX)$ consists of at most one element.

Indeed, in the light of (2.3.6) and (2.3.8), it is sufficient to connect arrows $f, g : T_\bullet \to EX$ by an s-homotopy h. But inside EX everything is glued up canonically and h is constructed tautologically. More accurate argument uses the adjointness from 2.3.5.

2.3.10. Let Y be a smooth scheme over S, where $S = \operatorname{Spec} k$. Then the following conditions are equivalent ([15, Lemma 3.8, Remark after Lemma 3.8]):

$$\text{for any extension of fields } L/k: \quad Y(L) \neq \emptyset \Rightarrow X(L) \neq \emptyset; \tag{1}$$

$$\mathcal{H}_s(EY, EX) \neq \emptyset; \tag{2}$$

For irreducible Y, the condition (1) is equivalent to: $X(L) \neq \emptyset$ for $L = k(Y)$.

2.3.11. It follows from (2.3.10) that the following conditions are equivalent:

$$\text{for each extension of fields } L/k: \quad Y(L) \neq \emptyset \Leftrightarrow X(L) \neq \emptyset; \tag{3}$$

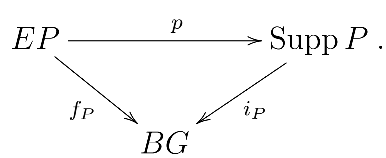
$$EY \simeq EX \text{ in } \mathcal{H}_s. \tag{4}$$

In particular, the projection $EX \to S$ is an $\mathcal{H}_s$-isomorphism if and only if $X(k) \neq \emptyset$.

2.3.12. Any algebraic G-torsor P induces a G-torsor over $\operatorname{Supp} P$ and so (see 2.3.3), an $\mathcal{H}_s$-arrow $i_P : \operatorname{Supp} P \to BG$. The pair $(\operatorname{Supp} P, i_P)$ is exactly the invariant of P we are looking for. The knowledge of this invariant permits to recover P.

There are explicit $\mathcal{H}_s$-models for the $\operatorname{Supp} P$ (see 2.3.4) and the classificator, namely, EP and BG. Therefore i_P induces the $\mathcal{H}_s$-arrow between them. It appears that this arrow can be constructed explicitly already on the level of spaces. Namely, consider the map $f_P : EP \to BG, (x_0, \dots, x_n) \mapsto (x_0 x_1^{-1}, x_1 x_2^{-1}, \dots, x_{n-1} x_n^{-1})$.

2.3.13 Proposition. *The following $\mathcal{H}_s$-diagram is commutative (the arrow p is described in 2.3.4):*



Proof. Due to Theorem of F. Morel and V. Voevodsky (see 2.3.3) it is sufficient to show that $p^* P \simeq f_P^* EG$. Here $p^* P = EP \times P$, and the needed isomorphism of torsors is given by: $EP \times P \to EP \times_{BG} EG$: $(x_0, \dots, x_n) \times x \mapsto (x_0, \dots, x_n) \times (x_n x^{-1}, \dots, x_0 x^{-1})$. □

2.3.14 Proposition. *Let X be a G-torsor. Then in* Spc *we have: $G \backslash (EG \times X) = EX$.*

Proof. It is easy to see that the map $G \backslash (EG \times X) \to EX$ defined by: $(g_0, \dots, g_n, x) \mapsto (g_n^{-1} x, \dots, g_0^{-1} x)$ is an isomorphism of simplicial schemes. □

Below EX can be denoted as $\mathcal{X}_X$ (which corresponds to the notations of [16] and [11]).

2.4 Some remarks

2.4.1. To simplify computations it would be desirable to pass from $\mathcal{H}_s$ to the motivic homotopic category of Morel–Voevodsky $\mathcal{H}$. The category $\mathcal{H}$ is the localization of $\mathcal{H}_s$, identifying $\mathbb{A}^1$ with the point (see [7]). For computations in $\mathcal{H}$ it is useful to keep in mind that $\mathcal{H}(X,Y) = \mathcal{H}_s(X,Y)$ for $\mathbb{A}^1$-local Y [7, Theorem 2.3.2, p. 86].

The $\mathcal{H}_s$-type of S is clearly $\mathbb{A}^1$-local as the final object of $\mathcal{H}_s$ (see 2.3.7). Therefore, S is also the final object of $\mathcal{H}$. The $\mathcal{H}_s$-type of EX is also $\mathbb{A}^1$-local, as a subobject of a local one (see 2.3.9).

Moreover, EX is a subobject of S in $\mathcal{H}$, that is, the unique $\mathcal{H}$-arrow $EX \to S$ is a monomorphism. Indeed, for arbitrary $Z_\bullet$, the set $\mathcal{H}(Z_\bullet, EX)$ consists of at most one element. This follows from the $\mathbb{A}^1$-locality of EX and the fact that EX is a subobject of S in $\mathcal{H}_s$ (see 2.3.9).

In the statements 2.3.10 and 2.3.11 one can replace $\mathcal{H}_s$ by $\mathcal{H}$. This follows from the $\mathbb{A}^1$-locality of EX.

2.5 Nisnevich and étale classifying spaces

2.5.1. Everything said above about the classification of torsors in the Nisnevich topology, can be also applied to other topologies, in particular, to the étale one. This way, we get a classificator in $\mathcal{H}_s(\mathrm{T})$ for the respective site. As a family of classificators and supports appears, the corresponding notations should contain references to T.

For the étale topology, $BG_{et} \in \mathcal{H}_s((\mathrm{Sm}_S)_{\mathrm{et}})$. But, in [7] the notation BG_{et} denotes something else, namely, $R\pi_*(BG_{et})$, where

$$\begin{array}{c} \mathcal{H}_s((\mathrm{Sm}_S)_{et}) \\ {\scriptstyle L\pi^*=\pi^*}\big\uparrow\big\downarrow{\scriptstyle R\pi_*} \\ \mathcal{H}_s((\mathrm{Sm}_S)_{Nis}) \end{array}$$

is the pair of conjugate functors induced by the morphism of sites $\pi : (\mathrm{Sm}_S)_{et} \to (\mathrm{Sm}_S)_{Nis}$. Therefore, to avoid confusion, we set $BG_{et/Nis} = R\pi_*(BG_{et})$.

Besides, the $\mathcal{H}_s$-type of $BG_{et/Nis}$ is defined in [7] slightly differently, namely, as $R\pi_*(\pi^* BG)$. The result is the same though, since $BG_{et} = \pi^*(BG)$.

2.5.2. The $\mathcal{H}_s$-types BG and $BG_{et/Nis}$ are quite different. For example, there is only one $\mathcal{H}_s$-arrow from $\operatorname{Spec} k$ to BG, while $\mathcal{H}_s$-arrows $\operatorname{Spec} k \to BG_{et/Nis}$ correspond to the isomorphism classes of algebraic G-torsors. For $G = O(n)$ it is the set of the isomorphism classes of n-dimensional quadratic forms over k, which is very non-trivial.

2.5.3. The discussion of 2.5.1 shows that the identity map on BG_{et}, by adjunction, induces an $\mathcal{H}_s$-arrow

$$\varepsilon : BG \to BG_{et/Nis}.$$

Any algebraic G-torsor P induces $\mathcal{H}_s$-arrow $i_P : \operatorname{Supp} P \to BG$ (see 2.3.12). On the other hand it induced an $\mathcal{H}_s((\mathrm{Sm}_S)_{et})$-arrow $i_{et,P} : S \to BG_{et}$ since the corresponding etale support coincides with S. By adjunction, we get the corresponding $\mathcal{H}_s$-arrow $i_{et/Nis,P} : S \to BG_{et/Nis}$. The arrows i_P and $i_{et/Nis,P}$ are related by means of ε. Namely, the following $\mathcal{H}_s$-diagram is commutative.

$$\begin{array}{ccc} \operatorname{Supp} P & \xrightarrow{i_P} & BG \\ \downarrow{\scriptstyle\subset} & & \downarrow{\scriptstyle\varepsilon} \\ S & \xrightarrow{i_{et/Nis,P}} & BG_{et/Nis} \end{array}$$

We know an $\mathcal{H}_s$-model for $\operatorname{Supp} P$ (see 2.3.12). In addition, there are nice geometric $\mathcal{H}$-models for $BG_{et/Nis}$ (see 2.5.4). We are going to relate these models by means of ε. Moreover, we are going to get a similar diagram not for an individual algebraic torsor, but rather for a family of such torsors.

2.5.4. For any exact representation $\rho : G \to GL(V)$ a geometric model $B(G,\rho)_{gm}$ for the $\mathcal{H}$-type of $BG_{et/Nis}$ is constructed in [7, Prop. 4.2.6]. As a space, that is as an object of Spc, it is defined as the quotient $G\backslash E(G,\rho)_{gm}$, where $E(G,\rho)_{gm}$ is an open subscheme of V^∞ consisting of the points where the diagonal action of G is free. It is proved that $E(G,\rho)_{gm}$ is $\mathcal{H}$-contractible and that $B(G,\rho)_{gm}$ is $\mathcal{H}$-isomorphic to $BG_{et/Nis}$. The choice of the representation ρ will not be important for us, so below we denote the respective spaces simply as EG_{gm} and BG_{gm}.

This isomorphism can be described explicitly. Indeed, the algebraic torsor EG_{gm} over BG_{gm} induces a canonical $\mathcal{H}_s((\mathrm{Sm}_S)_{et})$-arrow i_{et} from its base BG_{gm} to the classificator BG_{et}. By adjunction, we get the corresponding $\mathcal{H}_s$-arrow $i_{et/Nis} : BG_{gm} \to BG_{et/Nis}$. This arrow becomes the $\mathcal{H}$-isomorphism we look for.

Let $x : S \to BG_{gm}$ be a rational point of the (inductive) scheme BG_{gm}. Denote by P_x the algebraic G-torsor $\pi^{-1}(x)$, where π is the projection $EG_{gm} \to BG_{gm}$.

2.5.5. To relate the homotopic models and ε we need more spaces and arrows. Consider the Spc-diagram

$$BG \xleftarrow{p} G\backslash\left(EG \times EG_{gm}\right) \xrightarrow{\tilde{\varepsilon}_\rho} BG_{gm},$$

where the action on the middle term is diagonal and the arrows p and $\tilde{\varepsilon}_\rho$ are induced by the projections. It follows from [7, Prop. 4.2.3] that p is an $\mathcal{H}$-isomorphism. Set

$$\widetilde{B}G = G\backslash(EG \times EG_{gm}).$$

Let $\varphi : \mathcal{U} \to \mathcal{V}$ be a Spc-arrow represented by a termwise smooth morphism. Then we have the well-defined Spc-product $\mathcal{U} \times_{\mathcal{V}} \times \cdots \times_{\mathcal{V}} \mathcal{U}$. Denote it by $(\mathcal{U}/\mathcal{V})^n$. Consider the following simplicial space $sE_\bullet(\mathcal{U}/\mathcal{V}) \in \mathrm{sSpc}$, where $sE_n(\mathcal{U}/\mathcal{V}) = (\mathcal{U}/\mathcal{V})^{n+1} \in \mathrm{Spc}$, and the face and degeneration morphisms are partial projections and partial diagonals. Applying the diagonal functor $\mathrm{sSpc} \to \mathrm{Spc}$ to $sE_\bullet(\mathcal{U}/\mathcal{V})$ we get a space $E_\bullet(\mathcal{U}/\mathcal{V}) = \mathcal{X}_{\mathcal{U}/\mathcal{V}}$ together with the structure Spc-arrow $\mathcal{X}_{\mathcal{U}/\mathcal{V}} \to \mathcal{V}$. Denote this map as $\mathcal{X}(\mathcal{U} \to \mathcal{V})$.

2.5.6 Proposition.

$$\mathcal{X}\left(EG_{gm} \xrightarrow{\pi} BG_{gm}\right) = \widetilde{B}G \xrightarrow{\tilde{\varepsilon}_\rho} BG_{gm}.$$

Proof. It follows from the fact that $(EG_{gm}/BG_{gm})^{r+1} = EG_{gm} \times G^r$ and faces and degeneration maps are as in $\widetilde{B}G$ (which on the fibers can be seen from Prop. 2.3.14). $\square$

In particular, the fiber $(\tilde{\varepsilon}_\rho)^{-1}(x)$ over the rational point $x \in BG_{gm}$ is $\mathcal{X}_P = EP_x$ (see 2.5.4 for the notation P_x), and the composition $E(P_x) \hookrightarrow \widetilde{B}G \xrightarrow{p} BG$ is exactly the map i_P from 2.3.12. Abusing notations somewhat we will use the name "Nisnevich-étale fiber" for this classifying map.

2.6 Groups and groupoids

Let G be a linear algebraic group over a field k. An anti-automorphism: $g \mapsto g^{-1}$ of G identifies the set of left and right G-torsors: $X \mapsto X^{-1}$.

If X is a left algebraic G-torsor, the functor $U \mapsto \mathrm{Aut}_G(X \times U \to U)$, by étale descent is represented by some group scheme G_X over k, and X has a natural structure of the right G_X-torsor. We get a *torsor triple*, i. e. the triple (G, X, H), where G and H are group schemes, and X is a left G-torsor, and a right H-torsor, and these structures commute. Such triples can be composed: $(G, X, H) \circ (H, Y, K) := (G, X \times_H Y, K)$, and inverted: $(G, X, H)^{-1} := (H, X^{-1}, G)$, and form a *groupoid*. In particular, $(G, X, H) \circ (G, X, H)^{-1} = id_G = (G, G, G)$. Thus, torsor triples are just morphisms of our groupoid.

The following statement shows how the classifying spaces corresponding to two different groups from the same groupoid are related.

2.6.1 Proposition. *For any torsor triple (G, Y, H) there is a natural $\mathcal{H}_s$-identification*

$$\begin{array}{ccc} \mathcal{X}_Y \times BH^R & = & BG^L \times \mathcal{X}_Y \\ & \searrow \quad \swarrow & \\ & \mathcal{X}_Y & \end{array}$$

such that the natural projections $\mathcal{X}_Y \times BH^R \to BH^R$ and $BG^L \times \mathcal{X}_Y \to BG^L$ map "the other" copy of $\mathcal{X}_Y$ to the Nisnevich-étale fibers over $[Y]$ (the map from 2.3.12).

Proof. Consider $EG \times Y \times EH$ with the left G-action on the first two factors and the right H-action on the last two. From Proposition 2.3.14, in Spc we have an identification:

$$G\backslash(EG \times Y \times EH) = \mathcal{X}_Y \times EH \quad \text{and} \quad (EG \times Y \times EH)/H = EG \times \mathcal{X}_Y$$

with the standard right H and left G-action, respectively. And $G\backslash(EG \times \mathcal{X}_Y)$ and $(\mathcal{X}_Y \times EH)/H$ are just homotopic quotients of $\mathcal{X}_Y$ by these actions. But, by 2.3.9,

$$\mathrm{Hom}_{\mathcal{H}_s(k)}(G \times \mathcal{X}_Y, \mathcal{X}_Y) = * = \mathrm{Hom}_{\mathcal{H}_s(k)}(\mathcal{X}_Y \times H, \mathcal{X}_Y).$$

Thus, our actions on $\mathcal{X}_Y$ are homotopically trivial. Hence, $G\backslash(EG \times \mathcal{X}_Y) = BG^L \times \mathcal{X}_Y$ and $(\mathcal{X}_Y \times EH)/H = \mathcal{X}_Y \times BH^R$ in $\mathcal{H}_s(k)$. Thus, we get an identification in $\mathcal{H}_s(k)$:

$$\mathcal{X}_Y \times BH^R = G\backslash(EG \times Y \times EH)/H = BG^L \times \mathcal{X}_Y.$$

Since $\mathcal{X}_Y$ is a subobject of the final object $\bullet$ of $\mathcal{H}_s(k)$, this identification is over $\mathcal{X}_Y$.

Consider the $G-H$-equivariant projection: $EG \times Y \times EH \to EG \times EH$ giving the map: $G\backslash(EG \times Y \times EH)/H \to BG^L \times BH^R$. Since the maps $G\backslash(EG \times Y \times H)/H \to (\mathcal{X}_Y \times \bullet)$ and $G\backslash(G \times Y \times EH)/H \to (\bullet \times \mathcal{X}_Y)$ are isomorphisms (here $\bullet$ is the only homotopic rational point on BH^R and BG^L), we see that the map $(\mathcal{X}_Y \times \bullet) \to BG^L$ is induced by the G-equivariant map $EG \times Y \to EG$, while the map $(\bullet \times \mathcal{X}_Y) \to BH^R$ is induced by the H-equivariant map $Y \times EH \to EH$. By Lemma 2.3.14 these are exactly the Nisnevich-étale fibers over $[Y]$. □

2.6.2. Note, that left and right classifying spaces BG^L and BG^R are canonically isomorphic.

In the above situation, BG is not isomorphic to BH, in general. The situation with the étale classifying spaces is different.

2.6.3 Proposition. *For any torsor triple (G, Y, H) there is canonical $\mathcal{H}_s$-isomorphism*

$$BH_{et/Nis} \stackrel{\theta_Y}{\to} BG_{et/Nis},$$

which acts on homotopic rational points by $[X] \mapsto [Y \times_H X]$.

Proof. By Proposition 2.6.1, we have a natural identification $\mathcal{X}_Y \times BH \stackrel{\cong}{\to} BG \times \mathcal{X}_Y$. Since $BG_{et/Nis} = R\pi_* \circ \pi^* BG$, and $\pi^*(\mathcal{X}_Y) = \bullet_{et}$ and π^* respects products, we get an identification $BH_{et/Nis} \stackrel{\theta_Y}{\to} BG_{et/Nis}$.

Since the fibers $\mathcal{X}_X \to BH$ and $\mathcal{X}_{X \times_H Y} \to BG$ are given by the H, respectively, G-equivariant maps $EH \times X \to EH$ and $EG \times (X \times_H Y) \to EG$, and $(Y \times EH \times X)/H = (Y \times_H X) \times \mathcal{X}_Y$ we get that $[X] \mapsto [Y \times_H X]$. □

The above map does not preserve a base-point (given by a trivial torsor).

2.7 Some invariants of torsors

2.7.1. As soon as we have got an arrow $\operatorname{Supp} P \to BG$, we can apply cohomology theories to it. For example, to such a theory A we can associate an invariant:

$$\operatorname{Ker}_A(P) := \operatorname{Ker}[A(BG) \to A(\operatorname{Supp} P)].$$

The task is to determine which information on P this invariant carries, and to describe the possible values of it.

2.7.2. For an algebraic subgroup $H \subset G$ and an algebraic H-torsor Q we have an algebraic G-torsor P defined by $P = G \times_H Q$. This fits into the commutative $\mathcal{H}_s$-diagram

$$\begin{array}{ccc} \operatorname{Supp} Q & \longrightarrow & BH \\ \downarrow & & \downarrow \\ \operatorname{Supp} P & \longrightarrow & BG \end{array}.$$

It gives the commutative diagram

$$\begin{array}{ccc} A(BG) & \longrightarrow & A(\operatorname{Supp} P) \\ \downarrow & & \downarrow \\ A(BH) & \longrightarrow & A(\operatorname{Supp} Q). \end{array} \qquad (5)$$

This enables one to get information about $\operatorname{Ker}_A(P)$ from $\operatorname{Ker}_A(Q)$ for induced torsors.

3 Subtle Stiefel–Whitney Classes

We would like to apply the technique developed above to the case of an orthogonal group with the aim of classifying quadratic forms. The first step is to choose an appropriate cohomology theory A and to compute the cohomology ring of the classificator. Our theory will be motivic cohomology $\mathrm{H}_{\mathcal{M}}^{*,*'}$. From now on let $\operatorname{char} k \neq 2$. Set $H = \mathrm{H}_{\mathcal{M}}^{*,*'}(\operatorname{Spec} k, \mathbb{Z}/2)$. By the result of V. Voevodsky [15], $H = K_*^M(k)/2[\tau]$, where τ is the only non-zero element of degree $(1)[0]$.

3.1 Motivic cohomology of $BO(n)$

Everywhere below $\sqrt{-1} \in k$. For $n = 1, 2, \ldots$ the form $q_n = x_1^2 + \cdots + x_n^2$ will be called the standard one, and the respective automorphism group will be denoted $O(n)$. Then n-dimensional quadratic forms will correspond to the (left) $O(n)$-torsors via the rule: $q \leftrightarrow X_q = \mathrm{Iso}(q \to q_n)$. Let

$$w_i \in \mathrm{H}^{i,i}_{\mathcal{M}}(BO(n)_{et/Nis}, \mathbb{Z}/2), \quad c_i \in \mathrm{H}^{2i,i}_{\mathcal{M}}(BO(n)_{et/Nis}, \mathbb{Z}/2)$$

be the Stiefel–Whitney and Chern classes. Slightly abusing the notation denote ε^* of them also as w_i and c_i, where $\varepsilon : BO(n) \to BO(n)_{et/Nis}$ is the arrow from 2.5.3.

3.1.1 Theorem. *There is a unique set $u_1, \ldots, u_n$ of classes in the motivic $\mathbb{Z}/2$-cohomology for $BO(n)$ such that $\deg u_i = ([i/2])[i]$,*

$$\mathrm{H}^{*,*'}_{\mathcal{M}}(BO(n), \mathbb{Z}/2) = H[u_1, \ldots, u_n], \quad w_i = u_i \tau^{[(i+1)/2]}, \ \textit{and}\ c_i = u_i^2 \tau^{\delta(i)}.$$

Here $\delta(i)$ is the indicator of oddness, that is $\delta = 0$, for i even, and $\delta = 1$, for i odd.

The Theorem will be proven in 3.1.6. We start with some preliminary observations.

Let p, q be quadratic forms. Denote as $I(p, q)$ the (smooth) variety of isometric embeddings from p to q. For example, $A_q = I(q_1, q)$ is the affine quadric $q = 1$.

3.1.2 Proposition. *Let q be a quadratic form of dimension n. Then*

$$O(n-1)\backslash X_q \cong A_q, \quad \textit{in particular,} \quad O(n-1)\backslash O(n) \cong I(q_1, q_n).$$

Proof. An isomorphism is given by: $x \mapsto x^{-1}(0, ..., 0, 1)$. □

3.1.3 Proposition. *Let $M : \mathcal{H} \to \mathrm{DM}^-_{eff}(k)$ be the motivic functor, $A = I(q_1, q_n)$. Then*

$$M(A) = \mathbb{Z} \oplus \mathbb{Z}([n/2])[n-1].$$

Proof. Consider $r = \langle -1 \rangle \perp q_n$ with the respective quadric R. Then $Q \subset R$ is a codimension one subquadric with complement A. In $\mathrm{DM}^-_{eff}(k)$ we have Gysin's exact triangle:

$$M(A) \to M(R) \xrightarrow{j} M(Q)(1)[2] \to M(A)[1].$$

Since the quadrics R and Q are split, $\mathrm{Cone}[-1](j) = \mathbb{Z} \oplus \mathbb{Z}([n/2])[n-1]$ (we use the fact that $\sqrt{-1} \in k$.). □

3.1.4. Let $Y_\bullet \in \mathrm{Spc}$, R be a commutative ring. The category $\mathrm{DM}^-_{eff}(Y_\bullet)$ is introduced in [19]. The notation there is slightly different (minus is a subscript), but we prefer to denote it the above way, since it reflects the fact that the cohomological indices of the non-trivial terms of a complex are bounded from above. This is coherent with the derived category notations. We need the category $\mathrm{DM}^-_{eff}(Y_\bullet, R)$. This category is not introduced in [19], but it is mentioned there in §7 that all the results can be extended to the case with coefficients. So, we will use the R-analogues referring to the respective $\mathbb{Z}$-formulations.

By definition, $\mathrm{DM}^-_{eff}(Y_\bullet, R)$ is the localization of $D(Y_\bullet, R)$ [19, Def 4.2], that is ([19, after Lemma 2.3]), of the derived category $D^-(\mathrm{PST}(Y_\bullet, R))$, where $\mathrm{PST}(Y_\bullet, R)$ is the category of the presheaves of R-modules with transfers on $Y_\bullet$. The objects of $\mathrm{DM}^-_{eff}(Y_\bullet, R)$ are complexes $\dots \to C_{i+1} \to C_i \to \dots \to 0 \to 0 \to \dots$, where $C_j \in \mathrm{PST}(Y_\bullet, R)$.

The category $\mathrm{PST}(Y_\bullet, R)$ is defined as in [19, Def 2.1]) with the replacement of Abelian groups by R-modules. An object $K \in \mathrm{PST}(Y_\bullet, R)$ is represented by the system $\{K_n, f_\theta\}$, where $K_n \in \mathrm{PST}(Y_n, R)$ and $f_\theta : (Y_\theta^*)(K_n) \to K_m$, for $\theta : [n] \to [m]$ is a coherent system of arrows.

We need also functors

$$r_i^* : \mathrm{DM}^-_{eff}(Y_\bullet, R) \to \mathrm{DM}^-_{eff}(Y_i, R), \quad r_i^*(N) = N_i.$$

Following [19], for a space $Y_\bullet$ we denote as $CC(Y_\bullet)$ the simplicial set, where CC is the functor commuting with the coproducts and sending a connected scheme to the point.

3.1.5 Proposition. *Suppose that $H^1(CC(Y_\bullet), R) = 0$. Let $M = T(u)[v] \in \mathrm{DM}^-_{eff}(S, R)$ be the Tate-motive. Let $N \in \mathrm{DM}^-_{eff}(Y_\bullet, R)$ be such a motive that its graded components $N_i \in \mathrm{DM}^-_{eff}(Y_i, R)$ are isomorphic to M. Then N is isomorphic to M.*

Proof. Consider the conjugate pair of functors (see [19]):

$$Lc_\# : \mathrm{DM}^-_{eff}(Y_\bullet, R) \to \mathrm{DM}^-_{eff}(S, R) \quad \text{and} \quad c^* : \mathrm{DM}^-_{eff}(S, R) \to \mathrm{DM}^-_{eff}(Y_\bullet, R).$$

$Lc_\#(N)$ is naturally a complex obtained from the simplicial object in the category of (bounded from above) complexes of Nisnevich sheaves with transfers, where each component corresponds to $L(c_i)_\#(N_i)$. We get an "infinite Postnikov tower" with graded pieces $L(c_i)_\#(N_i)[i] = M(Y_i)(u)[v]$, and natural filtration $Lc_\#(N)_{\geqslant j}$, $j = 0, 1, \dots$. In particular,

$$\mathrm{Hom}_{\mathrm{DM}^-_{eff}(S,R)}(Lc_\#(N)_{\geqslant 1}, T(u)[v]) = 0;$$
$$\mathrm{Hom}_{\mathrm{DM}^-_{eff}(S,R)}(Lc_\#(N)_{\geqslant 2}, T(u)[v+k]) = 0, \quad \text{for } k = 0, 1.$$

For finite pieces of the tower it is evident from the description of graded pieces, and for the whole (infinite) thing it follows from the fact that it is a *colimit* of finite ones (see [16, proof of Prop.8.1]). Then we have an exact sequence:

$$0 \to \mathrm{Hom}(Lc_\#(N), T(u)[v]) \to \mathrm{Hom}(Lc_\#(N)_0, T(u)[v]) \to \mathrm{Hom}(Lc_\#(N)_1, T(u)[v]).$$

For each $\theta : [i] \to [j] \in Mor(\Delta)$, we have the natural map $f_\theta : (\mathcal{Y}_\theta^*)(N_j) \to N_i$, and each N_i can be identified with the $(T(u)[v])_i$. So, the choice of θ and of connected component of Y_i gives us an element of $R^\times$ (as an automorphism of a Tate-motive on a connected variety). Thus, the obstruction to identifying the coherent system (N_i, f_θ) with the similar system for $T(u)[v]$ lies in $\mathrm{Hom}_{gr}(\pi_1(CC(Y_\bullet)), R^\times)$. Since $H^1(CC(Y_\bullet), R) = 0$, the pair $Lc_\#(N)_1 \to Lc_\#(N)_0$ can be identified with $Lc_\#(T(u)[v])_1 \to Lc_\#(T(u)[v])_0$. The Cone of the latter map has a natural morphism to $T(u)[v]$ (as a $(0,1)$-floor of the Postnikov tower corresponding to $Lc_\#(T(u)[v])$). Thus, we get the map $Lc_\#(N) \to T(u)[v]$ which, by conjugation, gives us map $\varphi : N \to T(u)[v]$. From the construction, φ_0 is an isomorphism. Since all components of N are also isomorphic to $T(u)[v]$, we get from the existence of morphisms $[0] \stackrel{\alpha_i}{\to} [i] \stackrel{\beta_i}{\to} [0]$ that all the morphisms φ_i are isomorphisms as well, and so is φ by [19, Lemma 4.4]. □

In particular, the above result works if $R = \mathbb{Z}/2$, or if R is a field of characteristic p and $\pi_1(CC(Y_\bullet))$ is a p-group.

3.1.6. Proof of Theorem 3.1.1. Set $u_0 = 1$ and use induction on n. As the base take $n = 0$. For $n > 0$, consider the transition $(n-1) \to n$. Identifying $q_n = q_{n-1} \perp \langle 1 \rangle$, we get an embedding $O(n-1) \hookrightarrow O(n)$. Denote as $\widetilde{B}O(n-1)$ (respectively, $\hat{B}O(n-1)$) the quotient $O(n-1)\backslash(EO(n-1) \times EO(n))$ with the diagonal action (respectively, $O(n-1)\backslash EO(n)$) in Spc. Then we have natural maps in Spc (see the very beginning of 2.3):

$$BO(n-1) \xleftarrow{\varphi} \widetilde{B}O(n-1) \xrightarrow{\psi} \hat{B}O(n-1)$$

Then φ is an isomorphism in $\mathcal{H}_s(k)$. In particular, it induces an isomorphism on motivic cohomology. On the other hand, in the category $\mathrm{DM}^-_{eff}(\hat{B}O(n-1); \mathbb{Z}/2)$ of motives over $\hat{B}O(n-1)$ we have: $M(\widetilde{B}O(n-1) \stackrel{\psi}{\to} \hat{B}O(n-1)) = T$.

We have a natural fibration $\hat{B}O(n-1) \to BO(n) = O(n)\backslash EO(n)$ with fibers $O(n-1)\backslash O(n)$, which is trivial over the simplicial components.

Let $M(\hat{B}O(n-1) \to BO(n)) \stackrel{g}{\to} T$ be the natural projection in $\mathrm{DM}^-_{eff}(BO(n); \mathbb{Z}/2)$. Then it follows from Lemmas 3.1.2, 3.1.3, and 3.1.5 (taking into account that our coefficients are $\mathbb{Z}/2$) that $\mathrm{Cone}[-1](g) = T([n/2])[n-1]$. Recalling the fact about ψ above, we obtain an exact triangle in $\mathrm{DM}^-_{eff}(BO(n), \mathbb{Z}/2)$:

$$\begin{array}{ccc} & M(\widetilde{B}O(n-1) \to BO(n)) & \\ {}^{h_n}_{[1]}\nearrow & \star & \searrow^{g_n} \\ T([n/2])[n] & \xleftarrow{\quad f_n \quad} & T. \end{array}$$

Using the property of φ we get an induced diagram:

$$\begin{array}{ccc} & \mathrm{H}^{*,*'}_{\mathcal{M}}(BO(n-1), \mathbb{Z}/2) & \\ {}^{h_n^*}_{[1]}\swarrow & \star & \nwarrow^{g_n^*} \\ \mathrm{H}^{*-n,*'-[n/2]}_{\mathcal{M}}(BO(n), \mathbb{Z}/2) & \xrightarrow{\quad f_n^* \quad} & \mathrm{H}^{*,*'}_{\mathcal{M}}(BO(n), \mathbb{Z}/2), \end{array}$$

where f_n^* is multiplication by some non-zero $u_n \in \mathrm{H}^{n,[n/2]}_{\mathcal{M}}(BO(n), \mathbb{Z}/2)$. By induction, $\mathrm{H}^{*,*'}_{\mathcal{M}}(BO(n-1), \mathbb{Z}/2)$ is generated by $u_1, \ldots, u_{n-1}$. Since $\mathrm{H}^{?,<0}_{\mathcal{M}}(BO(n), \mathbb{Z}/2) = 0$, and f_n^* is injective on $\mathrm{H}^{0,0}_{\mathcal{M}}(BO(n), \mathbb{Z}/2) = \mathbb{Z}/2$, we get that $h_n^*(u_i) = 0$, for $i = 0, \ldots, n-1$. In particular, $u_i, i = 1, \ldots n-1$ can be uniquely lifted to $\mathrm{H}^{*,*'}_{\mathcal{M}}(BO(n), \mathbb{Z}/2)$. Since g_n^* is a ring homomorphism, it is surjective. Hence, $h_n^* = 0$ and $\mathrm{H}^{*,*'}_{\mathcal{M}}(BO(n), \mathbb{Z}/2) = H[u_1, \ldots, u_n]$.

Let us compare u_i's with ω_i's and c_i's. Start with $n = 1$ case: $BO(1) = B(\mathbb{Z}/2)$. Let $BO(1) \stackrel{\varepsilon}{\to} BO(1)_{et/Nis}$ be the Nisnevich-étale $\mathcal{H}_s$-map, and $\omega_1 \in \mathrm{H}^{1,1}_{\mathcal{M}}(BO(1)_{et/Nis}, \mathbb{Z}/2)$, $c_1 \in \mathrm{H}^{2,1}_{\mathcal{M}}(BO(1)_{et/Nis}, \mathbb{Z}/2)$ be the usual Stiefel–Whitney and Chern classes. Then $\varepsilon^*(\omega_1) = \tau \cdot u_1 + \{a\}$, for some $\{a\} \in K^M_1(k)/2$. But the only homotopic rational point $\bullet$ of $BO(1)$ is mapped to the fixed rational point of $BO(1)_{et/Nis}$ (corresponding to the trivial torsor $\mathrm{Iso}(\langle 1 \rangle \to \langle 1 \rangle)$), so the restriction of ω_1 to this point is zero. On the other hand, it is equal to $\{a\}$. Thus, $\varepsilon^*(\omega_1) = \tau \cdot u_1$. Analogously, $\varepsilon^*(c_1) = \tau \cdot u_1^2 + \{b\} \cdot u_1$, for some $\{b\} \in K^M_1(k)/2$. Since $\tau \cdot c_1 = \omega_1^2$ (as -1 is a square in k), we obtain that $\{b\} = 0$ and $\varepsilon^*(c_1) = \tau \cdot u_1^2$.

In particular, this shows that ω_i restricts non-trivially to $\mathrm{H}^{*,*'}_{\mathcal{M}}(B(\times_{i=1}^n O(1)), \mathbb{Z}/2)$. Comparing the Nisnevich and étale classificators for $O(i)$ and $O(i-1)$ we obtain that

$\varepsilon^*(\omega_i)$ is divisible by u_i. By degree consideration, we must have $\varepsilon^*(\omega_i) = \tau^{[\frac{i+1}{2}]} \cdot u_i$. But looking at the restriction to $\mathrm{H}_{\mathcal{M}}^{*,*'}(B(\times_{i=1}^n O(1)), \mathbb{Z}/2)$ we also obtain:

3.1.7 Proposition. *Let* $\left(\times_{j=1}^n O(1)\right) \stackrel{\delta}{\to} O(n)$ *be the standard embedding. Then*

$$\delta^*(u_i) = \tau^{[i/2]} \cdot \sigma_i(x_1, \ldots, x_n),$$

where x_j *is* u_1 *from the* j*-th component, and* σ_i *is the* i*-th elementary symmetric function. In particular, the map* $\delta^* : \mathrm{H}_{\mathcal{M}}^{*,*'}(BO(n), \mathbb{Z}/2) \to \mathrm{H}_{\mathcal{M}}^{*,*'}(B(\times_{i=1}^n O(1)), \mathbb{Z}/2)$ *is injective.*

Proof. The formula is clear from the description of $\delta^*\varepsilon^*(\omega_i)$. It remains to observe that these elements are algebraically independent over H. $\square$

Taking into account that c_i restricted to $B(\times_{i=1}^n O(1))_{et/Nis}$ is equal to the i-th elementary symmetric function in c_1's from components (and the fact that $2 = 0$), we obtain also that $\varepsilon^*(c_i)$ is either u_i^2, or $\tau \cdot u_i^2$, depending on parity. Theorem 3.1.1 is proven. $\square$

3.1.8 Proposition. *In the category* $\mathrm{DM}_{eff}^-(BO(n); \mathbb{Z}/2)$ *of motives over* $BO(n)$ *with* $\mathbb{Z}/2$*-coefficients we have:*

$$M(EO(n) \to BO(n)) = \otimes_{i=1}^n \mathrm{Cone}[-1]\left(T \stackrel{u_i}{\to} T([i/2])[i]\right),$$

where $T = T_{BO(n)}$ *is the Tate-motive.*

Proof. It follows by induction on n from the exact triangles involving $\widetilde{B}O(n-1)$ and $BO(n)$, the fact that the functor $\varphi^* : \mathrm{DM}_{eff}^-(BO(n-1); \mathbb{Z}/2) \to \mathrm{DM}_{eff}^-(\widetilde{B}O(n-1); \mathbb{Z}/2)$ maps $M(EO(n-1) \to BO(n-1))$ to $M(EO(n-1) \times EO(n) \to \widetilde{B}O(n-1))$, and the fact that $u_i \in \mathrm{H}_{\mathcal{M}}^{i,[i/2]}(BO(i), \mathbb{Z}/2)$ comes from $BO(n)$. $\square$

We call u_i the *subtle Stiefel–Whitney classes*. Clearly, under the topological realization functor these project to the topological Stiefel–Whitney classes just as the usual Stiefel–Whitney classes ω_i. Notice that the motivic cohomology (with $\mathbb{Z}/2$-coefficients) for $BO(n)$ look much simpler than for $BO(n)_{et/Nis}$ (computed by N.Yagita in [21, Theorem 8.1]).

The action of the Steenrod algebra is also as simple as in the topological case (provided that -1 is a square in k).

3.1.9 Proposition.

$$Sq^k(u_m) = \sum_{j=0}^{k} \binom{m-k}{j} u_{k-j} u_{m+j}.$$

Proof. For $n = 1$ we have $O(1) \cong \mathbb{Z}/2$, $Sq^1(u_1) = u_1^2$, and $Sq^k(u_1) = 0$, for $k > 1$. The general case can be obtained using Proposition 3.1.7. By [17, §9], we have a multiplicative operation $R^\bullet = \sum_i (Sq^{2i} + Sq^{2i+1} \cdot \tau^{\frac{1}{2}})$. Since $R^\bullet(u_1) = u_1 + u_1^2 \cdot \tau^{\frac{1}{2}}$, and $R^\bullet(\tau) = \tau$ (as -1 is a square in k), we get:

$$R^\bullet(\delta^*(u_m)) = \tau^{[m/2]} \cdot \sigma_m(x_1(1 + x_1\tau^{\frac{1}{2}}), \ldots, x_n(1 + x_n\tau^{\frac{1}{2}})).$$

It implies what we need since $\binom{m-k}{j} = 0 \bmod 2$, if $[k/2] + [m/2] \neq [k-j/2] + [m+j/2]$. $\square$

An important role in the computation of the map $\mathrm{H}^{*,*'}_{\mathcal{M}}(BO(n),\mathbb{Z}/2)\to\mathrm{H}^{*,*'}_{\mathcal{M}}(\mathcal{X}_X,\mathbb{Z}/2)$ will be played by the restrictions induced by the embedding of groups $O(m)\times O(l)\subset O(n)$, where $n=m+l$.

3.1.10 Proposition. *The map* $\mathrm{H}^{*,*'}_{\mathcal{M}}(BO(n),\mathbb{Z}/2)\to\mathrm{H}^{*,*'}_{\mathcal{M}}(B(O(m)\times O(l)),\mathbb{Z}/2)$ *is given by:*

$$u_r\mapsto\sum_{i=0}^{r}u_i\otimes u_{r-i}\cdot\tau^{[\frac{r}{2}]-[\frac{i}{2}]-[\frac{r-i}{2}]}.$$

Proof. It follows immediately from Proposition 3.1.7. □

3.2 Towards classification of quadratic forms

Let us see how the above techniques can be used to classify torsors for an orthogonal group $G=O(n)$, that is, quadratic forms.

Let $G=O(n)=O(q_n)$, where $q_n=\perp_{i=1}^{n}\langle(-1)^{i-1}\rangle$ is the standard split form of dimension n. Everywhere below we assume that (-1) is a square in k, so this form coincides with the one considered in 3.1. G-torsors are in 1-to-1 correspondence with quadratic forms q, where $X_q:=Iso(q\to q_n)$ is the variety of isomorphisms (of course, it has no rational point unless q is standard split). It has the natural left action of $G=Iso(q_n\to q_n)$, as well as the right action of $G_q=Iso(q\to q)$ (and it is a torsor under both).

To a quadratic form q we can associate the variety of complete isotropic flags F_q. If $\dim(q)$ is even, then it has the property that, for any field extension L/k,

$$F_q(L)\neq\emptyset\;\Leftrightarrow\;q\cong\perp_j\mathbb{H}\;\Leftrightarrow\;(X_q)_L\text{ is trivial}\;\Leftrightarrow\;X_q(L)\neq\emptyset.$$

Thus, by (2.3.11), in the case of even-dimensional q, $\mathcal{X}_{X_q}=\mathcal{X}_{F_q}$ (canonically).

If $\dim(q)$ is odd, let $a=\det_{\pm}(q)\in k^*/(k^*)^2$ be it's signed determinant. Then

$$F_q(L)\neq\emptyset\text{ and }X_{\langle a\rangle}(L)\neq\emptyset\;\Leftrightarrow\;q\cong(\perp_j\mathbb{H})\perp\langle1\rangle\;\Leftrightarrow\;(X_q)_L\text{ is trivial}\;\Leftrightarrow\;X_q(L)\neq\emptyset.$$

Thus, in this case, $\mathcal{X}_{X_q}=\mathcal{X}_{F_q}\times\mathcal{X}_{X_{\langle a\rangle}}$, where $a=\det_{\pm}(q)$.

We can see that the object $\mathcal{X}_{X_q}$ (of $\mathcal{H}_s(k)$, or $\mathcal{H}(k)$) itself carries the information of where q is split, but it does not remember q.

3.2.1 Example. *Let* $q=\langle\langle a\rangle\rangle\cdot p$, *where* $\dim(p)$ *is odd. Then, for any extension* L/k,

$$q_L\textit{ is split}\;\Leftrightarrow\;\langle\langle a\rangle\rangle_L\textit{ is split.}$$

Thus, $\mathcal{X}_{X_q}=\mathcal{X}_{X_{\langle\langle a\rangle\rangle}}$.

Still $\mathcal{X}_{X_q}$ remembers various interesting things, for example, the J-invariant $J(q)$ (see [12, Definition 5.11] for the definition). Recall, that the $\mathcal{H}_s$-map $\mathcal{X}_{X_q}\stackrel{\alpha_{X_q}}{\to}BO(n)$ does remember q itself.

Now we can use u_i's to reconstruct the motive of a torsor X out of the motive of $\mathcal{X}_X$. We have the following diagram with cartesian squares in Spc:

$$\begin{array}{ccccc} X\times EO(n) & \longrightarrow & EO(n)_{gm}\times EO(n) & \longrightarrow & EO(n)\\ \downarrow{\scriptstyle p} & & \downarrow & & \downarrow\\ \mathcal{X}_X & \longrightarrow & O(n)\backslash(EO(n)_{gm}\times EO(n)) & \longrightarrow & BO(n), \end{array}\qquad(6)$$

where the map p is given by: $(x, g_0, g_1, \ldots, g_m) \mapsto (g_m^{-1}x, \ldots, g_0^{-1}x)$.

3.2.2 Theorem. *Let X be an $O(n)$-torsor. Then in $\mathrm{DM}^-_{eff}(k;\mathbb{Z}/2)$,*

$$M(X) = \otimes_{i=1}^n \mathrm{Cone}[-1]\left(M(\mathcal{X}_X) \xrightarrow{u_i(X)} M(\mathcal{X}_X)([i/2])[i]\right).$$

Proof. It follows from (6) that $M(X \times EO(n) \to \mathcal{X}_X)$ is just the pull-back of $M(EO(n) \to BO(n))$ (whose description we know from Proposition 3.1.8) under the restriction functor:

$$\mathrm{DM}^-_{eff}(BO(n);\mathbb{Z}/2) \to \mathrm{DM}^-_{eff}(\mathcal{X}_X;\mathbb{Z}/2).$$

This functor respects tensor products. It remains to apply the forgetful functor

$$\mathrm{DM}^-_{eff}(\mathcal{X}_X;\mathbb{Z}/2) \to \mathrm{DM}^-_{eff}(k;\mathbb{Z}/2)$$

which sends our motive to $M(X)$ and also respects tensor products since the diagonal map $M(\mathcal{X}_X) \to M(\mathcal{X}_X) \otimes M(\mathcal{X}_X)$ is an isomorphism (see [19, Lemma 6.8, Example 6.3]). □

If $n = \dim(q)$ is even, we have an action of $\mathbb{G}_m(k)$ on $\mathrm{H}^{*,*'}_{\mathcal{M}}(BO(n),\mathbb{Z}/2)$. Let $q' = \lambda \cdot q$ be two proportional forms of the same (even) dimension n. Then we have the canonical identification $\mathcal{X}_{X_q} \cong \mathcal{X}_{X_{q'}}$.

3.2.3 Proposition. *There is a commutative diagram:*

$$\begin{array}{ccc}
\mathrm{H}^{*,*'}_{\mathcal{M}}(BO(n),\mathbb{Z}/2) & \xrightarrow{\alpha^*_{X_q}} & \mathrm{H}^{*,*'}_{\mathcal{M}}(\mathcal{X}_{X_q},\mathbb{Z}/2) \\
\downarrow{\scriptstyle\varphi_\lambda} & & \| \\
\mathrm{H}^{*,*'}_{\mathcal{M}}(BO(n),\mathbb{Z}/2) & \xrightarrow[\alpha^*_{X_{q'}}]{} & \mathrm{H}^{*,*'}_{\mathcal{M}}(\mathcal{X}_{X_{q'}},\mathbb{Z}/2),
\end{array}$$

where φ_λ is an automorphism over H s. t. $\varphi_\lambda(u_{2i+1}) = u_{2i+1}$, $\varphi_\lambda(u_{2i}) = u_{2i} + \{\lambda\} \cdot u_{2i-1}$.

Proof. We start with the 1-dimensional case. Consider quadratic forms $\langle 1\rangle$ and $\langle\lambda\rangle$. We have the following generalization of Theorem 3.1.1, which can be proven in exactly the same way.

3.2.4 Proposition. *Let $\mathcal{Y}$ be smooth simplicial scheme. Then*

$$\mathrm{H}^{*,*'}_{\mathcal{M}}(\mathcal{Y} \times BO(n),\mathbb{Z}/2) = \mathrm{H}^{*,*'}_{\mathcal{M}}(\mathcal{Y},\mathbb{Z}/2)[u_1,\ldots,u_n],$$

where u_i are subtle Stiefel–Whitney classes.

We have a torsor triple $(O(\langle 1\rangle), Y, O(\langle\lambda\rangle))$, where $Y = Iso(\langle\lambda\rangle \to \langle 1\rangle)$. $\mathcal{X}_Y = \mathcal{X}_{\langle\langle\lambda\rangle\rangle}$. By Proposition 3.2.4, $\mathrm{H}^{*,*'}_{\mathcal{M}}(\mathcal{X}_Y \times BO(\langle 1\rangle),\mathbb{Z}/2) = \mathrm{H}^{*,*'}_{\mathcal{M}}(\mathcal{X}_{\langle\langle\lambda\rangle\rangle},\mathbb{Z}/2)[u_1]$. Our groups can be identified: $O(\langle\lambda\rangle) = O(\langle 1\rangle) = \mathbb{Z}/2$, and by Proposition 2.6.1 we have an identification:

$$\mathcal{X}_Y \times BO(\langle\lambda\rangle) \overset{\theta_Y^{Zar}}{\cong} BO(\langle 1\rangle) \times \mathcal{X}_Y.$$

3.2.5 Lemma. $\mathrm{H}^{*,*'}_{\mathcal{M}}(\mathcal{X}_{\langle\langle\lambda\rangle\rangle},\mathbb{Z}/2) = H[\gamma]/(\tau\cdot\gamma = \{\lambda\}; \gamma\cdot Ann_{\{\lambda\}} = 0)$, *where $\gamma \in \mathrm{H}^{1,0}_{\mathcal{M}}(\mathcal{X}_{\langle\langle\lambda\rangle\rangle},\mathbb{Z}/2)$ and $Ann_{\{\lambda\}}$ is the annulator of $\{\lambda\}$ in $K^M_*(k)/2$.*

Proof. In $\mathrm{DM}^{-}_{eff}(k;\mathbb{Z}/2)$ we have an exact triangle (cf. [18, Theorem 4.4]):

$$\begin{array}{ccc} & M(\mathrm{Spec}(k\sqrt{\lambda})) & \\ \nearrow & \star & \searrow \\ M(\mathcal{X}_{\langle\langle\lambda\rangle\rangle}) & \xleftarrow[{[1]}]{\gamma} & M(\mathcal{X}_{\langle\langle\lambda\rangle\rangle}). \end{array}$$

The map γ here is given by the same-named element $\gamma \in \mathrm{H}^{1,0}_{\mathcal{M}}(\mathcal{X}_{\langle\langle\lambda\rangle\rangle},\mathbb{Z}/2)$, and since $\mathrm{Spec}(k\sqrt{\lambda})$ is a zero-dimensional pure motive, multiplication by γ is an isomorphism on all diagonals starting from the 1-st one, and a surjection on the 0-th diagonal. On the other hand, it follows from [18] that multiplication by τ identifies the 1-st diagonal with the $\mathrm{Ker}(K^M_*(k)/2 \to K^M_*(k\sqrt{\lambda})/2)$, which is a principal ideal in $K^M_*(k)/2$ generated by $\{\lambda\}$. Thus, for each $i \geqslant 1$, the i-th diagonal is a cyclic module over $K^M_*(k)/2$ generated by γ^i and isomorphic to $\{\lambda\}\cdot K^*(k)/2$. Clearly, $\tau\cdot\gamma = \{\lambda\}$. The rest of the description follows. □

Let $\mathcal{X}_Y \stackrel{\alpha_Y}{\to} BO(\langle 1\rangle)$ be the fiber over $[Y]$.

3.2.6 Lemma. *We have: $\alpha_Y^*(u_1) = \gamma$. In particular, the map*

$$\alpha_Y^* : \mathrm{H}^{*,*'}_{\mathcal{M}}(BO(\langle 1\rangle),\mathbb{Z}/2) \to \mathrm{H}^{*,*'}_{\mathcal{M}}(\mathcal{X}_{\langle\langle\lambda\rangle\rangle},\mathbb{Z}/2)$$

is surjective.

Proof. We know that $\alpha_Y^*(\omega_1) = \{\lambda\}$. Hence, $\alpha_Y^*(u_1) = \gamma$. □

It follows from Proposition 2.6.1 that $(\theta_Y^{Zar})^*(u_1) = u_1 + \gamma$.

Let now $q_n = \perp_{i=1}^{n} \langle 1\rangle$, and $q'_n = \lambda\cdot q_n$. Consider the torsor triple $(O(q_n), X, O(q'_n))$, where $X = Iso(q'_n \to q_n)$. Since q_n is even-dimensional split, the torsor X is trivial. Hence, by Proposition 2.6.1, we have an identification $BO(q'_n) \xrightarrow[=]{\theta_X^{Zar}} BO(q_n)$. It can be extended to a commutative diagram:

$$\begin{array}{ccc} \mathcal{X}_{\langle\langle\lambda\rangle\rangle} \times BO(q'_n) & \xrightarrow{\theta} & BO(q_n) \times \mathcal{X}_{\langle\langle\lambda\rangle\rangle} \\ \uparrow & & \uparrow \\ \mathcal{X}_{\langle\langle\lambda\rangle\rangle} \times \times_{j=1}^{n} BO(\langle\lambda\rangle) & \xrightarrow[\tilde{\theta}]{} & \times_{j=1}^{n} BO(\langle 1\rangle) \times \mathcal{X}_{\langle\langle\lambda\rangle\rangle} \end{array}$$

Computing the induced maps on u_r we get:

$$\begin{array}{ccc} \sum_{j=0}^{r}\binom{n-j}{r-j} u_j\cdot\gamma^{r-j}\cdot\tau^{[r/2]-[j/2]} & & u_r \\ \downarrow & & \downarrow \\ \sigma_r(x_1+\gamma,\ldots,x_n+\gamma)\cdot\tau^{[r/2]} & \longleftarrow & \sigma_r(x_1,\ldots,x_n)\cdot\tau^{[r/2]} \end{array}$$

Therefore,

$$\theta^*(u_r) = \sum_{j=0}^{r}\binom{n-j}{r-j} u_j\cdot\gamma^{r-j}\cdot\tau^{[r/2]-[j/2]}.$$

Notice that $\gamma^2\cdot\tau = \gamma\cdot\{-1\}$, and since -1 is a square in k, we obtain: $\theta^*(u_{2m+1}) = u_{2m+1}$ and $\theta^*(u_{2m}) = u_{2m} + u_{2m-1}\cdot\{\lambda\}$. Since the map:

$$\mathrm{H}^{*,*'}_{\mathcal{M}}(BO(q'_n),\mathbb{Z}/2) \to \mathrm{H}^{*,*'}_{\mathcal{M}}(\mathcal{X}_{\langle\langle\lambda\rangle\rangle}\times BO(q'_n),\mathbb{Z}/2)$$

is injective by Proposition 3.2.4, the same formulas work for the map $(\theta_X^{Zar})^*$. If now q is an arbitrary n-dimensional form, and $q' = \lambda \cdot q$, then the torsor $Z' = Iso(q' \to q'_n)$ can be canonically identified with the torsor $Z = Iso(q \to q_n)$ so that we have a commutative diagram:

$$\begin{array}{ccc} \mathcal{X}_{Z'} & = & \mathcal{X}_Z \\ {\scriptstyle \alpha^*_{Z'}}\downarrow & & \downarrow{\scriptstyle \alpha^*_Z} \\ BO(q'_n) & = & BO(q_n), \end{array}$$

where we use the standard identification $O(q'_n) = O(q_n)$ (not θ_X^{Zar}). It remains to observe that the composition $\mathcal{X}_{X_q} \stackrel{\alpha^*_{Z'}}{\to} BO(q'_n) \stackrel{\theta_X^{Zar}}{\longrightarrow} BO(q_n)$ is equal to the composition: $\mathcal{X}_{X_q} = \mathcal{X}_{X_{q'}} \stackrel{\alpha^*_{X_{q'}}}{\longrightarrow} BO(q_n)$. Proposition 3.2.3 is proven. □

We can use the *subtle Stiefel–Whitney classes* to reconstruct the motive of the highest quadratic Grassmanian corresponding to $X = X_q$. Let $d = [n/2]$ and $P_d = P_d^n$ be the stabilizator in $O(n)$ of the totally-isotropic subspace of maximal dimension (equal to d). Then $G_d(X) = P_d \backslash X$ is the variety of projective subspaces of maximal dimension on the projective quadric Q (defined by the form q). We have a natural embedding $GL_d \subset P_d$, for even n, and $GL_d \times \mathbb{Z}/2 \subset P_d$, for odd n, coming from the presentation of V_{q_n} as $V \oplus V^*$ and $V \oplus V^* \oplus V_{\langle 1 \rangle}$, respectively.

The cohomology of $B(GL_d)$ can be computed in the same way as for $BO(n)$.

3.2.7 Proposition.

$$\mathrm{H}_{\mathcal{M}}^{*,*'}(B(GL_d), \mathbb{Z}) = \mathrm{H}_{\mathcal{M}}^{*,*'}(\mathrm{Spec}(k), \mathbb{Z})[c_1, c_2, \ldots, c_d],$$

where $c_i \in \mathrm{H}_{\mathcal{M}}^{2i,i}(B(GL_d), \mathbb{Z})$ *coincides with the pull-back of the i-th Chern class from* $B(GL_d)_{et/Nis}$. *Also, in* $\mathrm{DM}^-_{eff}(B(GL_d))$,

$$M(E(GL_d) \to B(GL_d)) = \otimes_{j=1}^d \mathrm{Cone}[-1] \left(T \stackrel{c_j}{\longrightarrow} T(j)[2j] \right),$$

where $T = T_{B(GL_d)}$ *is the Tate-motive.*

Proof. From the tower of subgroups

$$\{e\} \to GL_1 \to GL_2 \to \ldots \to GL_{d-1} \to GL_d,$$

we get a tower of fibrations:

$$E(GL_d) \stackrel{g_1}{\longrightarrow} \widetilde{B}(GL_1) \stackrel{g_2}{\longrightarrow} \widetilde{B}(GL_2) \longrightarrow \ldots \longrightarrow \widetilde{B}(GL_{d-1}) \stackrel{g_d}{\longrightarrow} B(GL_d),$$

(where $\widetilde{B}(GL_i)$ is $\mathcal{H}_s$-isomorphic to $B(GL_i)$) which is trivial over simplicial components. Since $GL_{d-1} \backslash GL_d \cong \mathbb{A}^d \backslash 0$, whose motive in $\mathrm{DM}^-_{eff}(k)$ is $\mathbb{Z} \oplus \mathbb{Z}(d)[2d-1]$, we get that in $\mathrm{DM}^-_{eff}(B(GL_d))$ there is a distinguished triangle

$$\begin{array}{ccccc} & & M(\widetilde{B}(GL_{d-1})) & & \\ & \stackrel{[1]}{\nearrow} & \star & \stackrel{g_d}{\searrow} & \\ M(B(GL_d))(d)[2d] & & \longleftarrow & & M(B(GL_d)) \end{array}$$

And again, by induction on d and degree considerations, we get that g_d^* is surjective on $\mathrm{H}_{\mathcal{M}}^{*,*'}(-,\mathbb{Z})$ with the kernel generated by $c_d \in \mathrm{H}_{\mathcal{M}}^{2i,i}(B(GL_d),\mathbb{Z})$, and $c_1,\dots,c_{d-1}$ are lifted uniquely to $\mathrm{H}_{\mathcal{M}}^{*,*'}(B(GL_d),\mathbb{Z})$. Simultaneously, we get that in $\mathrm{DM}_{eff}^{-}(B(GL_d))$,

$$M(\widetilde{B}(GL_{d-1}) \to B(GL_d)) = \mathrm{Cone}[-1]\left(T \xrightarrow{c_d} T(d)[2d]\right).$$

Recalling that c_i comes from $B(GL_d)$, we get the description of $M(E(GL_d) \to B(GL_d))$. $\square$

3.2.8 Remark. *In particular, the map* $\varepsilon^* : \mathrm{H}_{\mathcal{M}}^{*,*'}(B(GL_d)_{et/Nis},\mathbb{Z}) \to \mathrm{H}_{\mathcal{M}}^{*,*'}(B(GL_d),\mathbb{Z})$ *is an isomorphism. But, actually, the very map* $\varepsilon : B(GL_d) \to B(GL_d)_{et/Nis}$ *is an isomorphism in* $\mathcal{H}_s(k)$ *by* [7, Lemma 4.1.18].

3.2.9 Proposition. *In* $\mathcal{H}(k)$*, we have an identification:* $B(P_d^n) = B(GL_d)$*, for even* n*, and* $B(P_d^n) = B(GL_d \times \mathbb{Z}/2)$*, for odd* n*. In particular,*

$$\mathrm{H}_{\mathcal{M}}^{*,*'}(B(P_d^n),\mathbb{Z}/2) = \begin{cases} H[c_1,\dots,c_d], & \text{if } n \text{ is even;} \\ H[u_1,c_1,\dots,c_d], & \text{if } n \text{ is odd.} \end{cases}$$

Proof. For n even, we have a decomposition: $P_d^n = GL_d \cdot U$, where U consists of transformations: $(v,v^*) \mapsto (v+f(v^*),v^*)$, where $f: V^* \to V$ is a linear map with the property: $< f(v^*),v^* >= 0$.

For n odd, we have a decomposition: $P_d^n = (GL_d \times \mathbb{Z}/2)\cdot U'$, where U' consists of transformations: $(v,v^*,\alpha) \mapsto (v+f(v^*)+\alpha\cdot w, v^*, \alpha - \frac{1}{2} < w,v^* >)$, where $w \in V$ is an arbitrary vector, and $f : V^* \to V$ is a linear map with the property:
$< f(v^*),v^* >= -\frac{1}{4}(< w,v^* >)^2$.

Since U and U' are isomorphic to affine spaces, we have an identification in $\mathcal{H}(k)$:

$$B(P_d^n) = \begin{cases} B(GL_d), & \text{if } n \text{ is even;} \\ B(GL_d \times \mathbb{Z}/2), & \text{if } n \text{ is odd.} \end{cases}$$

$\square$

We have the following natural diagram in Spc:

$$BP_d^n \xleftarrow{\varphi} \widetilde{B}P_d^n \xrightarrow{\psi} \hat{B}P_d^n,$$

where $\widetilde{B}P_d^n = P_d^n \backslash (EP_d^n \times EO(n))$, and $\hat{B}P_d^n = P_d^n \backslash EO(n)$. Since the projection $O(n) \to P_d^n \backslash O(n)$ is split over each point of the base, the maps φ and ψ are isomorphisms in $\mathcal{H}_s(k)$. In particular, $M(\widetilde{B}P_d^n \xrightarrow{\psi} \hat{B}P_d^n) = T \in \mathrm{DM}_{eff}^{-}(\hat{B}P_d^n;\mathbb{Z}/2))$.

3.2.10 Proposition. *Under the natural projection* $\hat{B}(P_d^n) \xrightarrow{f} BO(n)$*, we have:* $f^*(u_{2i}) = c_i$*, and* $f^*(u_{2i+1}) = 0$*, for even* n*, and* $f^*(u_{2i+1}) = c_i \cdot u_1$*, for odd* n*. In particular,*

$$\mathrm{H}_{\mathcal{M}}^{*,*'}(B(P_d^n),\mathbb{Z}/2) = \begin{cases} \mathrm{H}_{\mathcal{M}}^{*,*'}(BO(n),\mathbb{Z}/2)/(u_{2i+1},\ 0 \leqslant i < n/2), & \text{if } n \text{ is even;} \\ \mathrm{H}_{\mathcal{M}}^{*,*'}(BO(n),\mathbb{Z}/2)/(u_{2i+1} - u_{2i}\cdot u_1,\ 0 < i < n/2), & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Consider the diagram of group embeddings:

$$\begin{array}{ccc} P^{n-2}_{d-1} & \longrightarrow & P^n_d \\ \downarrow & & \downarrow \\ O(n-2) & \longrightarrow & O(n). \end{array}$$

If n is even, then the composition

$$GL_{d-1}\backslash GL_d \to P^{n-2}_{d-1}\backslash P^n_d \to O(n-2)\backslash O(n) \to O(n-1)\backslash O(n)$$

is an isomorphism in $\mathrm{DM}^-_{eff}(k)$. If n is odd, then the composition

$$GL_{d-1}\backslash GL_d \to P^{n-2}_{d-1}\backslash P^n_d \to O(n-2)\backslash O(n)$$

factors through the map $GL_{d-1}\backslash GL_d \to O(n-2)\backslash O(n-1)$, which is an isomorphism in $\mathrm{DM}^-_{eff}(k)$. This implies that $c_i = f^*(u_{2i})$. The fact that f^* of odd subtle Stiefel–Whitney classes is zero, for even n, follows from the fact that there are no elements of such grading in $\mathrm{H}^{*,*'}_{\mathcal{M}}(B(GL_d), \mathbb{Z}/2)$. And for odd n, we observe that the map $GL_d \times \mathbb{Z}/2 \to O(n)$ factors through $O(n-1) \times O(1) \to O(n)$. It remains to apply Proposition 3.1.10. □

Note that the fibration $\hat{B}(P^n_d) \to BO(n)$ is trivial over the graded components with fibers - the split Grassmannian $P^n_d\backslash O(n)$. In particular, the graded components

$$r_i^* M(\hat{B}(P^n_d) \to BO(n)) \in \mathrm{DM}^-_{eff}((BO(n))_i; \mathbb{Z}/2)$$

belong to the thick subcategory $DT((BO(n))_i)$ generated by Tate-motives. We have the following general fact:

3.2.11 Proposition. *Let F be a field of characteristic p, $\mathcal{Y}$ be smooth simplicial scheme over smooth scheme S such that $\pi = \pi_1(CC(\mathcal{Y}))$ is a p-group. Let $N \in \mathrm{DM}^-_{eff}(\mathcal{Y}, F)$ be such motive that its graded components $N_i \in \mathrm{DM}^-_{eff}(\mathcal{Y}_i, F)$ belong to $DT(\mathcal{Y}_i)$, and for each $\theta : [i] \to [j] \in Mor(\Delta)$ the natural map $f_\theta : (\mathcal{Y}^*_\theta)(N_j) \to N_i$ is an isomorphism. Then $N \in DT(\mathcal{Y})$.*

Proof. By [19, Lemma 5.9] we have a slice filtration on each of N_i with only finitely many nontrivial graded pieces $s_m(N_i) \in DT_m(\mathcal{Y}_i)$. Moreover, for each $\theta : [i] \to [j] \in Mor(\Delta)$, we have the natural map $f_\theta : (\mathcal{Y}^*_\theta)(N_j) \to N_i$ which uniquely extends to the filtration according to [19, Lemma 5.11]. It follows from [19, Remarks 5.19, 5.21] and our conditions on F and $\mathcal{Y}$ (which guarantee that any $F[\pi]$-module is an extension of trivial ones) that $DT'_0(\mathcal{Y}) = DLC(\mathcal{Y})$ (loc. cit.). Let $l = \min(m |\, s_m(N_i) \neq 0)$ (for some = for any i). Then $s_l(N_i)$ considered as an element of $DT'_0(\mathcal{Y}_i)$ (see [19, Proposition 5.20]) has natural filtration coming from the t-structure with the heart $LC(\mathcal{Y}_i) = F - mod$ ([19, Remark 5.21]). This filtration is respected by the maps f_θ, which provide the action of π on the t-graded pieces $(s_l(N_i))_k$. If $r = \min(k |\, (s_l(N_i))_k \neq 0)$, then representing $(s_l(N_i))_r$ as an extension of trivial π-modules, we get the natural map $(s_l(N_i))_r \to \oplus T(l)[r]$ (the coinvariants of the π-action) which is respected by the maps f_θ and so gives the map $Lc_\# N \to \oplus T(l)[r]$ in $\mathrm{DM}^-_{eff}(S, F)$ (here we are using arguments from the proof of Proposition 3.1.5). By conjugation we get the map $\varphi : N \to \oplus T(l)[r]$. Clearly the motive $N' := Cone[-1](\varphi)$ still satisfies the conditions of our Proposition. Repeating this process we will eventually kill the $(l)[r]$-component of N, and the induction on l and r finishes the proof. □

Now we can compute the motive of $\hat{B}(P^n_d)$ in $\mathrm{DM}^-_{eff}(BO(n);\mathbb{Z}/2)$.

3.2.12 Proposition. *In* $\mathrm{DM}^-_{eff}(BO(n);\mathbb{Z}/2)$*, we have the natural identification:*

$$M(\hat{B}(P^n_d) \to BO(n)) = \otimes_{\delta\leqslant j<n/2}\,\mathrm{Cone}[-1]\left(T \stackrel{v_{2j+1}}{\longrightarrow} T(j)[2j+1]\right),$$

where $v_{2j+1} = u_{2j+1}$*, for even* n*,* $= u_{2j+1} - u_{2j}\cdot u_1$*, for odd* n*, and* $\delta = n-2d$*.*

Proof. Since $f^*(v_{2j+1}) = 0$, and v_{2j+1}'s form a regular sequence, it follows that the map

$$M(\hat{B}(P^n_d) \stackrel{f}{\to} BO(n)) \to T$$

factors through $\otimes_{\delta\leqslant j<n/2}\,\mathrm{Cone}[-1]\left(T \stackrel{v_{2j+1}}{\longrightarrow} T(j)[2j+1]\right)$, and the respective map induces an isomorphism on $\mathrm{H}^{*,*'}_{\mathcal{M}}(-,\mathbb{Z}/2)$. Observing that $CC(BO(n)) = K(\mathbb{Z}/2,1)$, from Proposition 3.2.11 we obtain that $M(\hat{B}(P^n_d) \to BO(n))$ belongs to $DT(BO(n))$. So, we have a morphism between two objects of DT which gives an isomorphism on cohomology. It must be an isomorphism by [19, Lemma 5.2]. Thus,

$$M(\hat{B}(P^n_d) \to BO(n)) = \otimes_{\delta\leqslant j<n/2}\,\mathrm{Cone}[-1]\left(T \stackrel{v_{2j+1}}{\longrightarrow} T(j)[2j+1]\right),$$

in $\mathrm{DM}^-_{eff}(BO(n);\mathbb{Z}/2)$. □

Now we can compute the motive of G_d.

3.2.13 Theorem. *Let* q *be a quadratic form of even dimension* $n = 2d$*, and* $G_d(q)$ *be it's highest quadratic Grassmannian. Then, in* $\mathrm{DM}^-_{eff}(k;\mathbb{Z}/2)$*,*

$$M(G_d(q)) = \otimes_{0\leqslant j\leqslant d-1}\,\mathrm{Cone}[-1]\left(M(\mathcal{X}_{G_d(q)}) \stackrel{u_{2j+1}(q)}{\longrightarrow} M(\mathcal{X}_{G_d(q)})(j)[2j+1]\right).$$

Proof. Let $X = X_q$ be the respective $O(n)$-torsor. We have a diagram with cartesian squares in Spc:

$$\begin{array}{ccccc} EO(n) & \longrightarrow & \hat{B}(P^n_d) & \longrightarrow & BO(n) \\ \uparrow & & \uparrow & & \uparrow \\ X\times EO(n) & \longrightarrow & P^n_d\backslash(X\times EO(n)) & \longrightarrow & \mathcal{X}_X, \end{array}$$

coming from the P^n_d and $O(n)$ actions. Denote $\tilde{G}_d := P^n_d\backslash(X\times EO(n))$. We obtain that $M(\tilde{G}_d \to \mathcal{X}_X)$ in $\mathrm{DM}^-_{eff}(\mathcal{X}_X;\mathbb{Z}/2)$ is simply the image of the $M(\hat{B}(P^n_d) \to BO(n))$ under the natural functor $\mathrm{DM}^-_{eff}(BO(n);\mathbb{Z}/2) \to \mathrm{DM}^-_{eff}(\mathcal{X}_X;\mathbb{Z}/2)$. Since this functor respects the tensor product, Proposition 3.2.12 implies that

$$M(\tilde{G}_d \to \mathcal{X}_X) = \otimes_{0\leqslant j\leqslant d-1}\,\mathrm{Cone}[-1]\left(T_{\mathcal{X}_{X_q}} \stackrel{u_{2j+1}(q)}{\longrightarrow} T_{\mathcal{X}_{X_q}}(j)[2j+1]\right).$$

It remains to apply the forgetful functor $\mathrm{DM}^-_{eff}(\mathcal{X}_X;\mathbb{Z}/2) \to \mathrm{DM}^-_{eff}(k;\mathbb{Z}/2)$, which also respects the tensor product, since the diagonal map $M(\mathcal{X}_X) \stackrel{\Delta}{\to} M(\mathcal{X}_X)\otimes M(\mathcal{X}_X)$ is an isomorphism (by [19, Lemma 6.8, Example 6.3]). The result will be $M(\tilde{G}_d)$ which coincides with $M(G_d)$ since the projection $X \to G_d$ is split over every point (notice, that this would not work for other Grassmannians, or for odd n). □

There is an odd-dimensional variant as well. Let q be a form of odd dimension $n = 2d+1$, and $p = q \perp \langle a \rangle$, where $a = \det_{\pm}(q)$ be an $(n+1)$-dimensional form from I^2, containing it. Then $\mathcal{X}_{X_q} = \mathcal{X}_{X_p} \times \mathcal{X}_{\{a\}}$, and it follows from Proposition 3.1.10 that, for $\mathcal{X}_{X_q} \stackrel{\nu}{\to} \mathcal{X}_{X_p}$, $\nu^*(u_{2j+1}(p)) = u_{2j+1}(q) + u_{2j}(q) \cdot u_1(q)$. Taking into account that $G_{d+1}(p) = G_d(q) \coprod G_d(q)$, and $\mathcal{X}_{X_p} = \mathcal{X}_{G_d(q)}$ we get:

3.2.14 Proposition. *Let q be a form of odd dimension $n = 2d+1$, and $p = q \perp \langle \det_{\pm}(q) \rangle$. Then the motive of the highest Grassmannian of q can be presented as:*

$$M(G_d(q)) = \otimes_{1 \leqslant j \leqslant d} \operatorname{Cone}[-1] \left(M(\mathcal{X}_{G_d(q)}) \stackrel{u_{2j+1}(p)}{\longrightarrow} M(\mathcal{X}_{G_d(q)})(j)[2j+1] \right).$$

3.2.15 Example. *Let $q = \langle a, b, -ab, -c, -d, cd \rangle$ be an Albert form. Then $d = 3$, and $G_3(q) = S \coprod S$, where $S = SB(\{a,b\} + \{c,d\})$ is the Severi-Brauer variety corresponding to the element $\{a,b\} + \{c,d\} \in K_2^M(k)/2$. It follows from the above that*

$$M(S) = \operatorname{Cone}[-1] \left(M(\mathcal{X}_S) \stackrel{u_3(q)}{\longrightarrow} M(\mathcal{X}_S)(1)[3] \right) \otimes \operatorname{Cone}[-1] \left(M(\mathcal{X}_S) \stackrel{u_5(q)}{\longrightarrow} M(\mathcal{X}_S)(2)[5] \right).$$

Even in this simple case, the decomposition into tensor product of binary motives was unknown (though, expected ... for 18 years).

3.2.16 Remark. *Another case where the presentation of the motive of a variety as an extension of motives of Chech simplicial schemes is known is the case of a quadric. The canonical decomposition there was obtained in* [11, Theorems 3.1, 3.7]*. Though, in the case of the highest quadratic Grassmannian above we get nice poly-binary structure with the precise description of connections involved and all elementary pieces of the same kind (as opposed to the case of a quadric), which is related to the fact that $G_d(q)$ is generically split.*

We have the following "flexible" versions of Proposition 3.1.8, Proposition 3.2.12, and Theorem 3.2.2, Theorem 3.2.13 (Proposition 3.2.14), respectively.

3.2.17 Proposition. *Let $\widetilde{u}_i = u_i +$ decomposable terms $\in \mathrm{H}_{\mathcal{M}}^{i,[i/2]}(BO(n), \mathbb{Z}/2)$, for $i = 1, \ldots, n$ be some elements. Then in $\mathrm{DM}_{eff}^{-}(BO(n); \mathbb{Z}/2)$ and in $\mathrm{DM}_{eff}^{-}(k; \mathbb{Z}/2)$, respectively:*

(1) $$M(EO(n) \to BO(n)) = \otimes_{i=1}^{n} \operatorname{Cone}[-1] \left(T \stackrel{\widetilde{u}_i}{\to} T([i/2])[i] \right),$$

(2) $$M(X) = \otimes_{i=1}^{n} \operatorname{Cone}[-1] \left(M(\mathcal{X}_X) \stackrel{\widetilde{u}_i(X)}{\longrightarrow} M(\mathcal{X}_X)([i/2])[i] \right).$$

Proof. Since the sequence $\widetilde{u}_i$, $i = 1, \ldots, n$ is regular, the natural map $M(EO(n) \to BO(n)) \to T$ can be factored through $\otimes_{i=1}^{n} \operatorname{Cone}[-1] \left(T \stackrel{\widetilde{u}_i}{\to} T([i/2])[i] \right)$ inducing an isomorphism on $\mathrm{H}_{\mathcal{M}}^{*,*'}$. By Proposition 3.1.8, $M(EO(n) \to BO(n))$ belongs to the thick subcategory $DT(BO(n))$ generated by Tate-motives. By [19, Lemma 5.2], our map is an isomorphism. This settles 1). Then 2) follows as in the proof of Theorem 3.2.2. □

The case of Grassmannians can be done in exactly the same way (we formulate the even dimensional case only, the other one is analogous):

3.2.18 Proposition. *Let $n = 2d$ be even, and*

$$\widetilde{u_{2j+1}} = u_{2j+1} + \textit{decomposable terms} \in \mathrm{H}^{2j+1,j}_{\mathcal{M}}(BO(n), \mathbb{Z}/2), \quad (j = 0, \dots, d-1).$$

Then in $\mathrm{DM}^{-}_{eff}(BO(n); \mathbb{Z}/2)$ and in $\mathrm{DM}^{-}_{eff}(k; \mathbb{Z}/2)$, respectively:

$$(1) \qquad M(\hat{B}(P^n_d) \to BO(n)) = \otimes_{0 \leqslant j \leqslant d-1} \mathrm{Cone}[-1]\left(T \stackrel{\widetilde{u_{2j+1}}}{\longrightarrow} T(j)[2j+1]\right),$$

$$(2) \qquad M(G_d(q)) = \otimes_{0 \leqslant j \leqslant d-1} \mathrm{Cone}[-1]\left(M(\mathcal{X}_{G_d(q)}) \stackrel{\widetilde{u_{2j+1}}(q)}{\longrightarrow} M(\mathcal{X}_{G_d(q)})(j)[2j+1]\right).$$

Theorem 3.2.13 and Proposition 3.2.14 permit us to connect the subtle Stiefel–Whitney classes with the J-invariant of q (see [12]).

3.2.19 Proposition. *Let q be a quadratic form of dimension n, and $p = q$, if n is even, and $p = q \perp \langle \det_{\pm}(q) \rangle$, if n is odd. Then:*

$$\min\{j |\, j \notin J(q)\} = \min\{j |\, u_{2j+1}(p) \neq 0\}.$$

Proof. By the Main Theorem of [12], $\min\{j |\, j \notin J(q)\}$ is equal to the minimal codimension of non-rational class in $\mathrm{CH}^*(G_d(q)|_{\overline{k}})/2$. By Theorem 3.2.13 (Proposition 3.2.14, respectively), this number is also equal to $\min\{j |\, u_{2j+1}(p) \neq 0\}$. □

But there are much more precise statements. In $\mathrm{DM}^{-}_{eff}(k; \mathbb{Z}/2)$ consider objects:

$$C_{2l+1} := \mathrm{Cone}[-1]\left(\mathcal{X}_{X_q} \stackrel{u_{2l+1}(q)}{\longrightarrow} \mathcal{X}_{X_q}(l)[2l+1]\right).$$

Let $N_{j-1} := \otimes_{0 \leqslant l < j} C_{2l+1}$, and f_j be the composition $N_{j-1} \to \mathcal{X}_{X_q} \stackrel{u_{2j+1}(q)}{\longrightarrow} \mathcal{X}_{X_q}(j)[2j+1]$.

3.2.20 Proposition. *Let q be an $n = 2d$-dimensional quadratic form, and $0 \leqslant j < d$. Then the following conditions are equivalent:*

(1) *$j \in J(q)$;*

(2) *The map $f_j : N_{j-1} \to \mathcal{X}_{X_q}(j)[2j+1]$ is zero.*

Proof. In the category $\mathrm{DM}^{-}_{eff}(BO(n); \mathbb{Z}/2)$ of motives over $BO(n)$ consider objects: $\hat{C}_{2l+1} := \mathrm{Cone}[-1]\left(T \stackrel{u_{2l+1}}{\longrightarrow} T(l)[2l+1]\right)$. Then the natural map $M(\hat{B}(P^n_d) \to BO(n)) \to T$ can be lifted to the map

$$\rho_l : M(\hat{B}(P^n_d) \to BO(n)) \to \hat{C}_{2l+1}.$$

This lifting is defined up to the choice of element of

$$\mathrm{H}^{2l,l}_{\mathcal{M}}(B(P^n_d), \mathbb{Z}/2) = H[u_2, u_4, \dots, u_{2d}]_{(l)[2l]} = \mathbb{Z}/2[u_2, u_4, \dots, u_{2d}]_{(l)[2l]}.$$

The composition:

$$M(\hat{B}(P^n_d) \to BO(n)) \stackrel{\Delta_d}{\longrightarrow} M(\hat{B}(P^n_d) \to BO(n))^{\otimes d} \stackrel{\otimes \rho_l}{\longrightarrow} \otimes_{0 \leqslant l < d} \hat{C}_{2l+1}$$

is a choice of isomorphism of Proposition 3.2.12. Let $(BO(n))_0 = \mathrm{Spec}(k) = \bullet$ be the 0-th graded component of the simplicial scheme $BO(n)$. Then $(\hat{C}_{2l+1})_0$ uniquely splits as $T \oplus T(l)[2l]$, since the Subtle Stiefel–Whitney classes are trivial when restricted to $(BO(n))_0$, and there are no maps between the Tate-motives involved (both by degree consideration, for example). At the same time, $(M(\hat{B}(P^n_d) \to BO(n)))_0$ is the motive $M(G_d(q_n))$ of the split (highest) Grassmannian in $\mathrm{DM}^-_{eff}(k;\mathbb{Z}/2)$. Thus, we get the canonical map $\phi_l : M(G_d(q_n)) \to T(l)[2l]$ giving the class in $\mathrm{CH}^l(G_d(q_n))/2$.

The Chow ring of the split quadratic Grassmannian is generated by the special "elementary classes" z_l - see [12, Proposition 2.4], or [14, Section 2].

3.2.21 Lemma. *We have:* $\phi_l = z_l$.

Proof. We have a commutative diagram in Spc:

$$\begin{array}{ccccc} \hat{B}P^{2l}_l & \longrightarrow & BO(2l) & \xleftarrow{g} & \bullet \\ \downarrow & & \downarrow f & \swarrow h & \\ \hat{B}P^n_d & \longrightarrow & BO(n), & & \end{array}$$

which induces the map: $M(\hat{B}P^{2l}_l \to BO(2l)) \longrightarrow f^*M(\hat{B}P^n_d \to BO(n))$. If we apply g^* to it we will get the natural map $M(G_l(q_{2l})) \xrightarrow{p} M(G_d(q_n))$. We have a natural identification $f^*(\hat{C}_{2l+1}) = \hat{C}_{2l+1}$. Thus, our lifting $f^*M(\hat{B}P^n_d \to BO(n)) \to f^*(\hat{C}_{2l+1})$ will restrict to the lifting $M(\hat{B}P^{2l}_l \to BO(2l)) \to \hat{C}_{2l+1}$. But $\hat{C}_{2l+1}$ in $\mathrm{DM}^-_{eff}(BO(2l);\mathbb{Z}/2)$ is split since $f^*u_{2l+1} = 0$. Thus, the projection to $T(l)[2l]$ is defined already on the level of $M(\hat{B}P^{2l}_l \to BO(2l))$ and so is a polynomial in u_{2i}, $1 \leqslant i \leqslant l$ with $\mathbb{Z}/2$-coefficients. But all these classes vanish under g^*. Hence, $p^*(\phi_l) = 0$, and so $\phi_l = z_l \in \mathrm{CH}^l(G_d(q_n))/2$ by [12, Proposition 2.4(3)]. □

Apply the motivic restriction $\alpha^*_X : \mathrm{DM}^-_{eff}(BO(n);\mathbb{Z}/2) \to \mathrm{DM}^-_{eff}(\mathcal{X}_{X_q};\mathbb{Z}/2)$, corresponding to the map $\alpha_X : \mathcal{X}_{X_q} \to BO(n)$. Notice that this map respects tensor products. Denote the image of $M(\hat{B}(P^n_d) \to BO(n))$ as M, and the image of $\hat{C}_{2l+1}$ as C_{2l+1}. Consider $N := \otimes_{0 \leqslant l < j} C_{2l+1}$ and $N' := \otimes_{j<l<d} C_{2l+1}$. Then N' is an extension of T and $T(r)[*]$, where $r > j$. Since $M = N \otimes C_{2j+1} \otimes N'$, we get an exact triangle: $R \to M \to N \otimes C_{2j+1} \to R[1]$, where R is an extension of $T(r)[*]$ with $r > j$. In particular, $\mathrm{Hom}(M, T(j)[2j]) = \mathrm{Hom}(N \otimes C_{2j+1}, T(j)[2j])$, and we have an exact sequence:

$$\mathrm{Hom}(M, T(j)[2j]) \to \mathrm{Hom}(N(j)[2j], T(j)[2j]) \to \mathrm{Hom}(N, T(j)[2j+1]).$$

Identifying $\mathrm{Hom}(N(j)[2j], T(j)[2j])$ with $\mathrm{Hom}(T(j)[2j], T(j)[2j]) = \mathbb{Z}/2$ we get an exact sequence:

$$Hom(M, T(j)[2j]) \xrightarrow{\varphi} \mathbb{Z}/2 \xrightarrow{\psi} Hom(N, T(j)[2j+1]), \qquad (7)$$

where ψ sends $1 \in \mathbb{Z}/2$ to the composition $N \to T \xrightarrow{u_{2j+1}(q)} T(j)[2j+1]$, and

$$\mathrm{Hom}(M, T(j)[2j]) = \mathrm{CH}^j(G_d(q))/2$$

(since the restriction of M to $\mathrm{DM}^-_{eff}(k;\mathbb{Z}/2)$ is $M(G_d(q))$, and the restriction functor is a full embedding by [19, Lemma 6.7]).

From (6) we have the following commutative diagram:

$$\begin{array}{ccccc} X_q & \longleftarrow & \operatorname{Spec} k(X_q) & \longrightarrow & \bullet \\ \downarrow & & & & \downarrow \\ X_q \times EO(n) & & \longrightarrow & & EO(n) \\ \downarrow & & & & \downarrow \\ \mathcal{X}_{X_q} & & \longrightarrow & & BO(n). \end{array}$$

Let $\overline{C}_{2l+1}$ be the restriction of C_{2l+1} to the category $\mathrm{DM}^{-}_{eff}(\operatorname{Spec} k(X_q); \mathbb{Z}/2)$. It follows from our diagram and Lemma 3.2.21 that the natural lifting $M_{k(X_q)} \to T(l)[2l]$ of the projection to $\overline{C}_{2l+1}$ is given by $z_l \in \mathrm{CH}^l(G_d(q)|_{k(X_q)})/2$. In particular, the map $\varphi : Hom(M, T(j)[2j]) \to \mathbb{Z}/2$ will be surjective if and only if in $\mathrm{CH}^j(G_d(q))/2$ there is an element whose restriction to $\mathrm{CH}^*(G_d(q)|_{k(X_q)})/2 = \Lambda_{\mathbb{Z}/2}(z_0, \ldots, z_{d-1})$ (additive isomorphism - see [12, Proposition 2.4]) has a non-zero z_j-coordinate. By [12, Main Theorem 5.8] this is equivalent to: z_j is defined over k, or in other words, $j \in J(q)$. It follows from (7) that this condition is equivalent to the fact that the composition $N \to \mathcal{X}_{X_q} \stackrel{u_{2j+1}(q)}{\longrightarrow} \mathcal{X}_{X_q}(j)[2j+1]$ is zero. $\square$

We immediately obtain:

3.2.22 Corollary. *Let q be an n-dimensional quadratic form, and $p = q$, for even n, and $p = q \perp \langle \det_{\pm}(q) \rangle$, for odd n. Then*

$$u_{2j+1}(p) \in (u_{2l+1}(p) \,|\, 0 \leqslant l < j) \cdot \mathrm{H}^{*,*'}_{\mathcal{M}}(\mathcal{X}_{X_p}, \mathbb{Z}/2) \quad \Rightarrow \quad j \in J(q).$$

Proof. Since, for odd n, $J(q) = J(p) \backslash \{0\}$ - see [12, Definition 5.11], we can assume that $n = 2d$ is even, and $p = q$. Our result follows from Proposition 3.2.20 taking into account that the projection $N_{j-1} \to \mathcal{X}_{X_q}$ factors through

$$C_{2l+1} = \mathrm{Cone}[-1] \left(\mathcal{X}_{X_q} \stackrel{u_{2l+1}(q)}{\longrightarrow} \mathcal{X}_{X_q}(l)[2l+1] \right),$$

for each $0 \leqslant l < j$. $\square$

3.2.23 Question. *Are the following conditions equivalent?*

(1) $j \in J(q)$;

(2) $u_{2j+1}(p) = f(u_1(p), \ldots, u_{2j}(p))$, *for some* $f \in H[u_1, \ldots, u_{2j}]$;

(3) $u_{2j+1}(p) \in (u_{2i+1}(p) \,|\, 0 \leqslant i < j) \cdot \mathrm{H}^{*,*'}_{\mathcal{M}}(\mathcal{X}_{X_p}, \mathbb{Z}/2)$.

3.2.24 Remark. *Note, that the condition $j \in J(q)$ is not equivalent to $u_{2j+1}(p) = 0$, even when $q \in I^2$. Consider $q = \langle\langle a, b \rangle\rangle \cdot \langle 1, c, d \rangle$, where a, b, c, d are "generic". Then it follows from the computations of the Example 3.2.33 that $u_{11}(q) = \mu^3_{\{a,b\}} \cdot \{c, d\} \neq 0 \in \mathrm{H}^{*,*'}_{\mathcal{M}}(\mathcal{X}_{X_q}, \mathbb{Z}/2)$ (by Proposition 3.2.25, under the identification of the 6-th diagonal in $\mathrm{H}^{*,*'}_{\mathcal{M}}(\mathcal{X}_{\{a,b\}}, \mathbb{Z}/2)$ with the ideal in $K^M_*(k)/2$, this element corresponds to $\{a, b, c, d\} \neq 0$). At the same time, $J(q) = \{0, \ldots, 5\} \backslash 1$ contains 5. But $u_{11}(q) = u_3(q) \cdot u_8(q)$.*

We can compute the map α_q^* completely in the case of a Pfister form due to the fact that it is the rare case where the motivic cohomology of $\mathcal{X}_{X_q}$ is known. The following computations were performed in the original version of [8], and later by N.Yagita in [20, Theorem 5.8].

3.2.25 Theorem. *Let $\alpha = \{a_1, \ldots, a_n\} \in K_*^M(k)/2$, and $\mathcal{X}_\alpha = \mathcal{X}_{Q_\alpha}$, where Q_α is a Pfister quadric corresponding to α. Then the $\leqslant 0$ diagonal part of $\mathrm{H}_{\mathcal{M}}^{*,*'}(\mathcal{X}_\alpha, \mathbb{Z}/2)$ is identified with $\mathrm{H}_{\mathcal{M}}^{*,*'}(\mathrm{Spec}(k), \mathbb{Z}/2)$ by the restriction $\mathcal{X}_\alpha \to \mathrm{Spec}(k)$. The > 0 diagonal part of $\mathrm{H}_{\mathcal{M}}^{*,*'}(\mathcal{X}_\alpha, \mathbb{Z}/2)$ as a $K_*^M(k)/2$-module is isomorphic to*

$$\mathbb{Z}/2[\mu] \otimes \Lambda(Q_0, \ldots, Q_{n-2})(\gamma) \otimes L,$$

where Λ is the external algebra (over $\mathbb{Z}/2$), Q_i is the i-th Milnor operation (of degree $(2^i - 1)[2^{i+1} - 1]$), $\gamma \in \mathrm{H}_{\mathcal{M}}^{n,n-1}(\mathcal{X}_\alpha, \mathbb{Z}/2)$ is the unique element such that $\tau \cdot \gamma = \alpha$, $\mu = Q_{n-2} \circ \ldots \circ Q_0(\gamma) \in \mathrm{H}_{\mathcal{M}}^{2^n-1,2^{n-1}-1}(\mathcal{X}_\alpha, \mathbb{Z}/2)$, and multiplication by τ identifies the 1-st diagonal $\gamma \otimes L$ with $\alpha \cdot K_*^M(k)/2 = \mathrm{Ker}(K_*^M(k)/2 \to K_*^M(k(Q_\alpha))/2)$. Moreover, for any $Q_I \in \Lambda(Q_0, \ldots, Q_{n-2})$, $Q_{n-1}(Q_I(\gamma)) = Q_I(\gamma) \cdot \mu$.

Now it is not difficult to compute the subtle Stiefel–Whitney classes.

3.2.26 Theorem. *Let $\alpha = \{a_1, \ldots, a_n\} \in K_n^M(k)/2$ be a non-zero pure symbol, and $q_\alpha = \langle\langle a_1, \ldots, a_n \rangle\rangle$ be the respective Pfister form. Then*

$$u_i(q_\alpha) = \begin{cases} Q_{n-2} \circ \ldots \circ \widehat{Q_{r-1}} \circ \ldots \circ Q_0(\gamma_\alpha), & \text{if } i = 2^n - 2^r,\ 0 \leqslant r \leqslant n-1; \\ 0, & \text{otherwise} \end{cases}$$

Proof. Induction on n. (base) For $n = 1$, we know that $u_1(q_{\{a\}}) = \gamma_{\{a\}}$ and $u_2(q_{\{a\}}) = 0$.

(step) Consider $\beta = \{a_1, \ldots, a_{n-1}\}$, so that $\alpha = \beta \cdot \{a_n\}$. Then we have the canonical (unique) map $\mathcal{X}_\beta \xrightarrow{f} \mathcal{X}_\alpha$, and it follows from Theorem 3.2.25 that $f^*(\gamma_\alpha) = \gamma_\beta \cdot \{a_n\}$. We have: $q_\alpha = q_\beta \perp -a_n \cdot q_\beta$. We have the respective embedding $O(2^{n-1}) \times O(2^{n-1}) \overset{j}{\hookrightarrow} O(2^n)$, and by Proposition 3.1.10 and (5), we get:

$$f^*(u_i(q_\alpha)) = \sum_{j=0}^{i} u_j(q_\beta) \cdot u_{i-j}(-a_n \cdot q_\beta) \cdot \tau^{[i/2]-[j/2]-[i-j/2]}.$$

By Proposition 3.2.3 and inductive assumption, $u_l(-a_n \cdot q_\beta) = u_l(q_\beta)$, for $l < 2^{n-1}$, while $u_{2^{n-1}}(-a_n \cdot q_\beta) = \{a_n\} \cdot u_{2^{n-1}-1}(q_\beta)$. This implies that $f^*(u_i(q_\alpha)) = 0$, if $i \neq 2^n - 2^r$, for $0 \leqslant r \leqslant n-1$, and

$$\begin{aligned} &f^*(u_{2^n-2^r}(q_\alpha)) = u_{2^{n-1}-2^r}(q_\beta) \cdot u_{2^{n-1}-1}(q_\beta) \cdot \{a_n\} = \\ &= Q_{n-3} \circ \ldots \circ \widehat{Q_{r-1}} \circ \ldots \circ Q_0(\gamma_\beta) \cdot \mu_\beta \cdot \{a_n\} = f^*(Q_{n-2} \circ \ldots \circ \widehat{Q_{r-1}} \circ \ldots \circ Q_0(\gamma_\alpha)), \end{aligned}$$

for $0 \leqslant r \leqslant n-2$.

For $r = n-1$, $f^*(u_{2^{n-1}}(q_\alpha)) = u_{2^{n-1}-1}(q_\beta) \cdot \{a_n\} = Q_{n-3} \circ \ldots \circ Q_0(\gamma_\beta) \cdot \{a_n\} = f^*(Q_{n-3} \circ \ldots \circ Q_0(\gamma_\alpha))$. Since f^* is injective on all the diagonals up to 2^{n-1} (follows from Theorem 3.2.25), we obtain: $u_i(q_\alpha) = 0$, for $i \neq 2^n - 2^r$, for $0 \leqslant r \leqslant n-1$ and $u_{2^n-2^r}(q_\alpha) = Q_{n-2} \circ \ldots \circ \widehat{Q_{r-1}} \circ \ldots \circ Q_0(\gamma_\alpha)$ (recall that u_i lives on the diagonal with number $[i+1/2]$). □

Now we can use *subtle Stiefel–Whitney classes* to describe the powers of the fundamental ideal I^n in $W(k)$.

3.2.27 Theorem. *The following conditions are equivalent:*

1) $q \in I^n$;

2) $u_i(q) = 0$, *for* $1 \leqslant i \leqslant 2^{n-1} - 1$;

3) $u_i(q) = 0$, *for* $i = 2^r$, $0 \leqslant r \leqslant n-2$.

Proof. The implication $(2 \to 3)$ is evident. $(3 \to 1)$: Suppose, $q \notin I^n$. Then by [8, Theorem 4.3] (the J-filtration Conjecture), there exists a field extension L/k such that $(q_L)_{an}$ is an r-fold Pfister form, with $r < n$. By Theorem 3.2.26, $u_{2^{r-1}}(q)|_L \neq 0$ - a contradiction.

$(1 \to 2)$: We will show that the respective cohomology group is zero.

3.2.28 Proposition. *Let* $q_1, \ldots, q_s$ *be forms from* I^n. *Then*

$$\mathrm{H}^{b,a}_{\mathcal{M}}(\mathcal{X}_{X_{q_1}} \times \ldots \times \mathcal{X}_{X_{q_s}}, \mathbb{Z}/2) = 0 \quad \text{for :}$$

1) $\frac{b}{a} > 2 + \frac{1}{2^{n-1}-1}$; *and for*

2) $\frac{b+l-n+1}{a+l-n+1} > 2 + \frac{1}{2^l - 1}$, *where* $n-1 \geqslant l = [log_2(b-a)]$ *and* $b > a$.

Proof. Let $\mathcal{R}$ denote the set of pairs (a,b) satisfying the conditions 1), or 2) of Proposition 3.2.28 union with $\{(a,b) | \, b \leqslant a\}$. Denote as $\partial\mathcal{R}$ the set of such (a,b) that $(a,b) \notin \mathcal{R}$, but $(a, b+1) \in \mathcal{R}$.

We can safely assume that $n > 1$. Use increasing induction on a. For $a < 0$, the groups in question are, clearly, zero.

3.2.29 Lemma. *Let* $p_1, \ldots, p_s \in I^n$, *and* q_α *be an* n*-fold Pfister form. Suppose* $(a,b) \in \mathcal{R}$, *and Proposition 3.2.28 is valid for all* $(a', b') \in \mathcal{R}$ *with* $b' > a' < a$. *Then the natural map*

$$\mathrm{H}^{b,a}_{\mathcal{M}}(\mathcal{X}_{X_{p_1}} \times \ldots \times \mathcal{X}_{X_{p_s}}, \mathbb{Z}/2) \to \mathrm{H}^{b,a}_{\mathcal{M}}(\mathcal{X}_{X_{p_1}} \times \ldots \times \mathcal{X}_{X_{p_s}} \times \mathcal{X}_\alpha, \mathbb{Z}/2)$$

is an isomorphism for the given (a,b).

Proof. In $\mathrm{DM}^{-}_{eff}(k; \mathbb{Z}/2)$ we have a distinguished triangle: $\widetilde{\mathcal{X}}_\alpha \to \mathcal{X}_\alpha \to \mathbb{Z}/2 \to \widetilde{\mathcal{X}}_\alpha[1]$.

3.2.30 Lemma. *Let* Y *be any smooth variety, and* q_α *be an* n*-fold Pfister form. Then* $\mathrm{H}^{b,a}_{\mathcal{M}}(Y \times \widetilde{\mathcal{X}}_\alpha, \mathbb{Z}/2) = 0$, *for* $(a,b) \in \mathcal{R}$, *and the map* $\mathrm{H}^{b,a}_{\mathcal{M}}(Y \times \mathcal{X}_\alpha, \mathbb{Z}/2) \twoheadrightarrow \mathrm{H}^{b,a}_{\mathcal{M}}(Y \times \widetilde{\mathcal{X}}_\alpha, \mathbb{Z}/2)$ *is surjective for* $(a,b) \in \partial\mathcal{R}$. *In particular, the map*

$$\mathrm{H}^{b,a}_{\mathcal{M}}(Y, \mathbb{Z}/2) \to \mathrm{H}^{b,a}_{\mathcal{M}}(Y \times \mathcal{X}_\alpha, \mathbb{Z}/2)$$

is an isomorphism, for $(a,b) \in \mathcal{R}$.

Proof. For any field extension L/k, $\mathrm{H}^{b,a}_{\mathcal{M}}(\widetilde{\mathcal{X}}_\alpha|_L, \mathbb{Z}/2) = 0$ for $(a,b) \in \mathcal{R}$, by Theorem 3.2.25 (the $b \leqslant a$ case follows already from the Beilinson-Lichtenbaum Conjecture proven by V.Voevodsky [18] using A.Suslin-V.Voevodsky [10]). Since $\mathcal{R}$ is stable under $(a,b) \mapsto (a-m, b-2m)$, for any $m \geqslant 0$, it follows from the localization sequence for motivic cohomology that $\mathrm{H}^{b,a}_{\mathcal{M}}(Y \times \widetilde{\mathcal{X}}_\alpha, \mathbb{Z}/2) = 0$, for any smooth variety Y, for $(a,b) \in \mathcal{R}$. Theorem 3.2.25

implies also that, for any field extension L/k, the map $\mathrm{H}^{b,a}_{\mathcal{M}}(\widetilde{\mathcal{X}}_\alpha, \mathbb{Z}/2) \twoheadrightarrow \mathrm{H}^{b,a}_{\mathcal{M}}(\widetilde{\mathcal{X}}_\alpha|_L, \mathbb{Z}/2)$ is surjective, for $(a,b) \in \partial\mathcal{R}$. Since the map $\mathrm{H}^{b,a}_{\mathcal{M}}(\mathcal{X}_\alpha, \mathbb{Z}/2) \twoheadrightarrow \mathrm{H}^{b,a}_{\mathcal{M}}(\widetilde{\mathcal{X}}_\alpha, \mathbb{Z}/2)$ is surjective, for all (a,b), it follows from the localization sequence (and the fact that $\mathcal{R}$ is stable under: $(a,b) \mapsto (a-2m, b-2m)$) again that the map $\mathrm{H}^{b,a}_{\mathcal{M}}(Y \times \mathcal{X}_\alpha, \mathbb{Z}/2) \twoheadrightarrow \mathrm{H}^{b,a}_{\mathcal{M}}(Y \times \widetilde{\mathcal{X}}_\alpha, \mathbb{Z}/2)$ is surjective for $(a,b) \in \partial\mathcal{R}$. □

Let q be one of our forms $p_1, \ldots, p_s$. We know that $\mathcal{X}_{X_q} = \mathcal{X}_{G_d(q)}$, where $G_d(q)$ is the variety of totally isotropic subspaces in V_q of maximal dimension. By Theorem 3.2.13, $M(G_d(q))$ is an extension of $M(\mathcal{X}_{G_d(q)})(j)[2j]$, for some j's. We have the following description of I^n in terms of the J-invariant.

3.2.31 Proposition. *The following conditions are equivalent:*

1) $q \in I^n$;

2) $\{0, \ldots, 2^{n-1}-2\} \subset J(q)$;

3) $2^r - 1 \in J(q)$, *for* $0 \leqslant r \leqslant n-2$.

Proof. $(2 \to 3)$ is evident. $(1 \to 2)$ follows from [13, Corollary 3.5]. $(3 \to 1)$ follows from the J-filtration Conjecture ([8, Theorem 4.3]) and [14, Example 3.2]. □

It follows from Proposition 3.2.19 and Proposition 3.2.31 that $u_{2j+1}(q) = 0$, for $0 \leqslant j < 2^{n-1}-1$. This implies that we have a direct summand N of $M(G_d(q))$ such that $N_{\overline{k}} = \otimes_j(\mathbb{Z}/2 \oplus \mathbb{Z}/2(j)[2j])$, where $j \geqslant 2^{n-1}-1$.

Then we have a distinguished triangle: $P \to N \to M(\mathcal{X}_{F_q}) \to P[1]$ in $\mathrm{DM}^-_{eff}(k;\mathbb{Z}/2)$, where P is an extension of $M(\mathcal{X}_{F_q})(j)[2j]$, for $j \geqslant 2^{n-1}-1$. Thus, if we know that $\mathrm{H}^{b,a}_{\mathcal{M}}(F_{p_1} \times \ldots \times F_{p_s} \times \widetilde{\mathcal{X}}_\alpha, \mathbb{Z}/2) = 0$, and $\mathrm{H}^{b-2j-1,a-j}_{\mathcal{M}}(\mathcal{X}_{F_{p_1}} \times \ldots \times \mathcal{X}_{F_{p_s}} \times \widetilde{\mathcal{X}}_\alpha, \mathbb{Z}/2) = 0$, for all $j \geqslant 2^{n-1}-1$, then $\mathrm{H}^{b,a}_{\mathcal{M}}(\mathcal{X}_{F_{p_1}} \times \ldots \times \mathcal{X}_{F_{p_s}} \times \widetilde{\mathcal{X}}_\alpha, \mathbb{Z}/2) = 0$. Hence, it follows by the induction on the degree from Lemma 3.2.30 and the fact that $\mathcal{R}$ is stable under $(a,b) \mapsto (a-j, b-2j-1)$, for $j \geqslant 2^{n-1}-1$, that $\mathrm{H}^{b,a}_{\mathcal{M}}(\mathcal{X}_{F_{p_1}} \times \ldots \times \mathcal{X}_{F_{p_s}} \times \widetilde{\mathcal{X}}_\alpha, \mathbb{Z}/2) = 0$, for $(a,b) \in \mathcal{R}$. By the inductive assumption, $\mathrm{H}^{b,a}_{\mathcal{M}}(P, \mathbb{Z}/2) = 0$, for our pair $(a,b) \in \mathcal{R}$. Thus, the map $\mathrm{H}^{b,a}_{\mathcal{M}}(\mathcal{X}_{F_{p_1}} \times \ldots \times \mathcal{X}_{F_{p_s}}, \mathbb{Z}/2) \hookrightarrow \mathrm{H}^{b,a}_{\mathcal{M}}(F_{p_1} \times \ldots \times F_{p_s}, \mathbb{Z}/2)$ is injective, and so, by Lemma 3.2.30, the map $\mathrm{H}^{b,a}_{\mathcal{M}}(\mathcal{X}_{F_{p_1}} \times \ldots \times \mathcal{X}_{F_{p_s}}, \mathbb{Z}/2) \to \mathrm{H}^{b,a}_{\mathcal{M}}(\mathcal{X}_{F_{p_1}} \times \ldots \times \mathcal{X}_{F_{p_s}} \times \mathcal{X}_\alpha, \mathbb{Z}/2)$ is an isomorphism. This proves Lemma 3.2.29. □

We can present each of the forms $p_1, \ldots, p_s$ as sum of n-fold Pfister forms. Let $\pi_{\alpha_1}, \ldots, \pi_{\alpha_t}$ be all the Pfister forms involved. Then it follows from Lemma 3.2.29 that (for the given pair (a,b)) the map:

$$\mathrm{H}^{b,a}_{\mathcal{M}}(\mathcal{X}_{X_{p_1}} \times \ldots \times \mathcal{X}_{X_{p_s}}, \mathbb{Z}/2) \to \mathrm{H}^{b,a}_{\mathcal{M}}(\mathcal{X}_{X_{p_1}} \times \ldots \times \mathcal{X}_{X_{p_s}} \times \mathcal{X}_{\alpha_1} \times \ldots \times \mathcal{X}_{\alpha_t}, \mathbb{Z}/2)$$

is an isomorphism. But $\mathcal{X}_{X_{p_1}} \times \ldots \times \mathcal{X}_{X_{p_s}} \times \mathcal{X}_{\alpha_1} \times \ldots \times \mathcal{X}_{\alpha_t} = \mathcal{X}_{\alpha_1} \times \ldots \times \mathcal{X}_{\alpha_t}$. It remains to apply Lemma 3.2.29 again reducing the set $\alpha_1, \ldots, \alpha_t$ to an empty one. Proposition 3.2.28 is proven. □

It follows from Proposition 3.2.28 that $u_i(q)$, for $i \leqslant 2^{n-1}-1$, lives in a zero group, which proves the implication $(1 \to 2)$. □

The above results imply that *subtle Stiefel–Whitney classes* do distinguish the triviality of the torsor.

3.2.32 Corollary. *The following conditions are equivalent:*

1) $q \cong q_n$;

2) $u_i(q) = 0$, *for all* i;

3) $u_{2^r}(q) = 0$, *for all* r;

4) $u_{2^r-1}(q) = 0$, *for all* r.

Proof. The implications $(1 \to 2)$, $(2 \to 3)$, and $(2 \to 4)$ are evident. On the other hand, by adding $\langle 1 \rangle$ to our form, if needed, we can assume that it is even-dimensional. If $q \not\cong q_n$, then using the tower of M.Knebusch (see [4]) we can find a field extension L/k such that $(q_L)_{an}$ is a Pfister form (using the result of A.Pfister: any even-dimensional form of height one is proportional to a Pfister form - see [9]). Then, by Theorem 3.2.26, $u_{2^r}(q_L) \neq 0$, and $u_{2^{r+1}-1}(q_L) \neq 0$, for some r. This proves $(3 \to 1)$ and $(4 \to 1)$. □

Although, the *subtle Stiefel–Whitney classes* distinguish the triviality of the torsor, these do not, unfortunately, distinguish torsors among themselves.

3.2.33 Example. *Let* $q_\alpha = \langle\langle a_1, \ldots, a_d \rangle\rangle$ *be a* d*-fold anisotropic Pfister form, and* p *be an odd-dimensional form. Consider* $q = q_\alpha \cdot p$*. Then* q *is even-dimensional, and, for any field extension* L/k,

$$(X_q)|_L \text{ is trivial } \Leftrightarrow \alpha|_L = 0 \in K^M_d(k)/2.$$

Thus, $\mathcal{X}_{X_q} = \mathcal{X}_\alpha$*. Moreover, if* $p = p_1 \perp p_2$*, and* $q_i = q_\alpha \cdot p_i$*, then* $\mathcal{X}_{X_{q_1}} \times \mathcal{X}_{X_{q_2}} = \mathcal{X}_\alpha$*. If* $p = \langle b_1, \ldots, b_m \rangle$*, then, by the proof of Theorem 3.2.26,* $\sum_i u_i(b_l \cdot q_\alpha) = \sum_{s=0}^{n-2} Q_{[0,\ldots,n-2]\backslash s}(\gamma_\alpha) + \mu_\alpha \cdot (1 + \{b_l\})$*. It is also known (is contained in the original version of [8]) that (provided* -1 *is a square in* k*):*

$$Q_I(\gamma_\alpha) \cdot Q_J(\gamma_\alpha) = \begin{cases} \mu_\alpha \cdot Q_{I\cap J}(\gamma_\alpha), & \text{if } I \cup J = \{0, \ldots, n-2\}; \\ 0 \text{ otherwise.} \end{cases}$$

By Theorem 3.2.25 each positive diagonal of $\mathrm{H}^{*,*'}_{\mathcal{M}}(\mathcal{X}_\alpha, \mathbb{Z}/2)$ *can be identified with* $\alpha \cdot K^M_*(k)/2 \subset K^M_*(k)/2$*. It follows from Proposition 3.1.10, that under this identification, subtle Stiefel–Whitney classes of* q *are identified with (some multiples of)* $\alpha \cdot \omega_j(p)$*, for some* j*. Notice, that it is more "informative" than* $\omega_l(q)$*! At the same time, if* $p - p' \in I^3$*, then subtle Stiefel–Whitney classes of* $q = q_\alpha \cdot p$ *and* $q' = q_\alpha \cdot p'$ *will be the same.*

But subtle Stiefel–Whitney classes carry a lot of information about q. In particular, they contain *Arason invariant* (see [1]) and all higher invariants $e_r : I^r/I^{r+1} \to K^M_r(k)/2$. Indeed, since, for any $q \in I^n$, $u_{2j+1}(q) = 0$, for all $j < 2^{n-1} - 1$, it follows from Theorem 3.2.13 and the fact that $\mathrm{H}^{b,a}_{\mathcal{M}}(G_d(q), \mathbb{Z}/2) = 0$, for all $b > 2a$, that $\mathrm{H}^{2^n-1, 2^{n-1}-1}_{\mathcal{M}}(\mathcal{X}_{X_q}, \mathbb{Z}/2) = \mathbb{Z}/2 \cdot u_{2^n-1}(q)$. On the other hand, by [8, Proposition 2.3], we have an exact sequence:

$$0 \to \mathrm{H}^{n,n-1}_{\mathcal{M}}(\mathcal{X}_{X_q}, \mathbb{Z}/2) \xrightarrow{\cdot\tau} K^M_n(k)/2 \to K^M_n(k(X_q))/2.$$

Clearly, $e_n(q) \in \mathrm{Ker}(K^M_n(k)/2 \to K^M_n(k(X_q))/2) = \mathrm{Ker}(K^M_n(k)/2 \to K^M_n(k(F_d(q)))/2)$, and it follows from [8, Theorems 3.2 and 4.2] that $\mathrm{H}^{n,n-1}_{\mathcal{M}}(\mathcal{X}_{X_q}, \mathbb{Z}/2) = \mathbb{Z}/2 \cdot \gamma$, where $\tau \cdot \gamma = e_n(q)$. By the J-filtration Conjecture ([8, Theorem 4.3]), there exists such field extension L/k that $(q_L)_{an}$ is an n-fold Pfister form q_α (if $q \notin I^{n+1}$). And we know from Theorem 3.2.26 that $u_{2^n-1}(q_\alpha) = Q_{n-2} \circ \ldots \circ Q_0(\gamma_\alpha)$. Hence, the same is true for q. Thus, we obtain:

3.2.34 Theorem. *Let $q \in I^n$. Then the map*

$$Q_{n-2} \circ \ldots \circ Q_0 : \mathrm{H}_{\mathcal{M}}^{n,n-1}(\mathcal{X}_{X_q}, \mathbb{Z}/2) \xrightarrow{\cong} \mathrm{H}_{\mathcal{M}}^{2^n-1,2^{n-1}-1}(\mathcal{X}_{X_q}, \mathbb{Z}/2)$$

is an isomorphism, and

$$(Q_{n-2} \circ \ldots \circ Q_0)^{-1}(u_{2^n-1}(q)) \cdot \tau = e_n(q).$$

3.2.35 Remark. *One can also show that $e_n(q) = (Q_{n-3} \circ \ldots \circ Q_0)^{-1}(u_{2^{n-1}}(q)) \cdot \tau$, but it requires a bit more of work.*

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